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Research article

On asymptotics of solutions for superdiffusion and subdiffusion equations with the Riemann-Liouville fractional derivative

Zhiqiang Li* **and Yanzhe Fan**

Department of Mathematics, Lyuliang University, Lvliang, Shanxi 033001, China

* **Correspondence:** Email: lizhiqiang0914@126.com.

Abstract: In the present paper, we focus on the study of the asymptotic behaviors of solutions for the Cauchy problem of time-space fractional superdiffusion and subdiffusion equations with integral initial conditions, where the Riemann-Liouville derivative is used in the temporal direction and the integral fractional Laplacian is applied in the spatial variables. The fundamental solutions of the considered equations, which can be represented in terms of the Fox H-function, are constructed and investigated by using asymptotic expansions of the Fox H-function. Then, we obtain the asymptotic behaviors of solutions in the sense of $L^p(\mathbb{R}^d)$ and $L^{p,\infty}(\mathbb{R}^d)$ norms, where Young's inequality for convolution plays a very important role. Finally, gradient estimates and large time behaviors of solutions are also provided. In particular, we derive the optimal L^2- decay estimate for the subdiffusion equation.

Keywords: Riemann-Liouville derivative; integral fractional Laplacian; Fox H-function; asymptotic behavior; gradient estimate; large time behavior
Mathematics Subject Classification: 26A33, 35R11, 35B40

1. Introduction

The aim of this paper is to consider asymptotic behaviors of solutions for the following time-space fractional superdiffusion equation with integral initial conditions and $\alpha \in (1, 2)$:

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{0,t} u(x,t) + (-\Delta)^s u(x,t) = f(x,t), \ x \in \mathbb{R}^d, \ t > 0, \\ {}_{RL}\mathrm{D}^{\alpha-2}_{0,t} u(x,0) = \varphi(x), \ x \in \mathbb{R}^d, \\ {}_{RL}\mathrm{D}^{\alpha-1}_{0,t} u(x,0) = \psi(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

and subdiffusion equation with integral initial condition and $\alpha \in (0, 1)$:

$$\begin{cases} {}_{RL}\mathrm{D}^{\alpha}_{0,t} u(x,t) + (-\Delta)^s u(x,t) = g(x,t), \ x \in \mathbb{R}^d, \ t > 0, \\ {}_{RL}\mathrm{D}^{\alpha-1}_{0,t} u(x,0) = \phi(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where ${}_{RL}\mathrm{D}_{0,t}^{\alpha}u$ is the Riemann-Liouville derivative of u, $(-\Delta)^s$ denotes the integral fractional Laplace operator with $s \in (0,1)$, and $\varphi(x)$, $\psi(x)$, $\phi(x)$, $f(x,t)$, and $g(x,t)$ are given functions. Moreover, the symbols ${}_{RL}\mathrm{D}_{0,t}^{\alpha-2}$ in Eq (1.1) and ${}_{RL}\mathrm{D}_{0,t}^{\alpha-1}$ in Eq (1.2) are Riemann-Liouville integral operators, and the symbol ${}_{RL}\mathrm{D}_{0,t}^{\alpha-1}$ in Eq (1.1) is Riemann-Liouville derivative operator.

It is well known [14, 19, 28] that the Riemann-Liouville fractional integral of a function $f(t) \in L^1[a,b]$ $(-\infty < a < b < +\infty)$ can be defined by

$$ {}_{RL}\mathrm{D}_{a,t}^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau,\ \alpha > 0,\ a < t < b, \tag{1.3} $$

and the Riemann-Liouville fractional derivative may be represented in the form

$$ \begin{aligned} {}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_{RL}\mathrm{D}_{a,t}^{-(n-\alpha)}f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau,\ a < t < b, \end{aligned} \tag{1.4} $$

where $n-1 < \alpha < n \in \mathbb{N}$ and $f(t) \in AC^n[a,b]$, here $AC^n[a,b]$ denotes the set of functions with an absolutely continuous $(n-1)$st derivative.

For a function $v(x) \in H^{2s}(\mathbb{R}^d) = \{v \in \mathcal{S}(\mathbb{R}^d) \,|\, (-\Delta)^s v \in L^2(\mathbb{R}^d), s \in (0,1)\}$ with $\mathcal{S}(\mathbb{R}^d)$ being the Schwartz space, the integral fractional Laplacian of the function $v(x)$ is given by [8]

$$ (-\Delta)^s v(x) = C(d,s)\ \mathrm{P.V.}\int_{\mathbb{R}^d}\frac{v(x)-v(y)}{|x-y|^{d+2s}}\mathrm{d}y,\ x \in \mathbb{R}^d, \tag{1.5} $$

where P.V. denotes the Cauchy principal value and $C(d,s)$ is a dimensional constant

$$ C(d,s) = \Big(\int_{\mathbb{R}^d}\frac{1-\cos y_1}{|y|^{d+2s}}\mathrm{d}y\Big)^{-1},\ y = (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d. $$

Superdiffusion and subdiffusion equations in the forms of Eqs (1.1) and (1.2) have drawn much interest in developing existence, uniqueness, stability as well as asymptotics of the solutions, due to their excellent modelling capability for various applications such as theory of viscoelasticity [23], signal and image processing [31], anomalous diffusion [24], control theory [26], epidemic phenomena in biology [1], economics [4], etc. For more widespread applications on fractional differential equations we refer the reader to other works [3, 9, 10, 12, 14, 19, 27, 28, 33] and the references cited therein.

As far as the asymptotic behaviors of solutions of fractional partial differential equations is concerned, we review some results on this topic in the current literatures. For the fractional superdiffusion (or call diffusion-wave) equation, the authors in [25] first studied the asymptotics of solutions in the sense of L^∞ norm, where time derivatives are the Riemann-Liouville and Caputo ones respectively and spatial derivative is the standard Laplace operator. After that, the article [7] investigated the asymptotic estimates of solution under L^p norm with $1 \le p \le \infty$, where the Riemann-Liouville derivative replaced by Caputo derivative in Eq (1.1) and the initial conditions are written as $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$. Recently, Li and Li [21, 22] discussed the same problem as above and derived similar asymptotic behaviors, in which the temporal derivative are taken as Caputo-Hadamard and ψ-Caputo fractional ones.

On the other hand, concerning the fractional subdiffusion equation, Ma et al. [25] considered the asymptotic properties of such equation in the cases of the Riemann-Liouville and Caputo derivatives for Eq (1.2) when the force term is equal to zero, where the spatial direction is the standard Laplacian and the initial value is $u(x, 0) = u_0(x)$. Subsequently, the paper [17] generalized these conclusions of [25] and they established the asymptotic analysis of solution in terms of L^p norm in which the Caputo derivative is used as temporal one. Shortly after, the results in [17] are further extended to time-space fractional subdiffusion equation [18] with the Caputo derivative and integral fractional Laplacian. Very recently, Li et al. [20, 22] devoted to asymptotic properties of solution of Eq (1.2), where the Caputo-Hadamard and ψ-Caputo derivatives substituted for the Riemann-Liouville one. For other related studies we refer the reader to [16, 32]. However, to the best of our knowledge, the asymptotic behaviors of solutions for Eqs (1.1) and (1.2) with the Riemann-Liouville derivative have been less studied and the literature [25] only considered very special cases for which the results obtained there can also be further improved.

Based on the above reasons and existing research works, the goal of this paper is to study the asymptotic behaviors of solutions of Eqs (1.1) and (1.2) in the sense of more general L^p or weak L^p norms. Specifically, we first investigate asymptotic estimates of the solution to Eq (1.1). Using the technique of integral transforms the solution of convolutional form of Eq (1.1) is constructed and the fundamental solutions are also explicitly expressed by the Fox H-function. Then we estimate the fundamental solutions by means of asymptotic expansions of the H-function and further obtain the asymptotics of solution with the help of Young's inequality for convolution. By applying similar argument we can derive gradient estimates and large time behaviors of solution to Eq (1.1). For the subdiffusion Eq (1.2), we likewise discuss the asymptotic properties, gradient estimates and large time behaviors of solution. In particular we obtain the optimal decay rate in the sense of L^2 norm. We find that these results with the Riemann-Liouville derivative in time are different from the Caputo case, for example, see Theorem 3.1 of this paper and Proposition 5.7 in [18].

The remaining part of this article is organized as follows. In Section 2, the asymptotic behaviors of solution of the fractional superdiffusion Eq (1.1) are studied by means of Young's inequality for convolution. Further, the gradient estimates and large time behaviors of the solution are also presented. By using the almost same methods, Section 3 discusses decay estimates of the solution for the fractional subdiffusion Eq (1.2) and the optimal L^2-decay rate is particularly derived. Some conclusions and remarks are presented in Section 4. At last, the Appendix recalls several integral transforms and concept of the Fox H-function. Throughout the paper we denote by C a generic positive constant whose value may vary from line to line.

2. Asymptotic estimates of solution for Eq (1.1)

In this section, we shall study asymptotic analysis of the solution to Eq (1.1). First, the solution of convolutional form for Eq (1.1) is constructed in terms of Fourier and Laplace transforms, where the fundamental solutions are written via the Fox H-functions. We subsequently investigate asymptotic behaviors and estimations of L^p-norm for the fundamental solutions. Then, the asymptotic estimates of solution of Eq (1.1) are established by means of Young's inequality for convolution. Finally, we present gradient estimates and large time behaviors of the solution to Eq (1.1) by using the almost same argument.

2.1. Asymptotic behaviors of the solution

We first deduce the fundamental solutions and solution of Eq (1.1) by using integral transforms. Making use of the standard Laplace transform for temporal variable t and the Fourier transform for spatial variable x, and taking the formulas (A2) and (A4) into account, it follows that

$$\lambda^{\alpha}\overline{\widehat{u}}(\omega,\lambda)-\lambda\widehat{\varphi}(\omega)-\widehat{\psi}(\omega)+|\omega|^{2s}\overline{\widehat{u}}(\omega,\lambda)=\overline{\widehat{f}}(\omega,\lambda). \tag{2.1}$$

Furthermore,

$$\overline{\widehat{u}}(\omega,\lambda)=\overline{\widehat{G}}_{\varphi}(\omega,\lambda)\widehat{\varphi}(\omega)+\overline{\widehat{G}}_{\psi}(\omega,\lambda)\widehat{\psi}(\omega)+\overline{\widehat{G}}_{f}(\omega,\lambda)\overline{\widehat{f}}(\omega,\lambda), \tag{2.2}$$

where $\overline{\widehat{G}}_{\varphi}(\omega,\lambda)=\dfrac{\lambda}{\lambda^{\alpha}+|\omega|^{2s}}$ and $\overline{\widehat{G}}_{\psi}(\omega,\lambda)=\overline{\widehat{G}}_{f}(\omega,\lambda)=\dfrac{1}{\lambda^{\alpha}+|\omega|^{2s}}$.

Applying the inverse Fourier transform and inverse Laplace transform to the identity (2.2) we obtain

$$\begin{aligned} u(x,t)=&G_{\varphi}(x,t)*\varphi(x)+G_{\psi}(x,t)*\psi(x)+G_f(x,t)\star f(x,t)\\ =&\int_{\mathbb{R}^d}G_{\varphi}(x-y,t)\varphi(y)\mathrm{d}y+\int_{\mathbb{R}^d}G_{\psi}(x-y,t)\psi(y)\mathrm{d}y\\ &+\int_0^t\int_{\mathbb{R}^d}G_f(x-y,t-\tau)f(y,\tau)\mathrm{d}y\mathrm{d}\tau, \end{aligned} \tag{2.3}$$

where the character $*$ denotes the standard convolution with respect to spatial variable, and the symbol $\star$ is used as a convolution in time and space directions.

