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Research article

Hölder and Schauder estimates for weak solutions of a certain class of non-divergent variation inequality problems in finance

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Abstract: This article studies a class of variational inequality problems composed of non-divergence type parabolic operators. In comparison with traditional differential equations, this study focuses on overcoming inequality constraints to obtain Hölder and Schauder estimates for weak solutions. The results indicate that the weak solution of the variational inequality possesses the C^α continuity and the Schauder estimate on the $W^{1,p}$ space, where $\alpha \in (0, 1)$ and $p \geq 2$.

Keywords: variation-inequality problems; non-divergence type parabolic operator; Hölder and Schauder estimate
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

Variational inequalities are often used in American-style option valuation analysis, and they provide a good description of early exercise provisions in the presence of uncertain equities. As an application of variational inequality, we investigate the pricing problem of American-style options on two risky assets. Assuming the existence of two risky assets in the financial market, their prices follow:

$$\mathrm{d}S_i(t) = \mu_i S_i(t)\mathrm{d}t + \sigma_i S_i(t)\mathrm{d}W_i(t), i = 1, 2,$$

where μ_i represents the return rate of the i-th asset, and σ_i represents its volatility, $i = 1, 2$. $\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$ are two standard 1-D Brownian motions used to describe the background noise of the financial market. Investors who purchase American-style options have the right to choose between risk assets $\{S_1(t), t \geq 0\}$ and $\{S_2(t), t \geq 0\}$ for holding, in terms of value, expected return rate, turnover rate, and asset volatility, with the aim of maximizing their gains while minimizing fluctuations. According to the literature [1–3], the value v of American-style option is suitable for the following variational inequality model:

$$\begin{cases} \min\{-Lv, v - \max\{S_1, S_2\}\} = 0, \\ v(S_1, S_2, T) = \max\{S_1, S_2\}, \end{cases} \tag{1}$$

where r represents risk-free interest rate and

$$Lv = \partial_t v + \frac{1}{2}\sigma_1^2 S_1^2 \partial_{S_1 S_1} v + \frac{1}{2}\sigma_2^2 S_2^2 \partial_{S_2 S_2} v + rS_1 \partial_{S_1} v + rS_2 \partial_{S_2} v - rv.$$

Many documents indicate that when there are costs in stock trading, sigma in Lv is often related to the first spatial gradient of v. The famous Leland model can rewrite sigma as

$$\sigma^2 = \sigma_0^2 (1 + Le \cdot \mathrm{sign}(\mathrm{div}(|V_S|^{p-2} V_S)). \tag{2}$$

In this equation, the variable σ_0^2 denotes the initial volatility, while Le represents the Leland number. The value of p is greater than or equal to 2.

Taking inspiration from Leland's fee model for American-style options, this paper examines a variational inequality initial-boundary value problem:

$$\begin{cases} -Lu \geq 0, & (x,t) \in \Omega_T, \\ u - u_0 \geq 0, & (x,t) \in \Omega_T, \\ Lu(u - u_0) = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial v} = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{3}$$

incorporating the non-divergence parabolic operator

$$Lu = \partial_t u - u\Delta_p u - \gamma |\nabla u|^p, \ \Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u), \ p > 2. \tag{4}$$

Different from (1), we restrict Ω to be a bounded and open subset of R^N, and $\Omega_T = \Omega \times (0, T)$. In terms of the parameter γ, we continue to employ the hypothesis conditions from the study in reference [4,5] to verify the presence of weak solutions, which is represented by $\gamma \in (0, 1)$.

Theoretical research on variational inequalities has been extensively expanded in many aspects. Some examples include the study of the existence and uniqueness of solutions for 2-D variational inequalities in [4], and with fourth-order $p(x)$-Kirchhoff operators in [5]. The existence of solutions in whole R_N can be found in [6]. Furthermore, the existence and uniqueness of solutions for mixed variational-hemivariational inequality were discovered in [7], without Lipschitz continuity in [8] and with nonlocal fractional operators in [9]. Additionally, stability analysis for variational inequalities, hemivariational inequalities and variational-hemivariational inequalities was analyzed in [10], as well as in reflexive Banach spaces in [11]. Finally, the regularity of weak solutions to a class of fourth-order parabolic variational inequalities was proved in [12].

