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Research article

On finite-dimensional irreducible modules for the universal Askey-Wilson algebra

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Abstract: Let Δ_q be the universal Askey-Wilson algebra. If q is not a root of unity, it is shown in the Huang's earlier paper that an $(n+1)$-dimensional irreducible Δ_q-module is a quotient $V_n(a,b,c)$ of a Δ_q-Verma module with

$$\textbf{Condition A:}\ \ abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \left\{q^{n-2i+1} | 1 \leq i \leq n\right\}.$$

The aim of this paper is to discuss the structures of $(n+1)$-dimensional Δ_q-modules $V_n(a,b,c)$ when the given triples (a,b,c) do not satisfy **Condition A**.

Keywords: universal Askey-Wilson algebra; irreducible module; Verma module
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1. Introduction

In 1991, Zhedanov in [15] introduced the Askey-Wilson algebra AW_q which has attracted a lot of attentions. The Askey-Wilson algebra AW_q is defined by three generators K_0, K_1, K_2. It satisfies the relations

$$\begin{aligned}
[K_0, K_1]_q &= K_2,\\
[K_1, K_2]_q &= BK_1 + C_0K_0 + D_0,\\
[K_1, K_2]_q &= BK_0 + C_1K_1 + D_1,
\end{aligned}$$

where $[L, M]_q = qLM - q^{-1}ML$, and B, C_0, C_1, D_0, D_1 are the structural constants of the algebra.

In the course of the research, it has been found that the Askey-Wilson algebra AW_q plays an important role in the quantum integrable systems (see [1, 9]) and in the theory of Leonard pairs (see [12–14]) and Leonard triples (see [2, 7]). The algebra AW_q has very closed relations with the quantum algebra $U_q(\mathfrak{sl}_2)$ (see [11]) and the non-standard deformation $U_q'(\mathfrak{so}_3)$ (see [3]).

In 2011, Terwilliger in [10] introduced the universal Askey-Wilson algebra Δ_q. It is an associative $\mathbb{F}$-algebra generated by A, B, C such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

are all central in Δ_q. In 2015, Huang in [5] gave the classification of finite-dimensional irreducible modules of Δ_q when q is not a root of unity. It is shown that an $(n+1)$-dimensional irreducible Δ_q-module is a quotient $V_n(a, b, c)$ of a Δ_q-Verma module with

$$abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \left\{q^{n-2i+1} | 1 \leq i \leq n\right\}. \tag{1.1}$$

He also established the connections between this new classification and those of finite-dimensional irreducible modules of $U'_q(\mathfrak{so}_3)$ (see [4]) and $U_q(\mathfrak{sl}_2)$ (see [8]) respectively. Huang in [6] classified the finite-dimensional irreducible modules of Δ_q at roots of unity.

The aim of this paper is to discuss the structures of $(n+1)$-dimensional Δ_q-module $V_n(a, b, c)$ when the given triples (a, b, c) do not satisfy the condition (1.1) under the assumption that q is not a root of unity.

In Section 1, we recall that the definition of the universal Askey-Wilson algebra Δ_q and some known results about the construction of the Verma module $M_\lambda(a, b, c)$ and its $(n+1)$-dimensional quotient $V_n(a, b, c)$, respectively. In Section 2, for a given triple (a, b, c), and the condition (1.1) is not satisfied, we show that there is at most four isomorphism classes of $(n+1)$-dimensional irreducible $\Delta_q(\alpha,\beta,\gamma)$-module. In these cases, we also discuss the structures of $(n+1)$-dimensional $\Delta_q(\alpha,\beta,\gamma)$-module $V_n(a, b, c)$. In Section 3, we give some examples satisfying various conditions of the theorems.

2. Some known results

First of all, let $\mathbb{F}$ be an algebraically closed field with char $\mathbb{F} = 0$, $\mathbb{Z}$ the ring of integers, $\mathbb{N}$ the set of the nonnegative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. All algebras, vector spaces etc. are defined over the field $\mathbb{F}$.

2.1. The universal Askey-Wilson algebra Δ_q

Fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. Let us recall the concepts of the universal Askey-Wilson algebra Δ_q, which was introduced by Terwilliger in [10, Definition 1.3].

Definition 2.1. *The universal Askey-Wilson algebra Δ_q is an associative $\mathbb{F}$-algebra generated by A, B, C with the following relations*

$$\begin{aligned} A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} &= \frac{\alpha}{q + q^{-1}}, \\ B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} &= \frac{\beta}{q + q^{-1}}, \\ C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} &= \frac{\gamma}{q + q^{-1}}, \end{aligned} \tag{2.1}$$

where each of α, β, γ is central in Δ_q.

The algebra Δ_q has other presentation as follows (see [10, Theorem 2.6]).