In the following part, we present the explicit expressions of fundamental solutions $G_{\varphi}(x,t)$, $G_{\psi}(x,t)$, and $G_f(x,t)$ in (2.3). In terms of the relation (A3) for Laplace and Mellin transforms, one has

$$\begin{aligned} \widetilde{\widehat{G}}_{\varphi}(\omega,\xi)=&\mathscr{M}[\widehat{G}_{\varphi}(\omega,t),\xi]=\frac{1}{\Gamma(1-\xi)}\mathscr{M}[\mathscr{L}[\widehat{G}_{\varphi}(\omega,t),\lambda],1-\xi]\\ =&\frac{1}{\Gamma(1-\xi)}\mathscr{M}[\overline{\widehat{G}}_{\varphi}(\omega,\lambda),1-\xi]\\ =&\frac{1}{\Gamma(1-\xi)}\mathscr{M}\left[\frac{\lambda}{\lambda^{\alpha}+|\omega|^{2s}},1-\xi\right]\\ =&\frac{1}{\alpha\Gamma(1-\xi)}(|\omega|^{2s})^{\frac{2-\xi}{\alpha}-1}\Gamma\Big(\frac{2-\xi}{\alpha}\Big)\Gamma\Big(1-\frac{2-\xi}{\alpha}\Big). \end{aligned}$$

The inverse Fourier transform of the above equality yields

$$\begin{aligned} \widetilde{G}_{\varphi}(x,\xi)=&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widetilde{\widehat{G}}_{\varphi}(\omega,\xi)e^{-i\omega\cdot x}\mathrm{d}\omega\\ =&\frac{1}{(2\pi)^d}\frac{1}{\alpha\Gamma(1-\xi)}\Gamma\Big(\frac{2-\xi}{\alpha}\Big)\Gamma\Big(1-\frac{2-\xi}{\alpha}\Big)\int_{\mathbb{R}^d}(|\omega|^{2s})^{\frac{2-\xi}{\alpha}-1}e^{-i\omega\cdot x}\mathrm{d}\omega\\ =&\frac{1}{(2\pi)^d}\frac{1}{\alpha\Gamma(1-\xi)}\Gamma\Big(\frac{2-\xi}{\alpha}\Big)\Gamma\Big(1-\frac{2-\xi}{\alpha}\Big)\frac{(2\pi)^{\frac{d}{2}}}{|x|^{\frac{d-2}{2}}}\times\int_0^{\infty}(\rho^{2s})^{\frac{2-\xi}{\alpha}-1}\rho^{\frac{d}{2}}J_{\frac{d}{2}-1}(\rho|x|)\mathrm{d}\rho, \end{aligned}$$

where $J_{\frac{d}{2}-1}(\rho|x|)$ is the first kind of Bessel function, see [14] for related definition and property. Observing that the formula (2.6.4) in [15], one has

$$\int_0^\infty (\rho^{2s})^{\frac{2-\xi}{\alpha}-1}\rho^{\frac{d}{2}}J_{\frac{d}{2}-1}(\rho|x|)\mathrm{d}\rho = \int_0^\infty \rho^{\frac{d}{2}+(\frac{2-\xi}{\alpha}-1)2s}J_{\frac{d}{2}-1}(\rho|x|)\mathrm{d}\rho$$
$$=|x|^{-\frac{d}{2}-(\frac{2-\xi}{\alpha}-1)2s-1}\frac{2^{\frac{d}{2}+(\frac{2-\xi}{\alpha}-1)2s}\Gamma(\frac{d}{2}+(\frac{2-\xi}{\alpha}-1)s)}{\Gamma(-(\frac{2-\xi}{\alpha}-1)s)}.$$

Therefore, we arrive at

$$\widetilde{G}_\varphi(x,\xi) = \frac{|x|^{(1-\frac{2-\xi}{\alpha})2s}}{\alpha|x|^d\pi^{\frac{d}{2}}2^{(1-\frac{2-\xi}{\alpha})2s}}\frac{\Gamma(\frac{2-\xi}{\alpha})\Gamma(1-\frac{2-\xi}{\alpha})\Gamma(\frac{d}{2}-s(1-\frac{2-\xi}{\alpha}))}{\Gamma(-(\frac{2-\xi}{\alpha}-1)s)\Gamma(1-\xi)}.$$

Finally, it follows from the inverse Mellin transform that

$$\begin{aligned}
G_\varphi(x,t) =&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widetilde{G}_\varphi(x,\xi)t^{-\xi}\mathrm{d}\xi\\
=&\frac{1}{|x|^d\pi^{\frac{d}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{2-\xi}{\alpha})\Gamma(1-\frac{2-\xi}{\alpha})\Gamma(\frac{d}{2}-s(1-\frac{2-\xi}{\alpha}))}{\Gamma(-(\frac{2-\xi}{\alpha}-1)s)\Gamma(1-\xi)}\times\left(\frac{|x|^{2s}}{2^{2s}}\right)^{1-\frac{2-\xi}{\alpha}}t^{-\xi}\mathrm{d}\left(1-\frac{2-\xi}{\alpha}\right)\\
=&-\frac{t^{\alpha-2}}{|x|^d\pi^{\frac{d}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{d}{2}+s(\frac{2-\xi}{\alpha}-1))\Gamma(1+(\frac{2-\xi}{\alpha}-1))\Gamma(1-1-(\frac{2-\xi}{\alpha}-1))}{\Gamma(\alpha-1+\alpha(\frac{2-\xi}{\alpha}-1))\Gamma(1-1-s(\frac{2-\xi}{\alpha}-1))}\\
&\times\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-(\frac{2-\xi}{\alpha}-1)}\mathrm{d}\left(\frac{2-\xi}{\alpha}-1\right)\\
=&\frac{t^{\alpha-2}}{|x|^d\pi^{\frac{d}{2}}}H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle|\begin{array}{l}(1,1);(\alpha-1,\alpha)\\(1,1),(\frac{d}{2},s);(1,s)\end{array}\right),
\end{aligned}$$

that is,

$$G_\varphi(x,t) = \frac{t^{\alpha-2}}{|x|^d\pi^{\frac{d}{2}}}H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle|\begin{array}{l}(1,1);(\alpha-1,\alpha)\\(1,1),(\frac{d}{2},s);(1,s)\end{array}\right). \tag{2.4}$$

Similarly, we can derive the expression of fundamental solutions $G_\psi(x,t) = G_f(x,t)$ in the form

$$G_\psi(x,t) = G_f(x,t) = \frac{t^{\alpha-1}}{|x|^d\pi^{\frac{d}{2}}}H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle|\begin{array}{l}(1,1);(\alpha,\alpha)\\(1,1),(\frac{d}{2},s);(1,s)\end{array}\right). \tag{2.5}$$

By using asymptotic expansions of the Fox H-function at infinity and zero [15], we can prove the following lemma on the fundamental solutions $G_\varphi(x,t)$ in (2.4) and $G_\psi(x,t) = G_f(x,t)$ in (2.5).

Lemma 2.1. *Let $d\in\mathbb{N}$, $1<\alpha<2$ and $0<s<1$. Suppose $R = t^{-\alpha}|x|^{2s}$. For the fundamental solutions $G_\varphi(x,t)$ in* (2.4) *and $G_\psi(x,t) = G_f(x,t)$ in* (2.5)*, the following asymptotic behaviors hold.*
(1) *If $R>1$, then*

$$|G_\varphi(x,t)| \le Ct^{2\alpha-2}|x|^{-d-2s}, \tag{2.6}$$

and if $R\le 1$, then

$$|G_\varphi(x,t)| \le \begin{cases} Ct^{-\alpha-2}|x|^{-d+4s},\ d>4s,\\ Ct^{-\alpha-2}\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d=4s,\\ Ct^{\alpha-2-\frac{\alpha d}{2s}},\ d<4s.\end{cases} \tag{2.7}$$

(2) *If* $R>1$, *then*

$$|G_\psi(x,t)| = |G_f(x,t)| \le Ct^{2\alpha-1}|x|^{-d-2s}, \tag{2.8}$$

and if $R \le 1$, *then*

$$|G_\psi(x,t)| = |G_f(x,t)| \le \begin{cases} Ct^{-\alpha-1}|x|^{-d+4s},\ d>4s, \\ Ct^{-\alpha-1}\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d=4s, \\ Ct^{\alpha-1-\frac{\alpha d}{2s}},\ d<4s. \end{cases} \tag{2.9}$$

Proof. (1) For the fundamental solution $G_\varphi(x,t)$ given by (2.4), we need to estimate asymptotic expansions of the H-function $H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)$ which is the most important step. Noting that

$$H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle| \begin{array}{l} (1,1);(\alpha-1,\alpha) \\ (1,1),(\frac{d}{2},s);(1,s) \end{array}\right),\ x\neq 0,$$

then we find that $a^* = 2-\alpha > 0$.

We first prove (2.6) for $R>1$. Using Theorems 1.4 and 1.7 in [15] gives

$$H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = \sum_{l=1}^{1}\sum_{k=0}^{\infty} h_{lk}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{1-1-k}{1}} = \sum_{k=0}^{\infty} h_{1k}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-k},$$

where

$$h_{10} = \frac{\Gamma(1)\Gamma(\frac{d}{2})}{\Gamma(\alpha-1)\Gamma(0)} = 0,\quad h_{11} = -\frac{\Gamma(2)\Gamma(\frac{d}{2}+s)}{\Gamma(2\alpha-1)\Gamma(-s)} > 0.$$

Hence, one get

$$H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = h_{11}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1} + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1}\right],\ \frac{|x|^{2s}}{2^{2s}t^\alpha}\to\infty.$$

Furthermore, there holds

$$|G_\varphi(x,t)| \le C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2}h_{11}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1} \le Ct^{2\alpha-2}|x|^{-d-2s},\ R\to\infty.$$

This illustrates that there is a positive constant M satisfying

$$|G_\varphi(x,t)| \le Ct^{2\alpha-2}|x|^{-d-2s},\ R>M. \tag{2.10}$$

In light of analyticity of the H-function $H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)$, we find that it is bounded for $1<R\le M$. Hence,

$$\begin{aligned} |G_\varphi(x,t)| &\le C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2} = C\left(\frac{|x|^{2s}}{t^\alpha}\right)t^{2\alpha-2}|x|^{-d-2s} \\ &\le CMt^{2\alpha-2}|x|^{-d-2s} \le Ct^{2\alpha-2}|x|^{-d-2s},\ 1<R\le M. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we obtain

$$|G_\varphi(x,t)| \le Ct^{2\alpha-2}|x|^{-d-2s}.$$

Next, we show (2.7) with $R \leq 1$. If $d > 4s$, then $b_{1\sigma} = -\frac{1+\sigma}{1} = -(\sigma+1)$ and $b_{2k} = -\frac{d/2+k}{s}$ for $\sigma, k = 0, 1, 2, \ldots$. Therefore, b_{10} is a simple pole and Theorems 1.3 and 1.11 in [15] implies

$$H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right) = \sum_{l=0}^{\infty} h_{1l}^{*}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{1+l}.$$

Since

$$h_{10}^{*} = \frac{\Gamma(\frac{d}{2}-s)\Gamma(1)}{\Gamma(-1)\Gamma(s)} = 0, \quad h_{11}^{*} = -\frac{\Gamma(\frac{d}{2}-2s)\Gamma(2)}{\Gamma(-\alpha-1)\Gamma(2s)} > 0,$$

then it follows that

$$H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right) = h_{11}^{*}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2} + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2}\right], \; \frac{|x|^{2s}}{2^{2s}t^{\alpha}} \to 0,$$

which indicates

$$|G_{\varphi}(x,t)| \leq C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2}h_{11}^{*}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2} \leq Ct^{-\alpha-2}|x|^{-d+4s}, \; R \to 0.$$

Consequently, there exists a positive constant δ such that

$$|G_{\varphi}(x,t)| \leq Ct^{-\alpha-2}|x|^{-d+4s}, \; R < \delta. \tag{2.12}$$

Exploiting again analyticity of the H-function $H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)$ we get

$$\begin{aligned} |G_{\varphi}(x,t)| &\leq C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2} = C\left(\frac{|x|^{2s}}{t^{\alpha}}\right)^{-2} t^{-\alpha-2}|x|^{-d+4s} \\ &= \frac{C}{R^{2}}t^{-\alpha-2}|x|^{-d+4s} \leq Ct^{-\alpha-2}|x|^{-d+4s}, \; \delta \leq R \leq 1. \end{aligned} \tag{2.13}$$

Using (2.12) and (2.13) yields

$$|G_{\varphi}(x,t)| \leq Ct^{-\alpha-2}|x|^{-d+4s}$$

for $d > 4s$, and which is the first inequality in (2.7).

