Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 19121–19141.
DOI: 10.3934/math.2023976
Received: 06 February 2023
Revised: 29 April 2023
Accepted: 07 May 2023
Published: 07 June 2023

Research article

Hamiltonian elliptic system involving nonlinearities with supercritical exponential growth

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Abstract: In this paper, we deal with the existence of nontrivial solutions to the following class of strongly coupled Hamiltonian systems:

$$\begin{cases} -\mathrm{div}(w(x)\nabla u) = g(x,v), & x \in B_1(0), \\ -\mathrm{div}(w(x)\nabla v) = f(x,u), & x \in B_1(0), \\ u = v = 0 & x \in \partial B_1(0), \end{cases}$$

where $w(x) = \left(\log 1/|x|\right)^{\gamma}$, $0 \leq \gamma < 1$, and the nonlinearities f and g possess exponential growth ranges above the exponential critical hyperbola. Our approach is based on Trudinger-Moser type inequalities for weighted Sobolev spaces and variational methods.

Keywords: Hamiltonian system; Trudinger-Moser inequality; supercritical exponential growth; variational methods; linking theorem
Mathematics Subject Classification: 35J20, 35J47, 35J50, 26D10

1. Introduction

In the literature, the existence of nontrivial solutions for strongly coupled Hamiltonian systems has been extensively studied by many authors [3, 6, 15–18, 20, 25, 29, 34, 38–41]. A Hamiltonian system is a mathematical expression of the following form:

$$\begin{cases} -\Delta u = H_v(x,u,v), & x \in \Omega, \\ -\Delta v = H_u(x,u,v), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $H(x,u,v)$ is a nonlinear function.

In the case $N \geq 3$. A classical model for H is given by $H(x,u,v) = |u|^{p+1}/(p+1) + |v|^{q+1}/(q+1)$

and the maximal growth for the exponents p and q are related to the curve [17, 20, 29]:

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{N-2}{N}. \tag{1.2}$$

If the couple (p,q) lies on (1.2), some features of noncompactness arises, this motivates one to name (1.2) as the critical hyperbola, and we say that the nonlinearities $H_v=|v|^{q-1}v$ and $H_u=|u|^{p-1}u$ possess critical growth; alternatively, if the couple (p,q) is below (1.2) the growth of the nonlinearities are denominated subcritical. We want to point out that the critical hyperbola results from the borderline between existence and nonexistence of solutions for (1.1) (see [6]).

In the case when $N=2$, the critical hyperbola is not defined. Notice that, if Ω is a bounded domain in $\mathbb{R}^N$, the Sobolev embeddings state $W_0^{1,2}(\Omega)\subset L^q(\Omega)$ for all $1\leq q\leq 2^*=2N/(N-2)$ for $N\geq 3$. In dimension $N=2$, one has $2^*=+\infty$ and $W_0^{1,2}(\Omega)\not\subset L^\infty(\Omega)$. Therefore, H_u and H_v may have any arbitrary polynomial growth. It was shown independently by Yudovich [44] , Pohožaev [32], and Trudinger [43] that the growth is of exponential type. More precisely, $e^{\alpha u^2}\in L^1(\Omega)$ for all $u\in H_0^1(\Omega)$ and $\alpha>0$. Furthermore, Moser [30] proved the existence of a positive constant $C=C(\alpha,\Omega)$ such that

$$\sup_{\substack{u\in H_0^1(\Omega)\\ \|\nabla u\|_2\leq 1}}\int_\Omega e^{\alpha u^2}\,dx\begin{cases}\leq C, & \alpha\leq 4\pi,\\ =+\infty, & \alpha>4\pi.\end{cases} \tag{1.3}$$

From now on, the estimate of the type (1.3) will be referred to as the Trudinger-Moser inequality. These inequalities have been extended in many directions (see [3, 10, 12, 14, 23, 25, 28, 31, 36, 42] among others). The above results motivate us to say that the function f has subcritical exponential growth if

$$\lim_{s\to+\infty}\frac{f(s)}{e^{\alpha s^2}}=0,\quad \text{for all }\ \alpha>0,$$

and critical exponential growth if there exists $\alpha_0>0$ such that

$$\lim_{s\to+\infty}\frac{f(s)}{e^{\alpha s^2}}=\begin{cases}0, & \alpha>\alpha_0,\\ +\infty, & \alpha<\alpha_0.\end{cases} \tag{1.4}$$

Nonlinear equations considering nonlinearities involving subcritical and critical exponential growth were treated by Adimurthi [1], Adimurthi-Yadava [2], de Figueiredo, Miyagaki, and Ruf [19] (see also [10, 12, 24, 31, 35]). We recall that a nonlinear equation in a domain $\Omega\subset\mathbb{R}^N$ with $N\geq 3$ a classical assumption on the nonlinearity is given by $|f(s)|\leq c(1+|s|^{q-1})$, with $1<q\leq 2^*=2N/(N-2)$ (see [5, 7, 8, 11, 26, 27] among others). If there exist positive constants k and s_0 such that $g_1(s)\leq g_2(ks)$ for $s\geq s_0$, we shall write $g_1(s)\prec g_2(s)$. Additionally, we shall say that g_1 and g_2 are equivalent and write $g_1(s)\sim g_2(s)$ if $g_1(s)\prec g_2(s)$ and $g_2(s)\prec g_1(s)$. Therefore, f possesses critical exponential growth if and only if $f(s)\sim e^{|s|^2}$.

The existence of a nontrivial solution of the system (1.1) under $H_v\sim e^{v^2}$ and $H_u\sim e^{u^2}$ and considering $H_0^1(\Omega)\times H_0^1(\Omega)$ as the setting space was proved by de Figueiredo, do Ó, and Ruf [18].

Now, we recall some facts about Lorentz-Sobolev spaces. Let $1<r<+\infty$, $1\leq s<+\infty$ and Ω subset of $\mathbb{R}^N$, the Lorentz space $L^{r,s}(\Omega)$ is the collection of all measurable and finite almost everywhere functions on Ω such that $\|\phi\|_{r,s}<+\infty$, where

$$\|\phi\|_{r,s}=\left(\int_0^{+\infty}[\phi^*(t)t^{1/r}]^s\frac{dt}{t}\right)^{1/s},$$

where $\phi*$ denotes the spherically symmetric decreasing rearrangement of ϕ. In addition, if Ω is an open bounded domain in $\mathbb{R}^N$, the Lorentz-Sobolev space $W_0^1 L^{r,s}(\Omega)$ is defined to be the closure of the compactly supported smooth functions on Ω, with respect to the quasinorm

$$\|u\|_{W_0^1 L^{r,s}} := \|\nabla u\|_{r,s}.$$

Brezis and Wainger [9] proposed the following Trudinger-Moser inequality version on Lorentz-Sobolev spaces: If Ω be a bounded domain in $\mathbb{R}^2$ and $s > 1$, then $e^{\alpha|u|^{\frac{s}{s-1}}}$ belongs to $L^1(\Omega)$ for all $u \in W_0^1 L^{2,s}(\Omega)$ and $\alpha > 0$. Furthermore, Alvino [4] proved the following refinement of (1.3), there exists a positive constant $C = C(\Omega, s, \alpha)$ such that

$$\sup_{\substack{u\in W_0^1 L^{2,s}(\Omega)\\ \|\nabla u\|_{2,s}\leq 1}} \int_\Omega e^{\alpha|u|^{\frac{s}{s-1}}}\,dx \begin{cases} \leq C, & \alpha \leq (4\pi)^{s/(s-1)},\\ = +\infty, & \alpha > (4\pi)^{s/(s-1)}. \end{cases} \tag{1.5}$$

Ruf [34] showed that, if the setting space of the system (1.1) is given by the product space $W_0^1 L^{2,q}(\Omega) \times W_0^1 L^{2,p}(\Omega)$, the maximal growth of the nonlinearities can be considered like $H_u \sim e^{|u|^p}$ and $H_v \sim e^{|v|^q}$ with $p, q > 1$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.6}$$

In analogy to (1.2), the curve (1.6) is called exponential critical hyperbola. The existence of solutions of the system (1.8) for $p = q = 2$ has been treated in many works [3, 18, 38, 40, 41] among others, and the case where (p, q) lies on the exponential critical hyperbola given by (1.6) was studied in [15,25,39].