Theorem 2.2. *The algebra Δ_q can be presented by generators A, B, γ with relations*

$$A^3B - BA^3 + \left(q^2 + q^{-2} + 1\right)\left(ABA^2 - A^2BA\right) = \left(q^2 - q^{-2}\right)^2 (BA - AB), \tag{2.2}$$

$$B^3A - AB^3 + \left(q^2 + q^{-2} + 1\right)\left(BAB^2 - B^2AB\right) = \left(q^2 - q^{-2}\right)^2 (AB - BA), \tag{2.3}$$

$$A^2B^2 - B^2A^2 + \left(q^2 + q^{-2}\right)(BABA - ABAB) = \left(q - q^{-1}\right)^2 (BA - AB)\gamma, \tag{2.4}$$

$$A\gamma = \gamma A, \qquad B\gamma = \gamma B. \tag{2.5}$$

2.2. The Verma Δ_q-module $M_\lambda(a, b, c)$

Now we recall the concept of Verma Δ_q-module $M_\lambda(a, b, c)$ of the universal Askey-Wilson algebra Δ_q for a given triple (a, b, c).

Let us list some notations for convenience firstly.

Let Λ, Q, X, Y, Z denote five mutually commuting indeterminate elements over $\mathbb{F}$. Define

$$\begin{aligned}
\theta_i(\Lambda, Q; X) =& \Lambda Q^{-2i}X^{-1} + \Lambda^{-1}Q^{2i}X, \qquad \text{for } i \in \mathbb{Z},\\
\phi_i(\Lambda, Q; X, Y, Z) =& \Lambda Q X^{-1}Y^{-1}\left(Q^i - Q^{-i}\right)\left(\Lambda^{-1}Q^{i-1} - \Lambda Q^{1-i}\right)\\
&\times\left(Q^{-i} - \Lambda^{-1}Q^{i-1}XYZ\right)\left(Q^{-i} - \Lambda^{-1}Q^{i-1}XYZ^{-1}\right), \qquad \text{for } i \in \mathbb{Z},\\
\omega(\Lambda, Q; X, Y, Z) =& \left(\Lambda Q + \Lambda^{-1}Q^{-1}\right)\left(Z + Z^{-1}\right) + \left(X + X^{-1}\right)\left(Y + Y^{-1}\right).
\end{aligned}$$

Let $L(\Lambda, Q; X) = (L_{ij})$ be the $\mathbb{N} \times \mathbb{N}$ matrix with

$$L_{ij} = \begin{cases} 1, & \text{if } j = i - 1;\\ \theta_j(\Lambda, Q; X), & \text{if } j = i;\\ 0, & \text{otherwise.} \end{cases}$$

Let $U(\Lambda, Q; X, Y, Z) = (U_{ij})$ be the $\mathbb{N} \times \mathbb{N}$ matrix with

$$U_{ij} = \begin{cases} \theta_j(\Lambda, Q; Y), & \text{if } j = i;\\ \phi_j(\Lambda, Q; X, Y, Z), & \text{if } j = i + 1;\\ 0, & \text{otherwise.} \end{cases}$$

Let $T(\Lambda, Q; X, Y, Z) = (T_{ij})$ be the $\mathbb{N} \times \mathbb{N}$ matrix with

$$T_{ij} = \begin{cases} \frac{Q^{-1}\theta_{j+1}(\Lambda,Q;Y) - Q\theta_j(\Lambda,Q;Y)}{Q^2 - Q^{-2}}, & \text{if } j = i - 1;\\[2ex] \frac{Q^{-1}\phi_{j+1}(\Lambda,Q;X,Y,Z) - Q\phi_j(\Lambda,Q;X,Y,Z)}{Q^2 - Q^{-2}} \\ \qquad + \frac{\omega(\Lambda,Q;X,Y,Z) - \theta_j(\Lambda,Q;X)\theta_j(\Lambda,Q;Y)}{Q + Q^{-1}}, & \text{if } j = i;\\[2ex] \frac{Q^{-1}\theta_j(\Lambda,Q;X) - Q\theta_{j-1}(\Lambda,Q;X)}{Q^2 - Q^{-2}}\phi_j(\Lambda, Q; X, Y, Z), & \text{if } j = i + 1;\\ 0, & \text{otherwise.} \end{cases}$$

All notations given in the above are fixed unless specified.

Let a, b, c, λ be given nonzero scalars in $\mathbb{F}$ and $M_\lambda(a,b,c)$ an $\mathbb{F}$-vector space with a basis $\{m_i\}_{i\in\mathbb{N}}$. It is easy to see that there is a Δ_q-module structure on $M_\lambda(a,b,c)$ on which the matrices of A, B, C acting are

$$L(\lambda,q;a),\quad U(\lambda,q;a,b,c),\quad T(\lambda,q;a,b,c) \tag{2.6}$$

respectively, and the actions of α,β,γ are as follows:

$$\alpha m_i=\omega(\lambda,q;b,c,a)m_i,\quad \beta m_i=\omega(\lambda,q;c,a,b)m_i,\quad \gamma m_i=\omega(\lambda,q;a,b,c)m_i, \tag{2.7}$$

for all $i\in\mathbb{N}$.