If $d = 4s$, then the poles b_{10} is simple and the poles $b_{11} = b_{20} = -2$ are coincided. In view of Theorems 1.5 and 1.12 in [15] we have

$$H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right) = H_{201}^{*}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2}\log\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right) + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2}\log\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)\right], \; \frac{|x|^{2s}}{2^{2s}t^{\alpha}} \to 0,$$

where $H_{201}^{*} = \frac{\Gamma(2)}{s\Gamma(-\alpha-1)\Gamma(2s)} \neq 0$. As a result,

$$\begin{aligned} |G_{\varphi}(x,t)| &\leq C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2}|H_{201}^{*}|\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{2}\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)\right| \\ &\leq Ct^{-\alpha-2}\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)\right|, \; R \to 0. \end{aligned}$$

That is to say that there exists a positive constant δ_1 such that

$$|G_\varphi(x,t)| \le Ct^{-\alpha-2}\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)\right|,\ R<\delta_1.$$

We further derive

$$|G_\varphi(x,t)| \le Ct^{-\alpha-2}\left(1+\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)\right|\right)$$

for $d=4s$, and the second inequality in (2.7) is proved.

Finally, we show that the third inequality in (2.7) holds when $d<4s$. To do this, we consider three cases respectively. If $d=2s$, then the poles $b_{10}=-1$ and $b_{20}=-\frac{d}{2s}$ are coincide, but the coefficients $H^*_{100}=H^*_{101}=0$ by a direct calculation in terms of Theorems 1.5 and 1.12 in [15]. If $d>2s$, then $b_{10}=-1$ is a simple pole, but we find $h^*_{10}=0$ in this case. If $d<2s$, then $b_{20}=-\frac{d}{2s}$ is a simple pole, by using Theorem 1.11 in [15] one has

$$h^*_2=\frac{\Gamma(1-\frac{d}{2s})\Gamma(\frac{d}{2s})}{s\Gamma(\alpha-1-\frac{\alpha d}{2s})\Gamma(\frac{d}{2})}>0.$$

In either case, we can obtain

$$H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)=h^*_2\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{d}{2s}}+o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{d}{2s}}\right],\ \frac{|x|^{2s}}{2^{2s}t^\alpha}\to 0.$$

Consequently,

$$|G_\varphi(x,t)|\le C\pi^{-\frac{d}{2}}|x|^{-d}t^{\alpha-2}|h^*_2|\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{d}{2s}}\le Ct^{\alpha-2-\frac{\alpha d}{2s}},\ R\to 0.$$

Furthermore it holds that

$$|G_\varphi(x,t)|\le Ct^{\alpha-2-\frac{\alpha d}{2s}}$$

for $d<4s$, and the third inequality holds.

Similarly, we can prove (2.8) and (2.9) by using the same technique as the above (1) and omit them. The proof is now completed. □

In our further consideration, $\|\cdot\|_p$ and $\|\cdot\|_{p,\infty}$ are used to simplify $\|\cdot\|_{L^p(\mathbb{R}^d)}$ and $\|\cdot\|_{L^{p,\infty}(\mathbb{R}^d)}$ respectively, where $L^{p,\infty}(\mathbb{R}^d)$ means weak $L^p(\mathbb{R}^d)$ space on $\mathbb{R}^d$, for example, see [11]. We can also introduce

$$\kappa(d,s)=\begin{cases}\frac{d}{d-4s}, & d>4s,\\ \infty, & d\le 4s,\end{cases}$$

and

$$\kappa^*(d,s)=\begin{cases}\frac{d}{d+1-4s}, & d+2>4s,\\ \infty, & d+2\le 4s.\end{cases}$$

The following estimates of the fundamental solution $G_\varphi(x,t)$ and $G_\psi(x,t)=G_f(x,t)$ in $L^p(\mathbb{R}^d)$ and $L^{p,\infty}(\mathbb{R}^d)$ norms are crucial in proving the asymptotic behaviors of the solution of Eq (1.1).

Lemma 2.2. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Then for any $t > 0$, it holds that $G_\varphi(x,t) \in L^p(\mathbb{R}^d)$ and*

$$\|G_\varphi(x,t)\|_p \leq Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})}, \tag{2.14}$$

for every $1 \leq p < \kappa(d,s)$. Moreover, if $p = \frac{d}{d-4s}$ for $d > 4s$, we have $G_\varphi(x,t) \in L^{\frac{d}{d-4s},\infty}(\mathbb{R}^d)$ and

$$\|G_\varphi(x,t)\|_{\frac{d}{d-4s},\infty} \leq Ct^{-\alpha-2}. \tag{2.15}$$

Proof. Firstly, we prove (2.14). Note that

$$\|G_\varphi(x,t)\|_p^p = \int_{R>1} |G_\varphi(x,t)|^p \mathrm{d}x + \int_{R\leq 1} |G_\varphi(x,t)|^p \mathrm{d}x.$$

From (2.6) one can get

$$\begin{aligned}\int_{R>1} |G_\varphi(x,t)|^p \mathrm{d}x &\leq C \int_{R>1} t^{(2\alpha-2)p} |x|^{-dp-2sp} \mathrm{d}x \\ &\leq Ct^{(2\alpha-2)p} \int_{t^{\frac{\alpha}{2s}}}^{\infty} \rho^{-dp-2sp} \rho^{d-1} \mathrm{d}\rho \\ &\leq Ct^{\alpha p-2p-\frac{\alpha d}{2s}(p-1)},\end{aligned}$$

namely,

$$\Big(\int_{R>1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \leq Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ 1 \leq p < \infty. \tag{2.16}$$

On the other hand, when $d > 4s$ and $1 \leq p < \kappa(d,s)$, it follows from the first inequality in (2.7) that

$$\begin{aligned}\int_{R\leq 1} |G_\varphi(x,t)|^p \mathrm{d}x &\leq C \int_{R\leq 1} t^{(-\alpha-2)p} |x|^{-dp+4sp} \mathrm{d}x \\ &\leq Ct^{(-\alpha-2)p} \int_0^{t^{\frac{\alpha}{2s}}} \rho^{(4s-d)p} \rho^{d-1} \mathrm{d}\rho \\ &\leq Ct^{\alpha p-2p-\frac{\alpha d}{2s}(p-1)},\end{aligned}$$

i.e.,

$$\Big(\int_{R\leq 1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \leq Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ 1 \leq p < \kappa(d,s). \tag{2.17}$$

If $d = 4s$, applying the second inequality in (2.7) we obtain

$$\begin{aligned}\int_{R\leq 1} |G_\varphi(x,t)|^p \mathrm{d}x &\leq C \int_{R\leq 1} t^{(-\alpha-2)p} \Big(1 + |\log(|x|/2)^{2s} t^{-\alpha}|\Big)^p \mathrm{d}x \\ &\leq Ct^{(-\alpha-2)p+2\alpha} \int_0^{\frac{1}{2^{2s}}} \eta(1 + |\log \eta|)^p \mathrm{d}\eta \\ &\leq Ct^{(-\alpha-2)p+2\alpha}\end{aligned}$$

for $1 \leq p < \infty$. Consequently,

$$\Big(\int_{R\leq 1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \leq Ct^{-\alpha-2+\frac{2\alpha}{p}} \leq Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ 1 \leq p < \infty. \tag{2.18}$$

For $d < 4s$, we use the third inequality in (2.7) to derive

$$\int_{R\le 1} |G_\varphi(x,t)|^p \mathrm{d}x \le C\int_{R\le 1} t^{\alpha p-2p-\frac{\alpha d}{2s}p}\mathrm{d}x \le C\int_0^{t^{\frac{\alpha}{2s}}} t^{\alpha p-2p-\frac{\alpha d}{2s}p}\mathrm{d}x \le Ct^{\alpha p-2p-\frac{\alpha d}{2s}p+\frac{\alpha d}{2s}}$$

for $1 \le p < \infty$, which leads to

$$\Big(\int_{R\le 1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \le Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ 1 \le p < \infty. \tag{2.19}$$

Collecting the above estimates (2.16)–(2.19), it follows that

$$\|G_\varphi(x,t)\|_p \le \Big(\int_{R>1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{R\le 1} |G_\varphi(x,t)|^p \mathrm{d}x\Big)^{\frac{1}{p}} \le Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})}$$

for $1 \le p < \kappa(d,s)$ with $d \ge 1$ and $0 < s < 1$.

We next show (2.15). Let $R = t^{-\alpha}|x|^{2s}$ and $p = \frac{d}{d-4s}$ for $d > 4s$. Due to the fact

$$\begin{aligned}\|G_\varphi(x,t)\|_{p,\infty} &= (\|G_\varphi(x,t)\chi_{\{R>1\}}(t) + G_\varphi(x,t)\chi_{\{R\le 1\}}(t)\|_{p,\infty})\\ &\le 2(\|G_\varphi(x,t)\chi_{\{R>1\}}(t)\|_{p,\infty} + \|G_\varphi(x,t)\chi_{\{R\le 1\}}(t)\|_{p,\infty}),\end{aligned}$$

where $\chi_{\{E\}}(t)$ means the characteristic function of the set E. In terms of (2.16), there holds

$$\|G_\varphi(x,t)\chi_{\{R>1\}}(t)\|_{p,\infty} \le \|G_\varphi(x,t)\chi_{\{R>1\}}(t)\|_p \le Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})} = Ct^{-\alpha-2}. \tag{2.20}$$

To estimate $\|G_\varphi(\mathrm{x},t)\chi_{\{R\le 1\}}(t)\|_{p,\infty}$, we may use the first inequality in (2.7) to obtain

$$\begin{aligned}d_{G_\varphi(x,t)\chi_{\{R\le 1\}}(t)}(\gamma) &= \varrho(\{x\in\mathbb{R}^d : |G_\varphi(x,t)| > \gamma \text{ and } R \le 1\})\\ &\le \varrho\Big(\Big\{x\in\mathbb{R}^d : \gamma < Ct^{-\alpha-2}|x|^{4s-d}\Big\}\Big)\\ &\le C\Big(t^{-\alpha-2}\gamma^{-1}\Big)^p,\end{aligned}$$

where ϱ stands for the measure on $\mathbb{R}^d$. Thus we have

$$\gamma(d_{G_\varphi(\mathrm{x},t)\chi_{\{R\le 1\}}(t)}(\gamma))^{\frac{1}{p}} \le Ct^{-\alpha-2}.$$

That is

$$\|G_\varphi(x,t)\chi_{\{R\le 1\}}(t)\|_{p,\infty} \le Ct^{-\alpha-2}. \tag{2.21}$$

Therefore the required result follows by using (2.20) and (2.21) and the proof is thus completed. □

Remark 2.1. *If $d < 4s$, we infer from the third inequality of* (2.7) *in Lemma 2.1 that $G_\varphi(\cdot,t) \in L^\infty(\mathbb{R}^d)$ and $\|G_\varphi(x,t)\|_\infty \le Ct^{\alpha-2-\frac{\alpha d}{2s}}$ for all $t > 0$.*

Lemma 2.3. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. If $1 \le p < \kappa(d,s)$, then $G_\psi(x,t) = G_f(x,t) \in L^p(\mathbb{R}^d)$ for any $t > 0$ and*

$$\|G_\psi(x,t)\|_p = \|G_f(x,t)\|_p \le Ct^{\alpha-1-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ t > 0. \tag{2.22}$$

Moreover, if $p = \frac{d}{d-4s}$ and $d > 4s$, then $G_\psi(x,t) = G_f(x,t) \in L^{\frac{d}{d-4s},\infty}(\mathbb{R}^d)$ for any $t > 0$ and

$$\|G_\psi(x,t)\|_{\frac{d}{d-4s},\infty} = \|G_f(x,t)\|_{\frac{d}{d-4s},\infty} \le Ct^{-\alpha-1},\ t > 0. \tag{2.23}$$

Proof. The proof is similar to that of Lemma 2.2 above. □

Remark 2.2. *For the case $d < 4s$, it follows from the third inequality of* (2.9) *in Lemma 2.1 that* $G_\psi(\cdot,t) = G_f(\cdot,t) \in L^\infty(\mathbb{R}^d)$ *and* $\|G_\psi(x,t)\|_\infty = \|G_f(x,t)\|_\infty \le Ct^{\alpha-1-\frac{\alpha d}{2s}}$ *for any* $t > 0$.

Let us now turn our attention to the asymptotic estimates of the solution to Eq (1.1) when the force term $f \equiv 0$ and the initial values $\varphi = \psi = 0$, respectively.