Inspired by the literature [4–6], we studied the Hölder estimate and Schauder estimate for weak solutions of variational inequalities formed by a class of non-divergence parabolic operators. On one hand, starting from the weak solutions constructed in [4–6], we obtained several gradient estimates using techniques such as Hölder and Young inequalities, and obtained the Hölder estimate results based on [13]. On the other hand, we constructed test functions for weak solutions using time and space truncation factors, and obtained the Caccioppoli inequality that is suitable for the variational inequality (3), which serves as the cornerstone for proving the Schauder estimate. By varying different parameters in the Caccioppoli inequality, we obtained the Schauder estimate for weak solutions of the variational inequality (3).

In summary, this section provides a definition of the weak solution for the variational inequality based on references [4–6], along with a set of maximal monotone maps specified in [1–3,5,6]:

$$G = \{u|u(x) = 0, x > 0; u(x) \in [-M_0, 0], \quad x = 0\}. \tag{5}$$

Here M_0 is a positive constant.

Definition 1.1. *A pair (u, ξ) is said to be a generalized solution of variation-inequality (1), if (u, ξ) satisfies $u \in L^\infty(0, T, W^{1,p}(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$ and $\xi \in G$ for any $(x, t) \in \Omega_T$,*
(a) $u(x, t) \geq u_0(x)$, $u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,
(b) for every test-function $\varphi \in C_0^\infty(\Omega_T)$ and $t \in (0, T]$, there admits the equality

$$\int\int_{\Omega_t} \partial_t u \cdot \varphi + u|\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t + (1-\gamma)\int\int_{\Omega_t} |\nabla u|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t. \tag{6}$$

It should be noted by the reader that the above formula implies

$$\int\int_{\Omega_t} \partial_t u \cdot \varphi + u\Delta_p u \cdot \varphi \mathrm{d}x\mathrm{d}t - \gamma\int\int_{\Omega_t} |\nabla u|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t. \tag{7}$$

Note that $u_0 \geq 0$ in Ω_T . Then, using the second condition of inequality (3) and with ease, we can see that $u \geq 0$ in Ω_T.

2. Hölder estimates of solution

This section considers the Hölder estimate of weak solutions, and first provides several gradient energy estimates for weak solutions. For any $t \in (0, T]$, define $\Omega_t = \Omega \times (0, t)$. By choosing u as the test function in (6), we can obtain

$$\int\int_{\Omega_t} \partial_t u \cdot \varphi + u|\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t + (1-\gamma)\int\int_{\Omega_t} |\nabla u|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t.$$

After simplification, the equation can be written as follows:

$$\frac{1}{2}\int\int_{\Omega_t} \partial_t u^2 \mathrm{d}x\mathrm{d}t + (2-\gamma)\int\int_{\Omega_t} |\nabla u|^p u \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot u \mathrm{d}x\mathrm{d}t. \tag{8}$$

Now using the Hölder and Young inequalities to analyze $\int\int_{\Omega_t} \xi \cdot u \mathrm{d}x\mathrm{d}t$, we have

$$\int\int_{\Omega_t} \xi \cdot u \mathrm{d}x\mathrm{d}t \leq M_0^2 T + \frac{1}{4}\int\int_{\Omega_t} u^2 \mathrm{d}x\mathrm{d}t \tag{9}$$

from (5). Thus, substituting (9) into (8) and integrating with respect to $\int\int_{\Omega_t} \partial_t u^2 \mathrm{d}x\mathrm{d}t$, it is not difficult to obtain

$$\frac{1}{2}\int_{\Omega} u^2 \mathrm{d}x + (2-\gamma)\int\int_{\Omega_t} |\nabla u|^p u \mathrm{d}x\mathrm{d}t \le M_0^2 T + \frac{1}{4}\int\int_{\Omega_t} u^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\Omega} u_0^2 \mathrm{d}x. \tag{10}$$

On the one hand, it should be noted that $2-\gamma > 0$. By removing the non-positive term $(2-\gamma)\int\int_{\Omega_t} |\nabla u|^p u \mathrm{d}x\mathrm{d}t$, we obtain

$$\int_{\Omega} u^2 \mathrm{d}x - 2\int_0^t \int_{\Omega_t} u^2 \mathrm{d}x\mathrm{d}t \le 2M_0^2 T + \int_{\Omega} u_0^2 \mathrm{d}x.$$

