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Complete solutions of the simultaneous Pell's equations $(a^2 + 2)x^2 - y^2 = 2$ and $x^2 - bz^2 = 1$

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Abstract: In this paper, we consider the simultaneous Pell equations $(a^2+2)x^2-y^2 = 2$ and $x^2-bz^2 = 1$ where a is a positive integer and $b > 1$ is squarefree and has at most three prime divisors. We obtain the necessary and sufficient conditions that the above simultaneous Pell equations have positive integer solutions by using only the elementary methods of factorization, congruence, the quadratic residue and fundamental properties of Lucas sequence and the associated Lucas sequence. Moreover, we prove that these simultaneous Pell equations have at most one solution in positive integers. When a solution exists, assuming the positive solutions of the Pell equation $(a^2 + 2)x^2 - y^2 = 2$ are $x = x_m$ and $y = y_m$ with $m \geq 1$ odd, then the only solution of the system is given by $m = 3$ or $m = 5$ or $m = 7$ or $m = 9$.

Keywords: diophantine equations; simultaneous Pell equations; minimal solutions; Lehmer sequences
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1. Introduction

The study of positive integer solutions of Pell equations has a wide range of applications in finding integer points on elliptic curves, which is an important part of many scientific problems, as shown in the reference [1–3]. In [4], A. Thue showed that the system Diophantine equations

$$a_1x^2 - b_1y^2 = c_1, \quad a_2y^2 - b_2z^2 = c_2, \quad a_1b_2 \neq a_2b_1 \tag{1.1}$$

have at most finitely many solutions. Let $sqf(b)$ denote the square-free part of positive integer b. In [5], Cipu proved the following

Theorem CIPU. Let a and b be integers greater than 1, with b not a perfect square.

a) Assume b is odd and its square-free part has at most two prime divisors. Then the system

$$x^2 - (a^2 - 1)y^2 = y^2 - bz^2 = 1 \tag{1.2}$$

is solvable in positive integers if and only if b divides $4a^2-1$ and the quotient is a perfect square. When it exists, this solution is

$$(x, y, z) = (2a^2 - 1, 2a, \sqrt{(4a^2 - 1)/b}).$$

b) Assume $sqf(b) = 2p$ with p either prime or equal to 1. Then the system (1.2) is solvable in positive integers if and only if $(2a^2 - 1)/p$ is a perfect square and b divides $8a^2(2a^2 - 1)$ and the quotient is a perfect square. When it exists, this solution is

$$(x, y, z) = (4a^3 - 3a, 4a^2 - 1, \sqrt{8a^2(2a^2 - 1)/b}).$$

Bennett [6] showed that the system Pell equations

$$x^2 - ay^2 = y^2 - bz^2 = 1 \tag{1.3}$$

has at most three solutions, where a and b are distinct positive integers. Yuan [7] conjectured that for any positive integers a and b, (1.1) has at most one solution and he proved that the conjecture holds for $a = 4m(m + 1)$. Walsh [8] proved that the system Pell equations

$$x^2 - (m^2 - c)y^2 = c, \quad y^2 - bz^2 = 1, \quad c \in \{\pm1, \pm2, \pm4\} \tag{1.4}$$

has at most one solution, where m and $b > 1$ are positive integers with b squarefree, and $m^2 - c$ is a positive nonsquare integer. In [9], the authors considered the simultaneous Pell equations

$$(a^2 + 1)x^2 - y^2 = x^2 - bz^2 = 1 \tag{1.5}$$

where $a > 0$ is an integer and $b > 1$ is squarefree and has at most three prime divisors. Assuming the positive integer solutions of the Pell equation $(a^2 + 1)x^2 - y^2 = 1$ are $x = x_m$ and $y = y_m$ with $m \geq 1$ an odd integer, they proved that the only possible solution of system (1.5) is given by $m = 3$ or $m = 5$ or $m = 7$ or $m = 9$.

In this paper, we consider the simultaneous Pell equations

$$(a^2 + 2)x^2 - y^2 = 2, \quad x^2 - bz^2 = 1 \tag{1.6}$$

where a is a positive integer and $b > 1$ is squarefree and has at most three prime divisors. By the results of Walsh [8], we know that (1.5) and (1.6) has at most one solution for any positive integer a and $b > 1$ squarefree. Assuming the positive integer solutions of the Pell equation $(a^2 + 2)x^2 - y^2 = 2$ are $x = x_m$ and $y = y_m$ with $m \geq 1$ an odd integer, we prove that system (1.6) has solutions only when $m = 3$ or $m = 5$ or $m = 7$ or $m = 9$. We prove the following results.

Theorem 1.1. *Let p be a prime and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2 + 2)x^2 - y^2 = 2, \quad x^2 - pz^2 = 1 \tag{1.7}$$

have positive integer solutions if and only if $a^2 + 1$ is a product of p and a square integer. When a solution exists there is exactly one solution. The only one solution is given by

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{p}}\right) = \left(2a^2 + 1, 2a^3 + 3a, 2a\sqrt{\frac{a^2 + 1}{p}}\right).$$

Theorem 1.2. *Let p and q be two distinct primes and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2+2)x^2-y^2=2, \quad x^2-pqz^2=1 \tag{1.8}$$

have at most one positive integer solution. Moreover, the solution exists if and only if one of the following two conditions holds:

α) a^2+1 *is a product of pq and a square integer.*

β) $a \equiv 2 \pmod 4$, $a^2+1=pb^2$, $2a^2+1=c^2$, and $2a^2+3=qd^2$.

When it exists, the solution is given by formula

$$(x,y,z)=\left(x_3,y_3,\sqrt{\frac{x_3^2-1}{pq}}\right)=\left(2a^2+1,2a^3+3a,2a\sqrt{\frac{a^2+1}{pq}}\right) \quad \textit{in} \quad \textit{case} \quad \alpha).$$

$$(x,y,z)=\left(x_5,y_5,\sqrt{\frac{x_5^2-1}{pq}}\right) \quad \textit{in} \quad \textit{case} \quad \beta).$$

We shall denote by □ an unspecified perfect square.

Theorem 1.3. *Let p, q and r be distinct primes and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2+2)x^2-y^2=2, \quad x^2-pqrz^2=1 \tag{1.9}$$

have at most one positive integer solution. Moreover, the solution exists if and only if one of the following conditions holds:

α) $a^2+1=pqr\square$.

β) $a^2+1=p\square$, $2a^2+1=q\square$, and $2a^2+3=r\square$ or

$$a^2+1=p\square,\ 2a^2+1=\square, \text{ and } 2a^2+3=qr\square \text{ or}$$

$$a\equiv 2 \pmod 4,\ a^2+1=pq\square,\ 2a^2+1=\square, \text{ and } 2a^2+3=r\square.$$

γ) $a^2+1=2\square$, $2a^4+4a^2+1=p\square$, $2a^2+1=q\square$, and $2a^2+3=r\square$ or

$$a\equiv 2 \pmod 4,\ p=2,\ a^2+1=q\square,\ 2a^4+4a^2+1=\square,\ 2a^2+1=\square, \text{ and } 2a^2+3=r\square.$$

δ) $a^2+1=2\square$, $2a^4+4a^2+1=p\square$, $4a^4+6a^2+1=q\square$, and $4a^4+10a^2+5=r\square$.