Theorem 2.1. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Suppose $f \equiv 0$. Then the solution $u(x,t) = G_\varphi(x,t) * \varphi(x) + G_\psi(x,t) * \psi(x)$ to Eq* (1.1)*, where $\varphi, \psi \in L^q(\mathbb{R}^d)$ for $1 \le q \le \infty$, has the following asymptotic estimates:*
(1) *If $q = \infty$, then*

$$\|u(x,t)\|_\infty \le Ct^{\alpha-2}\|\varphi(x)\|_\infty + Ct^{\alpha-1}\|\psi(x)\|_\infty,\ t > 0. \tag{2.24}$$

(2) *If $1 \le q < \infty$, then*

$$\|u(x,t)\|_r \le Ct^{\alpha-2-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\varphi(x)\|_q + Ct^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\psi(x)\|_q,\ t > 0, \tag{2.25}$$

for any

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-4sq}\right), & \text{if } d > 4sq, \\ r \in [q, \infty), & \text{if } d = 4sq, \\ r \in [q, \infty], & \text{if } d < 4sq. \end{cases}$$

Moreover, it holds that

$$\|u(x,t)\|_{\frac{dq}{d-4sq},\infty} \le Ct^{-\alpha-2}\|\varphi(x)\|_q + Ct^{-\alpha-1}\|\psi(x)\|_q,\ t > 0, \tag{2.26}$$

if $d > 4sq$.

Proof. Let $1 \le p, q, r \le \infty$ satisfy the relation

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{2.27}$$

In view of Young's inequality for convolution, see (57) in [21], we get

$$\|u(x,t)\|_r \le \|G_\varphi(x,t)\|_p\|\varphi(x)\|_q + \|G_\psi(x,t)\|_p\|\psi(x)\|_q. \tag{2.28}$$

(1) If $q = \infty$, then $r = \infty$ and $p = 1$ by (2.27). Observe that (2.14) in Lemma 2.2 and (2.22) in Lemma 2.3 for $p = 1$, one has

$$\|G_\varphi(x,t)\|_1 \le Ct^{\alpha-2},\ t > 0,$$

and

$$\|G_\psi(x,t)\|_1 \le Ct^{\alpha-1},\ t > 0,$$

which together with (2.28) yields

$$\|u(x,t)\|_\infty \le Ct^{\alpha-2}\|\varphi(x)\|_\infty + Ct^{\alpha-1}\|\psi(x)\|_\infty,\ t > 0.$$

(2) If $1 \le q < \infty$, then for $r \in \left[q, \frac{dq}{d-4sq}\right)$ when $d > 4sq$, we have $1 \le p < \frac{d}{d-4s}$ when $d > 4s$. Therefore, substituting (2.14) and (2.22) into (2.28) there holds

$$\|u(x,t)\|_r \le Ct^{\alpha-2-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\varphi(x)\|_q + Ct^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\psi(x)\|_q,\ t > 0,$$

as required (2.25). Similarly, we can prove (2.25) for $r \in [q,\infty)$ if $d = 4sq$ and $r \in [q,\infty]$ if $d < 4sq$.

Finally we show (2.26) for $d > 4sq$. Recalling that Young's inequality for weak L^p-norm, see (58) in [21], combining (2.15), (2.23) and (2.27), it follows that

$$\|u(x,t)\|_{\frac{dq}{d-4sq},\infty} \le Ct^{-\alpha-2}\|\varphi(x)\|_q + Ct^{-\alpha-1}\|\psi(x)\|_q,\ t > 0,$$

which is expected inequality (2.26) and the proof is thus complete. □

Theorem 2.2. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. And let $1 \le q < \infty$. Assume that $f(\cdot,t) \in L^q(\mathbb{R}^d)$ for any $t > 0$ and there exists some $\gamma > 0$ such that*

$$\|f(x,t)\|_q \le C(1+t)^{-\gamma},\ t > 0. \tag{2.29}$$

Then for every

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-2sq}\right), & \text{for } 1 \le q < \infty \text{ and } d > 2sq, \\ r \in [q,\infty), & \text{for } 1 < q < \infty \text{ and } d \le 2sq, \end{cases}$$

the solution $u(x,t) = G_f(x,t) \star f(x,t)$ has the following estimates:

$$\|u(x,t)\|_r \le Ct^{\alpha-\min\{1,\gamma\}-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})},\ t > 0,$$

if $\gamma \neq 1$, and

$$\|u(x,t)\|_r \le Ct^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\log(1+t),\ t > 0,$$

if $\gamma = 1$.

Proof. The proof of this theorem can be referred to that of Proposition 5.15 in [18] or Theorem 3.9 in [20]. □

2.2. Gradient estimates and large time behaviors

This subsection will develop gradient estimates and large time behaviors of the solution for Eq (1.1). Let us start with estimates of the derivatives for the fundamental solutions.

Lemma 2.4. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Suppose $R = t^{-\alpha}|x|^{2s}$. Then the spatial derivatives of $G_\varphi(x,t)$ and $G_\psi(x,t)$ and the temporal derivatives of $G_f(x,t)$ have the following estimates:*
(1) *If $R > 1$, then*

$$|\nabla G_\varphi(x,t)| \le Ct^{2\alpha-2}|x|^{-(d+1)-2s}, \tag{2.30}$$

and if $R \le 1$, then

$$|\nabla G_\varphi(x,t)| \le \begin{cases} Ct^{-\alpha-2}|x|^{-(d+1)+4s},\ d+2 > 4s, \\ Ct^{-\alpha-2}|x|\left(1 + \left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d+2 = 4s, \\ Ct^{\alpha-2-\frac{\alpha(d+2)}{2s}}|x|,\ d+2 < 4s. \end{cases} \tag{2.31}$$

(2) *If* $R > 1$, *then*

$$|\nabla G_\psi(x,t)| \le Ct^{2\alpha-1}|x|^{-(d+1)-2s}, \tag{2.32}$$

and if $R \le 1$, *then*

$$|\nabla G_\psi(x,t)| \le \begin{cases} Ct^{-\alpha-1}|x|^{-(d+1)+4s},\ d+2 > 4s, \\ Ct^{-\alpha-1}|x|\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d+2 = 4s, \\ Ct^{\alpha-1-\frac{\alpha(d+2)}{2s}}|x|,\ d+2 < 4s. \end{cases} \tag{2.33}$$

(3) *If* $R > 1$, *then*

$$|\partial_t G_f(x,t)| \le Ct^{2\alpha-2}|x|^{-d-2s}, \tag{2.34}$$

and if $R \le 1$, *then*

$$|\partial_t G_f(x,t)| \le \begin{cases} Ct^{-\alpha-2}|x|^{-d+4s},\ d > 4s, \\ Ct^{-\alpha-2}|x|\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d = 4s, \\ Ct^{\alpha-2-\frac{\alpha d}{2s}},\ d < 4s. \end{cases} \tag{2.35}$$

Proof. We only give the proof of (1), while (2) and (3) can be handled by similar method. Recalling that Property 2.8 in [15] results in

$$\begin{aligned}
&\nabla G_\varphi(x,t) \\
=&\frac{-t^{\alpha-2}}{|x|^{d+1}\pi^{\frac{d}{2}}}H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle| \begin{matrix} (1,1);(\alpha-1,\alpha),(d,2s) \\ (d+1,2s),(1,1),(\frac{d}{2},s);(1,s) \end{matrix}\right)\left(\frac{x_1}{|x|},\frac{x_2}{|x|},\cdots,\frac{x_d}{|x|}\right) \\
=&\frac{-t^{\alpha-2}}{|x|^{d+1}\pi^{\frac{d}{2}}}H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle| \begin{matrix} (1,1);(\alpha-1,\alpha),(d,2s) \\ (1,1),(\frac{d}{2},s),(d+1,2s);(1,s) \end{matrix}\right)\left(\frac{x_1}{|x|},\frac{x_2}{|x|},\cdots,\frac{x_d}{|x|}\right),\ x \ne 0.
\end{aligned}$$

Thus the modulus of $\nabla G_\varphi(x,t)$ is

$$|\nabla G_\varphi(x,t)| = \frac{t^{\alpha-2}}{|x|^{d+1}\pi^{\frac{d}{2}}}\left|H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\middle| \begin{matrix} (1,1);(\alpha-1,\alpha),(d,2s) \\ (1,1),(\frac{d}{2},s),(d+1,2s);(1,s) \end{matrix}\right)\right|,\ x \ne 0. \tag{2.36}$$

First of all, we prove (2.30). By means of Theorems 1.4 and 1.7 in [15] we find that

$$H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = \sum_{k=0}^{\infty} h_{1k}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-k},$$

where

$$h_{10} = \frac{\Gamma(1)\Gamma(d+1)\Gamma(\frac{d}{2})}{\Gamma(\alpha-1)\Gamma(d)\Gamma(0)} = 0,\ \ h_{11} = -\frac{\Gamma(2)\Gamma(\frac{d}{2}+s)\Gamma(d+1+2s)}{\Gamma(2\alpha-1)\Gamma(d+2s)\Gamma(-s)} > 0.$$

Hence, it follows that

$$H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = h_{11}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1} + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1}\right],\ \frac{|x|^{2s}}{2^{2s}t^\alpha} \to \infty,$$

from which one has

$$|\nabla G_\varphi(x,t)| \le C\pi^{-\frac{d}{2}}|x|^{-d-1}t^{\alpha-2}h_{11}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{-1} \le Ct^{2\alpha-2}|x|^{-(d+1)-2s},\ R\to\infty.$$

Using this inequality and the analyticity of the H-function $H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)$, we conclude that there exists a positive constant C such that

$$|\nabla G_\varphi(x,t)| \le Ct^{2\alpha-2}|x|^{-(d+1)-2s}$$

for $R>1$ and which gives (2.30).

Next we show (2.31). According to Theorems 1.3 and 1.11 in [15], we find $h^*_{10} = h^*_{20} = h^*_{30} = 0$. Hence, $b_{11} = -2$ is a simple pole when $d+2>4s$ and

$$H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = h^*_{11}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2 + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2\right],\ \frac{|x|^{2s}}{2^{2s}t^\alpha}\to 0,$$

where

$$h^*_{11} = -\frac{\Gamma(\frac{d}{2}-2s)\Gamma(d+1-4s)\Gamma(2)}{\Gamma(-\alpha-1)\Gamma(d-4s)\Gamma(2s)}.$$

Thus there is a positive constant C such that

$$|\nabla G_\varphi(x,t)| \le C\pi^{-\frac{d}{2}}|x|^{-d-1}t^{\alpha-2}|h^*_{11}|\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2 \le Ct^{-\alpha-2}|x|^{-(d+1)+4s},\ R\to 0.$$

We further obtain

$$|\nabla G_\varphi(x,t)| \le Ct^{-\alpha-2}|x|^{-(d+1)+4s} \tag{2.37}$$

for $R\le 1$ and $d+2>4s$.

If $d+2=4s$, we see that the poles $b_{11}=b_{21}=b_{31}=-2$ are coincided, then

$$H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = H^*_{111}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)\right],\ \ \frac{|x|^{2s}}{2^{2s}t^\alpha}\to 0,$$

where $H^*_{111}\ne 0$. Then one gets

$$|\nabla G_\varphi(x,t)| \le \frac{Ct^{\alpha-2}}{|x|^{d+1}\pi^{\frac{d}{2}}}|H^*_{111}|\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^2\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)\right| \le Ct^{-\alpha-2}|x|\left|\log\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)\right|,\ R\to 0.$$

Furthermore, it follows that

$$|\nabla G_\varphi(x,t)| \le Ct^{-\alpha-2}|x|\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right) \tag{2.38}$$

for $R\le 1$ and $d+2=4s$.

It remains to show the case $d+2<4s$. Since the poles $b_{21}=b_{31}=-\frac{d+2}{2s}$ are coincided, then

$$H^{31}_{34}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right) = H^*_{210}\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{d+2}{2s}} + o\left[\left(\frac{|x|^{2s}}{2^{2s}t^\alpha}\right)^{\frac{d+2}{2s}}\right],\ \frac{|x|^{2s}}{2^{2s}t^\alpha}\to 0,$$

with $H_{210}^{*} \neq 0$, which leads to

$$|\nabla G_{\varphi}(x,t)| \leq \frac{Ct^{\alpha-2}}{|x|^{d+1}\pi^{\frac{d}{2}}}|H_{210}^{*}|\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\right)^{\frac{d+2}{2s}} \leq Ct^{\alpha-2-\frac{\alpha(d+2)}{2s}}|x|,\ R \to 0.$$

So we have

$$|\nabla G_{\varphi}(x,t)| \leq Ct^{\alpha-2-\frac{\alpha(d+2)}{2s}}|x| \tag{2.39}$$

for $R \leq 1$ and $d+2 < 4s$.