Trudinger-Moser type inequalities for radial Sobolev spaces with logarithmic weights were considered by Calanchi and Ruf [12] . Denote by $H^1_{0,\mathrm{rad}}(B_1, w)$, the subspace of the radially symmetric functions in the closure of $C_0^\infty(B_1)$ with respect to the norm

$$\|u\| := \Big(\int_{B_1} |\nabla u|^2 w(x)\,dx\Big)^{\frac{1}{2}},$$

where

$$w(x) = \Big[\log\Big(\frac{1}{|x|}\Big)\Big]^\gamma, \quad 0 \leq \gamma < 1. \tag{1.7}$$

Calanchi and Ruf [12] found that

$$\int_{B_1} e^{\alpha|u|^{\frac{2}{1-\gamma}}}\,dx < +\infty, \quad \text{for all} \quad u \in H^1_{0,\mathrm{rad}}(B_1, w) \ \text{ and } \ \alpha > 0.$$

Furthermore, if $\alpha \leq \alpha^*_\gamma = 2\Big[2\pi(1-\gamma)\Big]^{\frac{2}{2-\gamma}}$, there exists a positive constant C such that

$$\sup_{\|u\|\leq 1} \int_{B_1} e^{\alpha|u|^{\frac{2}{1-\gamma}}}\,dx \leq C.$$

The above results represent an increase in the maximal growth of the exponential type. For $\lambda = 1$, the weight given by (1.7) allows us to consider double exponential growth, see [12, 13, 37] for more details. In this paper, we deal with the existence of solutions to the following Hamiltonian system:

$$\begin{cases} -\mathrm{div}(w(x)\nabla u) = g(x,v), & x \in B_1, \\ -\mathrm{div}(w(x)\nabla v) = f(x,u), & x \in B_1, \\ u = v \equiv 0, & x \in \partial B_1, \end{cases} \tag{1.8}$$

where w is given by (1.7) and B_1 denotes the unit open ball center at the origin in $\mathbb{R}^2$. In order to use variational methods, we consider an associated functional defined on the space $H^1_{0,\,\mathrm{rad}}(B_1, w) \times H^1_{0,\,\mathrm{rad}}(B_1, w)$, which allows us to have nonlinearities of the form $f(u) \sim e^{|u|^{2/(1-\gamma)}}$ and $g(v) \sim e^{|v|^{2/(1-\gamma)}}$.

We assume the following conditions on the nonlinerities f and g:

(H_1) $f, g \in C(\overline{B}_1 \times \mathbb{R})$ and $f(x,s) = g(x,s) = o(s)$ as $s \to 0^+$ and $f(x,s) = g(x,s) = 0$ for all $x \in B_1$ and $s \le 0$.

(H_2) There exist constants $\mu > 2$, $\nu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(x,s) \le s f(x,s) \quad \text{and} \quad 0 < \nu G(x,s) \le s g(x,s), \quad \text{for all} \quad x \in B_1 \text{ and } s > s_0,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$ and $G(x,s) = \int_0^s g(x,t)\,dt$.

(H_3) There exist constants $M > 0$ and $s_1 > 0$ such that

$$0 < F(x,s) \le M f(x,s) \quad \text{and} \quad 0 < G(x,s) \le M g(x,s), \quad \text{for all} \quad x \in B_1 \text{ and } s > s_1.$$

(H_4) There exist constants $\alpha_0 > 0$ and $\beta_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(x,s)}{e^{\alpha s^{2/(1-\gamma)}}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases} \quad \text{and} \quad \lim_{s\to\infty} \frac{g(x,s)}{e^{\beta s^{2/(1-\gamma)}}} = \begin{cases} 0, & \beta > \beta_0, \\ +\infty, & \beta < \beta_0, \end{cases}$$

uniformly on $x \in B_1$.

(H_5) There exist constants $p > 2$ and $C_p > 0$ such that

$$f(x,s) \ge C_p s^{p-1} \quad \text{and} \quad g(s) \ge C_p s^{p-1}, \quad \text{for all} \quad s \ge 0,$$

where

$$C_p > \frac{(p-2)^{(p-2)/2} (\max\{\alpha_0, \beta_0\})^{(p-2)(1-\gamma)/2} S_p^p}{p^{(p-2)/2} (\alpha_\gamma^*)^{(p-2)(1-\gamma)/2}}$$

and

$$S_p := \inf_{0 \ne u \in H^1_{0,rad}(B_1,w)} \frac{\left(\int_{B_1} w(x) |\nabla u|^2 \, dx \right)^{1/2}}{\left(\int_{B_1} |u|^p \, dx \right)^{1/p}}.$$

Setting the product space

$$E = H^1_{0,\,\mathrm{rad}}(B_1, w) \times H^1_{0,\,\mathrm{rad}}(B_1, w),$$

which is a Hilbert space endowed with the inner product

$$\langle (u,v), (\phi,\psi) \rangle_E = \int_{B_1} w(x) (\nabla u \nabla \phi + \nabla v \nabla \psi) \, dx,$$

for all $(u, v), (\phi, \psi) \in E$, to which corresponds the norm

$$\|(u, v)\| = \|(u, v)\|_E := \left(\|u\|^2 + \|v\|^2\right)^{1/2}.$$

Additionally, we denote the dual space of E with its usual norm by E^*. We say that $(u, v) \in E$ is a weak solution of (1.8) if

$$\int_{B_1} w(x)(\nabla u \nabla \psi + \nabla v \nabla \phi)\, dx = \int_{B_1} (f(x, u)\phi + g(x, v)\psi)\ dx, \quad \text{for all} \quad (\phi, \psi) \in E. \tag{1.9}$$

Under the assumption on f and g, we establish the Euler-Lagrange functional $J : E \to \mathbb{R}$ defined by

$$J(u, v) = \int_{B_1} w(x)\nabla u \nabla v\, dx - \int_{B_1} F(x, u)\, dx - \int_{B_1} G(x, v)\, dx,$$

for all $(u, v) \in E$. Furthermore, using standard arguments [21] , $J \in C^1(E, \mathbb{R})$ and, for all (u, v), $(\phi, \psi) \in E$, it holds

$$J'(u, v)(\phi, \psi) = \int_{B_1} w(x)(\nabla u \nabla \psi + \nabla v \nabla \phi)\, dx - \int_{B_1} f(x, u)\phi\, dx - \int_{B_1} g(x, v)\psi\, dx.$$

In particular, $(u, v) \in E$ is a nontrivial weak solution of the system (1.8) if only if $(u, v) \in E$ is a nontrivial critical point of the functional J. Next, we present our existence result for the system (1.8).