When it exists, the solution is given by formula

$$(x,y,z)=\left(x_3,y_3,\sqrt{\frac{x_3^2-1}{pqr}}\right)=\left(2a^2+1,2a^3+3a,2a\sqrt{\frac{a^2+1}{pqr}}\right) \quad \textit{in} \quad \textit{case} \quad \alpha).$$

$$(x,y,z)=\left(x_5,y_5,\sqrt{\frac{x_5^2-1}{pqr}}\right) \quad \textit{in} \quad \textit{case} \quad \beta).$$

$$(x,y,z)=\left(x_7,y_7,\sqrt{\frac{x_7^2-1}{pqr}}\right) \quad \textit{in} \quad \textit{case} \quad \gamma).$$

$$(x,y,z)=\left(x_9, y_9, \sqrt{\frac{x_9^2-1}{pqr}}\right) \quad in \quad case \quad \delta).$$

We organize this paper as follows. In Section 2, we present some basic definitions and some lemmas which are needed in the proofs of our main results. Consequently, in Sections 3–5, we give the proofs of Theorems 1.1 to 1.3, respectively. In Section 6, we give some examples of applications of Theorems 1.1–1.3.

2. Some tools and basic definitions and some lemmas

In the proof of our main result, Lehmer sequences and the associated Lehmer sequences play an essential role. So, we need to recall them. Let $P > 0$, Q be nonzero coprime integers, let $D = P - 4Q$ be called discriminant, and assume that $D > 0$. Consider the polynomial $x^2 - \sqrt{P}x + Q$, called characteristic polynomial, which has the roots

$$\alpha = \frac{\sqrt{P}+\sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{\sqrt{P}-\sqrt{D}}{2}.$$

For each $n \geq 0$, define the Lehmer sequence $U_n = U_n(P, Q)$ and the associated Lehmer sequence $V_n = V_n(P, Q)$ as follows:

$$U_n = \begin{cases} \frac{\alpha^n-\beta^n}{\alpha-\beta}, & \text{if } 2 \not| n, \\ \frac{\alpha^n-\beta^n}{\alpha^2-\beta^2}, & \text{if } 2|n, \end{cases}$$

and

$$V_n = \begin{cases} \frac{\alpha^n+\beta^n}{\alpha+\beta}, & \text{if } 2 \not| n, \\ \alpha^n+\beta^n, & \text{if } 2|n. \end{cases}$$

Consider the Pell equation

$$kx^2 - ly^2 = c, \quad c = 1, 2, \tag{2.1}$$

with $k > 1$ when $c = 1$. Let $\alpha = \frac{x_1\sqrt{k}+y_1\sqrt{l}}{\sqrt{c}}$, $\beta = \frac{x_1\sqrt{k}-y_1\sqrt{l}}{\sqrt{c}}$, where $x_1\sqrt{k} + y_1\sqrt{l}$ is the minimal positive integer solution of the Eq (2.1), then all positive integer solutions of this equation are given by

$$\frac{x_n\sqrt{k}+y_n\sqrt{l}}{\sqrt{c}} = \alpha^n$$

with $n \geq 1$ an odd integer. Moreover, α^2 is the fundamental solution of the equation

$$x^2 - kly^2 = 1, \tag{2.2}$$

all positive integer solutions of this equation are given by

$$X_n + Y_n\sqrt{kl} = \alpha^{2n}$$

with $n \geq 1$.

The next results are classical, so well known and frequently employed that it is very difficult to locate their first appearance in print.

Lemma 2.1. *Let $x_1\sqrt{k}+y_1\sqrt{l}$ be the minimal positive integer solution of the Eq (2.1). Then all positive integer solutions of the Eq (2.1) are given by*

$$x_n = x_1 V_n(\frac{4kx_1^2}{c}, 1) \quad and \quad y_n = y_1 U_n(\frac{4kx_1^2}{c}, 1)$$

with n an odd integer. All positive integer solutions of the Eq (2.2) are given by

$$X_n = \frac{V_{2n}(\frac{4kx_1^2}{c}, 1)}{2} \quad and \quad Y_n = x_1 y_1 U_{2n}(\frac{4kx_1^2}{c}, 1)$$

with $n \geq 1$.

The following identities are fairly well known and valid for the numbers $U_n = U_n(P, 1)$ and $V_n = V_n(P, 1)$:

$$\text{If} \quad d = \gcd(m, n), \quad \text{then} \quad \gcd(U_m, U_n) = U_d, \tag{2.3}$$

$$U_{2n} = U_n V_n. \tag{2.4}$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = \gcd(m, n)$.

$$\gcd(U_m, V_n) = \begin{cases} V_d, & \text{if} \quad a > b, \\ 1 \quad \text{or} \quad 2, & \text{if} \quad a \leq b. \end{cases} \tag{2.5}$$

If P is even, then V_n is always even and U_m is even iff m is even. When P is even and $a \leq b$, we get $\gcd(U_m, V_n) = 2$ if m is even and $\gcd(U_m, V_n) = 1$ if m is odd. Moreover, if n is odd, we have

$$U_n^2 - 1 = (\alpha + \beta)^2 U_{n-1} U_{n+1}, \quad V_n^2 - 1 = (\alpha - \beta)^2 U_{n-1} U_{n+1}. \tag{2.6}$$

We omit the proofs of the following lemmas, as they are based on straightforward induction. The details can be also seen in the references [10–13].

Lemma 2.2. *Let* $\alpha = \frac{x_1\sqrt{k}+y_1\sqrt{l}}{\sqrt{c}}, \beta = \frac{x_1\sqrt{k}-y_1\sqrt{l}}{\sqrt{c}}$, *then we have*

$$v_2(U_n(P, 1)) = \begin{cases} 0, & \text{if} \quad 2 \nmid n, \\ v_2(n) - 1, & \text{if} \quad 2 | n, \end{cases}$$

and

$$v_2(V_n(P, 1)) = \begin{cases} 0, & \text{if} \quad 2 \nmid n, \\ 1, & \text{if} \quad 2 | n. \end{cases}$$

Lemma 2.3. ([14]) *Let the minimal positive integer solution of the equation $Ax^2 - By^2 = 1$ be $\varepsilon = x_0\sqrt{A} + y_0\sqrt{B}$, where $A > 1$ and B are coprime positive integers with $d = AB$ not a square. Then the only possible solution of the equation $Ax^2 - By^4 = 1$ is given by $x\sqrt{A} + y^2\sqrt{B} = \varepsilon^l$ where $y_0 = lf^2$ for some odd squarefree integer l.*

Lemma 2.4. ([15]) *Let $A > 1$ and B be coprime positive integers with $d = AB$ not a square. The Diophantine equation*

$$AX^4 - BY^2 = 1 \tag{2.7}$$

has at most two positive integer solutions. Moreover, (2.7) is solvable if and only if x_0 is a square, where $\varepsilon = x_0\sqrt{A} + y_0\sqrt{B}$ is the minimal positive integer solution of the equation $AU^2 - BV^2 = 1$. And if $x^2\sqrt{A} + y\sqrt{B} = \varepsilon^k$, then $k = 1$ or $k = p \equiv 3 \pmod 4$ is a prime.