Based on (2.37)–(2.39), the desired assertion (2.31) is obtained and the proof is now completed. □

According to Lemma 2.4, we can establish estimates of $\|\nabla G_{\varphi}(x,t)\|_p$, $\|\nabla G_{\psi}(x,t)\|_p$, and $\|\nabla G_f(x,t)\|_p$ and further derive asymptotic properties of $\nabla u(x,t)$ to Eq (1.1), whose proofs are very similar to those of counterparts in the previous subsection and omitted.

Lemma 2.5. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Assume that $1 \leq p < \kappa^{*}(d,s)$. Then it holds that $\nabla G_{\varphi}(x,t) \in L^{p}(\mathbb{R}^d;\mathbb{R}^d)$ for any $t > 0$ and*

$$\|\nabla G_{\varphi}(x,t)\|_p \leq Ct^{\alpha-2-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ t > 0. \tag{2.40}$$

Moreover, if $p = \frac{d}{d+1-4s}$ for $d+2 > 4s$, then $\nabla G_{\varphi}(x,t) \in L^{\frac{d}{d+1-4s},\infty}(\mathbb{R}^d;\mathbb{R}^d)$ for all $t > 0$ and

$$\|\nabla G_{\varphi}(x,t)\|_{\frac{d}{d+1-4s},\infty} \leq Ct^{-\alpha-2},\ t > 0. \tag{2.41}$$

Lemma 2.6. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Assume that $1 \leq p < \kappa^{*}(d,s)$. Then it holds that $\nabla G_{\psi}(x,t) = \nabla G_f(x,t) \in L^{p}(\mathbb{R}^d;\mathbb{R}^d)$ for all $t > 0$ and*

$$\|\nabla G_{\psi}(x,t)\|_p = \|\nabla G_f(x,t)\|_p \leq Ct^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ t > 0. \tag{2.42}$$

Moreover, if $p = \frac{d}{d+1-4s}$ for $d+2 > 4s$, then $\nabla G_{\psi}(x,t) = \nabla G_f(x,t) \in L^{\frac{d}{d+1-4s},\infty}(\mathbb{R}^d;\mathbb{R}^d)$ for all $t > 0$ and

$$\|\nabla G_{\psi}(x,t)\|_{\frac{d}{d+1-4s},\infty} = \|\nabla G_f(x,t)\|_{\frac{d}{d+1-4s},\infty} \leq Ct^{-\alpha-1},\ t > 0. \tag{2.43}$$

Theorem 2.3. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Suppose that $1 \leq q \leq \infty$ and $f \equiv 0$. Then the following estimates hold on the gradient of solution $\nabla u(x,t) = \nabla G_{\varphi}(x,t) * \varphi(x) + \nabla G_{\psi}(x,t) * \psi(x)$ with $\varphi, \psi \in L^{q}(\mathbb{R}^d)$.*

(1) *If $q = \infty$, then*

$$\|\nabla u(x,t)\|_\infty \leq Ct^{\alpha-2-\frac{\alpha}{2s}}\|\varphi(x)\|_\infty + Ct^{\alpha-1-\frac{\alpha}{2s}}\|\psi(x)\|_\infty,\ t > 0.$$

(2) *If $1 \leq q < \infty$, then for any*

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-(4s-1)q}\right), & \text{if } d > (4s-1)q, \\ r \in [q,\infty), & \text{if } d \leq (4s-1)q, \end{cases}$$

one has

$$\|\nabla u(x,t)\|_r \leq Ct^{\alpha-2-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\varphi(x)\|_q + Ct^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\psi(x)\|_q,\ t > 0.$$

Moreover, if $d > (4s-1)q$, then

$$\|\nabla u(x,t)\|_{\frac{dq}{d-(4s-1)q},\infty} \leq Ct^{-\alpha-2}\|\varphi(x)\|_q + Ct^{-\alpha-1}\|\psi(x)\|_q,\ t > 0.$$

Theorem 2.4. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Let $1 \le q < \infty$ and the condition (2.29) be satisfied. Then for every*

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-(4s-1)q}\right), & \text{for } 1 \le q < \infty \text{ and } d > (4s-1)q, \\ r \in [q, \infty), & \text{for } 1 < q < \infty \text{ and } d \le (4s-1)q, \end{cases}$$

the gradient of solution $\nabla u(x,t) = \nabla G_f(x,t) \star f(x,t)$ has the following relations:

$$\|\nabla u(x,t)\|_r \le C t^{\alpha - \min\{1,\gamma\} - \frac{\alpha}{2s} - \frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})},\ t > 0, \tag{2.44}$$

if $\gamma \neq 1$, and

$$\|\nabla u(x,t)\|_r \le C t^{\alpha - 1 - \frac{\alpha}{2s} - \frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})} \log(1+t),\ t > 0, \tag{2.45}$$

if $\gamma = 1$.

The last two theorems present the large time behavior of the solution $u(x,t)$ for Eq (1.1).

Theorem 2.5. *Let $d \in \mathbb{N}$, $1 < \alpha < 2$ and $0 < s < 1$. Denote $M_\varphi = \int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x$ and $M_\psi = \int_{\mathbb{R}^d} \psi(x) \mathrm{d}x$ with $\varphi, \psi \in L^1(\mathbb{R}^d)$. Assume that $f \equiv 0$ and $1 \le p < \kappa^*(d,s)$. Then we have the following results.*
(1) *If $\| |x|\varphi(x)\|_1 < \infty$ and $\| |x|\psi(x)\|_1 < \infty$, then*

$$t^{\frac{\alpha d}{2s}(1-\frac{1}{p})+1-\alpha} \|u(x,t) - M_\varphi G_\varphi(x,t) - M_\psi G_\psi(x,t)\|_p \le C t^{-\frac{\alpha}{2s}-1} + C t^{-\frac{\alpha}{2s}} \tag{2.46}$$

for any $t > 0$. Moreover, when $p = \frac{d}{d+1-4s}$, one gets

$$t^{\frac{\alpha(4s-1)}{2s}+1-\alpha} \|u(x,t) - M_\varphi G_\varphi(x,t) - M_\psi G_\psi(x,t)\|_{\frac{d}{d+1-4s},\infty} \le C t^{-\frac{\alpha}{2s}-1} + C t^{-\frac{\alpha}{2s}} \tag{2.47}$$

for any $t > 0$.
(2) *It follows that*

$$t^{\frac{\alpha d}{2s}(1-\frac{1}{p})+1-\alpha} \|u(x,t) - M_\varphi G_\varphi(x,t) - M_\psi G_\psi(x,t)\|_p \to 0 \tag{2.48}$$

as $t \to \infty$.

Proof. (1) Note that the conditions $\varphi, \psi \in L^1(\mathbb{R}^d)$ and $\| |x|\varphi(x)\|_1 < \infty$ and $\| |x|\psi(x)\|_1 < \infty$. It follows from the decomposition lemma (see Lemma 8.4 [18]) that there exists functions $\Phi, \Psi \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\varphi = M_\varphi \delta_\varphi + \mathrm{div}\Phi, \quad \psi = M_\psi \delta_\psi + \mathrm{div}\Psi,$$

where $\|\Phi\|_1 \le C\| |x|\varphi(x)\|_1$ and $\|\Psi\|_1 \le C\| |x|\psi(x)\|_1$. Therefore we find that

$$\begin{aligned} u(x,t) =& G_\varphi(x,t) * (M_\varphi \delta_\varphi + \mathrm{div}\Phi) + G_\psi(x,t) * (M_\psi \delta_\psi + \mathrm{div}\Psi) \\ =& M_\varphi G_\varphi(x,t) + \nabla G_\varphi(x,t) \bullet * \Phi(x) + M_\psi G_\psi(x,t) + \nabla G_\psi(x,t) \bullet * \Psi(x), \end{aligned}$$

where $\bullet *$ means the convolution between two vector functions. Further there holds

$$u(x,t) - M_\varphi G_\varphi(x,t) - M_\psi G_\psi(x,t) = \nabla G_\varphi(x,t) \bullet * \Phi(x) + \nabla G_\psi(x,t) \bullet * \Psi(x). \tag{2.49}$$

By using Young's inequality for convolution (57) in [21], and taking (2.40) and (2.42) into account, we obtain

$$\begin{aligned}\|u(x,t)-M_\varphi G_\varphi(x,t)-M_\psi G_\psi(x,t)\|_p \leq&\|\nabla G_\varphi(x,t)\|_p\|\Phi(x)\|_1+\|\nabla G_\psi(x,t)\|_p\|\Psi(x)\|_1\\ \leq& Ct^{\alpha-2-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})}+Ct^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})}\end{aligned}$$

for $1\leq p<\kappa^*(d,s)$, which gives

$$t^{\frac{\alpha d}{2s}(1-\frac{1}{p})+1-\alpha}\|u(x,t)-M_\varphi G_\varphi(x,t)-M_\psi G_\psi(x,t)\|_p\leq Ct^{-\frac{\alpha}{2s}-1}+Ct^{-\frac{\alpha}{2s}},$$

and the claim (2.46) holds. For the limit case $p=\frac{d}{d+1-4s}$, applying Young's inequality for convolution (58) in [21], (2.41) and (2.43) to (2.49) we immediately know that (2.47) is true.

(2) Let a sequence $\{\eta_m(x)\}\subseteq C_0^\infty(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d}\eta_m(x)\mathrm{d}x=M_\varphi$ for all m and $\eta_m(x)\to\varphi(x)$ as $m\to\infty$ in $L^1(\mathbb{R}^d)$. Likewise, set a sequence $\{\zeta_n(x)\}\subseteq C_0^\infty(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d}\zeta_n(x)\mathrm{d}x=M_\psi$ for all n and $\zeta_n(x)\to\psi(x)$ as $n\to\infty$ in $L^1(\mathbb{R}^d)$. Now we use Young's inequality for convolution (57) in [21], (2.14), (2.22) and the conclusion of (1) to derive for any m,n

$$\begin{aligned}&\|u(x,t)-M_\varphi G_\varphi(x,t)-M_\psi G_\psi(x,t)\|_p\\ \leq&\|G_\varphi(x,t)*\varphi(x)-M_\varphi G_\varphi(x,t)\|_p+\|G_\psi(x,t)*\psi(x)-M_\psi G_\psi(x,t)\|_p\\ \leq&\|G_\varphi(x,t)*(\varphi(x)-\eta_m(x))\|_p+\|G_\varphi(x,t)*\eta_m(x)-M_\varphi G_\varphi(x,t)\|_p\\ &+\|G_\psi(x,t)*(\psi(x)-\zeta_n(x))\|_p+\|G_\psi(x,t)*\zeta_n(x)-M_\psi G_\psi(x,t)\|_p\\ \leq& Ct^{\alpha-2-\frac{\alpha d}{2s}(1-\frac{1}{p})}\|\varphi-\eta_m\|_1+C_mt^{\alpha-2-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})}+Ct^{\alpha-1-\frac{\alpha d}{2s}(1-\frac{1}{p})}\|\psi-\zeta_n\|_1+C_nt^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})}.\end{aligned}$$

Consequently,

$$t^{\frac{\alpha d}{2s}(1-\frac{1}{p})+1-\alpha}\|u(x,t)-M_\varphi G_\varphi(x,t)-M_\psi G_\psi(x,t)\|_p\leq Ct^{-1}\|\varphi-\eta_m\|_1+C\|\psi-\zeta_n\|_1+C_mt^{-\frac{\alpha}{2s}-1}+C_nt^{-\frac{\alpha}{2s}},$$

from which we have

$$\limsup_{t\to\infty}t^{\frac{\alpha d}{2s}(1-\frac{1}{p})+1-\alpha}\|u(x,t)-M_\varphi G_\varphi(x,t)-M_\psi G_\psi(x,t)\|_p\leq C\|\psi-\zeta_n\|_1,$$

and the claimed result (2.48) can be achieved by letting $n\to\infty$. The proof of this theorem is now ended. □

Theorem 2.6. *Let $d\in\mathbb{N}$, $1<\alpha<2$ and $0<s<1$. Let $\varphi\equiv\psi\equiv0$ and denote $M_f=\int_0^\infty\int_{\mathbb{R}^d}f(x,t)\mathrm{d}x\mathrm{d}t$. Moreover, let us assume that $f(x,t)\in L^1\left(\mathbb{R}^d\times(0,\infty)\right)$ and there exists some $\gamma>1$ such that*

$$\|f(x,t)\|_1\leq C(1+t)^{-\gamma},\ t>0.$$

Then it holds that

$$t^{1-\alpha+\frac{\alpha d}{2s}(1-\frac{1}{p})}\|u(x,t)-M_fG_f(x,t)\|_p\to0$$

as $t\to\infty$ for any

$$\begin{cases}1\leq p\leq\infty, & \text{if } d<4s,\\ 1\leq p<\kappa(d,s), & \text{if } d\geq4s.\end{cases}$$

Proof. The proof can be completed by using the methods provided by Theorem 2.21 in [18] or Theorem 3.16 in [20] and the details are omitted here. □

3. Decay estimates of solution for Eq (1.2)

In this section we shall deal with decay behaviors of the solution for Eq (1.2). Similar to the previous section, we present the asymptotics of the fundamental solution, decay estimates of the solution, gradient estimates and large time behaviors. In particular the optimal L^2-decay estimates are provided by virtue of Plancherel's theorem and the boundedness of Mittag-Leffler function. For the most of theorems and lemmas in the section, we directly give results without proofs since their proof techniques are very similar to ones of corresponding conclusions in the previous section.