Theorem 1.1. *Suppose that f and g satisfy (H_1)–(H_5). Then, the Hamiltonian system (1.8) has a nontrivial weak solution.*

First, observe that if $\gamma = 0$, then the system (1.8) is reduced to $-\Delta u = g(x, v)$ and $-\Delta v = f(x, u)$ with Dirichlet conditions, and the growth of the functions are given by $f(x, u) \sim e^{|u|^2}$ and $g(x, v) \sim e^{|v|^2}$ uniformly on $x \in B_1$, whose existence of nontrivial weak solutions was found in [18]. In the case for $\gamma > 0$, the nonlinearities under the assumption (H_4) behaves like $f(x, u) \sim e^{|u|^p}$ and $g(x, v) \sim e^{|v|^q}$ uniformly on $x \in B_1$, where $1/p+1/q < 1$ and $p = q$, that is, the pair (p, q) lies in the diagonal direction above the exponential critical hyperbola. Therefore, our result treats the Hamiltonian system (1.8) involving nonlinearities with supercritical exponential growth. Consequently, our result complements the works which study nonlinearities $f(x, u) \sim e^{|u|^p}$ and $g(x, v) \sim e^{|v|^q}$ for values where (p, q) lies under and on the the curve (1.6), that is, for nonlinerities that possess subcritical and critical exponential growth, respectively [15, 18, 25, 38–41].

The paper is organized as follows: Section 2 contains some preliminaries results and properties our setting space. In Section 3, we show that the Euler-Lagrange energy functional possesses the geometry of the linking theorem. In Section 4, it is established the finite-dimensional approximation and estimated the minimax level of the functional. Finally, in Section 5, we present the proof of Theorem 1.1.

2. Preliminaries

Let $H^1_{0,rad}(B_1, w)$ be the subspace of the radially symmetric functions in the closure of $C^\infty_0(B_1)$ with respect to the norm

$$\|u\| = \|u\|_{H^1_{0,\,\mathrm{rad}}(B_1,w)} := \left(\int_{B_1} |\nabla u|^2 w(x)\, dx\right)^{1/2},$$

where $w(x) = (\log 1/|x|)^{\gamma}$ and $0 \le \gamma < 1$.

Proposition 2.1. *The Sobolev weighted space $H^1_{0,\,\mathrm{rad}}(B_1, w)$ is a separable Banach space.*

Proof. Let $0 \le \gamma < 1$ be fixed and consider $s(t) = (\log 1/(1-t))^{\gamma}$. Then, s is a continuous positive function defined for the interval $(0,1)$ and satisfies $\lim_{t\to 0} s(t) = 0$. Moreover, if $\Omega = B_1(0)$, we have $d(x) = \mathrm{dist}(x, \partial\Omega) = 1 - |x|$. Thus, $s \circ d = w$. Therefore, we get that $W^{1,2}(\Omega, s\circ d)$ and its restrictions to the radial symmetric functions are separable Banach spaces (see Theorem 3.9 in [22]). □

We note that $H^1_{0,\,\mathrm{rad}}(B, w)$ is a Hilbert space endowed with inner product

$$\langle u, v\rangle := \int_{B_1} w(x)\nabla u \nabla v\, dx, \quad \text{for all } u,\ v \in H^1_{0,\,\mathrm{rad}}(B, w).$$

Now, we state a compactness result.

Lemma 2.2. *The embedding $H^1_{0,\,\mathrm{rad}}(B_1, w) \hookrightarrow L^p(B_1)$ is continuous and compact for $1 \le p < \infty$.*

Proof. It follows from the Cauchy-Schwarz inequality that

$$\int_{B_1} |\nabla u|\, dx \le \Big(\int_{B_1} |\nabla u|^2 w(x)\, dx\Big)^{1/2} \cdot \Big(\int_{B_1} w(x)^{-1}\, dx\Big)^{1/2}.$$

Using the change of variable $|x| = e^{-s}$, we obtain

$$\frac{1}{2\pi}\int_{B_1} w(x)^{-1}\, dx = \int_0^{+\infty} e^{-2s} s^{-\gamma}\, ds = \int_0^1 e^{-2s} s^{-\gamma}\, ds + \int_1^{+\infty} e^{-2s} s^{-\gamma}\, ds,$$

where

$$\int_0^1 e^{-2s} s^{-\gamma}\, ds \le \int_0^1 s^{-\gamma}\, ds = \frac{1}{1-\gamma}$$

and

$$\int_1^{+\infty} e^{-2s} s^{-\gamma}\, ds \le \int_1^{+\infty} e^{-2s}\, ds = \frac{e^{-2}}{2}.$$

Therefore, there exists $C > 0$ such that

$$\|\nabla u\|_1 \le C\Big(\int_{B_1} |\nabla u|^2 w(x)\, dx\Big)^{1/2}.$$

Thus, $H^1_{0,\,\mathrm{rad}}(B_1, w) \hookrightarrow W^{1,1}_0(B_1)$ continuosly, which implies the continuous and compact embedding

$$H^1_{0,\,\mathrm{rad}}(B_1, w) \hookrightarrow L^p(B_1), \quad \text{for all} \quad p \ge 1.$$

□

Proposition 2.3. *(See [12]) Let w be the weight given by (1.7). Then,*

$$\int_{B_1} e^{\alpha |u|^{\frac{2}{1-\gamma}}}\, dx < +\infty, \quad \textit{for all } u \in H^1_{0,\,\mathrm{rad}}(B_1, w) \textit{ and } \alpha > 0. \tag{2.1}$$

Furthermore, if $\alpha \le \alpha^*_{\gamma} = 2\big[2\pi(1-\gamma)\big]^{\frac{1}{2-\gamma}}$, there exists a positive constant C such that

$$\sup_{\|u\|\le 1} \int_{B_1} e^{\alpha |u|^{\frac{2}{1-\gamma}}}\, dx \le C. \tag{2.2}$$

Lemma 2.4. *Let (u_n) be a sequence in $H^1_{0,\,\mathrm{rad}}(B_1, w)$ such that $u_n \rightharpoonup 0$ in $H^1_{0,\,\mathrm{rad}}(B_1, w)$ and $\|u_n\| = 1$ for every $n \in \mathbb{N}$. Then, for every $0 < \alpha < \alpha^*_\gamma$, there exists a subsequence, still denoted by (u_n) such that*

$$\int_{B_1} \left(e^{\alpha|u_n|^{\frac{2}{1-\gamma}}} - 1\right) dx = 0 \to 0, \quad as \quad n \to +\infty.$$

Proof. Choosing $\epsilon > 0$ such that $\alpha + \epsilon < \alpha^*_\gamma$. We have the following limits:

$$\lim_{|t|\to 0} \frac{e^{\alpha|t|^{\frac{2}{1-\gamma}}} - 1}{|t|} = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{e^{\alpha|t|^{\frac{2}{1-\gamma}}} - 1}{|t| e^{(\alpha+\epsilon)|t|^{\frac{2}{1-\gamma}}}} = 0.$$

Thus, we can find $C > 0$ such that

$$e^{\alpha|t|^{\frac{2}{1-\gamma}}} - 1 \leq C|t| + C|t|e^{(\alpha+\epsilon)|t|^{\frac{2}{1-\gamma}}}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Using the Hölder inequality with $r > 1$ such that $r(\alpha + \epsilon) < \alpha^*_\gamma$ and Proposition 2.3, we get

$$\int_{B_1} \left(e^{\alpha|u_n|^{\frac{2}{1-\gamma}}} - 1\right) dx \leq C\|u_n\|_1 + C\|u_n\|_{r'} \int_{B_1} e^{r(\alpha+\epsilon)|u_n|^{\frac{2}{1-\gamma}}} dx \leq C\|u_n\|_1 + C\|u_n\|_{r'}.$$

By the weakly convergence of $u_n \rightharpoonup 0$ in $H^1_{0,\,\mathrm{rad}}(B_1, w)$ and Lemma 2.2 for a subsequence, we have

$$\int_{B_1} \left(e^{\alpha|u_n|^{\frac{2}{1-\gamma}}} - 1\right) dx \to 0.$$

□

3. Linking geometry

This section is devoted to prove that the functional J possesses the geometry of the linking theorem. We start setting the following subspaces:

$$E^+ = \{(u, u) : u \in H^1_{0,\,\mathrm{rad}}(B_1, w)\} \quad \text{and} \quad E^- = \{(u, -u) : u \in H^1_{0,\,\mathrm{rad}}(B_1, w)\},$$

