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Research article

Chatterjea type theorems for complex valued extended b-metric spaces with applications

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Abstract: In this article, we establish common α-fuzzy fixed point theorems for Chatterjea type contractions involving rational expression in complex valued extended b-metric space. Our results generalize and extend some familiar results in the literature. Some common fixed point results for multivalued and single valued mappings are derived for complex valued extended b-metric space, complex valued b-metric space and complex valued metric space as consequences of our leading results. As an application, we investigate the solution of Fredholm integral inclusion.

Keywords: fuzzy mapping; common α-fuzzy fixed point; complex valued extended b-metric space; Fredholm integral inclusion
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Metric fixed point theory is one of the distinguished and traditional theories in the area of functional analysis which has broad applications in various fields of mathematics. The Banach contraction principle (BCP) [1] is a fundamental and pioneering result for this theory. It is a prominent and an outstanding tool to use to solve the existence problems in pure and applied sciences. Kannan [2] gave an analogous variety of contractive type conditions that endorsed the existence of fixed points. The elementary difference between the Banach contraction principle and Kannan's fixed point theorem is the contractive condition and continuity of mapping. In Kannan's fixed point theorem, the contractive mapping is not necessarily continuous. Later on, Chatterjea [3] commuted the terms for the contractive condition used by Kannan and proved an analogue fixed point result. In 1969, Nadler [4] used the notion of the Hausdorff metric to obtain fixed points of multivalued mappings. In all of these results, the metric space plays a significant role. Over the past few decades, different interesting generalizations of metric space have been invented by several researchers. Some of these well-known generalizations of metric space are the partial metric space constructed by

Mathew [5], b-metric space constructed by Czerwik [6] and extended b-metric space constructed by Kamran et al. [7]. After this such generalizations, Azam et al. [8] introduced the study of a complex valued metric space (CVMS) and generalized the classical metric space by replacing a set of real numbers $\mathbb{R}$ with the set of complex numbers $\mathbb{C}$ in the range. Rouzkard and Imdad [9] employed this idea of new space and manifested a result by adding more terms in the inequality to generalize Azam's results. In due course, Sitthikul and Saejung [10] extended the contractive condition of Rouzkard and Imdad [9] and established some new common fixed point theorems for self-mappings. Ahmad et al. [11, 12] gave the notion of generalized Housdorff metric function in the background of CVMS and proved common fixed points of multivalued mappings. In [13], Mukheimer extended the concept of CVMS to complex valued b-metric space (CVbMS) by using a constant $\pi \geq 1$ in the triangle inequality. Later on, Ullah et al. [14] gave the study of complex valued extended b-metric space (CVEbMS) and replaced that constant $\pi \geq 1$ with a control function $\varphi(x, y)$ in 2019. The notion of CVEbMS is an up-to-date and contemporary generalization of CVbMS and CVMS. Subsequently, Mohammed and Ullah [15] used the notion of a CVEbMS to obtain the common fixed points of two self mappings.

Alternatively, Zadeh [16] gave the theory of a fuzzy set (FS) to deal with irregularity which happens because of inaccuracy or ambiguity in preference to the abstraction in 1960. Heilpern [17] used this notion of a FS to give the concept of fuzzy mappings (FMs) in the context of a metric space and broadened the Nadler's fixed point theorem [4]. Several generalizations of Heilpern's fixed point theorem have been derived by researchers in different spaces. Kutbi et al. [18] obtained α-fuzzy fixed point theorems for CVMS and derived some results in the metric space and CVMS. Humaira et al. [19, 20] utilized the notion of a CVMS to prove fixed and common fixed points of FMs. Recently, Shammaky et al. [21] and Albargi and Ahmad [22] defined Banach and Kannan type contractions including rational expressions in CVEbMS and proved common α-fuzzy fixed point results. The results given by Shammaky et al. [21] and Albargi and Ahmad [22] are generalizations of Banach and Kannan type contractions results in CVbMS and CVMS. For further characteristics in this order, we refer the researchers to [23–29].

In this work, we utilize the concept of a CVEbMS and establish common α-fuzzy fixed point theorems for Chatterjea type contractions involving rational expressions. In this way, we generalize Chatterjea type contraction results in CVEbMS, CVbMS and CVMS. As outcomes of our main result, we derive the leading results of Azam et al. [8], Rouzkard and Imdad [9], Ahmad et al. [11] and Kutbi et al. [18] from our results. We investigate the solution of Fredholm integral inclusion as an application.

2. Preliminaries

In 1922, Banach [1] proved the following well-known fixed point result:

Theorem 1. *([1]) Let (X, d) be a CMS and $T\colon X \to X$. If there exists $\ell \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \ell d(x, y),$$

for all $x, y \in X$; then, there exists a unique point $x^* \in X$ such that $x^* = Tx^*$.

Kannan [2] established the following fixed point result:

Theorem 2. *([2]) Let (X, d) be a CMS and $T\colon X \to X$. If there exists $\ell \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \ell \left(d\left(x, Tx\right) + d\left(y, Ty\right)\right),$$

for all $x, y \in X$; then, there exists a unique point $x^* \in X$ such that $x^* = Tx^*$.

Chatterjea [3] presented a fixed point result in this aspect.

Theorem 3. *([3]) Let (X, d) be a CMS and $T\colon X \to X$. If there exists $\ell \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \ell \left(d\left(y, Tx\right) + d\left(x, Ty\right)\right),$$

for all $x, y \in X$; then, there exists a unique point $x^* \in X$ such that $x^* = Tx^*$.

In 1969, Nadler [4] introduced the concept of multivalued mapping and generalized single valued mapping.

Theorem 4. *([4]) Let (X, d) be a CMS and $T\colon X \to CB(X)$. If there exists $\ell \in (0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq \ell d\left(x, y\right),$$

for all $x, y \in X$; then, there exists a unique point $x^* \in X$ such that $x^* \in Tx^*$.

Azam et al. [8] defined the notion of a CVMS in this manner.

Definition 1. ([8]) A partial order $\precsim$ on $\mathbb{C}$ (set of complex numbers) is given as follows:

$$\varrho_1 \precsim \varrho_2 \iff \mathrm{R}\left(\varrho_1\right) \leqslant \mathrm{R}\left(\varrho_2\right),\ \mathrm{I}\left(\varrho_1\right) \leqslant \mathrm{I}\left(\varrho_2\right),$$

for all $\varrho_1, \varrho_2 \in \mathbb{C}$.