Lemma 2.5. ([16, 17]) *Let the fundamental solution of the equation* $v^2 - du^2 = 1$ *be* $a + b\sqrt{d}$. *Then the only possible solutions of the equation* $X^4 - dY^2 = 1$ *are given by* $X^2 = a$ *and* $X^2 = 2a^2 - 1$; *both solutions occur in the following cases:* $d = 1785, 7140, 28560$.

Lemma 2.6. ([18]) *Let* $D > 0$ *be a nonsquare integer. Define*

$$T_n + U_n\sqrt{D} = (T_1 + U_1\sqrt{D})^n,$$

where $T_1 + U_1\sqrt{D}$ *is the fundamental solution of the Pell equation*

$$X^2 - DY^2 = 1. \tag{2.8}$$

There are at most two positive integer solutions (X, Y) *to the equation*

$$X^2 - DY^4 = 1. \tag{2.9}$$

(1) If two solutions $Y_1 < Y_2$ *exist, then* $Y_1^2 = U_1$ *and* $Y_2^2 = U_2$, *except only if* $D = 1785$ *or* $D = 16 \cdot 1785$, *in which case* $Y_1^2 = U_1$ *and* $Y_2^2 = U_4$.

(2) If only one positive integer solution (X, Y) *to Eq (2.9) exists, then* $Y^2 = U_l$ *where* $U_1 = lv^2$ *for some squarefree integer l, and either* $l = 1$, $l = 2$ *or* $l = p$ *for some prime* $p \equiv 3 \pmod 4$.

Let (x_1, y_1) be the minimal positive integer solution to (2.1) with $c = 2$, and define

$$\alpha = \frac{x_1\sqrt{k} + y_1\sqrt{l}}{\sqrt{2}}.$$

Furthermore, for n odd, define

$$\alpha^n = \frac{x_n\sqrt{k} + y_n\sqrt{l}}{\sqrt{2}},$$

where x_n, y_n are positive integers.

Lemma 2.7. ([19]) *(1) If* y_1 *is not a square, then equation*

$$kx^2 - ly^4 = 2 \tag{2.10}$$

has no solutions.

(2) If y_1 *is a square and* y_3 *is not a square, then* (x_1, y_1) *is the only solution of (2.10).*

(3) If y_1 *and* y_3 *are both squares, then* (x_1, y_1) *and* (x_3, y_3) *are the only solutions of (2.10).*

Lemma 2.8. ([20]) *The Diophantine equation*

$$kx^4 - ly^2 = 2$$

has at most one solution in positive integers, and such a solution must arise from the minimal positive integer solution to (2.1) with $c = 2$.

Lemma 2.9. *The Diophantine equation*

$$x^2 - a^2(a^2 + 2)y^4 = 1 \tag{2.11}$$

has at most one positive integer solution other than $(x, y) = (a^2 + 1, 1)$, *which is*

$$(x, y^2) = (2(a^2 + 1)^2 - 1, 2(a^2 + 1)).$$

Proof. It is easy to see that $(x, y) = (a^2+1, 1)$ is the fundamental solution of (2.11) with $D = a^2(a^2+2)$. The result immediately follows by Lemma 2.5. □

Lemma 2.10. *The simultaneous Diophantine equations*

$$x^2 - 2y^2 = 1,\ 3z^2 - x^2 = 2 \tag{2.12}$$

has no positive integer solutions.

Proof. Assume that (x, y, z) is a positive integer solution of (2.12). By Lemma 2.1 we know that

$$z = V_{2m+1}, \quad x = U_{2m+1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $U_{4k+1}^2 - 1 = 2y^2$, it follows from (2.6) that $6U_{4k}U_{4k+2} = (\alpha+\beta)^2U_{4k}U_{4k+2} = 2y^2$, where $\alpha = \frac{\sqrt{3}+1}{\sqrt{2}}$, $\beta = \frac{\sqrt{3}-1}{\sqrt{2}}$. Using the fact that $\gcd(4k, 4k+2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 1$ by (2.3). Then either

$$U_{4k} = b^2, \quad U_{4k+2} = 3c^2 \tag{2.13}$$

or

$$U_{4k} = 3b^2, \quad U_{4k+2} = c^2 \tag{2.14}$$

for some positive integers b and c.

The former equation of (2.13) yields $U_{2k}V_{2k} = b^2$, so that $U_{2k} = b_1^2$, $V_{2k} = b_2^2$ for some positive integers b_1 and b_2, which is impossible since $v_2(V_{2k}) = 1$ by Lemma 2.2.

The latter equation of (2.14) yields that $\left(\frac{V_{4k+2}}{2}, c\right)$ is a solution of $X^2 - 3Y^4 = 1$. We get by Lemma 2.5 that $c = 1$, which leads to $U_{4k+2} = 1 = U_2$, a contradiction, or $U_{4k+2} = c^2 = 4 = V_2$, which is impossible. Hence, both of these are impossible.

The case m is odd, say $m = 2k+1$ for some nonnegative integer k. Since $U_{4k+3}^2 - 1 = 2y^2$, it follows from (2.6) that $6U_{4k+4}U_{4k+2} = (\alpha+\beta)^2U_{4k+4}U_{4k+2} = 2y^2$. Using the fact that $\gcd(4k+4, 4k+2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 1$ by (2.3). Then either

$$U_{4k+4} = b^2, \quad U_{4k+2} = 3c^2 \tag{2.15}$$

or

$$U_{4k+4} = 3b^2, \quad U_{4k+2} = c^2 \tag{2.16}$$

for some positive integers b and c.

The former equation of (2.15) yields $U_{2k+2}V_{2k+2} = b^2$, so that $U_{2k+2} = b_1^2$, $V_{2k+2} = b_2^2$ for some positive integers b_1 and b_2, which is impossible since $v_2(V_{2k+2}) = 1$ by Lemma 2.2.

According to the above discussion of (2.14), we know that (2.16) is impossible. Hence, both of these are impossible.

□

The first equation of (1.6)

$$(a^2+2)x^2 - y^2 = 2 \tag{2.17}$$

has the minimal positive integer solution $\sqrt{a^2+2}+a$ with a odd. Then all positive integer solutions of the Eq (2.17) are given by

$$x = V_{2m+1}(2(a^2+2),1), \quad y = aU_{2m+1}(2(a^2+2),1), \quad m \geq 0$$

by Lemma 2.1. If a is even, then let $a = 2a_1$, the equation

$$(2a_1^2+1)x^2 - 2y^2 = 1 \tag{2.18}$$

has the minimal positive integer solution $\sqrt{2a_1^2+1}+a_1\sqrt{2}$. Then all positive integer solutions of the Eq (2.18) are given by

$$x = V_{2m+1}(2(a^2+2),1), \quad y = a_1U_{2m+1}(2(a^2+2),1), \quad m \geq 0$$

by Lemma 2.1. In the sequel, we write V_m and U_m instead of $V_m(2(a^2+2),1)$ and $U_m(2(a^2+2),1)$, respectively. We have the following:

Lemma 2.11. *Let a be positive integer.*
(a) If $U_{2m} = 2\square$, then the equation there is no positive integer solutions.
(b) If, for $m > 0$, $U_{2m} = \square$, then $m = 1$ and $x = 1$ or $m = 2$ and $a^2+1 = 2\square$.