In particular, we have

$$E = E^+ \oplus E^-.$$

Lemma 3.1. *Assume that $(H_1), (H_4)$, and (H_5), are hold. Then, there exist positive constants σ and ρ such that $J(z) \geq \sigma$ for all $z \in \partial B_\rho \cap E^+$.*

Proof. Given $\epsilon > 0$ and $q > 2$, it follows from (H_1) and (H_4), the existence of some $c > 0$ such that

$$|G(x, s)|,\ |F(x, s)| \leq \epsilon|s|^2 + c|s|^q e^{2\alpha_0|s|^{\frac{2}{2-\gamma}}}, \quad \text{for all} \quad (x, s) \in B_1 \times \mathbb{R}.$$

Using the Cauchy-Schwarz inequality and (2.2), we obtain

$$\int_{B_1} F(x, u)\, dx \leq \epsilon\|u\|_2^2 + c\|u\|_{2q}^q \int e^{4\alpha_0|u|^{\frac{2}{2-\gamma}}} dx \leq \epsilon\|u\|_2^2 + c\|u\|_{2q}^q,$$

provided that $\|u\| \leq \rho_0$ for some $\rho_0 > 0$ such that $4\alpha_0\rho_0^{\frac{2}{2-\gamma}} < \alpha_\gamma^*$. A similar inequality holds for the function G. By Lemma 2.2, we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$J(u,u) \geq \|u\|^2 - \int_{B_1} F(x,u)\,dx - \int_{B_1} G(x,u)\,dx \geq \|u\|^2 - 2\epsilon\|u\|_2^2 - 2c\|u\|_{2q}^q \geq (1-2\epsilon c_1)\|u\|^2 - c_2\|u\|^q.$$

Therefore, taking $\epsilon > 0$ sufficiently small, we can choose $\rho > 0$ sufficiently small and $\sigma > 0$ such that $J(z) \geq \sigma$, for all $z \in \partial B_\rho \cap E^+$. □

Taking $e \in H^1_{0,rad}(B_1, w)$ such that $\|e\| = 1$, we define

$$Q_e = \{r(e,e) + (\omega,-\omega) : \ \|(\omega,-\omega)\| \leq R_0,\ 0 \leq r \leq R_1\},$$

where $R_0 >$ and $R_1 > 0$ will be determined by the following result.

Lemma 3.2. *Assume that* (H_1), (H_2) *and* (H_5), *are hold. Then, there exist positive constants* R_0 *and* R_1 *such that*

$$J(z) \leq 0, \quad \textit{for all} \quad z \in \partial Q_e.$$

Proof. We observe that the boundary ∂Q_e of the set Q_e consists of three parts that corresponds the following cases:

(i) Let $z = (\omega, -\omega) \in \partial Q_e \cap E^-$. Therefore,

$$J(z) = -\|\omega\|^2 - \int_{B_1} F(x,\omega)\,dx - \int_{B_1} G(x,-\omega)\,dx \leq 0.$$

(ii) Let $z = r(e,e) + (\omega,-\omega) = (re+\omega, re-\omega) \in \partial Q_e$, with $\|(\omega,-\omega)\| = R_0$ and $0 \leq r \leq R_1$, then

$$J(z) = r^2\|e\|^2 - \|\omega\|^2 - \int_{B_1} F(x, re+\omega)\,dx - \int_{B_1} G(x, re-\omega)\,dx \leq R_1^2\|e\|^2 - \|\omega\|^2 = R_1^2 - \frac{R_0^2}{2}.$$

Thus, $J(z) \leq 0$ if $R_0 = \sqrt{2}R_1$.

(iii) Let $z = R_1(e,e) + (\omega,-\omega) \in \partial Q_e$, with $\|(\omega,-\omega)\| \leq R_0$, it follows from (H_5) that

$$F(x,s), G(x,s) \geq \frac{C_p}{p}|s|^p \quad \text{for all } \ x \in B_1 \ \text{ and } \ s \geq 0.$$

Therefore,

$$\begin{aligned} J(z) &= R_1^2\|e\|^2 - \|\omega\|^2 - \int_{B_1} F(x, R_1e+\omega)\,dx - \int_{B_1} G(x, R_1e-\omega)\,dx \\ &\leq R_1^2 - \int_{B_1} F(x, R_1e+\omega)\,dx - \int_{B_1} G(x, R_1e-\omega)\,dx \\ &\leq R_1^2 - \frac{C_p}{p}\int_{B_1} (|R_1e+\omega|^p + |R_1e-\omega|^p)\,dx \\ &\leq R_1^2 - \frac{2C_pR_1^p}{p}\int_{B_1} |e|^p\,dx. \end{aligned}$$

Since $p > 2$, we can choose $R_1 > 0$ such that $J(z) \leq 0$.

□

4. Finite-dimensional approximation

Observe that the leading part of the functional J is strongly indefinite, that is, J can assume positives and negatives values on infinite-dimensional subspaces of E. Therefore, we can not use the linking theorem. To deal with this inconvenience, we follow the arguments developed by de Figueiredo, do Ó, and Ruf [16], that is, we use a finite dimensional approximation.

Let $e = u_p \in H^1_{0,rad}(B_1, w)$ be a nonnengative function with $\|u_p\|_p = 1$ where S_p is attained. We consider $\{e_i\}_{i\in\mathbb{N}}$ a Hilbert basis of $\langle e\rangle^{\perp}$ and setting

$$E_n^+ = \text{Span}\{(e_i, e_i) : \ i = 1, 2, \ldots, n\}$$

and

$$E_n^- = \text{Span}\{(e_i, -e_i) : \ i = 1, 2 \ldots, n\},$$

$$E_n = E_n^+ \oplus E_n^-.$$

we denote by

$$H_n = \mathbb{R}(e, e) \oplus E_n, \quad H_n^+ = \mathbb{R}(e, e) \oplus E_n^+ \quad \text{and} \quad H_n^- = \mathbb{R}(e, e) \oplus E_n^-.$$

Setting the following class of functions:

$$\Gamma_n = \{\gamma \in C(Q_n, H_n) : \ \gamma(z) = z, \forall z \in \partial Q_n\},$$

where $Q_n = Q_e \cap H_n$, and set

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} J(\gamma(z)). \tag{4.1}$$

Now, let J_n be the restriction of J to the finite-dimensional space H_n. Moreover, Lemmas 3.1 and 3.2 are still valid for J_n. Additionally, it follows from [16] that

$$\gamma(Q_n) \cap (\partial B_\rho \cap H_n^+) \neq \emptyset, \quad \text{for all} \quad \gamma \in \Gamma_n, \tag{4.2}$$

for ρ given by Lemma 3.1. Moreover, Lemma 3.1 and (4.2), implies that

$$c_n \geq \sigma > 0, \quad \text{for all} \quad n \geq 1.$$

Using the fact that the identity map $I_n : Q_n \to H_n$ belongs to Γ_n and the fact that F and G are nonnegative functions, we obtain

$$J(z) = r^2\|e\|^2 - \|u\|^2 - \int_{B_1} F(x, re + u)\, dx - \int_{B_1} G(x, re - u)\, dx \leq R_1^2, \tag{4.3}$$

for each $z = r(e, e) + (u, -u) \in Q_n$. Hence,

$$c_n \leq R_1^2, \quad \text{for all} \quad n \geq 1. \tag{4.4}$$

Next, this proposition follows from the linking theorem for J_n (see [33]).