It follows that $\varrho_1 \precsim \varrho_2$ if one of these assertions is satisfied:

$$\begin{aligned}
&\text{(a) } \mathrm{R}\left(\varrho_1\right) = \mathrm{R}\left(\varrho_2\right),\ \mathrm{I}\left(\varrho_1\right) < \mathrm{I}\left(\varrho_2\right),\\
&\text{(b) } \mathrm{R}\left(\varrho_1\right) < \mathrm{R}\left(\varrho_2\right),\ \mathrm{I}\left(\varrho_1\right) = \mathrm{I}\left(\varrho_2\right),\\
&\text{(c) } \mathrm{R}\left(\varrho_1\right) < \mathrm{R}\left(\varrho_2\right),\ \mathrm{I}\left(\varrho_1\right) < \mathrm{I}\left(\varrho_2\right),\\
&\text{(d) } \mathrm{R}\left(\varrho_1\right) = \mathrm{R}\left(\varrho_2\right),\ \mathrm{I}\left(\varrho_1\right) = \mathrm{I}\left(\varrho_2\right),
\end{aligned}$$

where $\mathrm{R}\left(\varrho\right)$ and $\mathrm{I}\left(\varrho\right)$ denote the real and imaginary parts of $\varrho \in \mathbb{C}$ respectively.

Definition 2. ([8]) Let $X \neq \emptyset$. A mapping $d\colon X \times X \to \mathbb{C}$ is called a CVM if the following conditions hold:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0 \iff x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) \precsim d(x, v) + d(v, y)$.

For all $x, y, v \in X$; then, (X, d) is called a CVMS.

Example 1. ([8]) Let $X = [0, 1]$ and $x, y \in X$. Define d: $X \times X \to \mathbb{C}$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{i}{2}, & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a CVMS.

In [13], Mukheimer gave the notion of a CVbMS as follows:

Definition 3. ([13]) Let $X \neq \emptyset$ and $\pi \geq 1$. A mapping d: $X \times X \to \mathbb{C}$ is said to be a CVbMS if the following conditions hold:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) \precsim \pi \left[d(x, v) + d(v, y)\right]$.

For all $x, y, v \in X$; then, (X, d) is called a CVbMS.

Example 2. ([13]) Let $X = [0, 1]$. Define d: $X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i|x - y|^2,$$

for all $x, y \in X$. Then (X, d) is a CVbMS with $\pi = 2$.

Ullah et al. [14] conducted the study of the CVEbMS and replaced the constant $\pi \geq 1$ with a control function $\varphi(x, y)$ in 2014.

Definition 4. ([14]) Let $X \neq \emptyset$ and φ: $X \times X \to [1, +\infty)$. A mapping d: $X \times X \to \mathbb{C}$ is called a CVEbMS if the following conditions hold:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) \precsim \varphi(x, y) \left[d(x, v) + d(v, y)\right]$.

For all $x, y, v \in X$; then, (X, d) is called a CVEbMS.

Example 3. ([14]) Let $X \neq \varnothing$ and φ: $X \times X \to [1, +\infty)$ be defined by

$$\varphi(x, y) = \frac{1 + x + y}{x + y},$$

and d: $X \times X \to \mathbb{C}$ by

(i) $d(x, y) = \frac{i}{xy}, \forall 0 < x, y \leq 1$.
(ii) $d(x, y) = 0 \Leftrightarrow x = y, \forall 0 \leq x, y \leq 1$.
(iii) $d(x, 0) = d(0, x) = \frac{i}{x}, \forall 0 < x \leq 1$.

Then (X, d) is a CVEbMS.

Example 4. Let $X = [0, +\infty)$ and φ: $X \times X \to [1, +\infty)$ be a function defined by $\varphi(x, y) = 1 + x + y$ and d: $X \times X \to \mathbb{C}$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ i, & \text{if } x \neq y. \end{cases}$$

Then (X, d) is a CVEbMS.

Lemma 1. *([14]) Let (X, d) be a CVEbMS and $\{x_n\} \subseteq X$. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to +\infty$.*

Lemma 2. *([14]) Let (X, d) be a CVEbMS and $\{x_n\} \subseteq X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$, as $n, m \to +\infty$.*

Let (X, d) be a CVEbMS; then, $\mathfrak{CB}(X)$ represents the class of all non-empty, bounded and closed subsets of X.

We apply $s(x_1) = \{x_2 \in \mathbb{C}\colon x_1 \preceq x_2\}$ for $x_1 \in \mathbb{C}$, and

$$s(x_1, \mathfrak{R}_2) = \bigcup_{x_2 \in \mathfrak{R}_2} s(d(x_1, x_2)) = \bigcup_{x_2 \in \mathfrak{R}_2} \{x \in \mathbb{C} : d(x_1, x_2) \preceq x\},$$

for $a \in X$ and $\mathfrak{R}_2 \in \mathfrak{CB}(X)$.

For $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathfrak{CB}(X)$, we denote

$$s(\mathfrak{R}_1, \mathfrak{R}_2) = \left(\bigcap_{x_1 \in \mathfrak{R}_1} s(x_1, \mathfrak{R}_2)\right) \cap \left(\bigcap_{x_2 \in \mathfrak{R}_2} s(x_2, \mathfrak{R}_1)\right).$$

Lemma 3. *([14]) Let (X, d) be a CVEbMS.*

(i) Let $x_1, x_2 \in \mathbb{C}$. If $x_1 \preceq x_2$, then $s(x_2) \subset s(x_1)$.

(ii) Let $x \in X$ and $\mathfrak{R} \in N(X)$. If $\theta \in s(x, \mathfrak{R})$, then it follows that $x \in \mathfrak{R}$.

(iii) Let $x \in \mathbb{C}$, $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathfrak{CB}(X)$ and $x_1 \in \mathfrak{R}_1$. If $x \in s(\mathfrak{R}_1, \mathfrak{R}_2)$, then $x \in s(x_1, \mathfrak{R}_2)$ for all $x_1 \in \mathfrak{R}_1$ or $x \in s(\mathfrak{R}_1, x_2)$ for all $x_2 \in \mathfrak{R}_2$.

Let T: $X \to \mathfrak{CB}(X)$ be a multivalued mapping. For $x \in X$ and $\mathfrak{R} \in \mathfrak{CB}(X)$, define

$$W_x(\mathfrak{R}) = \{d(x, x_1) : x_1 \in \mathfrak{R}\}.$$

Thus for $x, y \in X$

$$W_x(Ty) = \{d(x, x_1) : x_1 \in Ty\}.$$

Definition 5. ([14]) Let (X, d) be a CVEbMS. A subset $\mathfrak{R}$ of X is referred to as bounded below if there exists $x \in X$ such that $x \preceq x_1$, for all $x_1 \in \mathfrak{R}$.