Proof. (a) Assume that $U_{2m} = 2x^2$ for some positive integers m and x. We get by Lemma 2.2 that m is even. Write $m = 2k$. This yields $U_{2k}\frac{V_{2k}}{2} = x^2$, so that $U_{2k} = u^2$, $\frac{V_{2k}}{2} = v^2$ for some positive integers u and v odd since $v_2(V_{2k}) = 1$ by Lemma 2.2. Then (v^2, u) is a solution of (2.11) by Lemma 2.1. We get by Lemma 2.9 that $a^2+1 = v^2$, which is impossible for $a > 1$, or

$$2(a^2+1)^2 - 1 = v^2, \quad 2(a^2+1) = u^2.$$

Therefore $8\left(\frac{u}{2}\right)^4 - v^2 = 1$. It follows $v^2 \equiv -1 \pmod 8$, which is impossible.

(b) Assume that $U_{2m} = x^2$ for some positive integer m and some positive integer x. This yields $\left(\frac{V_{2m}}{2}, x\right)$ is a solution of (2.11). We get by Lemma 2.9 that $U_{2m} = 1 = U_2$ or $U_{2m} = x^2 = 2(a^2+1) = V_2$. The former case means that $m = 1$ and $x = 1$. The latter case yields $m = 2$, $a^2+1 = 2\square$. □

Lemma 2.12. *Let p be odd prime.*
(a) If, for $m > 0$, $U_{2m} = 2p\square$, then $m = 2$ or $m = 4$, $a^2+1 = 2\square$, $2a^4+4a^2+1 = p\square$.
(b) If, for $m > 0$, $U_{4m+2} = p\square$, then $2m+1 = P \equiv 3 \pmod 4$ is a prime and a is even.

Proof. (a) Assume that $U_{2m} = 2px^2$ for some positive integers m and x. We get by Lemma 2.2 that m is even. Write $m = 2k$. This yields $U_{2k}\frac{V_{2k}}{2} = px^2$, so that $U_{2k} = u^2$, $\frac{V_{2k}}{2} = pv^2$ or $U_{2k} = pu^2$, $\frac{V_{2k}}{2} = v^2$ for some positive integers u and v odd. By Lemma 2.11 and the former equation, we have $k = 1$ or $k = 2$, $a^2+1 = 2\square$ and $2a^4+4a^2+1 = \frac{V_4}{2} = p\square$. The latter equation yields (v, pu^2) is a solution of $x^4 - a^2(a^2+2)y^2 = 1$. Since $a^2(a^2+2) \neq 1785, 4\cdot 1785, 16\cdot 1785$, we get by Lemma 2.5 that $pu^2 = 1$, a contradiction, or $U_{2k} = pu^2 = 2(a^2+1) = U_4$. It follows that $k = 2$ and $2a^4+4a^2+1 = \frac{V_4}{2} = v^2$, $2 \nmid a$. Taking modulo 8 yields $7 \equiv v^2 \pmod 8$, which is impossible.

(b) Assume that $U_{2m+1}V_{2m+1} = U_{4m+2} = px^2$ for some positive integer m and some positive integer x. This yields $U_{2m+1} = pu^2$, $V_{2m+1} = v^2$ or $U_{2m+1} = u^2$, $V_{2m+1} = pv^2$.

We first consider the case $2 \nmid a$. If $U_{2m+1} = pu^2$, $V_{2m+1} = v^2$, then (v, pu^2) is a solution of $(a^2+2)x^4 - a^2y^2 = 2$. We get by Lemma 2.8 that $pu^2 = 1$, which is impossible. If $V_{2m+1} = pv^2$, $U_{2m+1} = u^2$, then (pv^2, u) is a solution of $(a^2+2)x^2 - a^2y^4 = 2$. We get by Lemma 2.7 that $pv^2 = 1$ or $u^2 = 2a^2+3$. Two cases that are obviously not true.

We now consider the case $2|a$. If $U_{2m+1} = u^2$, $V_{2m+1} = pv^2$, then (pv^2, u) is a solution of $(2a_1^2 + 1)X^2 - 2a_1^2Y^4 = 1$. Since $(1, 1)$ is the minimal positive integer solution of $(2a_1^2 + 1)U^2 - 2a_1^2V^2 = 1$, we know that is impossible by Lemma 2.3. If $U_{2m+1} = pu^2$, $V_{2m+1} = v^2$, then (v, pu^2) is a solution of $(2a_1^2 + 1)X^4 - 2a_1^2Y^2 = 1$. Then we get by Lemma 2.4 that $pu^2 = 1$, a contradiction, or $V_{2m+1} = v^2 = V_P$ for some prime $P \equiv 3 \pmod 4$. □

3. Proof of Theorem 1.1

Case 1: a is odd. Assume that (x, y, z) is a positive integer solution of (1.7). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = aU_{2m+1} \tag{3.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $V_{4k+1}^2 - 1 = pz^2$, it follows from (2.6) that $2a^2U_{4k}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k, 4k + 2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2|U_{4k}$, $2 \nmid U_{4k+2}$. Then we get by Lemma 2.10 that

$$U_{4k} = 2pb^2, \quad U_{4k+2} = c^2 \tag{3.2}$$

for some positive integers b and c.

We get from the latter equation of (3.2) and Lemma 2.11 that $k = 0$, which contradicts $k > 0$.

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. Since $V_{4k+3}^2 - 1 = pz^2$, it follows from (2.6) that $2a^2U_{4k+4}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k + 4, 4k + 2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2|U_{4k+4}$, $2 \nmid U_{4k+2}$. Then by Lemma 2.11, we have

$$U_{4k+4} = 2pb^2, \quad U_{4k+2} = c^2 \tag{3.3}$$

for some integers b and c. Again by Lemma 2.11 and the latter equation of (3.3), we get that $k = 0$. This means that $m = 1$ and $2(a^2 + 1) = U_2V_2 = U_4 = 2pb^2$, which implies $a^2 + 1 = pb^2$. Conversely, when $a^2 + 1 = pb^2$, by calculation one can easily find that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{p}}\right) = (2a^2 + 1, 2a^3 + 3a, 2ab)$$

is a solution of (1.7).

Case 2: a is even. Let $a = 2a_1$. Then (1.7) becomes

$$(2a_1^2 + 1)x^2 - 2y^2 = 1, \quad x^2 - pz^2 = 1. \tag{3.4}$$

Assume that (x, y, z) is a positive integer solution of (3.4). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = a_1U_{2m+1} \tag{3.5}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $V_{4k+1}^2 - 1 = pz^2$, it follows from (2.6) that $2a^2U_{4k}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k, 4k+2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2|U_{4k}$, $2 \nmid U_{4k+2}$. Then we get by Lemma 2.10 that

$$U_{4k} = 2pb^2, \quad U_{4k+2} = c^2 \tag{3.6}$$

for some positive integers b and c.

We get from the latter equation of (3.6) that $k = 0$, by Lemma 2.11, which contradicts the assumption $k > 0$. Hence (3.6) is impossible.

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. Since $V_{4k+3}^2 - 1 = pz^2$, it follows from (2.6) that $2a^2U_{4k+4}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k + 4, 4k + 2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2|U_{4k+4}$, $2 \nmid U_{4k+2}$. Then

$$U_{4k+4} = 2pb^2, \quad U_{4k+2} = c^2 \tag{3.7}$$

for some integers b and c.