Proposition 4.1. *Assume that f and g satisfy (H_1)–(H_4). Then, the functional J_n possesses a critical point $z_n = (u_n, v_n) \in H_n$ at level c_n for all $n \in \mathbb{N}$, satisfying*

$$J(z_n) = c_n \in [\sigma, R_1^2], \tag{4.5}$$

where σ and $R_1 > 0$ are given by Lemmas 3.1 and 3.2, respectively, and

$$J_n'(z_n)(\phi, \psi) = 0, \quad \text{for all } (\phi, \psi) \in H_n, \tag{4.6}$$

that is

$$\int_{B_1} w(x)\nabla u_n \nabla \psi \, dx = \int_{B_1} g(x, v_n)\psi \, dx \quad \text{and} \quad \int_{B_1} w(x)\nabla \phi \nabla v_n \, dx = \int_{B_1} f(x, u_n)\phi \, dx, \tag{4.7}$$

for each $(\phi, \psi) \in H_n$.

Lemma 4.2. *(See [15, Lemma 10]) If $r, r' > 1$ satisfy $1/r + 1/r' = 1$, then*

$$st \leq \begin{cases} (e^{t^r} - 1) + s(\ln s)^{1/r}, & \text{for all } t \geq 0 \text{ and } s \geq e^{1/r^{r'}}, \\ (e^{t^r} - 1) + \dfrac{s^{r'}}{r'}, & \text{for all } t \geq 0 \text{ and } 0 \leq s \leq e^{1/r^{r'}}. \end{cases}$$

Lemma 4.3. *Let (u_n, v_n) be a sequence in E satisfying $|J(u_n, v_n)| \leq d$ and*

$$|J'(u_n, v_n)(\phi, \psi)| \leq \epsilon_n \|(\phi, \psi)\|, \quad \text{for all } \phi, \psi \in \{0, u_n, v_n\}, \quad \text{where } \epsilon_n \to 0. \tag{4.8}$$

Then, the sequence (u_n, v_n) is bounded in E.

Proof. Taking $(\phi, \psi) = (u_n, v_n)$ in (4.8), we have

$$\int_{B_1} f(x, u_n)u_n \, dx + \int_{B_1} g(x, v_n)v_n \, dx \leq \left| 2 \int_{B_1} w(x)\nabla u_n \nabla v_n \, dx \right| + \epsilon_n \|(u_n, v_n)\|.$$

Since

$$\int_{B_1} w(x)\nabla u_n \nabla v_n \, dx = J(u_n, v_n) + \int_{B_1} F(x, u_n) \, dx + \int_{B_1} G(x, v_n) \, dx,$$

combined with the fact that $|J(u_n, v_n)| \leq d$, we obtain

$$\int_{B_1} f(x, u_n)u_n \, dx + \int_{B_1} g(x, v_n)v_n \, dx \leq 2d + 2\int_{B_1} F(x, u_n) \, dx + 2\int_{B_1} G(x, v_n) \, dx + \epsilon_n \|(u_n, v_n)\|. \tag{4.9}$$

From (H_2), we get

$$\begin{aligned} \int_{B_1} F(x, u_n) \, dx &= \int_{\{x \in B_1 : |u_n(x)| \leq s_0\}} F(x, u_n) \, dx + \int_{\{x \in B_1 : |u_n(x)| > s_0\}} F(x, u_n) \, dx \\ &\leq \int_{\{x \in B_1 : |u_n(x)| \leq s_0\}} F(x, u_n) \, dx + \frac{1}{\mu} \int_{\{x \in B_1 : |u_n(x)| > s_0\}} f(x, u_n)u_n \, dx \\ &= \int_{\{x \in B_1 : |u_n(x)| \leq s_0\}} \Big(F(x, u_n) - \frac{1}{\mu} f(x, u_n)u_n \Big) \, dx + \frac{1}{\mu} \int_{B_1} f(x, u_n)u_n \, dx \end{aligned}$$

$$\leq M_f|B_1| + \frac{1}{\mu}\int_{B_1} f(x,u_n)u_n\,dx, \tag{4.10}$$

where

$$M_f = \max_{(x,s)\in\overline{B}_1\times[0,s_0]}\Big(|F(x,s)| + \frac{1}{\mu}|f(x,s)s|\Big).$$

Similarly for the function g, there exists $M_g > 0$ such that

$$\int_{B_1} G(x,v_n)\,dx \leq M_g|B_1| + \frac{1}{\nu}\int_{B_1} g(x,v_n)v_n\,dx. \tag{4.11}$$

From (4.10) and (4.11) in (4.9), we obtain

$$\Big(1-\frac{2}{\mu}\Big)\int_{B_1} f(x,u_n)u_n\,dx + \Big(1-\frac{2}{\nu}\Big)\int_{B_1} g(x,v_n)v_n\,dx \leq 2d + 2(M_f+M_g)|B_1| + \epsilon_n\|(u_n,v_n)\|. \tag{4.12}$$

Taking $(\phi,\psi) = (v_n,0)$ and $(\phi,\psi) = (0,u_n)$ in (4.8), we get

$$\|v_n\|^2 = \int_{B_1} w(x)\nabla v_n\nabla v_n\,dx \leq \int_{B_1} f(x,u_n)v_n\,dx + \epsilon_n\|(v_n,0)\|$$

and

$$\|u_n\|^2 = \int_{B_1} w(x)\nabla u_n\nabla u_n\,dx \leq \int_{B_1} g(x,v_n)u_n\,dx + \epsilon_n\|(0,u_n)\|.$$

We define

$$V_n = \frac{v_n}{\|v_n\|} \quad \text{and} \quad U_n = \frac{u_n}{\|u_n\|}.$$

Thus,

$$\|v_n\| \leq \int_{B_1} f(x,u_n)V_n\,dx + \epsilon_n\|v_n\| \tag{4.13}$$

and

$$\|u_n\| \leq \int_{B_1} g(x,v_n)U_n\,dx + \epsilon_n\|u_n\|. \tag{4.14}$$

Let $\alpha_1 = \alpha_0 + \xi$. By assumption (H_4), there exists $\lambda > 0$ such that

$$|f(x,s)| \leq \lambda e^{\alpha_1|s|^{\frac{2}{1-\gamma}}}, \quad \text{for all} \quad (x,s)\in B_1\times\mathbb{R}. \tag{4.15}$$

Set $\alpha_2 = \alpha_\gamma^* - \xi$, using (4.13), we can write

$$\|v_n\| \leq \frac{\lambda}{\alpha_2^{\frac{1-\gamma}{2}}}\int_{B_1}\frac{|f(x,u_n(x))|}{\lambda}\alpha_2^{\frac{1-\gamma}{2}}|V_n(x)|\,dx.$$

From Lemma 4.2 with $s = |f(x,u_n(x))|/\lambda$, $t = \alpha_2^{\frac{1-\gamma}{2}}|V_n(x)|$, $r = 2/(1-\gamma)$ and $r' = 2/(1+\gamma)$, we obtain

$$\|v_n\| \leq \frac{\lambda}{\alpha_2^{\frac{1-\gamma}{2}}}\Big[\int_{B_1}(e^{\alpha_2|V_n|^{\frac{2}{1-\gamma}}}-1)\,dx + \frac{1+\gamma}{2\lambda^{\frac{2}{1+\gamma}}}\int_{\{x\in B_1:\frac{|f(x,u_n(x)|}{\lambda}\leq e^{\frac{1-\gamma}{2}\frac{2}{1+\gamma}}\}}|f(x,u_n)|^{\frac{2}{1+\gamma}}\,dx$$

$$+\int_{\{x\in B_1:\frac{|f(x,u_n(x)|}{\lambda}\geq e^{\frac{1-\gamma}{2}\frac{2}{1+\gamma}}\}}\frac{|f(x,u_n)|}{\lambda}\Big(\ln\frac{|f(x,u_n)|}{\lambda}\Big)^{\frac{1-\gamma}{2}}\,dx\Big]+\epsilon_n\|v_n\|. \tag{4.16}$$