3.1. Decay behaviors of the solution

We first construct the solution of Eq (1.2) by integral transforms. Applying Fourier and Laplace transforms to Eq (1.2), and noticing that formula (2.248) in [28] and equality (A4), there holds

$$\lambda^{\alpha}\bar{\bar{\widehat{u}}}(\omega,\lambda)-\widehat{\phi}(\omega)+|\omega|^{2s}\bar{\bar{\widehat{u}}}(\omega,\lambda)=\bar{\bar{\widehat{g}}}(\omega,\lambda). \tag{3.1}$$

Then we get

$$\begin{aligned}\bar{\bar{\widehat{u}}}(\omega,\lambda)&=\frac{1}{\lambda^{\alpha}+|\omega|^{2s}}\widehat{\phi}(\omega)+\frac{1}{\lambda^{\alpha}+|\omega|^{2s}}\bar{\bar{\widehat{g}}}(\omega,\lambda)\\&:=\bar{\bar{\widehat{\mathcal{G}}}}(\omega,\lambda)\widehat{\phi}(\omega)+\bar{\bar{\widehat{\mathcal{G}}}}(\omega,\lambda)\bar{\bar{\widehat{g}}}(\omega,\lambda).\end{aligned} \tag{3.2}$$

Performing the inverse Fourier transform and inverse Laplace transform on both sides of (3.2), the solution of Eq (1.2) reads as

$$\begin{aligned}u(x,t)=&\mathcal{G}(x,t)*\phi(x)+\mathcal{G}(x,t)\star g(x,t)\\=&\int_{\mathbb{R}^d}\mathcal{G}(x-y,t)\phi(y)\mathrm{d}y+\int_0^t\int_{\mathbb{R}^d}\mathcal{G}(x-y,t-\tau)g(y,\tau)\mathrm{d}y\mathrm{d}\tau,\end{aligned} \tag{3.3}$$

where the fundamental solution [18]

$$\mathcal{G}(x,t)=\frac{t^{\alpha-1}}{|x|^d\pi^{\frac{d}{2}}}H^{21}_{23}\left(\frac{|x|^{2s}}{2^{2s}t^{\alpha}}\left|\begin{array}{l}(1,1);(\alpha,\alpha)\\(1,1),(\frac{d}{2},s);(1,s)\end{array}\right.\right). \tag{3.4}$$

Lemma 3.1. *[18] Let $d\in\mathbb{N}$, $0<\alpha<1$ and $0<s<1$. Suppose $R=t^{-\alpha}|x|^{2s}$. Then for the fundamental solution $\mathcal{G}(x,t)$ in (3.4) we have*

$$|\mathcal{G}(x,t)|\le Ct^{2\alpha-1}|x|^{-d-2s} \tag{3.5}$$

for $R>1$, and

$$|\mathcal{G}(x,t)|\le\begin{cases}Ct^{-\alpha-1}|x|^{-d+4s},\ d>4s,\\Ct^{-\alpha-1}\left(1+\left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d=4s,\\Ct^{\alpha-1-\frac{\alpha d}{2s}},\ d<4s\end{cases} \tag{3.6}$$

for $R\le 1$.

Lemma 3.2. *[18] Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. If $1 \le p < \kappa(d,s)$, then we have $\mathcal{G}(x,t) \in L^p(\mathbb{R}^d)$ for all $t > 0$ and*

$$\|\mathcal{G}(x,t)\|_p \le Ct^{\alpha-1-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ t > 0. \tag{3.7}$$

And if moreover $p = \frac{d}{d-4s}$ for $d > 4s$, then it follows that $\mathcal{G}(x,t) \in L^{\frac{d}{d-4s},\infty}(\mathbb{R}^d)$ for all $t > 0$ and

$$\|\mathcal{G}(x,t)\|_{\frac{d}{d-4s},\infty} \le Ct^{-\alpha-1},\ t > 0. \tag{3.8}$$

Remark 3.1. *If $d < 4s$, by the third inequality of (3.6) in Lemma 3.1, it is clear that $\mathcal{G}(\cdot,t) \in L^\infty(\mathbb{R}^d)$ and $\|\mathcal{G}(x,t)\|_\infty \le Ct^{\alpha-1-\frac{\alpha d}{2s}}$ hold for all $t > 0$.*

Theorem 3.1. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $1 \le q \le \infty$ and $g \equiv 0$. Then the solution $u(x,t) = \mathcal{G}(x,t) * \phi(x)$ to Eq (1.2) with $\phi \in L^q(\mathbb{R}^d)$ has the following decay estimates:*
(1) *If $q = \infty$, then*

$$\|u(x,t)\|_\infty \le Ct^{\alpha-1}\|\phi(x)\|_\infty,\ t > 0.$$

(2) *If $1 \le q < \infty$, then*

$$\|u(x,t)\|_r \le Ct^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\phi(x)\|_q,\ t > 0$$

holds for any

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-4sq}\right), & \text{if } d > 4sq, \\ r \in [q,\infty), & \text{if } d = 4sq, \\ r \in [q,\infty], & \text{if } d < 4sq. \end{cases}$$

If moreover $d > 4sq$, then

$$\|u(x,t)\|_{\frac{dq}{d-4sq},\infty} \le Ct^{-\alpha-1}\|\phi(x)\|_q,\ t > 0.$$

Theorem 3.2. *[18] Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $1 \le q < \infty$ and $\phi \equiv 0$. Assume that $g(\cdot,t) \in L^q(\mathbb{R}^d)$ for all $t > 0$ and there is some $\gamma > 0$ such that*

$$\|g(x,t)\|_q \le C(1+t)^{-\gamma},\ t > 0. \tag{3.9}$$

Then the solution $u(x,t) = \mathcal{G}(x,t) \star g(x,t)$ satisfies the following

$$\|u(x,t)\|_r \le Ct^{\alpha-\min\{1,\gamma\}-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})},\ \gamma \ne 1,$$

and

$$\|u(x,t)\|_r \le Ct^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\log(1+t),\ \gamma = 1,$$

for any $t > 0$ and

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-2sq}\right), & \text{for } 1 \le q < \infty \text{ and } d > 2sq, \\ r \in [q,\infty), & \text{for } 1 < q < \infty \text{ and } d \le 2sq. \end{cases}$$

3.2. Gradient estimates and large time behaviors

The gradient estimates and large time behaviors of the solution for Eq (1.2) are provided in the subsection.

Lemma 3.3. *[18] Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Suppose $R = t^{-\alpha}|x|^{2s}$. Then the spatial and time derivatives of the fundamental solution $\mathcal{G}(x,t)$ in (3.4) have the following decay behaviors:*
(1) *If $R > 1$, then*

$$|\nabla\mathcal{G}(x,t)| \le Ct^{2\alpha-1}|x|^{-(d+1)-2s},\ d \ge 1,\ 0 < s < 1, \tag{3.10}$$

and if $R \le 1$, then

$$|\nabla\mathcal{G}(x,t)| \le \begin{cases} Ct^{-\alpha-1}|x|^{-(d+1)+4s},\ d+2 > 4s, \\ Ct^{-\alpha-1}|x|\left(1 + \left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d+2 = 4s, \\ Ct^{\alpha-1-\frac{\alpha(d+2)}{2s}}|x|,\ d+2 < 4s. \end{cases} \tag{3.11}$$

(2) *If $R > 1$, then*

$$|\partial_t\mathcal{G}(x,t)| \le Ct^{2\alpha-2}|x|^{-d-2s},\ d \ge 1,\ 0 < s < 1, \tag{3.12}$$

and if $R \le 1$, then

$$|\partial_t\mathcal{G}(x,t)| \le \begin{cases} Ct^{-\alpha-2}|x|^{-d+4s},\ d > 4s, \\ Ct^{-\alpha-2}|x|\left(1 + \left|\log\left((|x|/2)^{2s}t^{-\alpha}\right)\right|\right),\ d = 4s, \\ Ct^{\alpha-2-\frac{\alpha d}{2s}},\ d < 4s. \end{cases} \tag{3.13}$$

Lemma 3.4. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. For $1 \le p < \kappa^*(d,s)$, it holds that $\nabla\mathcal{G}(x,t) \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ for any $t > 0$ and*

$$\|\nabla\mathcal{G}(x,t)\|_p \le Ct^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(1-\frac{1}{p})},\ t > 0. \tag{3.14}$$

Moreover, if $p = \frac{d}{d+1-4s}$ for $d+2 > 4s$, we have $\nabla\mathcal{G}(x,t) \in L^{\frac{d}{d+1-4s},\infty}(\mathbb{R}^d;\mathbb{R}^d)$ for any $t > 0$ and

$$\|\nabla\mathcal{G}(x,t)\|_{\frac{d}{d+1-4s},\infty} \le Ct^{-\alpha-1},\ t > 0. \tag{3.15}$$

Theorem 3.3. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $1 \le q \le \infty$ and $g \equiv 0$. Then for $\nabla u(x,t) = \nabla\mathcal{G}(x,t) * \phi(x)$ with $\phi \in L^q(\mathbb{R}^d)$, we have:*
(1) *If $q = \infty$, then*

$$\|\nabla u(x,t)\|_\infty \le Ct^{\alpha-1-\frac{\alpha}{2s}}\|\phi(x)\|_\infty,\ t > 0.$$

(2) *If $1 \le q < \infty$, then*

$$\|\nabla u(x,t)\|_r \le Ct^{\alpha-1-\frac{\alpha}{2s}-\frac{\alpha d}{2s}(\frac{1}{q}-\frac{1}{r})}\|\phi(x)\|_q,\ t > 0$$

for any

$$\begin{cases} r \in \left[q, \dfrac{dq}{d-(4s-1)q}\right), & \text{if } d > (4s-1)q, \\ r \in [q,\infty), & \text{if } d \le (4s-1)q. \end{cases}$$

Moreover if $d > (4s-1)q$, then

$$\|\nabla u(x,t)\|_{\frac{dq}{d-(4s-1)q},\infty} \le Ct^{-\alpha-1}\|\phi(x)\|_q,\ t > 0.$$

Theorem 3.4. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $1 \leq q < \infty$ and $\phi \equiv 0$. Assume that $g(\cdot, t) \in L^q(\mathbb{R}^d)$ for any $t > 0$ and the assumption* (3.9) *holds. Then $\nabla u(x,t) = \nabla \mathcal{G}(x,t) \star g(x,t)$ has the estimates*

$$\|\nabla u(x,t)\|_r \leq Ct^{\alpha - \min\{1,\gamma\} - \frac{\alpha}{2s} - \frac{\alpha d}{2s}(\frac{1}{q} - \frac{1}{r})}, \ \gamma \neq 1,$$

and

$$\|\nabla u(x,t)\|_r \leq Ct^{\alpha - 1 - \frac{\alpha}{2s} - \frac{\alpha d}{2s}(\frac{1}{q} - \frac{1}{r})} \log(1+t), \ \gamma = 1,$$

for all $t > 0$ and for every

$$\begin{cases} r \in \left[q, \dfrac{dq}{d - (4s-1)q}\right), & \text{if } 1 \leq q < \infty \text{ and } d > (4s-1)q, \\ r \in [q, \infty), & \text{if } 1 < q < \infty \text{ and } d \leq (4s-1)q. \end{cases}$$

Theorem 3.5. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $1 \leq p < \kappa^*(d,s)$ and $g \equiv 0$. Assume $\phi \in L^1(\mathbb{R}^d)$ and denote $M_\phi = \int_{\mathbb{R}^d} \phi(x)\mathrm{d}x$. Then the following results hold.*
(1) *If $\| |x|\phi(x)\|_1 < \infty$, then*

$$t^{1-\alpha+\frac{\alpha d}{2s}(1-\frac{1}{p})} \|u(x,t) - M_\phi \mathcal{G}(x,t)\|_p \leq Ct^{-\frac{\alpha}{2s}}, \ t > 0.$$

When $p = \frac{d}{d+1-4s}$, one gets

$$t^{1-\alpha+\frac{\alpha(4s-1)}{2s}} \|u(x,t) - M_\phi \mathcal{G}(x,t)\|_{\frac{d}{d+1-4s},\infty} \leq Ct^{-\frac{\alpha}{2s}}, \ t > 0.$$

(2) *It follows that*

$$t^{1-\alpha+\frac{\alpha d}{2s}(1-\frac{1}{p})} \|u(x,t) - M_\phi \mathcal{G}(x,t)\|_p \to 0$$

as $t \to \infty$.