By Lemma 2.11, we get from the latter equation of (3.7) that $k = 0$. This means that $m = 1$ and $2(a^2 + 1) = U_2V_2 = U_4 = 2pb^2$, which implies $a^2 + 1 = pb^2$. Conversely, when $a^2 + 1 = pb^2$, by calculation one can easily find that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{p}}\right) = (2a^2 + 1, 8a_1^3 + 3a_1, 2ab)$$

is a solution of (3.4) and

$$(x, y, z) = (2a^2 + 1, 2a^3 + 3a, 2ab)$$

is a solution of (1.7). This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Case 1: a is odd. Assume that (x, y, z) is a positive integer solution of (1.8). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = aU_{2m+1} \tag{4.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $V_{4k+1}^2 - 1 = pqz^2$, it follows from (2.6) that $2a^2U_{4k}U_{4k+2} = pqz^2$. Using the fact that $\gcd(4k, 4k+2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2|U_{4k}$, $2 \nmid U_{4k+2}$. Then

$$U_{4k} = 2pqb^2, \quad U_{4k+2} = c^2 \tag{4.2}$$

or

$$U_{4k} = 2pb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = qc^2 \tag{4.3}$$

for some integers b and c.

Lemma 2.11 and the latter equation of (4.2) give $k = 0$, which contradicts $k > 0$. By Lemma 2.12 (2), we know that the latter equation of (4.3) is impossible. Hence, both (4.2) and (4.3) are impossible.

The case m is odd, say $m = 2k+1$ for some nonnegative integer k. Since $V_{4k+3}^2 - 1 = pqz^2$, it follows from (2.6) that $2a^2 U_{4k+4} U_{4k+2} = pqz^2$. Using the fact that $\gcd(4k+4, 4k+2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 1$ by (2.3). By Lemma 2.2, we have $2 | U_{4k+4}$, $2 \nmid U_{4k+2}$. Then

$$U_{4k+4} = 2pqb^2, \quad U_{4k+2} = c^2 \tag{4.4}$$

or

$$U_{4k+4} = 2pb^2, \quad U_{2k+1} V_{2k+1} = U_{4k+2} = qc^2 \tag{4.5}$$

for some integers b and c.

Lemma 2.12 (1) and the former equation of (4.5) give $k = 0$. Thus $U_2 = 1 = qc^2$, which is impossible. Hence, (4.5) is impossible.

Lemma 2.11 and the latter equation of (4.4) give $k = 0$. Substituting the value into the former Eq (4.4) gives $2(a^2 + 1) = U_2 V_2 = U_4 = 2pqb^2$. It follows that $a^2 + 1 = pqb^2$. Clearly, when $a^2 + 1 = pqb^2$, we get that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{pq}} \right) = (2a^2 + 1, 2a^3 + 3a, 2ab)$$

is a solution of (1.8).

Case 2: a is even. Let $a = 2a_1$. Then (1.8) becomes

$$(2a_1^2 + 1)x^2 - 2y^2 = 1, \quad x^2 - pqz^2 = 1. \tag{4.6}$$

Assume that (x, y, z) is a positive integer solution of (4.6). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = a_1 U_{2m+1} \tag{4.7}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Then, as see before, either

$$U_{4k} = 2pqb^2, \quad U_{4k+2} = c^2 \tag{4.8}$$

or

$$U_k V_k V_{2k} = U_{4k} = 2pb^2, \quad U_{2k+1} V_{2k+1} = U_{4k+2} = qc^2 \tag{4.9}$$

for some integers b and c.

Lemma 2.11 and the latter equation of Eq (4.8) give $k = 0$, which leads to a contradiction. Hence, (4.8) is impossible.

Lemma 2.12 (1) and the former equation of (4.9) yield $k = 1$. Substituting the value into Eq (4.9) gives $2(a^2 + 1) = V_2 = 2pb^2$ and $(2a^2 + 1)(2a^2 + 3) = U_3 V_3 = qc^2$ that implies $a^2 + 1 = pb^2$, $2a^2 + 1 = c_1^2$, $2a^2 + 3 = qc_2^2$ since $2a^2 + 3$ is never a square. We claim that $a \equiv 2 \pmod 4$. Otherwise $4|a$. We get by Lemmas 2.1 and 2.2 that $c_1 + a\sqrt{2} = (3 + 2\sqrt{2})^n$ for some even n. It follows that $c_1 + a\sqrt{2} = (17 + 12\sqrt{2})^{n_1}, n_1 = n/2$, so $3|a$. We get from the equation $2a^2 + 3 = qc_2^2$ that $q = 3$ and

$c_1^2 - 2a^2 = 1, 3c_2^2 - c_1^2 = 2$, which is impossible by Lemma 2.10. Clearly, when $a^2 + 1 = pb^2, 2a^2 + 1 = u^2, 2a^2 + 3 = qv^2$, we get

$$(x, y, z) = \left(x_5, y_5, \sqrt{\frac{x_5^2 - 1}{pq}}\right) = (4a^4 + 6a^2 + 1, 64a_1^5 + 80a_1^3 + 5a_1, 2abuv)$$

is a solution of (4.6) and

$$(x, y, z) = (4a^4 + 6a^2 + 1, 4a^5 + 10a^3 + 5a, 2abuv)$$

is a solution of (1.8).

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. From $V_{4k+3}^2 - 1 = pqz^2$ we get that one of the following holds:

$$U_{4k+4} = 2pqb^2, \quad U_{4k+2} = c^2 \tag{4.10}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2pb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = qc^2 \tag{4.11}$$

for some integers b and c.

Lemma 2.12 (1) and the former equation of (4.11) yield $k = 0$. Thus we get from the latter equation of (4.11) that $U_2 = 1 = qc^2$, which is impossible.

Lemma 2.11 and the latter equation of (4.10) give $k = 0$. Substituting the value into the former Eq (4.10) gives $2(a^2 + 1) = U_2V_2 = U_4 = 2pqb^2$. It follows that $a^2 + 1 = pqb^2$. Clearly, when $a^2 + 1 = pqb^2$, we get that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{pq}}\right) = (2a^2 + 1, 8a_1^3 + 3a_1, 2ab)$$

is a solution of (4.6) and

$$(x, y, z) = (2a^2 + 1, 2a^3 + 3a, 2ab)$$

is a solution of (1.8). This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Case 1: a is odd. Assume that (x, y, z) is a positive integer solution of (1.9). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = aU_{2m+1} \tag{5.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. From $V_{4k+1}^2 - 1 = pqrz^2$ we get that one of the following holds:

$$U_{4k} = 2pqrb^2, \quad U_{4k+2} = c^2 \tag{5.2}$$

or

$$U_{2k}V_{2k} = U_{4k} = 2pqb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = rc^2 \tag{5.3}$$

or

$$U_{4k} = 2pb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = qrc^2 \tag{5.4}$$

for some integers b and c.

By Lemma 2.12 (2), we know that the latter equation of (5.3) is impossible. Lemma 2.11 and the latter equation of (5.2) give $k = 0$, which contradicts the assumption $k > 0$.