By (4.15), we obtain

$$\int_{\{x\in B_1:\frac{|f(x,u_n(x)|}{\lambda}\geq e^{\frac{1-\gamma}{2}\frac{2}{1+\gamma}}\}}\frac{|f(x,u_n)|}{\lambda}\Big(\ln\frac{|f(x,u_n)|}{\lambda}\Big)^{\frac{1-\gamma}{2}}\,dx\leq\frac{\alpha_1^{\frac{1-\gamma}{2}}}{\lambda}\int_{B_1}f(x,u_n)u_n\,dx. \tag{4.17}$$

Observe that

$$\int_{\{x\in B_1:\frac{|f(x,u_n(x)|}{\lambda}\leq e^{\frac{1-\gamma}{2}\frac{2}{1+\gamma}}\}}|f(x,u_n)|^{\frac{2}{1+\gamma}}\,dx\leq\Big(\lambda e^{\frac{1-\gamma}{2}\frac{2}{1+\gamma}}\Big)^{\frac{2}{1+\gamma}}|B_1|. \tag{4.18}$$

From Proposition 2.3, we have

$$\int_{B_1}(e^{\alpha_2|V_n|^{\frac{2}{1-\gamma}}}-1)\,dx\leq C. \tag{4.19}$$

By replacing (4.17)–(4.19) in (4.16), we get $c_1>0$ and $c_2>0$ such that

$$\|v_n\|\leq c_1\int_{B_1}f(x,u_n)u_n\,dx+c_2+\epsilon_n\|v_n\|. \tag{4.20}$$

Similarly, we get

$$\|u_n\|\leq c_1\int_{B_1}g(x,v_n)v_n\,dx+c_2+\epsilon_n\|u_n\|. \tag{4.21}$$

Using (4.12), (4.20) and (4.21), we can find $c>0$ such that

$$\|v_n\|\leq c+\epsilon_n\|(u_n,v_n)\|+\epsilon_n\|v_n\|\quad\text{and}\quad\|u_n\|\leq c+\epsilon_n\|(u_n,v_n)\|+\epsilon_n\|u_n\|.$$

We finally obtain

$$\|(u_n,v_n\|\leq c+\epsilon_n\|(u_n,v_n)\|$$

which implies that $\|(u_n,v_n)\|\leq c$, for every $n\in\mathbb{N}$, for some positive constant c. □

Lemma 4.4. *Assuming the conditions* (H_1)–(H_5)*, are hold. Let* (u_n,v_n) *be a sequence in* E *and* $(u,v)\in E$ *such that* $(u_n,v_n)\rightharpoonup(u,v)$ *weakly in* E*,* $J(u_n,v_n)\to c$ *and* $\|J'(u_n,v_n)\|_{E^*}\to 0$*. Then,*

(i) $f(x,u_n)\to f(x,u)$ *in* $L^1(B_1)$ *and* $g(x,u_n)\to g(x,u)$ *in* $L^1(B_1)$*,*
(ii) $F(x,u_n)\to F(x,u)$ *in* $L^1(B_1)$ *and* $G(x,u_n)\to G(x,u)$ *in* $L^1(B_1)$*.*

Proof. From Lemma 2.2, we can suppose that u_n converges to u in $L^1(B_1)$. By Proposition 2.3, and the assumptions (H_1) and (H_4), we imply that $f(x,u_n)\in L^1(B_1)$. Moreover, using $J'(u_n,v_n)(u_n,v_n)=o_n(1)$, we can find $c>0$ such that

$$\int_{B_1}f(x,u_n)u_n\,dx+\int_{B_1}g(x,v_n)v_n\,dx\leq c.$$

According to [19, Lemma 2.10], we obtain the limit (i) . On the other hand, from (i), we obtain

$$\int_{B_1}f(x,u_n)\,dx\to\int_{B_1}f(x,u)\,dx.$$

Therefore, there exists $p \in L^1(B_1)$ such that

$$f(x,u_n) \leq p(x) \text{ almost everywhere in } B_1. \tag{4.22}$$

From (H_1) and (H_3), we obtain

$$F(x,t) \leq \max_{(x,t)\in\overline{B}_1\times[0,s_0]} F(x,t) + Mf(x,t), \quad \text{for all} \quad (x,t) \in B_1 \times \mathbb{R}. \tag{4.23}$$

Using (4.22) and (4.23), we have

$$F(x,u_n) \leq \max_{(x,t)\in\overline{B}_1\times[0,s_0]} F(x,t) + Mp(x), \quad \text{for all} \quad x \in B_1. \tag{4.24}$$

Therefore, $F(x,u_n) \to F(x,u)$ in $L^1(B_1)$, which follows from Lebesgue's dominated convergence theorem. □

Let recall that for $p > 2$, $u_p \in E$ denotes the nonnegative function such that $\|u_p\|_p = 1$ and

$$S_p = \inf_{0\neq u\in E} \frac{\Big(\int_{B_1} w(x)|\nabla u|^2\, dx\Big)^{1/2}}{\Big(\int_{B_1} |u|^p\, dx\Big)^{1/p}} = \|u_p\|. \tag{4.25}$$

Lemma 4.5. *Suppose that f and g satisfy $(H_1)-(H_5)$. Then, the following inequality holds:*

$$\sup_{z\in\mathbb{R}^+(u_p,u_p)+E^-} J(z) < \Big(\frac{\alpha_\gamma^*}{\max\{\alpha_0,\beta_0\}}\Big)^{1-\gamma}.$$

Proof. Let $z = t(u_p,u_p) + (v,-v)$ with $t \geq 0$, $v \in E$ and u_p is given by (4.25). Then,

$$J(z) = t^2\|u_p\|^2 - \|v\|^2 - \int_{B_1} F(x, tu_p + v)\, dx - \int_{B_1} G(x, tu_p - v)\, dx.$$

Using condition (H_5), we have

$$J(z) \leq t^2\|u_p\|^2 - \frac{C_p}{p}\int_{B_1}\Big(|tu_p + v|^p + |tu_p - v)|^p\Big)\, dx \leq t^2\|u_p\|^2 - \frac{2C_p t^p}{p}\int_{B_1} |u_p|^p\, dx.$$

Since $\|u_p\| = S_p$ and $\|u_p\|_p = 1$, we obtain

$$\sup_{z\in\mathbb{R}^+(u_p,u_p)+E^-} J(z) \leq \max_{t\geq 0}\Big\{t^2 S_p^2 - \frac{2C_p t^p}{p}\Big\}.$$

Since the function $\lambda(t) = t^2 S_p^2 - \dfrac{2C_p t^p}{p}$ achieves its maximum on $t_0 = \dfrac{S_p^{2/(p-2)}}{C_p^{1/(p-2)}}$ and using the estimate of C_p, we have

$$\sup_{z\in\mathbb{R}^+(u_p,u_p)+E^-} J(z) = \frac{(p-2)S_p^{2p/(p-2)}}{pC_p^{2/(p-2)}} < \Big(\frac{\alpha_\gamma^*}{\max\{\alpha_0,\beta_0\}}\Big)^{1-\gamma}.$$

□

Remark 4.6. *By Lemma 4.5, there exists $\delta > 0$ such that*

$$c_n \leq \max_{Q_n} J(z) \leq \sup_{\mathbb{R}^+(u_p,u_p)\oplus E_n^-} J(z) \leq \sup_{\mathbb{R}(e,e)\oplus E^-} J(z) \leq \Big(\frac{\alpha^*}{\max\{\alpha_0,\beta_0\}}\Big)^{1-\gamma} - \delta,$$

for every $n \in \mathbb{N}$.