Definition 6. ([14]) Let (X, d) be a CVEbMS. A mapping T: $X \to 2^{\mathbb{C}}$ is said to be bounded from below if for all $x \in X$, there exists $x_x \in \mathbb{C}$ such that

$$x_x \preceq u,$$

for all $u \in Tx$.

On the other hand, Heilpern [17] used the notion of a FS and gave the concept of FMs in metric space. A FS in X is a function with domain X and range $[0,1]$ and I^X is the family of all FSs in X. If the set $\mathfrak{R}$ is a FS and $x \in X$, then $\mathfrak{R}(x)$ is said to be the grade of membership of x in $\mathfrak{R}$. We express $[\mathfrak{R}]_\alpha$ as the α-level set of $\mathfrak{R}$ and define it in the following way:

$$[\mathfrak{R}]_\alpha = \{x : \mathfrak{R}(x) \geq \alpha\}, \text{ if } \alpha \in (0,1],$$
$$[\mathfrak{R}]_0 = \overline{\{x : \mathfrak{R}(x) > 0\}}.$$

Kutbi et al. [18] proved the following result for FMs in CVMS in this manner:

Theorem 5. *([18]) Let (X,d) be a complete CVMS and $S,T\colon X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0,1]$, such that for each $x \in X$, $[Sx]_\alpha$, $[Tx]_\alpha \in CB(X)$ and there exist $0 \leq \ell_1, \ell_2$ with*

$$2\ell_1 + \ell_2 < 1,$$

such that

$$\ell_1 \left(d\left(y,[Sx]_\alpha\right) + d\left(x,[Ty]_\alpha\right)\right) + \ell_2 \frac{d\left(x,[Sx]_\alpha\right) d\left(y,[Ty]_\alpha\right)}{1 + d(x,y)} \in s\left([Sx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$; then, there exists $x^* \in X$ such that

$$x^* \in [Sx^*]_\alpha \cap [Tx^*]_\alpha.$$

Definition 7. ([17]) Let X_1 be a non empty set and (X_2, d) be a metric space. A mapping T is called a FM if T is a mapping from X_1 into $\mathfrak{F}(X_2)$. A FM T is a fuzzy subset on $X_1 \times X_2$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 8. ([17]) Let (X,d) be a metric space and $S,T\colon X \to \mathfrak{F}(X)$. A point $x \in X$ is called a common α-fuzzy fixed point of S and T if and only if $x \in [Sx]_\alpha \cap [Tx]_\alpha$, for some $\alpha \in [0,1]$.

Ahmad et al. [11, 12] gave the notion of a generalized Hausdorff metric function for a CVMS and Kutbi et al. [18] used this study to prove fuzzy fixed point results in CVMS.

In this article, we utilize the notion of a CVEbMS and establish common α-fuzzy fixed point results for Chatterjea type contractions involving rational expressions. We implement our results to derive some well-known results in the literature.

3. Main results

Definition 9. Let (X,d) be a CVEbMS. A mapping $T\colon X \to \mathfrak{F}(X)$ is said to satisfy g.l.b. property on (X,d) if for any $x \in X$ and $\alpha \in (0,1]$, the greatest lower bound of $W_x([Ty]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$. We represent the greatest lower bound of $W_x([Ty]_\alpha)$ as $d(x,[Ty]_\alpha)$ which is defined as follows:

$$d(x,[Ty]_\alpha) = \inf\{d(x,v) : v \in [Ty]_\alpha\}.$$

Theorem 6. *Let (X,d) be a complete CVEbMS, $\varphi\colon X \times X \to [1,+\infty)$ and $S,T\colon X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0,1]$ such that for each $x \in X$, $[Sx]_\alpha$, $[Tx]_\alpha \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\varphi(x,y)\ell_1 + \ell_2 < 1$$

and

$$\lambda(1-\varphi(x,y)\ell_1-\ell_2)=\varphi(x,y)\ell_1,$$

where $\lambda\in[0,1)$ *such that*

$$\ell_1\left(d\left(y,[Sx]_\alpha\right)+d\left(x,[Ty]_\alpha\right)\right)+\ell_2\frac{d\left(x,[Sx]_\alpha\right)d\left(y,[Ty]_\alpha\right)}{1+d\left(x,y\right)}\in s\left([Sx]_\alpha,[Ty]_\alpha\right), \tag{3.1}$$

for all $x,y\in X$. *If for each* $x_0\in X$, $\lim_{n,m\to+\infty}\varphi\left(x_n,x_m\right)\lambda<1$, *then* S *and* T *have a common* α*-fuzzy fixed point.*

Proof. Let x_0 be an arbitrary point in X. By assumption, we can find that $x_1\in[Sx_0]_\alpha$. So, we have

$$\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}\in s\left([Sx_0]_\alpha,[Tx_1]_\alpha\right),$$

that is,

$$\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}\in\bigcap_{\omega\in[Sx_0]_\alpha}s\left(\omega,[Tx_1]_\alpha\right).$$

Since $x_1\in[Sx_0]_\alpha$, we have

$$\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}\in s\left(x_1,[Tx_1]_\alpha\right).$$

By definition

$$\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}\in\bigcup_{x\in[Tx_1]_\alpha}s\left(d(x_1,x)\right).$$

This implies that there exists $x=x_2\in[Tx_1]_\alpha$ such that

$$\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}\in s\left(d(x_1,x_2)\right),$$

that is,

$$d(x_1,x_2)\leq\ell_1\left(d\left(x_{1,},[Sx_{0,}]_\alpha\right)+d\left(x_0,[Tx_{1,}]_\alpha\right)\right)+\ell_2\frac{d\left(x_0,[Sx_0]_\alpha\right)d(x_{1,}\,[Tx_1]_\alpha)}{1+d\left(x_0,x_1\right)}.$$

By the definitions of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x,y\in X$, we get

$$\begin{aligned}d(x_1,x_2)&\leq\ell_1\left(d(x_{0,}x_2)\right)+\frac{\ell_2d\left(x_0,x_1\right)d\left(x_{1,}x_2\right)+\ell_3d\left(x_1,x_1\right)d\left(x_{0,}x_2\right)}{1+d\left(x_0,x_1\right)}\\&=\varphi(x_0,x_1)\ell_1\left(d\left(x_0,x_1\right)+d(x_{1,}x_2)\right)+\frac{\ell_2d\left(x_0,x_1\right)d\left(x_{1,}x_2\right)}{1+d\left(x_0,x_1\right)}\\&=\varphi(x_0,x_1)\ell_1\left(d\left(x_0,x_1\right)+d(x_{1,}x_2)\right)+\ell_2d\left(x_{1,}x_2\right)\left(\frac{d\left(x_0,x_1\right)}{1+d\left(x_0,x_1\right)}\right).\end{aligned}$$