Lemma 2.12 (1) and the former equation of (5.4) give $k = 1$ or $k = 2$. Substituting the value $k = 1$ into the (5.4) leads to

$$2(a^2 + 1) = U_2V_2 = U_4 = 2pb^2, \quad (2a^2 + 1)(2a^2 + 3) = U_3V_3 = U_6 = qrc^2.$$

Neither $2a^2 + 1$ nor $2a^2 + 3$ is square since a is odd. Therefore we get

$$a^2 + 1 = pb^2, \quad 2a^2 + 1 = qu^2, \quad 2a^2 + 3 = rv^2.$$

Clearly, when $a^2 + 1 = pb^2$, $2a^2 + 1 = qu^2$, $2a^2 + 3 = rv^2$, we get that

$$(x, y, z) = \left(x_5, y_5, \sqrt{\frac{x_5^2 - 1}{pqr}}\right) = (4a^4 + 6a^2 + 1, 4a^5 + 10a^3 + 5a, 2abuv)$$

is a solution of (1.9).

Substituting the value $k = 2$ into the latter Eq (5.4) gives $4a^4 + 6a^2 + 1 = V_5 = qu^2$, $4a^4 + 10a^2 + 5 = U_5 = rv^2$. Clearly, when $2(a^2 + 1) = b_1^2$, $2(a^2 + 1)^2 - 1 = pb_2^2$, $4a^4 + 6a^2 + 1 = V_5 = qu^2$, $4a^4 + 10a^2 + 5 = U_5 = rv^2$, we get that

$$\begin{aligned}(x, y, z) &= \left(x_9, y_9, \sqrt{\frac{x_9^2 - 1}{pqr}}\right)\\ &= (16a^8 + 56a^6 + 60a^4 + 20a^2 + 1, 16a^9 + 72a^7 + 108a^5 + 60a^3 + 9a, 2abuv)\end{aligned}$$

is a solution of (1.9).

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. As before, we get that one of the following holds:

$$U_{4k+4} = 2pqrb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = c^2 \tag{5.5}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2pqb^2, \quad U_{4k+2} = rc^2 \tag{5.6}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2pb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = qrc^2 \tag{5.7}$$

for some integers b and c.

By Lemma 2.12 (2), we know that the latter equation of (5.6) is impossible.

The latter equation of (5.5) yields $k = 0$ by Lemma 2.11. Substituting the value $k = 0$ into the former Eq (5.5) gives $2(a^2 + 1) = U_2V_2 = U_4 = 2pqrb^2$, so that $a^2 + 1 = pqrb^2$. Clearly, when $a^2 + 1 = pqrb^2$, we get that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{x_3^2 - 1}{pqr}}\right) = (2a^2 + 1, 2a^3 + 3a, 2ab)$$

is a solution of (1.9).

By Lemma 2.12 (1) and the former equation of (5.7), we get $k = 0$ or $k = 1, a^2 + 1 = 2\square, 2a^4 + 4a^2 + 1 = p\square$. If $k = 0$, substituting the value into the latter Eq (5.7) gives $1 = U_2 = qrc^2$, which is a contradiction. Substituting the value $k = 1, a^2 + 1 = 2\square, 2a^4 + 4a^2 + 1 = p\square$ into the Eq (5.7) leads to

$$a^2 + 1 = 2\square, \quad 2a^4 + 4a^2 + 1 = p\square, \quad a^2 + 1 = 2a^2 + 1 = qu^2, \quad 2a^2 + 3 = rv^2.$$

Clearly, when $a^2 + 1 = 2\square$, $2a^4 + 4a^2 + 1 = p\square$, $2a^2 + 1 = qu^2$, $2a^2 + 3 = rv^2$,we get that

$$(x, y, z) = \left(x_7, y_7, \sqrt{\frac{x_7^2 - 1}{pqr}}\right) = (8a^6 + 20a^4 + 12a^2 + 1, 8a^7 + 28a^5 + 28a^3 + 7a, 2abuv)$$

is a solution of (1.9).

Case 2: a is even. Let $a = 2a_1$. Then (1.9) becomes

$$(2a_1^2 + 1)x^2 - 2y^2 = 1, \quad x^2 - pqrz^2 = 1. \tag{5.8}$$

Assume that (x, y, z) is a positive integer solution of (5.8). By Lemma 2.1 we know that

$$x = V_{2m+1}, \quad y = a_1 U_{2m+1} \tag{5.9}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. As before, we get that one of the following holds:

$$U_{4k} = 2pqrb^2, \quad U_{4k+2} = c^2 \tag{5.10}$$

or

$$U_k V_k \frac{V_{2k}}{2} = \frac{U_{4k}}{2} = pqb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = rc^2 \tag{5.11}$$

or

$$U_{2k} \frac{V_{2k}}{2} = \frac{U_{4k}}{2} = pb^2, \quad U_{2k+1}V_{2k+1} = U_{4k+2} = qrc^2 \tag{5.12}$$

for some integers b and c.

Lemma 2.12 (2) and the latter equation of (5.11) give $2k + 1 = P \equiv 3 \pmod 4$ is a prime. We claim that $k = 1$. Otherwise $k > 1$, then we know that V_k is not a square again by Lemma 2.4. The former equation of (5.11) yields one of U_k and $\frac{V_{2k}}{2}$ is a square. If $U_k = u^2$, then (V_k, u) is a solution of $(2a_1^2 + 1)X^2 - 2a_1^2 Y^4 = 1$. Then we get by Lemma 2.3 that $V_k = p\square = 1$, which is a contradiction. If $\frac{V_{2k}}{2} = v^2$, then (v, U_{2k}) is a solution of $X^4 - a^2(a^2 + 2)Y^2 = 1$. Thus we get by Lemma 2.5 that $pq\square = U_{2k} = 1$, which is impossible or $U_{2k} = 2(a^2 + 1) = V_2 = U_4$. It follows that $k = 2$, which contradicts with $2k + 1 = P \equiv 3 \pmod 4$. Hence

$$k = 1, \quad a^2 + 1 = pqb^2, \quad 2a^2 + 1 = u^2, \quad 2a^2 + 3 = rv^2,$$

with $a \equiv 2 \pmod 4$ according to the discussion of (4.9). Clearly, when $a^2 + 1 = pqb^2$, $2a^2 + 1 = u^2$, $2a^2 + 3 = rv^2$, we get that

$$(x, y, z) = \left(x_5, y_5, \sqrt{\frac{x_5^2 - 1}{pqr}}\right) = (4a^4 + 6a^2 + 1, 64a_1^5 + 80a_1^3 + 5a_1, 2abuv)$$

is a solution of (5.8) and

$$(x, y, z) = (4a^4 + 6a^2 + 1, 4a^5 + 10a^3 + 5a, 2abuv)$$

is a solution of (1.9).

Lemma 2.11 and the latter equation of (5.10) give $k = 0$, which contradicts the assumption $k > 0$. Hence (5.10) cannot hold.