Using Gronwall's inequality,

$$\int_{\Omega} u^2 \mathrm{d}x \le \left(2M_0^2 T + \int_{\Omega} u_0^2 \mathrm{d}x\right)\exp\{2T\}. \tag{11}$$

On the other hand, by removing the non-negative term $\frac{1}{2}\int_{\Omega} u^2 \mathrm{d}x$ in (10), it is easy to obtain

$$(2-\gamma)\int\int_{\Omega_t} |\nabla u|^p u \mathrm{d}x\mathrm{d}t \le M_0^2 T + \frac{1}{4}\int\int_{\Omega_t} u^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\Omega} u_0^2 \mathrm{d}x. \tag{12}$$

Combining (11) and (12), it is easy to see that

$$\int\int_{\Omega_t} |\nabla u|^p u \mathrm{d}x\mathrm{d}t \le C(\gamma, M_0, T) + C(\gamma, M_0, T)\int_{\Omega} u_0^2 \mathrm{d}x. \tag{13}$$

It is worth noting that using integration by parts yields, $\int\int_{\Omega_t} u\Delta_p u \mathrm{d}x\mathrm{d}t = -\int\int_{\Omega_t} |\Delta u|^p \mathrm{d}x\mathrm{d}t$, and thus (13) can also be used to obtain $u|\Delta_p u| \in L^\infty(0,T;L^2(\Omega))$.We will now prove $u|\Delta_p u| \in L^\infty(0,T;L^2(\Omega))$. Assuming $\varphi = u\Delta_p u$ in (7), it is easy to see that

$$\begin{aligned}&\int\int_{\Omega_t} \partial_t u \cdot u\Delta_p u \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_t} u^2|\Delta_p u|^2 \mathrm{d}x\mathrm{d}t - \gamma\int\int_{\Omega_t} |\nabla u|^p u\Delta_p u \mathrm{d}x\mathrm{d}t\\ &= \int\int_{\Omega_t} \xi \cdot u\Delta_p u \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{14}$$

Here, we attempt to estimate $\int\int_{\Omega_t} u^2|\Delta_p u|^2 \mathrm{d}x\mathrm{d}t$ in (14). With the help of the Hölder and Young inequalities, it is easy to discover that

$$\left|\int\int_{\Omega_t} \partial_t u \cdot u\Delta_p u \mathrm{d}x\mathrm{d}t\right| \le \int\int_{\Omega_t} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{4}\int\int_{\Omega_t} |u\Delta_p u|^2 \mathrm{d}x\mathrm{d}t. \tag{15}$$

Combining with (5), we obtain

$$\left|\int\int_{\Omega_t} \xi \cdot u\Delta_p u \mathrm{d}x\mathrm{d}t\right| \le C(T, |\Omega|, M_0) + \frac{1}{4}\int\int_{\Omega_t} |u\Delta_p u|^2 \mathrm{d}x\mathrm{d}t. \tag{16}$$

From (9) we know that $u \in L^\infty(0,T;W^{1,p}(\Omega))$, such that we use Poincaré insert theory [14, P15] to arrive at $\sup_{\Omega_t} u \le \|u\|_{L^\infty(0,T;W^{1,p}(\Omega))}$, such that Using the multivariate mean value theorem and integrating by part gives

$$\begin{aligned}&\left|\int\int_{\Omega_t} |\nabla u|^p u\Delta_p u \mathrm{d}x\mathrm{d}t\right|\\ &\le C(p, \|u\|_{W^{1,p}(\Omega)})\left|\int\int_{\Omega_t} u\Delta_p u \mathrm{d}x\mathrm{d}t\right| = C(p, \|u\|_{W^{1,p}(\Omega)})\int\int_{\Omega_t} |\nabla u|^p \mathrm{d}x\mathrm{d}t\\ &\le C(p, \|u\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)} = C(p, T, \|u\|_{L^\infty(0,T;W^{1,p}(\Omega))}).\end{aligned} \tag{17}$$

Substituting (15)–(17) into (14), we obtain

$$\frac{1}{2}\int\int_{\Omega_t} u^2|\Delta_p u|^2 \mathrm{d}x\mathrm{d}t \le \int\int_{\Omega_t} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t + C(\gamma, p, T, |\Omega|, M_0, \|u\|_{L^\infty(0,T;W^1(\Omega))}). \tag{18}$$