This implies that

$$\begin{aligned}|d(x_1,x_2)| &\leq \varphi(x_0,x_1)\ell_1|d(x_0,x_1)| + \varphi(x_0,x_1)\ell_1|d(x_{1,}x_2)| + \ell_2|d(x_{1,}x_2)|\left|\frac{d(x_0,x_1)}{1+d(x_0,x_1)}\right| \\ &\leq \varphi(x_0,x_1)\ell_1|d(x_0,x_1)| + \varphi(x,y)\ell_1|d(x_{1,}x_2)| + \ell_2|d(x_{1,}x_2)|,\end{aligned}$$

which further implies that

$$\begin{aligned}|d(x_1,x_2)| &\leq \left(\frac{\varphi(x_0,x_1)\ell_1}{1-\varphi(x_0,x_1)\ell_1-\ell_2}\right)|d(x_0,x_1)| \\ &= \lambda|d(x_0,x_1)|. \end{aligned} \tag{3.2}$$

Similarly, for $x_2 \in [Tx_1]_\alpha$, we have

$$\ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)} \in s\left([Tx_1]_\alpha,[Sx_2]_\alpha\right),$$

that is,

$$\ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)} \in \bigcap_{\omega\in[Tx_1]_\alpha} s\left(\omega,[Sx_2]_\alpha\right).$$

Since $x_2 \in [Tx_1]_\alpha$, we have

$$\ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)} \in s\left(x_2,[Sx_2]_\alpha\right).$$

By definition, we have

$$\ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)} \in \bigcup_{d\in[Sx_2]_\alpha} s\left(d(x_2,d)\right).$$

By the definition of the "s" function, there exists $x_3 \in [Sx_2]_\alpha$, such that

$$\ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)} \in s\left(d(x_2,x_3)\right),$$

that is,

$$d(x_2,x_3) \leq \ell_1\left(d\left(x_{2,}[Tx_1]_\alpha\right)+d\left(x_1,[Sx_2]_\alpha\right)\right)+\ell_2\frac{d\left(x_{1,}[Tx_1]_\alpha\right)d\left(x_2,[Sx_2]_\alpha\right)}{1+d\left(x_2,x_1\right)}.$$

By the definitions of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x,y \in X$, we get

$$\begin{aligned}d(x_2,x_3) &\leq \ell_1\left(d\left(x_1,x_3\right)\right)+\frac{\ell_2 d\left(x_1,x_2\right)d\left(x_2,x_3\right)}{1+d\left(x_2,x_1\right)} \\ &= \varphi(x_1,x_2)\ell_1\left(d\left(x_{1,}x_2\right)+\varphi(x_1,x_2)d\left(x_2,x_3\right)\right)+\ell_2\frac{d\left(x_1,x_2\right)d\left(x_2,x_3\right)}{1+d\left(x_1,x_2\right)},\end{aligned}$$

which implies that

$$|d(x_2,x_3)| \leq \varphi(x_1,x_2)\ell_1|d(x_1,x_2)| + \varphi(x_1,x_2)\ell_1|d(x_2,x_3)| + \ell_2 d(x_2,x_3)\frac{|d(x_1,x_2)|}{|1+d(x_1,x_2)|}.$$

This yields

$$\begin{aligned}|d(x_2,x_3)| &\le \left(\frac{\varphi(x_1,x_2)\ell_1}{1-\varphi(x_1,x_2)\ell_1-\ell_2}\right)|d(x_1,x_2)| \\ &= \lambda|d(x_1,x_2)|.\end{aligned} \tag{3.3}$$

Continuing in this way, we get a sequence of points $\{x_n\}$ in X such that

$$\begin{aligned}|d(x_1,x_2)| &\le \lambda|d(x_0,x_1)|,\\ |d(x_2,x_3)| &\le \lambda^2|d(x_0,x_1)|,\\ &\ \ .\\ &\ \ .\\ &\ \ .\\ d(x_n,x_{n+1}) &\le \lambda^n d(x_0,x_1),\end{aligned}$$

for all $n\in\mathbb{N}$. Now for $m>n$ and by the triangle inequality, we have

$$\begin{aligned}d(x_n,x_m) &\le \varphi(x_n,x_m)\lambda^n d(x_0,x_1)\\ &+ \varphi(x_n,x_m)\varphi(x_{n+1},x_m)\lambda^{n+1}d(x_0,x_1)\\ &+ \cdots\\ &+ \varphi(x_n,x_m)\varphi(x_{n+1},x_m)\cdots\varphi(x_{m-2},x_m)\varphi(x_{m-1},x_m)\lambda^{m-1}d(x_0,x_1)\\ &\le d(x_0,x_1)\left[\begin{array}{c}\varphi(x_n,x_m)\lambda^n\\ +\varphi(x_n,x_m)\varphi(x_{n+1},x_m)\lambda^{n+1}+\cdots+\\ \varphi(x_n,x_m)\varphi(x_{n+1},x_m)\cdots\varphi(x_{m-2},x_m)\varphi(x_{m-1},x_m)\lambda^{m-1}\end{array}\right].\end{aligned}$$

Since

$$\lim_{n,m\to+\infty}\varphi(x_n,x_m)\lambda<1,$$

the series $\sum_{n=1}^{\infty}\lambda^n\prod_{i=1}^{p}\varphi(x_i,x_m)$ converges, according to the ratio test for each $m\in\mathbb{N}$. Let

$$S=\sum_{n=1}^{\infty}\lambda^n\prod_{i=1}^{p}\varphi(x_i,x_m),\quad S_n=\sum_{j=1}^{n}\lambda^j\prod_{i=1}^{p}\varphi(x_i,x_m).$$

Hence, the above inequality for $m>n$ can be written as

$$d(x_n,x_m)\le d(x_0,x_1)\left[S_{m-1}-S_n\right].$$

Letting $n\to+\infty$, we have

$$|d(x_n,x_m)|\to 0.$$

Thus the sequence $\{x_n\}$ is Cauchy in X according to Lemma 2. Because X is complete, there exists x^* such that $x_n\to x^*\in X$ as $n\to+\infty$. Now, we show that $x^*\in Sx^*$ and $x^*\in Tx^*$. By inequality (3.1), we have

$$\ell_1\left(d(x^*,[Sx_{2n}]_\alpha)+d(x_{2n},[Tx^*]_\alpha)\right)+\ell_2\frac{d(x_{2n},[Sx_{2n}]_\alpha)\,d(x^*,[Tx^*]_\alpha)}{1+d(x_{2n},x^*)}\in s\left([Sx_{2n}]_\alpha,[Tx^*]_\alpha\right),$$

that is,

$$\ell_1 \left(d\left(x^*, [Sx_{2n}]_\alpha\right) + d\left(x_{2n}, [Tx^*]_\alpha\right)\right) + \ell_2 \frac{d\left(x_{2n}, [Sx_{2n}]_\alpha\right) d\left(d^*, [Tx^*]_\alpha\right)}{1 + d\left(x_{2n}, x^*\right)} \in \bigcap_{\omega \in [Sx_{2n}]_\alpha} s\left(\omega, [Tx^*]_\alpha\right).$$