5. Proof of the Theorem 1.1

Let $(u_n, v_n) \in H_n$ be the sequence given by Proposition 4.1. From Lemma 4.3, this sequence is bounded in E. Thus, up to a subsequence, we can assume that $(u, v) \in E$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E, for some $(u, v) \in E$. Taking $(0, \psi)$ and $(\phi, 0)$ in (4.7), where ϕ and ψ belongs to $C^\infty_{0,\,\mathrm{rad}}(B_1) \cap H_n$. Therefore,

$$\int_{B_1} w(x)\nabla u_n \nabla\psi \, dx = \int_{B_1} g(x, v_n)\psi \, dx \tag{5.1}$$

and

$$\int_{B_1} w(x)\nabla v_n \nabla\phi \, dx = \int_{B_1} f(x, u_n)\phi \, dx. \tag{5.2}$$

Taking the limit in (5.1) and (5.2) as $n \to \infty$, by Lemma 4.4 and the density $C^\infty_{0,\,\mathrm{rad}}(B_1) \cap \left(\bigcup_{n\in\mathbb{N}} H_n \right)$ in $H^1_{0,\,\mathrm{rad}}(B_1, \omega)$, we obtain

$$\int_{B_1} w(x)\nabla u \nabla\psi \, dx = \int_{B_1} g(x, v)\psi \, dx, \quad \text{for all } \psi \in H^1_{0,\,\mathrm{rad}}(B_1, \omega) \tag{5.3}$$

and

$$\int_{B_1} w(x)\nabla v \nabla\phi \, dx = \int_{B_1} f(x, u)\phi \, dx, \quad \text{for all } \phi \in H^1_{0,\mathrm{rad}}(B_1, \omega). \tag{5.4}$$

Therefore, $(u, v) \in E$ is a critical point of J. Now, we prove that (u, v) is nontrivial. Since the system (1.8) is strongly coupled, if we assume that $u \equiv 0$ we get that $v \equiv 0$. Therefore, by Lemma 2.2, up to a subsequence, we have

$$u_n \to 0 \ \text{ and } v_n \to 0 \ \text{ in } L^p(B_1), \quad \text{for all } \ p \geq 1 \tag{5.5}$$

and

$$u_n \to 0 \ \text{ and } v_n \to 0 \ \text{ almost everywhere in } \mathbb{R}^2.$$

If we suppose that $\|u_n\|$ is not bounded below by a positive constant, we can get a subsequence of (u_n) such that $\|u_n\| \to 0$. Therefore,

$$\int_{B_1} w(x)\nabla u_n \nabla v_n \, dx \to 0. \tag{5.6}$$

Considering the pairs of functions $(\phi, \psi) = (u_n, 0)$ and $(\phi, \psi) = (0, v_n)$ in (4.7), we have

$$\int_{B_1} w(x)\nabla u_n \nabla v_n \, dx = \int_{B_1} f(x, u_n)u_n \, dx = \int_{B_1} g(x, v_n)v_n \, dx. \tag{5.7}$$

Using Lemma 4.4 and (5.5), we have

$$\int_{B_1} F(x, u_n) \, dx \to 0 \quad \text{and} \quad \int_{B_1} G(x, v_n) \, dx \to 0, \quad \text{as} \quad n \to +\infty. \tag{5.8}$$

Thus, by the above limits, we get that $J(u_n, v_n)$ tends to zero which contradicts (4.5); consequently, $\|u_n\|$ is bounded below by a positive constant, in particular, we can assume that $\|u_n\| \neq 0$ for all $n \in \mathbb{N}$. Now, taking $(\phi, \psi) = (0, u_n)$ in (4.7), we get

$$\|u_n\|^2 = \int_{B_1} g(x, v_n)u_n \, dx. \tag{5.9}$$

Thus,

$$\|u_n\| \le \int_{B_R} g(x, v_n)\frac{u_n}{\|u_n\|}\,dx. \tag{5.10}$$

We assume that $\max\{\alpha_0, \beta_0\} = \alpha_0$. Then, we can write

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \le \int_{B_1} |g(x, v_n)||\bar{u}_n|\,dx,$$

where

$$\bar{u}_n = \Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\frac{u_n}{\|u_n\|}. \tag{5.11}$$

Applying Lemma 4.2 with

$$s = \frac{|g(x, v_n(x))|}{\alpha_0^{\frac{1-\gamma}{2}}}, \quad t = \alpha_0^{\frac{1-\gamma}{2}}|\bar{u}_n(x)|, \quad r = \frac{2}{1-\gamma} \quad \text{and} \quad r' = \frac{2}{1+\gamma},$$

we have

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \le \Big[\int_{B_1}(e^{\alpha_0|\bar{u}_n|^{\frac{2}{1-\gamma}}} - 1)\,dx + \frac{1+\gamma}{2}\int_{\{x\in B_1:\frac{|g(x,v_n(x))|}{\alpha_0^{\frac{1-\gamma}{2}}}\le e^{\frac{1-\gamma}{2+\gamma}}\}}\frac{|g(x, v_n)|^{\frac{2}{1+\gamma}}}{\alpha_0^{\frac{1-\gamma}{1+\gamma}}}\,dx$$
$$+ \int_{\{x\in B_1:\frac{|g(x,v_n(x))|}{\alpha_0^{\frac{1-\gamma}{2}}}\ge e^{\frac{1-\gamma}{1+\gamma}}\}}\frac{|g(x, v_n)|}{\alpha_0^{\frac{1-\gamma}{2}}}\Big(\ln\frac{|g(x, v_n)|}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)^{\frac{1-\gamma}{2}}\,dx\Big]. \tag{5.12}$$

By Lemma 2.4 and (5.11) the first integral tends to zero, using dominated dominated theorem and the fact that $v_n \to 0$ almost everywhere in B_1 the second integral tends to zero. Hence,

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \le \int_{B_1}\frac{|g(x, v_n)|}{\alpha_0^{\frac{1-\gamma}{2}}}\Big(\ln\frac{|g(x, v_n)|}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)^{\frac{1-\gamma}{2}}\,dx + o_n(1). \tag{5.13}$$

Given $\epsilon \in (0, \dfrac{\alpha_0\delta}{4(\frac{\alpha_\gamma^*}{\alpha_0} - \delta)})$, where $\delta > 0$ is given by Remark 4.6. By (H_4) and the assumption $\alpha_0 \ge \beta_0$, we can find $C_\epsilon > 0$ such that

$$|g(x, s)| \le C_\epsilon e^{(\alpha_0+\epsilon)|s|^{\frac{2}{1-\gamma}}}, \quad \text{for all} \quad (x, s) \in B_1 \times \mathbb{R}.$$

Replacing the above inequality in (5.13), we get

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \le \frac{1}{\alpha_0^{\frac{1-\gamma}{2}}}\int_{B_1}|g(x, v_n)|\Big(\ln\frac{C_\epsilon e^{(\alpha_0+\epsilon)|v_n|^{\frac{2}{1-\gamma}}}}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)^{\frac{1-\gamma}{2}}\,dx + o_n(1).$$

Thus,

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \le \frac{1}{\alpha_0^{\frac{1-\gamma}{2}}}\int_{B_1}|g(x, v_n)|\left[\ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big) + (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}|v_n|\right] + o_n(1). \tag{5.14}$$