From reference [13], we have the following result established by Eqs (13) and (18):

Theorem 2.1. *For any $(x_1, t_1), (x_2, t_2) \in \Omega_T$ and any $\alpha \in (0, 1)$, there exists*

$$|u(x_1, t_1) - u(x_2, t_2)| \le C(|t_1 - t_2|^{\alpha/4} + |x_1 - x_2|^{\alpha}).$$

3. Schauder estimate of solution

This section considers the Schauder estimate of weak solutions. For any $(x_0, t_0) \in \Omega_T$, define

$$B_R = B_R(x_0) = \{x||x - x_0| < R\},$$

$$I_\rho = I_\rho(t_0) = (t_0 - \rho^2, \ \ t_0 + \rho^2), Q_\rho = B_R \times I_\rho,$$

where R and ρ are given positive numbers.

Lemma 3.1. *(Caccioppoli Type Inequality) If (u, ξ) is a weak solution to the variational inequality problem (3), then for any $B_R \subset \Omega_T$ and $\lambda \in \mathbb{R}$, there holds the following estimate:*

$$\sup_{t\in I_\rho}\int_{B_\rho}(u-\lambda)^2\mathrm{d}x + \frac{1}{p}\int\int_{Q_\rho}|\nabla u|^p \mathrm{d}x\mathrm{d}t \le \frac{C}{(R-\rho)^{4p}} + \frac{C}{(R-\rho)^2}\int_{Q_R}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.$$

Proof. For spatial variables, use the tangent function η on B_R relative to B_ρ, that is

$$\eta \in C_0^\infty(B_R), 0 \le \eta \le 1, \eta = 1 \text{ in } B_\rho, |\nabla\eta| \le \frac{C}{(R-\rho)^2}.$$

For a time variable, let $\kappa \in C_0^\infty(\mathbb{R})$ be defined such that $0 \le \kappa \le 1$. If $t \le t_0 - R^2$, we have $\kappa = 0$, and if $t \ge t_0 - \rho^2$, we have $\kappa = 1$. Moreover, $0 \le \kappa'(t) \le \frac{C}{(R-\rho)^2}$ in $[0,\ T]$.

Defining Q_R^s as

$$B_R \times (t_0 - R^2, s), s \in I_R$$

and selecting test function ϕ as $\eta^2\kappa^2(u - \lambda)$, it is easy to obtain

$$\begin{aligned}&\int\int_{Q_R^s}\partial_t u\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t + \int\int_{Q_R^s}u|\nabla u|^{p-2}\nabla u\cdot\nabla[\eta^2\kappa^2(u-\lambda)]\mathrm{d}x\mathrm{d}t\\ &\le(\gamma-1)\int\int_{Q_R^s}|\nabla u|^p\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t + \int\int_{Q_R^s}\xi\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{19}$$

Note that $u(t, x) = \frac{\partial u}{\partial\nu} = 0$ for any $(x, t) \in \partial\Omega \times (0, T)$. Thus, by applying integration by part on $\int\int_{Q_R^s}\partial_t[\eta^2\kappa^2(u-\lambda)^2]\mathrm{d}x\mathrm{d}t$ and $\int\int_{Q_R^s}u|\nabla u|^{p-2}\nabla u\cdot\nabla[\eta^2\kappa^2(u-\lambda)]\mathrm{d}x\mathrm{d}t$, we obtain

$$\int\int_{Q_R^s}\partial_t[\eta^2\kappa^2(u-\lambda)^2]\mathrm{d}x\mathrm{d}t = \int\int_{Q_R^s}\partial_t u\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t + 2\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t, \tag{20}$$

$$\begin{aligned}&\int\int_{Q_R^s} u|\nabla u|^{p-2}\nabla u\cdot\nabla[\eta^2\kappa^2(u-\lambda)]\mathrm{d}x\mathrm{d}t\\&=\int\int_{Q_R^s}\eta^2\kappa^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\kappa^2\eta\nabla\eta u|\nabla u|^{p-2}\nabla u(u-\lambda)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{21}$$