Since $x_{2n+1} \in [Sx_{2n}]_\alpha$, we have

$$\ell_1 \left(d\left(x^*, [Sx_{2n}]_\alpha\right) + d\left(x_{2n}, [Tx^*]_\alpha\right)\right) + \ell_2 \frac{d\left(x_{2n}, [Sx_{2n}]_\alpha\right) d\left(x^*, [Tx^*]_\alpha\right)}{1 + d\left(x_{2n}, x^*\right)} \in s\left(x_{2n+1}, [Tx^*]_\alpha\right),$$

$$\ell_1 \left(d\left(x^*, [Sx_{2n}]_\alpha\right) + d\left(x_{2n}, [Tx^*]_\alpha\right)\right) + \ell_2 \frac{d\left(x_{2n}, [Sx_{2n}]_\alpha\right) d\left(x^*, [Tx^*]_\alpha\right)}{1 + d\left(x_{2n}, x^*\right)} \in \bigcup_{x^/ \in [Tx^*]_\alpha} s\left(d\left(x_{2n+1}, x^/\right)\right).$$

This implies that there exists $\varsigma_n \in [Tx^*]_\alpha$ such that

$$\ell_1 \left(d\left(x^*, [Sx_{2n}]_\alpha\right) + d\left(x_{2n}, [Tx^*]_\alpha\right)\right) + \ell_2 \frac{d\left(x_{2n}, [Sx_{2n}]_\alpha\right) d\left(x^*, [Tx^*]_\alpha\right)}{1 + d\left(x_{2n}, x^*\right)} \in s\left(d\left(x_{2n+1}, \varsigma_n\right)\right),$$

that is,

$$d\left(x_{2n+1}, \varsigma_n\right) \leq \ell_1 \left(d\left(x^*, [Sx_{2n}]_\alpha\right) + d\left(x_{2n}, [Tx^*]_\alpha\right)\right) + \ell_2 \frac{d\left(x_{2n}, [Sx_{2n}]_\alpha\right) d\left(x^*, [Tx^*]_\alpha\right)}{1 + d\left(x_{2n}, x^*\right)}.$$

The g.l.b property of T yields

$$d\left(x_{2n+1}, \varsigma_n\right) \leq \ell_1 \left(d\left(x^*, x_{2n+1}\right) + d\left(x_{2n}, \varsigma_n\right)\right) + \ell_2 \frac{d\left(x_{2n}, x_{2n+1}\right) d\left(x^*, \varsigma_n\right)}{1 + d\left(x_{2n}, x^*\right)}.$$

From the triangle inequality, we have

$$d\left(x^*, \varsigma_n\right) \leq \theta\left(x^*, \varsigma_n\right) \left[d\left(x^*, x_{2n+1}\right) + d\left(x_{2n+1}, \varsigma_n\right)\right].$$

Hence

$$d\left(x^*, \varsigma_n\right) \leq \theta\left(x^*, \varsigma_n\right) d\left(x^*, x_{2n+1}\right) + \ell_1 \theta\left(x^*, \varsigma_n\right) \left(d\left(x^*, x_{2n+1}\right) + d\left(x_{2n}, \varsigma_n\right)\right) + + \ell_2 \frac{d\left(x_{2n}, x_{2n+1}\right) d\left(x^*, \varsigma_n\right)}{1 + d\left(x_{2n}, x^*\right)}.$$

It follows that

$$\begin{aligned} \left|d\left(x^*, \varsigma_n\right)\right| &\leq \theta\left(x^*, \varsigma_n\right) \left|d\left(x^*, x_{2n+1}\right)\right| + \ell_1 \theta\left(x^*, \varsigma_n\right) \left|d\left(x^*, x_{2n+1}\right)\right| + \ell_1 \theta\left(x^*, \varsigma_n\right) \left|d\left(x_{2n}, \varsigma_n\right)\right| \\ &+ \ell_2 \theta\left(x^*, \varsigma_n\right) \frac{\left|d\left(x_{2n}, x_{2n+1}\right)\right| \left|d\left(x^*, \varsigma_n\right)\right|}{\left|1 + d\left(x_{2n}, x^*\right)\right|}. \end{aligned}$$

Letting $n \to +\infty$, we get that $\left|d\left(x^*, \varsigma_n\right)\right| \to 0$. Thus, $\varsigma_n \to x^*$ according to Lemma 1. Because $[Tx]_\alpha$ is closed, $x^* \in [Tx]_\alpha$. Similarly, we can show that $x^* \in [Sx]_\alpha$. Thus there exists $x^* \in X$ such that $x^* \in [Sx]_\alpha \cap [Tx]_\alpha$. □

By taking $S = T$ in Theorem 6, we derive the following result:

Corollary 1. *Let (X, d) be a complete CVEbMS, φ: $X \times X \to [1, +\infty)$ and T: $X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0, 1]$, such that for each $x \in X$, $[Tx] \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\varphi(x, y)\ell_1 + \ell_2 < 1$$

and

$$\lambda(1 - \varphi(x, y)\ell_1 - \ell_2) = \varphi(x, y)\ell_1,$$

where $\lambda \in [0, 1)$ such that

$$\ell_1 \left(d\left(y, [Tx]_\alpha\right) + d\left(x, [Ty]_\alpha\right)\right) + \frac{\ell_2 d\left(x, [Tx]_\alpha\right) d\left(y, [Ty]_\alpha\right)}{1 + d\left(x, y\right)} \in s\left([Tx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$. If for each $x_0 \in X$,

$$\lim_{n,m\to+\infty} \varphi\left(x_n, x_m\right)\lambda < 1,$$

then T has an α-fuzzy fixed point.

Taking φ: $X \times X \to [1, +\infty)$ by $\varphi(x, y) = s \geq 1$ in Theorem 6, we get the following result:

Corollary 2. *Let (X, d) be a complete CVbMS with the coefficient $s \geq 1$, and S, T: $X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0, 1]$, such that for each $x \in X$, $[Sx]_\alpha$, $[Tx]_\alpha \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\ell_1 + \ell_2 < 1$$

and

$$\ell_1 \left(d\left(y, [Sx]_\alpha\right) + d\left(x, [Ty]_\alpha\right)\right) + \ell_2 \frac{d\left(x, [Sx]_\alpha\right) d\left(y, [Ty]_\alpha\right)}{1 + d\left(x, y\right)} \in s\left([Sx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$; then, S and T have a common α-fuzzy fixed point.