The former equation of (5.12) gives $k = 1, a^2 + 1 = pb^2$ according to the discussion of (4.9). Substituting $k = 1$ into (5.12) leads to

$$(2a^2 + 1)(2a^2 + 3) = U_3 V_3 = U_6 = qrc^2.$$

It is easy to see that $2a^2 + 3$ is not a square. Therefore we get $a^2 + 1 = pb^2, 2a^2 + 1 = u^2, 2a^2 + 3 = qrv^2$ or $a^2 + 1 = pb^2, 2a^2 + 1 = qu^2, 2a^2 + 3 = rv^2$. Clearly, when $a^2 + 1 = pb^2, 2a^2 + 1 = u^2, 2a^2 + 3 = qrv^2$ or $a^2 + 1 = pb^2, 2a^2 + 1 = qu^2, 2a^2 + 3 = rv^2$, we get that

$$(x, y, z) = \left(x_5, y_5, \sqrt{\frac{x_5^2 - 1}{pqr}} \right) = (4a^4 + 6a^2 + 1, 64a_1^5 + 80a_1^3 + 5a_1, 2abuv)$$

is a solution of (5.8) and

$$(x, y, z) = (4a^4 + 6a^2 + 1, 4a^5 + 10a^3 + 5a, 2abuv)$$

is a solution of (1.9).

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. Then

$$U_{4k+4} = 2pqrb^2, \quad U_{2k+1} V_{2k+1} = U_{4k+2} = c^2 \tag{5.13}$$

or

$$U_{2k+2} V_{2k+2} = U_{4k+4} = 2pqb^2, \quad U_{2k+1} V_{2k+1} = U_{4k+2} = rc^2 \tag{5.14}$$

or

$$U_{2k+2} V_{2k+2} = U_{4k+4} = 2pb^2, \quad U_{2k+1} V_{2k+1} = U_{4k+2} = qrc^2 \tag{5.15}$$

for some integers b and c.

According to the discussion of (5.11), we know that the Eq (5.14) leads to $k = 1$ and $2a^2 + 1 = U_3 = u^2$. Substituting the value $k = 1$ into the former Eq (5.14) yields

$$2a^2 + 1 = \square, \quad 2a^2 + 3 = r\square, \quad 4(a^2 + 1)(2a^4 + 4a^2 + 1) = 2pqb^2.$$

Therefore we get

$$p = 2, \quad a^2 + 1 = q\square, \quad 2a^4 + 4a^2 + 1 = \square.$$

Clearly, when $p = 2$, $a^2 + 1 = q\square$, $2a^4 + 4a^2 + 1 = \square$, $2a^2 + 1 = \square$, $2a^2 + 3 = rv^2$, we get that

$$(x, y, z) = \left(x_7, y_7, \sqrt{\frac{x_7^2 - 1}{pqr}} \right) = (8a^6 + 20a^4 + 12a^2 + 1, 512a_1^7 + 448a_1^5 + 112a_1^3 + 7a_1, 2abuv)$$

is a solution of (5.8) and

$$(x,y,z)=\left(x_7,y_7,\sqrt{\frac{x_7^2-1}{pqr}}\right)=(8a^6+20a^4+12a^2+1,8a^7+28a^5+28a^3+7a,2abuv)$$

is a solution of (1.9).

Lemma 2.11 and the latter equation of (5.13) gives $k=0$. Substituting the value into the former Eq (5.13) gives $2(a^2+1)=2pqrb^2$. It follows that $a^2+1=pqrb^2$. Thus in this case we proved that

$$(x,y,z)=\left(x_3,y_3,\sqrt{\frac{y_3^2-1}{pqr}}\right)=(2a^2+1,8a_1^3+3a_1,2ab)$$

is a solution of (5.8) with $a^2+1=pqrb^2$ and

$$(x,y,z)=(2a^2+1,2a^3+3a,2ab)$$

is a solution of (1.9) with $a^2+1=pqrb^2$.

Lemma 2.12 (1) and the former equation of (5.15) give $k=0$. Substituting the value into the latter Eq (5.15) gives $1=U_2=qrc^2$, which is a contradiction.

This completes the proof of Theorem 1.3.

6. Applications

In this section, we give some examples of applications of the results.

(1) Let p be a prime such that $x^2-py^2=-1$ has solution, and let (a_1,b_1) be its fundamental solution. Define

$$a_n+b_n\sqrt{p}=(a_1+b_1\sqrt{p})^n$$

for some odd integer. Let $a=a_n$. Then

$$(x,y,z)=\left(2a_n^2+1,2a_n^3+3a_n,2a_nb_n\right)$$

is the only solution of the simultaneous Pell equations (1.7).

Let $p=2$. Then $(a_1,b_1)=(1,1)$ is the fundamental solution of

$$x^2-2y^2=-1.$$

Define

$$a_n+b_n\sqrt{2}=(1+\sqrt{2})^n$$

for some odd integer n. Then

$$(x,y,z)=\left(2a_n^2+1,2a_n^3+3a_n,2a_nb_n\right)$$

is the only solution of the simultaneous Pell equations

$$(a_n^2+2)x^2-y^2=2,\quad x^2-2z^2=1$$

(results of $n = 1, 3, 5, 7$ see the following Table 1).

Table 1. Some examples of applications of Theorem 1.1.

n	$a = a_n$	$b = b_n$	$x = 2a^2 + 1$	$y = 2a^3 + 3a$	$z = 2ab$
1	1	1	3	5	2
3	7	5	99	707	70
5	41	29	3363	137965	2378
7	239	169	114243	27304555	80782

(2) Let p and q be two distinct primes such that $x^2 - pqy^2 = -1$ has solution, and let (a_1, b_1) be its fundamental solution. Define

$$a_n + b_n \sqrt{p} = (a_1 + b_1 \sqrt{pq})^n$$

for some odd integer n. Let $a = a_n$. Then

$$(x, y, z) = \left(2a_n^2 + 1, 2a_n^3 + 3a_n, 2a_nb_n\right)$$

is the only solution of the simultaneous Pell equations (1.8) satisfying the condition α) of Theorem 1.2.

Let $p = 2$, $q = 5$. Then $(a_1, b_1) = (3, 1)$ is the fundamental solution of $x^2 - 10y^2 = -1$. Define

$$a_n + b_n \sqrt{10} = (3 + \sqrt{10})^n$$

for some odd integer n. Then

$$(x, y, z) = \left(2a_n^2 + 1, 2a_n^3 + 3a_n, 2a_nb_n\right)$$

is the only solution of the simultaneous Pell equations

$$(a_n^2 + 2)x^2 - y^2 = 2, \quad x^2 - 10z^2 = 1$$

(results of $n = 1, 3, 5, 7$ see the following Table 2).

Table 2. Some examples of applications of Theorem 1.2 (satisfying the condition α).

n	$a = a_n$	$b = b_n$	$x = 2a^2 + 1$	$y = 2a^3 + 3a$	$z = 2ab$
1	3	1	19	63	6
3	117	37	27379	3203577	8658
5	4443	1405	39480499	175411865943	12484830
7	168717	53353	56930852179	9605202587421777	18003116202

(3) Let $a = 2$. Then $a^2 + 1 = 5$, $2a^2 + 1 = 3^2$, and $2a^2 + 3 = 11$. Thus

$$(a, p, q, x, y, z) = (2, 5, 11, 89, 218, 12)$$

is the only solution of the simultaneous Pell equations $6x^2 - y^2 = 2$, $x^2 - 55z^2 = 1$ satisfying the condition β) of Theorem 1.2.