Substituting (20) and (21) into (19), we can obtain

$$\begin{aligned}&\int\int_{Q_R^s}\partial_t[\eta^2\kappa^2(u-\lambda)^2]\mathrm{d}x\mathrm{d}t-2\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t\\&+\int\int_{Q_R^s}\eta^2\kappa^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\kappa^2\eta\nabla\eta\cdot u|\nabla u|^{p-2}\nabla u\cdot(u-\lambda)\mathrm{d}x\mathrm{d}t\\&\le(\gamma-1)\int\int_{Q_R^s}|\nabla u|^p\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t+\int\int_{Q_R^s}\xi\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t\end{aligned}$$

which means

$$\begin{aligned}&\int\int_{Q_R^s}\partial_t[\eta^2\kappa^2(u-\lambda)^2]\mathrm{d}x\mathrm{d}t+\int\int_{Q_R^s}\eta^2\kappa^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t\\&\le(\gamma-1)\int\int_{Q_R^s}|\nabla u|^p\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t\\&\quad-2\int\int_{Q_R^s}\kappa^2\eta\nabla\eta\cdot u|\nabla u|^{p-2}\nabla u\cdot(u-\lambda)\mathrm{d}x\mathrm{d}t+\int\int_{Q_R^s}\xi\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{22}$$

Using the Hölder and Young inequalities, and applying $|\nabla\eta|\le\frac{C}{(R-\rho)^2}$, we have

$$\int\int_{Q_R^s}|\nabla u|^p\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t\le\frac{C}{(R-\rho)^{4p}}+\frac{1}{2}\int\int_{Q_R^s}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x\mathrm{d}t,\tag{23}$$

$$\begin{aligned}&2\int\int_{Q_R^s}\kappa^2\eta\nabla\eta\cdot u|\nabla u|^{p-2}\nabla u\cdot(u-\lambda)\mathrm{d}x\mathrm{d}t\\&\le\frac{p-1}{p}\int\int_{Q_R^s}\kappa^2\eta^2|\nabla u|^p\mathrm{d}x\mathrm{d}t+\frac{1}{p}\int\int_{Q_R^s}\kappa^2|\nabla\eta|^2\cdot(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{24}$$

Substituting (23) and (24) into (22), we have that

$$\begin{aligned}&\int\int_{Q_R^s}\partial_t[\eta^2\kappa^2(u-\lambda)^2]\mathrm{d}x\mathrm{d}t+\int\int_{Q_R^s}\eta^2\kappa^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t\\&\le\frac{C}{(R-\rho)^{4p}}+\frac{1}{2}\int\int_{Q_R^s}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t\\&\quad+\frac{p-1}{p}\int\int_{Q_R^s}\kappa^2\eta^2|\nabla u|^p\mathrm{d}x\mathrm{d}t+\frac{1}{p}\int\int_{Q_R^s}\kappa^2|\nabla\eta|^2\cdot(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Rearranging the above inequality yields

$$\begin{aligned}&\int_{B_R}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x|_{t=s}+\frac{1}{p}\int\int_{Q_R^s}\eta^2\kappa^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t\\&\le\frac{C}{(R-\rho)^{4p}}+\frac{1}{2}\int\int_{Q_R^s}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t\\&+\frac{1}{p}\int\int_{Q_R^s}\kappa^2|\nabla\eta|^2\cdot(u-\lambda)^p\mathrm{d}x\mathrm{d}t+\int\int_{Q_R^s}\xi\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{25}$$

Since $0\le\xi\le M_0$, using the Hölder inequality to $\int\int_{Q_R^s}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x\mathrm{d}t$, $\int\int_{Q_R^s}\eta^2\kappa\kappa'(u-\lambda)^2\mathrm{d}x\mathrm{d}t$ and $\int\int_{Q_R^s}\xi\cdot\eta^2\kappa^2(u-\lambda)\mathrm{d}x\mathrm{d}t$, we have

$$\begin{aligned}&\int_{B_R}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x|_{t=s}+\frac{1}{p}\int\int_{Q_R^s}\eta^2\kappa^2|\nabla u|^p\mathrm{d}x\mathrm{d}t\\&\le\frac{C}{(R-\rho)^{4p}}+\frac{C}{(R-\rho)^2}\int\int_{Q_R^s}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{26}$$