By taking $S = T$ in the above corollary, we get the following result:

Corollary 3. *Let (X, d) be a complete CVbMS with the coefficient $s \geq 1$, and T: $X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0, 1]$, such that for each $x \in X$, $[Tx]_\alpha \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\ell_1 + \ell_2 < 1$$

and

$$\ell_1 \left(d\left(y, [Tx]_\alpha\right) + d\left(x, [Ty]_\alpha\right)\right) + \ell_2 \frac{d\left(x, [Tx]_\alpha\right) d\left(y, [Ty]_\alpha\right)}{1 + d\left(x, y\right)} \in s\left([Tx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$; then, there exists $x^* \in X$ such that $x^* \in [Tx^*]_\alpha$.

Taking φ: $X \times X \to [1, +\infty)$ by $\varphi(x, y) = 1$ in Theorem 6, we get the following result:

Corollary 4. *([18]) Let (X, d) be a complete CVMS, and let $S, T\colon X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0, 1]$ such that for each $x \in X$, $[Sx]_\alpha$, $[Tx]_\alpha \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\ell_1 + \ell_2 < 1$$

and

$$\ell_1 \left(d\left(y, [Sx]_\alpha\right) + d\left(x, [Ty]_\alpha\right)\right) + \ell_2 \frac{d\left(x, [Sx]_\alpha\right) d\left(y, [Ty]_\alpha\right)}{1 + d\left(x, y\right)} \in s\left([Sx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$; then, S and T have a common α-fuzzy fixed point.

By taking $S = T$ in the above corollary, we get the following result:

Corollary 5. *([18]) Let (X, d) be a complete CVMS, and $T\colon X \to \mathfrak{F}(X)$ satisfy the g.l.b property. Suppose that there exists $\alpha \in (0, 1]$ such that for each $x \in X$, $[Tx]_\alpha \in CB(X)$ and there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\ell_1 + \ell_2 < 1$$

and

$$\ell_1 \left(d\left(y, [Tx]_\alpha\right) + d\left(x, [Ty]_\alpha\right)\right) + \ell_2 \frac{d\left(x, [Tx]_\alpha\right) d\left(y, [Ty]_\alpha\right)}{1 + d\left(x, y\right)} \in s\left([Tx]_\alpha, [Ty]_\alpha\right),$$

for all $x, y \in X$; then, there exists $x^* \in X$ such that $x^* \in [Tx^*]_\alpha$.

4. Application

4.1. Multivalued mappings results

Theorem 7. *Let (X, d) be a complete CVEbMS, $\varphi\colon X \times X \to [1, +\infty)$ and $\Game_1, \Game_2\colon X \to CB(X)$ satisfy the g.l.b property. Suppose that there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\varphi(x, y)\ell_1 + \ell_2 < 1$$

and

$$\lambda(1 - \varphi(x, y)\ell_1 - \ell_2) = \varphi(x, y)\ell_1,$$

where $\lambda \in [0, 1)$ such that

$$\ell_1 \left(d\left(y, \Game_1 x\right) + d\left(x, \Game_2 y\right)\right) + \frac{\ell_2 d\left(x, \Game_1 x\right) d\left(y, \Game_2 y\right)}{1 + d\left(x, y\right)} \in s\left(\Game_1 x, \Game_2 y\right),$$

for all $x, y \in X$. If for each $x_0 \in X$,

$$\lim_{n,m \to +\infty} \varphi\left(x_n, x_m\right) \lambda < 1,$$

then there exists $x^* \in X$ such that $x^* \in \Game_1 x^* \cap \Game_2 x^*$.

Proof. Consider that $S, T\colon X \to \mathfrak{F}(X)$ is defined by

$$S(x)(t) = \begin{cases} \alpha, & t \in \Game_1 x \\ 0, & t \notin \Game_1 x \end{cases},$$

$$T(x)(t)=\begin{cases}\alpha, & t\in\eth_2 x\\ 0, & t\notin\eth_2 x\end{cases},$$

where $\alpha\in(0,1]$. Then

$$[Sx]_\alpha=\{t:S(x)(t)\geq\alpha\}=\eth_1 x,\quad [Tx]_\alpha=\eth_2 x.$$

Hence, we can get $x^*\in X$ by Theorem 6 such that

$$x^*\in[Sx^*]_\alpha\cap[Tx^*]_\alpha=\eth_1 x^*\cap\eth_2 x^*.$$

□

Corollary 6. *Let (X,d) be a complete CVEbMS, φ: $X\times X\to[1,+\infty)$ and $\eth$: $X\to CB(X)$ satisfy the g.l.b property. Suppose that there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\varphi(x,y)\ell_1+\ell_2<1$$

and

$$\lambda(1-\varphi(x,y)\ell_1-\ell_2)=\varphi(x,y)\ell_1,$$

where $\lambda\in[0,1)$ such that

$$\ell_1\left(d\left(y,\eth x\right)+d\left(x,\eth y\right)\right)+\frac{\ell_2 d\left(x,\eth x\right)d\left(y,\eth y\right)}{1+d\left(x,y\right)}\in s\left(\eth x,\eth y\right),$$

for all $x,y\in X$. If for each $x_0\in X$,

$$\lim_{n,m\to+\infty}\varphi\left(x_n,x_m\right)\lambda<1,$$

then there exists $x^*\in X$ such that $x^*\in\eth x^*$.

Proof. Take $\eth_1=\eth_2=\eth$ in Theorem 7. □

Corollary 7. *Let (X,d) be a complete CVEbMS, φ: $X\times X\to[1,+\infty)$ and $\eth_1,\eth_2$: $X\to CB(X)$ satisfy the g.l.b property. Suppose that there exists a non-negative constant ℓ_1 such that $2\varphi(x,y)\ell_1\in[0,1)$ and*

$$\ell_1\left(d\left(y,\eth_1 x\right)+d\left(x,\eth_2 y\right)\right)\in s\left(\eth_1 x,\eth_2 y\right),$$

for all $x,y\in X$. If for each $x_0\in X$,

$$\lim_{n,m\to+\infty}\varphi\left(x_n,x_m\right)\ell_1<1,$$

then there exists $x^*\in X$ such that $x^*\in\eth_1 x^*\cap\eth_2 x^*$.