Let $I_n = \int_{B_1} |g(x,v_n)|\Big[\ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big) + (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}|v_n|\Big]$ and set

$$Y_n := \{x \in B_1 : \ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big) \leq \Big((\alpha_0+2\epsilon)^{\frac{1-\gamma}{2}} - (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}\Big)|v_n|\}.$$

Hence,

$$\begin{aligned}
I_n &= \ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)\int_{B_1\setminus Y_n} |g(x,v_n)|\,dx + (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}\int_{B_1\setminus Y_n} g(x,v_n)v_n\,dx \\
&\quad + \int_{Y_n} |g(x,v_n)|\Big[\ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big) + (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}|v_n|\Big]\,dx \\
&\leq \ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)\int_{B_1\setminus Y_n} |g(x,v_n)|\,dx + (\alpha_0+\epsilon)^{\frac{1-\gamma}{2}}\int_{B_1\setminus Y_n} g(x,v_n)v_n\,dx + (\alpha_0+2\epsilon)^{\frac{1-\gamma}{2}}\int_{Y_n} g(x,v_n)v_n\,dx.
\end{aligned}$$

Then,

$$I_n \leq \ln^{\frac{1-\gamma}{2}}\Big(\frac{C_\epsilon}{\alpha_0^{\frac{1-\gamma}{2}}}\Big)\int_{B_1\setminus Y_n} |g(x,v_n)|\,dx + (\alpha_0+2\epsilon)^{\frac{1-\gamma}{2}}\int_{B_1} g(x,v_n)v_n\,dx. \tag{5.15}$$

Since $v_n \to 0$ almost everywhere in B_1 and g is bounded in $B_1\setminus Y_n$ for all $n \in \mathbb{N}$ (being independent of n), by the dominated convergence theorem, we get

$$\int_{B_1\setminus Y_n} |g(x,v_n)|\,dx = o_n(1). \tag{5.16}$$

Using (5.14)–(5.16), we have

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|u_n\| \leq \Big(1 + \frac{2\epsilon}{\alpha_0}\Big)^{\frac{1-\gamma}{2}}\int_{B_1} g(x,v_n)v_n\,dx + o_n(1). \tag{5.17}$$

Arguing similarly, we get

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{\frac{1-\gamma}{2}}\|v_n\| \leq \Big(1 + \frac{2\epsilon}{\alpha_0}\Big)^{\frac{1-\gamma}{2}}\int_{B_1} f(x,u_n)u_n\,dx + o_n(1). \tag{5.18}$$

On the other hand, using Proposition 4.1, Remark 4.6 and (5.8), we obtain

$$\Big|\int_{B_1} w(x)\nabla u_n\nabla v_n\,dx\Big| \leq o_n(1) + \Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{1-\gamma}. \tag{5.19}$$

Since $J_n'(u_n,v_n)(u_n,v_n) = 0$, we get

$$\int_{B_1} f(x,u_n)u_n\,dx + \int_{B_1} g(x,v_n)v_n\,dx = 2\Big|\int_{B_1} w(x)\nabla u_n\nabla v_n\,dx\Big|. \tag{5.20}$$

By (5.19) and (5.20), we find

$$\int_{B_R} f(x,u_n)u_n\,dx + \int_{B_R} g(x,v_n)v_n\,dx \leq 2\Big(\frac{\alpha_\gamma^*}{\alpha_0} - \delta\Big)^{1-\gamma} + o_n(1). \tag{5.21}$$

Combining (5.17), (5.18) and (5.21), we obtain

$$\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\delta\Big)^{\frac{1-\gamma}{2}}(\|u_n\|+\|v_n\|)\leq\Big(1+\frac{2\epsilon}{\alpha_0}\Big)^{\frac{1-\gamma}{2}}\Big(\int_{B_R}f(x,u_n)u_n\,dx+\int_{B_1}g(x,v_n)v_n\,dx\Big)+o_n(1)$$
$$\leq 2\Big(1+\frac{2\epsilon}{\alpha_0}\Big)^{\frac{1-\gamma}{2}}\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\delta\Big)^{1-\gamma}+o_n(1).$$

According to the election of ϵ, for every $n\in\mathbb{N}$, we obtain

$$\|u_n\|+\|v_n\|\leq 2\Big(1+\frac{2\epsilon}{\alpha_0}\Big)^{\frac{1-\gamma}{2}}\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\delta\Big)^{\frac{1-\gamma}{2}}+o_n(1)\leq 2\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{2}\Big)^{\frac{1-\gamma}{2}}+o_n(1).$$

Thus, there exists $n_0\in\mathbb{N}$ such that

$$\|u_n\|+\|v_n\|\leq 2\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{4}\Big)^{\frac{1-\gamma}{2}},\quad \text{for all } n\geq n_0.$$

Moreover, we can suppose that

$$\|u_n\|\leq\Big(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{4}\Big)^{\frac{1-\gamma}{2}},\quad \text{for all } n\geq n_0.$$

Taking $\xi>0$ such that $(\alpha_0+\xi)(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{4})<\alpha_\gamma^*$, by (H_4) there exists $c>0$ such that

$$|f(x,s)|\leq ce^{(\alpha_0+\xi)|s|^{\frac{2}{1-\gamma}}},\quad \text{for all}\quad (x,s)\in B_1\times\mathbb{R}.$$

Let $p>1$ close enough to 1 satisfying $p(\alpha_0+\xi)(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{4})<\alpha_\gamma^*$. By Proposition 2.3 and the Hölder inequality, we have

$$\int_{B_1}f(x,u_n)u_n\,dx\leq c\int_{B_1}|u_n|e^{(\alpha_0+\xi)|u_n|^{\frac{2}{1-\gamma}}}\,dx$$
$$\leq c\|u_n\|_{p'}\Big(\int_{B_1}e^{p(\alpha_0+\xi)|u_n|^{\frac{2}{1-\gamma}}}\,dx\Big)^{1/p}$$
$$\leq c\|u_n\|_{p'}\Big(\int_{B_1}e^{p(\alpha_0+\xi)(\frac{\alpha_\gamma^*}{\alpha_0}-\frac{\delta}{4})(\frac{|u_n|}{\|u_n\|})^{\frac{2}{1-\gamma}}}\,dx\Big)^{1/p}$$
$$\leq c\|u_n\|_{p'}.$$

Applying (5.5), we obtain

$$\int_{B_1}f(x,u_n)u_n\,dx\to 0,\quad \text{as}\ n\to+\infty.$$

Therefore, using (5.7) and (5.8), one has

$$\lim_{n\to+\infty}J(u_n,v_n)=0,$$

which represents a contradiction with the fact that $J(z_n)\geq\sigma$ for all $n\geq 1$. Therefore, (u,v) is a nontrivial weak solution. This complete the proof.

6. Conclusions

In this work, we apply variational methods to find a nontrivial solution for a Hamiltonian systems where the nonlinearities possess maximal growth related to Trudinger-Moser type inequalities. To the best of our knowledge, this is the first result to demonstrate the existence of nontrivial solutions for a Hamiltonian involving supercritical exponential growth in the sense of the exponential critical hyperbola in the literature. According to our definition of logarithmic weight, we restricted the domain to the unit ball. It is of interest to further our results to solutions for Hamiltonian systems involving supercritical exponential growth on the whole space $\mathbb{R}^2$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by CONCYTEC-PROCIENCIA within the call for proposal "Proyecto de Investigación Básica 2019-01[Contract Number 410-2019]" and the Universidad Nacional Mayor de San Marcos – RR N° 05753-R-21 and project number B21140091. Part of this work was done while the author was visiting the Universidade de São Paulo at São Carlos. He thanks all the faculty and staff of those Department of Mathematics for their support and kind hospitality.

Conflict of interest

The author declares no conflicts of interest.

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