Let $a = 70$. Then $a^2 + 1 = 29 \cdot 13^2$, $2a^2 + 1 = 99^2$, and $2a^2 + 3 = 9803$. Thus

$$(a, p, q, x, y, z) = (70, 29, 9803, 96069401, 6726230350, 180180)$$

is the only solution of the simultaneous Pell equations $4902x^2 - y^2 = 2$, $x^2 - 29 \cdot 9803z^2 = 1$ satisfying the condition β) of Theorem 1.2.

(4) Let p, q and r be three distinct primes such that $x^2 - pqry^2 = -1$ has solution, and let (a_1, b_1) be its fundamental solution. Define

$$a_n + b_n \sqrt{pqr} = (a_1 + b_1 \sqrt{pqr})^n.$$

for some odd integer n. Let $a = a_n$. Then

$$(x, y, z) = \left(2a_n^2 + 1, 2a_n^3 + 3a_n, 2a_n b_n\right)$$

is the only solution of the simultaneous Pell equations (1.9) satisfying the condition α) of Theorem 1.3.

Let $p = 2$, $q = 5$, $r = 17$. Then $(a_1, b_1) = (13, 1)$ is the fundamental solution of $x^2 - 170y^2 = -1$. Define

$$a_n + b_n \sqrt{170} = (13 + \sqrt{170})^n.$$

Then

$$(x, y, z) = \left(2a_n^2 + 1, 2a_n^3 + 3a_n, 2a_n b_n\right)$$

is the only solution of the simultaneous Pell equations

$$(a_n^2 + 2)x^2 - y^2 = 2, \quad x^2 - 170z^2 = 1$$

(results of $n = 1, 3, 5$ see the following Table 3).

Table 3. Some examples of applications of Theorem 1.3 (satisfying the condition α).

n	$a = a_n$	$b = b_n$	$x = 2a^2 + 1$	$y = 2a^3 + 3a$	$z = 2ab$
1	13	1	339	4433	26
3	8827	677	155831859	1375527837047	11951758
5	5984693	459005	71633100608499	428702115779991675193	5494008020930

(5) Let $a = 1$. Then $a^2 + 1 = 2$, $2a^2 + 1 = 3$ and $2a^2 + 3 = 5$. Thus

$$(a, p, q, r, x, y, z) = (1, 2, 3, 5, 11, 19, 2)$$

is the only solution of the simultaneous Pell equations $3x^2 - y^2 = 2$, $x^2 - 30z^2 = 1$ satisfying the condition β) of Theorem 1.3.

Let $a = 7$. Then $a^2 + 1 = 2 \cdot 5^2$, $2a^2 + 1 = 11 \cdot 3^2$ and $2a^2 + 3 = 101$. Thus

$$(a, p, q, r, x, y, z) = (7, 2, 11, 101, 9899, 10099, 210)$$

is the only solution of the simultaneous Pell equations $51x^2 - y^2 = 2$, $x^2 - 2222z^2 = 1$ satisfying the condition β) of Theorem 1.3.

Let $a = 6$. Then $a^2 + 1 = 37$, $2a^2 + 1 = 73$ and $2a^2 + 3 = 3 \cdot 5^2$. Thus

$$(a, p, q, r, x, y, z) = (6, 37, 73, 3, 33294, 5401, 60)$$

is the only solution of the simultaneous Pell equations $38x^2 - y^2 = 2$, $x^2 - 8103z^2 = 1$ satisfying the condition β) of Theorem 1.3.

Let $a = 110$. Then $a^2 + 1 = 12101$, $2a^2 + 1 = 2689 \cdot 3^2$ and $2a^2 + 3 = 24203$. Thus

$$(a, p, q, r, x, y, z) = (110, 12101, 2689, 24203, 64433710550, 585712601, 660)$$

is the only solution of the simultaneous Pell equations $12102x^2 - y^2 = 2$, $x^2 - 787555672567z^2 = 1$ satisfying the condition β) of Theorem 1.3.

Let $a = 160$. Then $a^2 + 1 = 25601$, $2a^2 + 1 = 5689 \cdot 3^2$ and $2a^2 + 3 = 51203$. Thus

$$(a, p, q, r, x, y, z) = (160, 25601, 5689, 51203, 419471360800, 2621593601, 960)$$

is the only solution of the simultaneous Pell equations $25602x^2 - y^2 = 2$, $x^2 - 7457414289067z^2 = 1$ satisfying the condition β) of Theorem 1.3.

(6) Let $a = 1$. Then $a^2 + 1 = 2$, $2a^4 + 4a^2 + 1 = 7$, $2a^2 + 1 = 3$ and $2a^2 + 3 = 5$. Thus

$$(a, p, q, r, x, y, z) = (1, 7, 3, 5, 41, 71, 16)$$

is the only solution of the simultaneous Pell equations $3x^2 - y^2 = 2$, $x^2 - 105z^2 = 1$ satisfying the condition γ) of Theorem 1.3.

Let $a = 2$. Then $a^2 + 1 = 5$, $2a^4 + 4a^2 + 1 = 7^2$, $2a^2 + 1 = 3^2$ and $2a^2 + 3 = 11$. Thus

$$(a, p, q, r, x, y, z) = (2, 2, 5, 11, 881, 2158, 989801, 84)$$

is the only solution of the simultaneous Pell equations $6x^2 - y^2 = 2$, $x^2 - 110z^2 = 1$ satisfying the condition γ) of Theorem 1.3.

Let $a = 7$. Then $2(a^2 + 1) = 10^2$,$2a^4 + 4a^2 + 1 = 4999$, $2a^2 + 1 = 11 \cdot 3^2$ and $2a^2 + 3 = 101$. Thus

$$(a, p, q, r, x, y, z) = (7, 4999, 11, 101, 7068593, 989801, 420)$$

is the only solution of the simultaneous Pell equations $51x^2 - y^2 = 2$, $x^2 - 5553889z^2 = 1$ satisfying the condition γ) of Theorem 1.3.

(7) Let $a = 1$. Then $a^2 + 1 = 2$, $2a^4 + 4a^2 + 1 = 7$, $4a^4 + 6a^2 + 1 = 11$ and $4a^4 + 10a^2 + 5 = 19$. Thus

$$(a, p, q, r, x, y, z) = (1, 7, 11, 19, 153, 265, 4)$$

is the only solution of the simultaneous Pell equations $3x^2 - y^2 = 2$, $x^2 - 1463z^2 = 1$ satisfying the condition δ) of Theorem 1.3.

7. Conclusions

Let a, b be distinct positive integers and b has at most three prime divisors. We proved that the system Pell equations

$$(a^2+2)x^2 - y^2 = 2, \quad x^2 - bz^2 = 1$$

has at most one solutions and get the sufficient and necessary conditions for it to have a solution. When a solution exists, assuming that $x_1\sqrt{a^2+2} + y_1$ is the fundamental solution of $(a^2+2)x^2 - y^2 = 2$, then the only solutions of the system is given by

$$x\sqrt{a^2+2} + y = x_m\sqrt{a^2+2} + y_m, \quad m \in \{3, 5, 7, 9\},$$

where $\frac{x_m\sqrt{a^2+2}+y_m}{\sqrt{2}} = \left(\frac{x_1\sqrt{a^2+2}+y_1}{\sqrt{2}}\right)^m$, m is odd.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest regarding the publication of this paper.

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