Proof. Take $\ell_2=0$ in Theorem 7. □

Taking $\varphi(x,y)=1$ in the above Theorem 7, then one can obtain the fundamental theorem of Ahmad et al. [11] in the following manner:

Corollary 8. *([11]) Let (X,d) be a complete CVMS and $\eth_1,\eth_2$: $X\to CB(X)$ satisfy the g.l.b property. Suppose that there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\ell_1+\ell_2<1$$

such that

$$\ell_1\left(d\left(y,\eth_1 x\right)+d\left(x,\eth_2 y\right)\right)+\ell_2\frac{d\left(x,\eth_1 x\right)d\left(y,\eth_2 y\right)}{1+d\left(x,y\right)}\in s\left(\eth_1 x,\eth_2 y\right),$$

for all $x,y\in X$; then, there exists $x^*\in X$ such that $x^*\in\eth_1 x^*\cap\eth_2 x^*$.

4.2. Self mapping results

Theorem 8. *Let (X, d) be a complete CVEbMS, φ: $X \times X \to [1, +\infty)$ and $\eth_1, \eth_2$: $X \to X$. Suppose that there exist non-negative constants ℓ_1 and ℓ_2 with*

$$2\varphi(x, y)\ell_1 + \ell_2 < 1$$

and

$$\lambda(1 - \varphi(x, y)\ell_1 - \ell_2) = \varphi(x, y)\ell_1,$$

where $\lambda \in [0, 1)$ such that

$$d(\eth_1 x, \eth_2 y) \leq \ell_1 (d(y, \eth_1 x) + d(x, \eth_2 y)) + \ell_2 \frac{d(x, \eth_1 x)\, d(y, \eth_2 y)}{1 + d(x, y)},$$

for all $x, y \in X$. If for each $x_0 \in X$,

$$\lim_{n,m \to +\infty} \varphi(x_n, x_m)\, \lambda < 1,$$

then there exists $x^* \in X$ such that $x^* = \eth_1 x^* = \eth_2 x^*$.

Taking $\varphi(x, y) = 1$ in the above Theorem 8, then one can establish the following corollary which is main result of Rouzkard and Imdad [9]:

Corollary 9. *([9]) Let (X, d) be a complete CVMS and $\eth_1, \eth_2$: $X \to X$. Suppose that there exist non-negative constants ℓ_1 and ℓ_2 with $2\ell_1 + \ell_2 < 1$ such that*

$$d(\eth_1 x, \eth_2 y) \leq \ell_1 (d(y, \eth_1 x) + d(x, \eth_2 y)) + \ell_2 \frac{d(x, \eth_1 x)\, d(y, \eth_2 y)}{1 + d(x, y)},$$

for all $x, y \in X$; then, there exists $x^* \in X$ such that $x^* = \eth_1 x^* = \eth_2 x^*$.

Taking $\varphi(x, y) = 1$ and $\ell_1 = 0$ in the above Theorem 8, then one can establish the following corollary which is one of the results of Azam et al. [8].

Corollary 10. *([8]) Let (X, d) be a complete CVMS and $\eth_1, \eth_2$: $X \to X$. Suppose that there exists a non-negative constant $\ell_2 \in [0, 1)$ such that*

$$d(\eth_1 x, \eth_2 y) \leq \ell_2 \frac{d(x, \eth_1 x)\, d(y, \eth_2 y)}{1 + d(x, y)},$$

for all $x, y \in X$; then, there exists $x^* \in X$ such that $x^* = \eth_1 x^* = \eth_2 x^*$.

5. Application

In this section, we investigate Fredholm type integral inclusion

$$x(t) \in \mathfrak{g}(t) + \int_a^b K(t, s, x(s))ds, \quad t \in [a, b], \tag{5.1}$$

where K: $[a, b] \times [a, b] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ (non-empty convex and compact subsets of $\mathbb{R}$), $\mathfrak{g} \in C[a, b]$ is given continuous function.

Define the complex valued extended b-metric d on $C[a,b]$ by

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|e^{it}, \tag{5.2}$$

for all $x, y \in C[a,b]$. Then $(C[a,b], d, \varphi)$ is a complete CVEbMS with $\varphi(x,y) = |x(t)| + |y(t)| + 2$.

The following conditions will be assumed in our next theorem:

(a) $\forall\ x \in C[a,b]$, the function K: $[a,b] \times [a,b] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is lower semicontinuous.

(b) There exist continuous functions $\mathfrak{O}, \mathfrak{P}$: $[a,b] \times [a,b] \to [0,+\infty)$ such that

$$d(K(t,s,x) - K(t,s,y) \preceq \mathfrak{O}(t,s)A(x,y) + \mathfrak{P}(t,s)B(x,y)$$

$\forall\ t, s \in [a,b]$, $x, y \in C[a,b]$, where

$$A(x,y) = \max_{t\in[a,b]} \left(|x - [Ty]_\alpha| + |y - [Tx]_\alpha|\right) e^{it},$$

$$B(x,y) = \frac{\max_{t\in[a,b]} \left|x - [Ty]_\alpha\right| e^{it} \max_{t\in[a,b]} |y - [Tx]_\alpha|| e^{it}}{1 + \max_{t\in[a,b]} |x - y| e^{it}}$$

and T: $C[a,b] \to \mathfrak{F}(X)$ is an FM given by

$$[Tx]_\alpha = \left\{ y \in X : y(t) \in \mathfrak{g}(t) + \int_a^b K(t,s,x(s))ds, \quad t \in [a,b] \right\},$$

for $\alpha \in (0,1]$,

(c) There exists some non-negative constants ℓ_1 and ℓ_2 such that

$$\max_{t\in[a,b]} \left(\int_a^b \mathfrak{O}(t,s)ds \right) \leq \ell_1$$

and

$$\max_{t\in[a,b]} \left(\int_a^b \mathfrak{P}(t,s)ds \right) \leq \ell_2$$

with

$$2\varphi(x,y)\ell_1 + \ell_2 < 1.$$

Theorem 9. *Under the assumptions (a)–(c), the integral inclusion (5.1) has a solution in* $C[a,b]$.