Theorem 3.6. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$ and $0 < s < 1$. Let $\phi \equiv 0$ and denote $M_g = \int_0^\infty \int_{\mathbb{R}^d} g(x,t)\mathrm{d}x\mathrm{d}t$. Let us assume $g(x,t) \in L^1\left(\mathbb{R}^d \times (0,\infty)\right)$ and there exists some $\gamma > 1$ such that*

$$\|g(x,t)\|_1 \leq C(1+t)^{-\gamma}, \ t > 0.$$

Then for all

$$\begin{cases} 1 \leq p \leq \infty, & \text{if } d < 4s, \\ 1 \leq p < \kappa(d,s), & \text{if } d \geq 4s, \end{cases}$$

it holds that

$$t^{1-\alpha+\frac{\alpha d}{2s}(1-\frac{1}{p})} \|u(x,t) - M_g \mathcal{G}(x,t)\|_p \to 0$$

as $t \to \infty$.

3.3. The optimal L^2-decay estimate

In this subsection our attention will be restricted to decay estimate of the solution for Eq (1.2) in the sense of L^2-norm when $g \equiv 0$. To do this, let us consider the solution of Eq (1.2) under the Fourier transform. By using the inverse Laplace transform on both sides of (3.1) we obtain

$$\widehat{u}(\omega, t) = \widehat{\mathcal{G}}(\omega, t)\widehat{\phi}(\omega) + \int_0^t \widehat{\mathcal{G}}(\omega, t-\tau)\widehat{g}(\omega, \tau)\mathrm{d}\tau,$$

where

$$\widehat{\mathcal{G}}(\omega, t) = t^{\alpha-1}E_{\alpha,\alpha}(-|\omega|^{2s}t^{\alpha}).$$

To derive optimal L^2 decay rate, we need investigate some properties of the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$.

Lemma 3.5. *[13] Let $0 < \alpha < 1$. Then the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta) > 0$ for $\eta \in (0, \infty)$.*

Lemma 3.6. *[13] Let $0 \le \alpha \le 1$ and $\beta \ge \alpha$. Then the Mittag-Leffler function $E_{\alpha,\beta}(-\eta)$ is completely monotone for $\eta \in (0, \infty)$.*

Lemma 3.7. *Let $0 < \alpha < 1$. Then there are positive constants C_1 and C_2 such that*

$$\frac{C_1}{1+\eta^2} \le E_{\alpha,\alpha}(-\eta) \le \frac{C_2}{1+\eta^2},\ \eta \ge 0.$$

Proof. First of all, we know that $E_{\alpha,\alpha}(-\eta) > 0$ for $0 < \alpha < 1$ and $\eta \ge 0$. In fact, one has $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha) > 0$. For $\eta > 0$, it is evident that $E_{\alpha,\alpha}(-\eta) > 0$ by Lemma 3.5.

In view of asymptotic expansions for the Mittag-Leffler function (see Theorem 1.4 in [28]) we obtain

$$E_{\alpha,\alpha}(-\eta) = -\frac{1}{\Gamma(-\alpha)}\frac{1}{\eta^2} + O(\eta^3),\ \eta \to \infty.$$

This implies the function $E_{\alpha,\alpha}(-\eta)$ behaves as C_0/η^2 when $\eta \to \infty$ with some positive constant C_0, i.e., there exist a positive real number X and two positive constants C_3 and C_4 such that

$$\frac{C_3}{1+\eta^2} \le E_{\alpha,\alpha}(-\eta) \le \frac{C_4}{1+\eta^2},\ \eta > X. \tag{3.16}$$

Since the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$ is analytic for any $\eta \in \mathbb{R}$, then it is bound on the interval $[0, X]$. Moreover, it follows from Lemma 3.6 that the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$ is monotonically decreasing for $\eta \in (0, \infty)$. Therefore, there are positive constants C_5 and C_6 satisfying $C_5 \le E_{\alpha,\alpha}(-\eta) \le C_6$ with $0 \le \eta \le X$. We further obtain

$$\frac{C_7}{1+\eta^2} \le E_{\alpha,\alpha}(-\eta) \le \frac{C_8}{1+\eta^2},\ \eta \in [0, X] \tag{3.17}$$

with positive constants C_7 and C_8.

The desired result follows from (3.16) and (3.17) and the proof is complete. □

We first present the estimates of lower bound for the solution $u(x, t) = \mathcal{G}(x, t) * \phi(x)$ to Eq (1.2) with $g \equiv 0$ in the sense of L^2-norm.

Theorem 3.7. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, $0 < s < 1$, and $d \neq 8s$. Let $g \equiv 0$ in Eq (1.2). If $\phi \in L^1(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \phi(x)\mathrm{d}x \neq 0$, then the solution $u(x,t) = \mathcal{G}(x,t) * \phi(x)$ to Eq (1.2) has the lower bound estimate*

$$\|u(x,t)\|_2 \geq Ct^{-\min\{\alpha+1, 1-\alpha+\frac{\alpha d}{4s}\}},\ t > t_0 > 0.$$

Proof. Let $\rho = \rho(t) \in (0, \rho_0]$ with $t > 0$ and $\rho_0 > 0$. According to Plancherel's theorem and the estimate for the Mittag-Leffler function in Lemma 3.7, it follows that

$$\begin{aligned}
\|u(x,t)\|_2^2 = \|\widehat{u}(\omega,t)\|_2^2 &= \int_{\mathbb{R}^d} |\widehat{\mathcal{G}}(\omega,t)|^2 |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&\geq \int_{B_\rho(0)} |t^{\alpha-1} E_{\alpha,\alpha}(-|\omega|^{2s} t^\alpha)|^2 |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&\geq \frac{Ct^{2\alpha-2}}{(1+\rho^{4s}t^{2\alpha})^2} \int_{B_\rho(0)} |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&= \frac{Ct^{2\alpha-2}}{(1+\rho^{4s}t^{2\alpha})^2} \rho^d \left(\rho^{-d} \int_{B_\rho(0)} |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega\right).
\end{aligned} \tag{3.18}$$

From the Plancherel's theorem and Riemann-Lebesgue lemma, it is easy to verify that $\widehat{\phi} \in C_0(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$ holds. By making use of Lebesgue differentiation theorem we have for a sufficient small ρ_0

$$\rho^{-d} \int_{B_\rho(0)} |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \geq \frac{|\widehat{\phi}(0)|^2}{2},\ \rho \in (0, \rho_0].$$

Substituting this into (3.18) leads to

$$\|u(x,t)\|_2^2 \geq \frac{Ct^{2\alpha-2}|\widehat{\phi}(0)|^2 \rho^d}{2(1+\rho^{4s}t^{2\alpha})^2}. \tag{3.19}$$

Now letting $\rho = \rho_0$ in (3.19) one gets

$$\|u(x,t)\|_2^2 \geq Ct^{-2\alpha-2},\ t > t_0 > 0. \tag{3.20}$$

On the other hand, we may take $\rho = \frac{\rho_0}{(1+t^{2\alpha})^{1/4s}}$ in (3.19) to derive

$$\|u(x,t)\|_2^2 \geq Ct^{2\alpha-2-\frac{\alpha d}{2s}},\ t > t_0 > 0. \tag{3.21}$$

Together with (3.20) and (3.21), the desired lower bound is established and the proof is thus completed. □

The upper bound for the solution $u(x,t) = \mathcal{G}(x,t) * \phi(x)$ to Eq (1.2) is estimated in the following theorem when $g \equiv 0$.

Theorem 3.8. *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, $0 < s < 1$, and $d \neq 8s$. Let $g \equiv 0$ in Eq (1.2). If $\phi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \phi(x)\mathrm{d}x \neq 0$, then the solution $u(x,t) = \mathcal{G}(x,t) * \phi(x)$ to Eq (1.2) satisfies the upper bound estimate*

$$\|u(x,t)\|_2 \leq Ct^{-\min\{\alpha+1, 1-\alpha+\frac{\alpha d}{4s}\}},\ t > 0. \tag{3.22}$$

For $d = 8s$ one has

$$\|u(x,t)\|_{2,\infty} \leq Ct^{-\alpha-1},\ t > 0. \tag{3.23}$$

Proof. We divide into two cases to prove the assertion (3.22). When $d < 8s$, applying Plancherel's theorem and Lemma 3.7 one gets

$$\begin{aligned}
\|u(x,t)\|_2^2 &= \|\widehat{u}(\omega,t)\|_2^2 = \int_{\mathbb{R}^d} |\widehat{\mathcal{G}}(\omega,t)|^2 |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&\leq \|\widehat{\phi}\|_\infty^2 \int_{\mathbb{R}^d} |\widehat{\mathcal{G}}(\omega,t)|^2 \mathrm{d}\omega \leq C\|\phi\|_1^2 \int_{\mathbb{R}^d} \frac{t^{2\alpha-2}}{(1+|\omega|^{4s}t^{2\alpha})^2} \mathrm{d}\omega \\
&= C\|\phi\|_1^2 t^{2\alpha-2} \int_0^\infty \frac{\rho^{d-1}\mathrm{d}\rho}{(1+\rho^{4s}t^{2\alpha})^2} \\
&= Ct^{2\alpha-2-\frac{\alpha d}{2s}} \|\phi\|_1^2 \int_0^\infty \frac{\rho_1^{d-1}\mathrm{d}\rho_1}{(1+\rho_1^{4s})^2},
\end{aligned}$$

i.e.,

$$\|u(x,t)\|_2 \leq Ct^{\alpha-1-\frac{\alpha d}{4s}}. \tag{3.24}$$

If $d > 8s$, then $\phi \in L^1(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$ implies $\phi \in L^{\frac{2d}{d+8s}}(\mathbb{R}^d)$ by interpolation. Furthermore, Theorem 8.5 in [18] gives

$$\|(-\Delta)^{-2s}\phi\|_2 \leq C\|\phi\|_{\frac{2d}{d+8s}} < \infty. \tag{3.25}$$

It follows from Plancherel's theorem and Lemma 3.7 that

$$\begin{aligned}
\|u(x,t)\|_2^2 &= \|\widehat{u}(\omega,t)\|_2^2 = \int_{\mathbb{R}^d} |\widehat{\mathcal{G}}(\omega,t)|^2 |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&\leq C \int_{\mathbb{R}^d} \frac{t^{2\alpha-2}}{(1+|\omega|^{4s}t^{2\alpha})^2} |\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&= Ct^{-2\alpha-2} \int_{\mathbb{R}^d} \frac{|\omega|^{8s}t^{4\alpha}}{(1+|\omega|^{4s}t^{2\alpha})^2} ||\omega|^{-4s}\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&\leq Ct^{-2\alpha-2} \int_{\mathbb{R}^d} ||\omega|^{-4s}\widehat{\phi}(\omega)|^2 \mathrm{d}\omega \\
&= Ct^{-2\alpha-2} \|(-\Delta)^{-2s}\phi\|_2^2.
\end{aligned} \tag{3.26}$$

Substituting (3.25) into (3.26) we have

$$\|u(x,t)\|_2 \leq Ct^{-\alpha-1}. \tag{3.27}$$

Based on (3.24) with (3.27), the expected inequality (3.22) is proved.