Removing the non-negative terms $\frac{1}{p}\int\int_{Q_R^s}\eta^2\kappa^2|\nabla u|^p\mathrm{d}x\mathrm{d}t$ from which implies that for any $s\in I_\rho$,

$$\begin{aligned}&\int_{B_R}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x|_{t=s}\le\int_{B_R}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x|_{t=s}\\&\le\frac{C}{(R-\rho)^{4p}}+\frac{C}{(R-\rho)^2}\int\int_{Q_R^s}(u-\lambda)^p\mathrm{d}x\mathrm{d}t\\&\le\frac{C}{(R-\rho)^{4p}}+\frac{C}{(R-\rho)^2}\int\int_{Q_R}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{27}$$

On the other hand, by removing the non-negative term $\int_{B_R}\eta^2\kappa^2(u-\lambda)^2\mathrm{d}x|_{t=s}$ in (26), we can easily obtain

$$\int\int_{Q_\rho}|\nabla u|^p\mathrm{d}x\mathrm{d}t \le \int\int_{Q_R}\eta^2\kappa^2|\nabla u|^p\mathrm{d}x\mathrm{d}t \le \frac{C}{(R-\rho)^{4p}} + \frac{C}{(R-\rho)^2}\int\int_{Q_R}(u-\lambda)^p\mathrm{d}x\mathrm{d}t. \tag{28}$$

Combining (27) and (28), the desired conclusion follows and the proof is completed. □

Now, we analyze the Schauder estimates for weak solutions of the variational inequality (3). On the one hand, choosing ρ and R to be $\frac{1}{2}R$ and $\frac{3}{4}R$, respectively, in the Lemma 4.1 and setting $\lambda = 0$, we obtain

$$\sup_{t\in I_{\frac{1}{2}R}}\int_{B_{\frac{1}{2}R}}u^2\mathrm{d}x + \frac{1}{p}\int\int_{Q_{\frac{1}{2}R}}|\nabla u|^p\mathrm{d}x\mathrm{d}t \le \frac{C}{R^2}\int\int_{Q_{\frac{3}{4}R}}u^2\mathrm{d}x\mathrm{d}t + \frac{C}{R^{4p}}. \tag{29}$$

On the other hand, in Lemma 4.1, we choose $\rho = \frac{3}{4}R$ and R, respectively, and set $\lambda = 0$,

$$\int\int_{Q_{\frac{3}{4}R}}|\nabla u|^2\mathrm{d}x\mathrm{d}t \le \frac{C}{R^2}\int\int_{Q_R}u^2\mathrm{d}x\mathrm{d}t + \frac{C}{R^{4p}}. \tag{30}$$

Combining (29) and (30), we can obtain the following Schauder estimates.

Theorem 3.2. *Let (u,ξ) be a weak solution of variational inequality (3), then*

$$\sup_{t\in I_{\frac{1}{2}R}}\int_{B_{\frac{1}{2}R}}u^2\mathrm{d}x + \frac{1}{p}\int\int_{Q_{\frac{1}{2}R}}|\nabla u|^p\mathrm{d}x\mathrm{d}t \le \frac{C}{R^2}\int\int_{Q_R}u^2\mathrm{d}x\mathrm{d}t + \frac{C}{R^{4p}}.$$

4. Conclusions

This paper studies the variational inequality problem associated with non-divergence type parabolic operators as follows:

$$Lu = \partial_t u - u\Delta_p u - \gamma|\nabla u|^p,\ \Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u),\ p > 2.$$

First, the Hölder estimate for the weak solution of the variational inequality (3) was analyzed. By using the maximal operator G to overcome the inequality constraints in model (1), combined with the Hölder inequality, Young's inequality, Gronwall's inequality, etc., the energy estimates for the spatial gradient and spatial second-order gradient of the weak solution of the variational inequality (3) were obtained, thus obtaining the Hölder estimate for the weak solution of the variational inequality (3).

Second, the Shauder estimate problem for weak solutions of the variational inequality (3) was studied. By utilizing weak solutions constructed with maximal operators, and combining spatial cutoff factors and time cutoff factors, the Caccioppoli inequality for the variational inequality (3) was obtained. Based on this, the Shauder estimate for the weak solutions of the variational inequality (3) was obtained by selecting different parameters.

There are still some areas for improvement in the current paper. The non-linear structure $\Delta_p u$ present in Lu (see (4), for details) restricts the possibility of obtaining higher-order Hölder and Schauder estimates through spatial partial derivatives. In addition, the inequality constraints in model (3) make it impossible to obtain higher-order Hölder and Schauder estimates through partial derivative operations. We will attempt to overcome these limitations in future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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