Proof. Let X=$C[a,b]$ and $x \in X$ be any arbitrary point. By Michael's selection theorem, there exists a continuous operator $k_x(t,s)$: $[a,b] \times [a,b] \to \mathbb{R}$ such that $k_x(t,s) \in K_x(t,s)$ for every $t, s \in [a,b]$ and set-valued function $K_x(t,s)$: $[a,b] \times [a,b] \to K_{cv}(\mathbb{R})$. This yields that

$$\mathfrak{g}(t) + \int_a^b k_x(t,s)ds \in [Tx]_\alpha .$$

Thus, $[Tx]_\alpha \neq \emptyset$. It is very simple to manifest that $[Tx]_\alpha$ is closed. Moreover, since the functions $\mathfrak{g}$ and $K_x(t,s)$ are continuous, the ranges of both functions are bounded. It also follows that $[Tx]_\alpha$ is bounded. Thus $[Tx]_\alpha \in P_{cb}(X)$. □

For this, let $x, y \in X$; then, there exist $[Tx]_\alpha$ and $[Ty]_\alpha$ such that $[Tx]_\alpha, [Ty]_\alpha \in P_{cb}(X)$. Let $u \in [Tx]_\alpha$ be an arbitrary point such that

$$u(t) \in \mathfrak{g}(t) + \int_a^b K(t, s, x(s))ds$$

for $t \in [a, b]$ holds. It means that $\forall\ t, s \in [a, b]$ and $\exists\ k_x(t, s) \in K_x(t, s) = K(t, s, x(s))$ such that

$$u(t) = \mathfrak{g}(t) + \int_a^b k_x(t, s)ds$$

for $t \in [a, b]$. From (b), we have

$$d(K(t, s, x) - K(t, s, y) \leq \mathfrak{O}(t, s)A\,(x(s), y(s)) + \mathfrak{P}(t, s)B\,(x(s), y(s))\,,$$

for all $x, y \in X$, where

$$A\,(x(s), y(s)) = \max_{t\in[a,b]} \left(|x(s) - [Ty(s)]_\alpha| + |y(s) - [Tx(s)]_\alpha|\right) e^{it},$$

$$B\,(x(s), y(s)) = \frac{\max_{t\in[a,b]} \left|x(s) - [Ty(s))]_\alpha\right| e^{it} \max_{t\in[a,b]} \left|y(s) - [Tx(s)]_\alpha\,|\right| e^{it}}{1 + \max_{t\in[a,b]} |x(s) - y(s)|\, e^{it}}.$$

This implies that $\exists\ z(t, s) \in K_y(t, s)$ such that

$$|k_x(t, s) - z(t, s)|^2 \leq \mathfrak{O}(t, s)A\,(x(s), y(s)) + \mathfrak{P}(t, s)B\,(x(s), y(s))\,,$$

for all $t, s \in [a, b]$ and

$$A\,(x(s), y(s)) = \max_{t\in[a,b]} \left(|x(s) - [Ty(s)]_\alpha| + |y(s) - [Tx(s)]_\alpha|\right) e^{it},$$

$$B\,(x(s), y(s)) = \frac{\max_{t\in[a,b]} \left|x(s) - [Ty(s))]_\alpha\right| e^{it} \max_{t\in[a,b]} \left|y(s) - [Tx(s)]_\alpha\,|\right| e^{it}}{1 + \max_{t\in[a,b]} |x(s) - y(s)|\, e^{it}}.$$

Now we consider U given as

$$U(t, s) = K_y(t, s) \cap \{w \in \mathbb{R} : |k_x(t, s) - w| \leq \mathfrak{O}(t, s)A\,(x(s), y(s)) + \mathfrak{P}(t, s)B\,(x(s), y(s))\},$$

which is a multivalued function. Thus, by (a), the multivalued function U is lower semicontinuous. This yields that there exists a continuous operator

$$k_y(t, s) : [a, b] \times [a, b] \to \mathbb{R}$$

such that $k_y(t, s) \in U(t, s)$ for $t, s \in [a, b]$. Then

$$v(t) = \mathfrak{g}(t) + \int_a^b k_x(t, s)ds$$

satisfies that

$$v(t) \in \mathfrak{g}(t) + \int_a^b K(t, s, y(s))ds, \quad t \in [a, b],$$

$t \in [a, b]$. That is $v \in [Ty]_\alpha$ and

$$\begin{aligned} |u(t) - v(t)| e^{it} &\leq \left(\int_a^b \left| k_x(t, s) - k_y(t, s) \right| e^{it} ds \right) \\ &\leq \left(\int_a^b \mathfrak{O}(t, s) A(x(s), y(s)) + \mathfrak{P}(t, s) B(x(s), y(s)) \right) \\ &\leq \left(\int_a^b \mathfrak{O}(t, s) A(x(s), y(s)) \right) ds + \left(\int_a^b \mathfrak{P}(t, s) B(x(s), y(s)) \right) ds \\ &\leq \max_{t \in [a,b]} \left(\int_a^b \mathfrak{O}(t, s) A(x(s), y(s)) \right) ds + \max_{t \in [a,b]} \left(\int_a^b \mathfrak{P}(t, s) B(x(s), y(s)) \right) ds \\ &\leq \ell_1 \left(d(x, [Ty]_\alpha) + d(y, [Tx]_\alpha) \right) + \ell_2 \frac{d(x, [Ty]_\alpha) d(y, [Tx]_\alpha)}{1 + d(x, y)}, \end{aligned}$$

$\forall t, s \in [a, b]$. Hence, we get

$$d(u, v) \leq \ell_1 \left(d(x, [Ty]_\alpha) + d(y, [Tx]_\alpha) \right) + \ell_2 \frac{d(x, [Ty]_\alpha) d(y, [Tx]_\alpha)}{1 + d(x, y)}.$$

Interchanging the roles of u and v, we obtain that

$$d([Tx]_\alpha, [Ty]_\alpha) \leq \ell_1 \left(d(x, [Ty]_\alpha) + d(y, [Tx]_\alpha) \right) + \ell_2 \frac{d(x, [Ty]_\alpha) d(y, [Tx]_\alpha)}{1 + d(x, y)},$$

by the definition of "s", we have

$$\ell_1 \left(d(x, [Ty]_\alpha) + d(y, [Tx]_\alpha) \right) + \ell_2 \frac{d(x, [Ty]_\alpha) d(y, [Tx]_\alpha)}{1 + d(x, y)} \in s\left([Tx]_\alpha, [Ty]_\alpha \right).$$

Hence all assumptions of Corollary 1 are fulfilled. Thus, there exists a solution of integral inclusion (5.1) by Corollary 1.

6. Conclusions

In this article, we utilized the notion of a CVEbMS and set up common α-fuzzy fixed point results for Chatterjea type contractions involving rational expressions. Some common α- fixed point theorems for self mappings and multivalued mappings have been established for CVEbMS as consequences of our leading results. In this way, we derived the leading results of Azam et al. [8], Rouzkard and Imdad [9], Ahmad et al. [11] and Kutbi et al. [18] from our results. As an application, we investigated the solution of Fredholm integral inclusion.

The study of L-FMs and common L-fuzzy fixed point results for CVEbMS can be the focus of our future work in this way.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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