If $d = 8s$, then by using Young's inequality for convolution and (2.15) in Lemma 2.2 it is evident that

$$\|u(x,t)\|_{2,\infty} = \|\mathcal{G}(x,t) * \phi(x)\|_{2,\infty} \leq C\|\mathcal{G}(x,t)\|_{2,\infty}\|\phi\|_1 \leq Ct^{-\alpha-1},\ t > 0,$$

which is the required result (3.23). This finishes the proof of theorem. □

Remark 3.2. *For* $1 < \alpha < 2$*, we have not give the optimal* L^2*-norm estimates for the solution of Eq* (1.1)*. The reason is as follows. We can apply the inverse Laplace transform to equality* (2.1) *to obtain the solution of Eq* (1.1) *in the Fourier domain*

$$\widehat{u}(\omega,t) = \widehat{G_\varphi}(\omega,t)\widehat{\varphi}(\omega) + \widehat{G_\psi}(\omega,t)\widehat{\psi}(\omega),$$

where

$$\widehat{G}_{\varphi}(\omega,t)=t^{\alpha-2}E_{\alpha,\alpha-1}(-|\omega|^{2s}t^{\alpha})$$

and

$$\widehat{G}_{\psi}(\omega,t)=t^{\alpha-1}E_{\alpha,\alpha}(-|\omega|^{2s}t^{\alpha}).$$

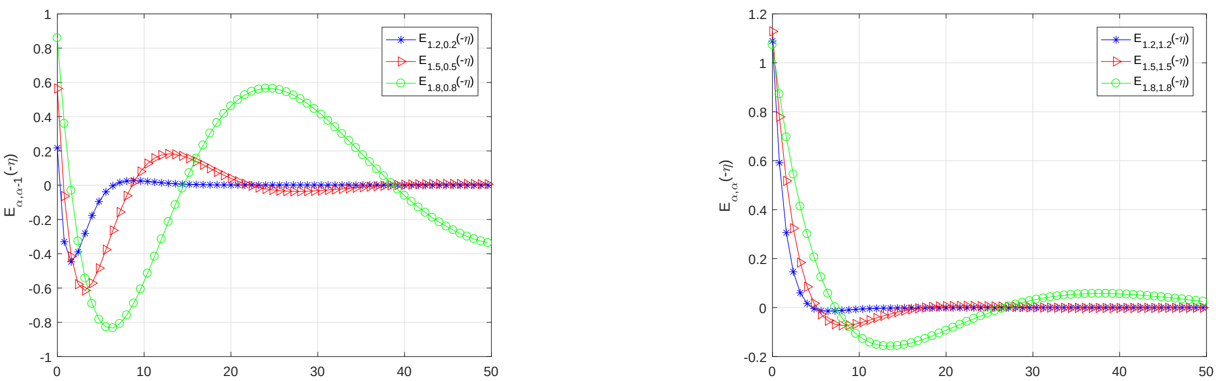
However, we can not derive positive lower bounds for the Mittag-Leffler functions $E_{\alpha,\alpha-1}(-\eta)$ and $E_{\alpha,\alpha}(-\eta)$ with $\eta \geq 0$ and $1<\alpha<2$ as Lemma 3.7 since these two functions have zeros on the real axis. In fact, using the result of Theorem 2 in [29] we immediately see that the Mittag-Leffler function $E_{\alpha,\alpha-1}(-\eta)$ exists real zeros. For the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$, it is obvious that $E_{\alpha,\alpha}(0)=1/\Gamma(\alpha)>0$. Additionally, In view of asymptotic expansion of the Mittag-Leffler function, see Theorem 1.4 in [28], we have

$$E_{\alpha,\alpha}(-\eta)=-\sum_{k=1}^{p}\frac{(-\eta)^k}{\Gamma(\alpha-\alpha k)}+O(\eta^{-1-p})=-\frac{1}{\Gamma(-\alpha)\eta^2}+O(\eta^{-3})$$

when $\eta\to\infty$. Since $1<\alpha<2$, then one gets $\Gamma(-\alpha)>0$. Hence the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$ behaves like

$$E_{\alpha,\alpha}(-\eta)\sim-\frac{1}{\eta^2},\ \eta\to\infty,$$

which implies $E_{\alpha,\alpha}(-\eta)<0$ for sufficiently large η. Combining the above analysis and taking the analyticity of the Mittag-Leffler function $E_{\alpha,\alpha}(-\eta)$ into account, we find that $E_{\alpha,\alpha}(-\eta)$ has one real zeros at least. In Figure 1, we depict graphs of the Mittag-Leffler functions $E_{\alpha,\alpha-1}(-\eta)$ and $E_{\alpha,\alpha}(-\eta)$ with $\eta\in[0,50]$, where the parameter α takes $\alpha=1.2,1.5,1.9$, respectively.



(a) $E_{\alpha,\alpha-1}(-\eta)$ (b) $E_{\alpha,\alpha}(-\eta)$

Figure 1. Plots of the Mittag-Leffler functions $E_{\alpha,\alpha-1}(-\eta)$ and $E_{\alpha,\alpha}(-\eta)$.

4. Conclusions

This work investigates asymptotic behaviors of solutions of Cauchy problems for superdiffusion equation (1.1) and subdiffusion equation (1.2) with integral initial conditions in the sense of $L^p(\mathbb{R}^d)$ and $L^{p,\infty}(\mathbb{R}^d)$. For these two kinds of equations, we construct their fundamental solutions and solutions, analyze asymptotic behaviors of solutions, and study gradient estimates and large time behaviors. In particular, the optimal L^2 decay estimate of solution is derived for Eq (1.2). Compared with the cases of Caputo derivative in the time direction [7, 18] in Eqs (1.1) and (1.2), it is not difficult to see that the asymptotic rates are faster in the cases of Riemann-Liouville derivative.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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Appendix

In the appendix we recall several concepts of integral transforms [5] such as Laplace transform, Mellin transform and Fourier transform, which play an important role on the process of deriving the solutions of Eqs (1.1) and (1.2). Moreover, we introduce the definition of the Fox H-function too.

The standard Laplace transform of a function $f(t)$ is defined by

$$\overline{f}(\lambda) = \mathscr{L}[f(t), \lambda] := \int_0^\infty e^{-\lambda t} f(t)\mathrm{d}t,\ \lambda \in \mathbb{C}.$$

Correspondingly, the inverse Laplace transform is given by

$$f(t) = \mathscr{L}^{-1}[\overline{f}(\lambda), t] := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\,\overline{f}(\lambda)\mathrm{d}\lambda,\ c = \mathrm{Re}(\lambda),\ t > 0.$$

In particular, the Laplace transforms of the Riemann-Liouville fractional integral (1.3) and derivative (1.4) with the starting point $a = 0$ can be written as [28]

$$\mathscr{L}[{}_{RL}\mathrm{D}_{0,t}^{-\alpha} f(t), \lambda] = \lambda^{-\alpha}\mathscr{L}[f(t), \lambda] \tag{A1}$$

and

$$\mathscr{L}[{}_{RL}\mathrm{D}_{0,t}^{\alpha} f(t), \lambda] = \lambda^{\alpha} \mathscr{L}[f(t), \lambda] - \sum_{k=0}^{n-1} \lambda^{k} [{}_{RL}\mathrm{D}_{0,t}^{\alpha-k-1} f(t)]|_{t=0} \tag{A2}$$

for $n-1 < \alpha < n \in \mathbb{N}$.

The standard Mellin transform of a function $f(t)$ is defined by

$$\widetilde{f}(\xi) = \mathscr{M}[f(t), \xi] := \int_0^{\infty} t^{\xi-1} f(t) \mathrm{d}t, \ \xi \in \mathbb{C}.$$

The inverse Mellin transform is correspondingly represented as

$$f(t) = \mathscr{M}^{-1}[\widetilde{f}(\xi), t] := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\xi} \widetilde{f}(\xi) \mathrm{d}\xi, \ c = \mathrm{Re}(\xi), \ t > 0.$$

The relation connecting the Laplace transform and the Mellin transform has the following form [6]:

$$\mathscr{M}[f(t), \xi] = \frac{1}{\Gamma(1-\xi)} \mathscr{M}[\mathscr{L}[f(t), \lambda], 1-\xi]. \tag{A3}$$

The Fourier transform of a function $f(x)$ is defined by

$$\widehat{f}(\omega) = \mathcal{F}[f(x), \omega] := \int_{\mathbb{R}^d} e^{i\omega \cdot x} f(x) \mathrm{d}x, \ \omega \in \mathbb{R}^d,$$

while the corresponding inverse Fourier transform can be written as

$$f(x) = \mathcal{F}^{-1}[\widehat{f}(\omega), x] := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\omega} \widehat{f}(\omega) \mathrm{d}\omega, \ x \in \mathbb{R}^d.$$

Then the Fourier transform of the integral fractional Laplacian (1.5) is given by [8]

$$\mathcal{F}[(-\Delta)^s v(x), \omega] = |\omega|^{2s} \mathcal{F}[v(x), \omega], \ x \in \mathbb{R}^d, \ s \in (0,1). \tag{A4}$$

Next, let us briefly recall the definition of the Fox H-function. The Fox H-function are special functions, where the fundamental solutions of Eqs (1.1) and (1.2) are written by means of these functions, and they paly a basic role in asymptotic analysis of the solutions.

According to the contour integral of Mellin-Barnes type, the Fox H-function can be represented as

$$\begin{aligned} H_{\mu\nu}^{mn}(z) &\equiv H_{\mu\nu}^{mn}\left(z \middle| \begin{matrix} (a_1,\alpha_1), \cdots, (a_n,\alpha_n); \ (a_{n+1},\alpha_{n+1}), \cdots, (a_\mu,\alpha_\mu) \\ (b_1,\beta_1), \ \cdots, (b_m,\beta_m); \ (b_{m+1},\beta_{m+1}), \cdots, (b_\nu,\beta_\nu) \end{matrix} \right) \\ &:= \frac{1}{2\pi i} \int_{\mathfrak{L}} \mathcal{H}_{\mu\nu}^{mn}(\tau) z^{-\tau} \mathrm{d}\tau, \end{aligned} \tag{A5}$$

where

$$\mathcal{H}_{\mu\nu}^{mn}(\tau) := \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j \tau) \prod_{l=1}^{n} \Gamma(1 - a_l - \alpha_l \tau)}{\prod_{l=n+1}^{\mu} \Gamma(a_l + \alpha_l \tau) \prod_{j=m+1}^{\nu} \Gamma(1 - b_j - \beta_j \tau)}, \tag{A6}$$

and m, n, μ, ν are nonnegative integers with $0 \le m \le \nu$ and $0 \le n \le \mu$, and α_l, β_j are positive real numbers and a_l, b_j are complex numbers for $l = 1, \dots, \mu;\ j = 1, \dots, \nu$. All the poles

$$b_{j\sigma} = -\frac{b_j + \sigma}{\beta_j},\ j = 1, \dots, m;\ \sigma = 0, 1, 2, \dots$$

of the gamma functions $\Gamma(b_j + \beta_j \tau)$ and

$$a_{lk} = \frac{1 - a_l + k}{\alpha_l},\ l = 1, \dots, n;\ k = 0, 1, 2, \dots$$

of the gamma functions $\Gamma(1 - a_l - \alpha_l \tau)$ are not equal, i.e.,

$$\alpha_l(b_j + \sigma) \neq \beta_j(a_l - k - 1),\ j = 1, \dots, m;\ l = 1, \dots, n;\ \sigma, k = 0, 1, 2, \dots$$

The contour $\mathfrak{L}$ is an infinite contour in the complex plane which separates all the poles $b_{j\sigma}$ from all the poles a_{lk}, and it may take $\mathfrak{L} = \mathfrak{L}_{-\infty}$ or $\mathfrak{L} = \mathfrak{L}_{+\infty}$ or $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$ with $\gamma \in \mathbb{R}$ and $i^2 = -1$. Besides, we denote

$$a^* = \sum_{l=1}^{n} \alpha_l - \sum_{l=n+1}^{\mu} \alpha_l + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{\nu} \beta_j.$$

A comprehensive and detailed description for the Fox H-function can be available from [2, 14, 15, 30].