2.3. *Finite-dimension Δ_q-module $V_n(a,b,c)$*

Under the condition that $\lambda=q^n$, we let $N_\lambda(a,b,c)$ be a submodule of $M_\lambda(a,b,c)$ spanned by the $\{m_i\}_{i\geqslant n+1}$, then

$$V_n(a,b,c)=M_\lambda(a,b,c)/N_\lambda(a,b,c)$$

is naturally an $(n+1)$-dimensional quotient Δ_q-module.

Let $\{v_i|0\le i\le n\}$ be a basis of Δ_q-module $V_n(a,b,c)$, $\mathbf{T}_n=\left\{q^{n-2i+1}|1\le i\le n\right\}$ and $\mathbf{T}$ the set of all triples (a,b,c) of nonzero scalars in $\mathbb{F}$ that satisfy

$$abc,\;\; a^{-1}bc,\;\; ab^{-1}c,\;\; abc^{-1}\notin\mathbf{T}_n.$$

For all $(a,b,c)\in\mathbf{T}$, the group $\{-1,1\}^3$ acts on $\mathbf{T}$ by

$$(a,b,c)^{(-1,1,1)}=\left(a^{-1},b,c\right), \tag{2.8}$$

$$(a,b,c)^{(1,-1,1)}=\left(a,b^{-1},c\right), \tag{2.9}$$

$$(a,b,c)^{(1,1,-1)}=\left(a,b,c^{-1}\right). \tag{2.10}$$

Let $\mathbf{T}/\{-1,1\}^3$ denote the set of the $\{-1,1\}^3$-orbits of $\mathbf{T}$ and $[a,b,c]$ the $\{-1,1\}^3$-orbit of $\mathbf{T}$ that contains (a,b,c) for $(a,b,c)\in\mathbf{T}$. Denote the isomorphism class of Δ_q-module $V_n(a,b,c)$ by $[V_n(a,b,c)]$.

The following results are referred to [5, Lemma 4.2; Theorem 4.4; Theorem 4.7].

Lemma 2.3. *The matrices of A, B, C acting on the basis $\left\{v_i^{(\varepsilon,g)}\right\}_{i=0}^n$ of $V_n(a,b,c)$ for each $(\varepsilon,g)\in K_4$ are as follows*:

	A	B	C
$\left\{v_i^{(1,1)}\right\}_{i=0}^n$	$L(q;a)$	$U(q;a,b,c)$	$T(q;a,b,c)$
$\left\{v_i^{(-1,1)}\right\}_{i=0}^n$	$L\left(q;a^{-1}\right)$	$U\left(q;a^{-1},b,c^{-1}\right)$	$T\left(q;a^{-1},b,c^{-1}\right)$
$\left\{v_i^{(1,\sigma)}\right\}_{i=0}^n$	$T\left(q^{-1};c^{-1},b^{-1},a^{-1}\right)$	$U\left(q^{-1};c^{-1},b^{-1},a^{-1}\right)$	$L\left(q^{-1};c^{-1}\right)$
$\left\{v_i^{(-1,\sigma)}\right\}_{i=0}^n$	$T\left(q^{-1};c,b^{-1},a\right)$	$U\left(q^{-1};c,b^{-1},a\right)$	$L\left(q^{-1};c\right)$

,

where $K_4=\{1,-1\}\times\{1,\sigma\}$ is a Klein group.

Theorem 2.4. *The Δ_q-module $V_n(a,b,c)$ is irreducible if and only if the following conditions hold*:

(1) $q^{2i} \neq 1$ *for all* $1 \leq i \leq n$;
(2) $abc,\ a^{-1}bc,\ ab^{-1}c,\ abc^{-1} \notin \left\{q^{n-2i+1} | 1 \leq i \leq n\right\}$.

Theorem 2.5. *Let* $\mathbf{M}$ *be the set of the isomorphism classes of irreducible* Δ_q*-modules with dimension* $n+1$. *Then there exists a bijection* $\mathbf{T}/\{-1,1\}^3 \longrightarrow \mathbf{M}$ *given by*

$$[a,b,c] \longmapsto [V_n(a,b,c)], \quad [a,b,c] \in \mathbf{T}/\{-1,1\}^3.$$

In the sequel, we always assume that q is not a root of unity, the values a, b, $c \in \mathbb{F}$ are fixed, and

$$\textbf{Condition A:} \quad abc,\ a^{-1}bc,\ ab^{-1}c,\ abc^{-1} \notin \left\{q^{n-2i+1} | 1 \leq i \leq n\right\}.$$

Fix a set

$$\mathfrak{S} = \{(1,1,1),\ (-1,1,1),\ (1,-1,1),\ (1,1,-1)\}.$$

The parameters a,b,c in the Verma Δ_q-module $M_\lambda(a,b,c)$ are related with corresponding parameters of the Askey-Wilson polynomials $\{p_i(X)\}_{i\in\mathbb{N}}$ in [5, Section 3.2]. If the **Condition A** fails, then there exist $j(1 \leq j \leq n)$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3) \in \mathfrak{S}$ such that $a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} = q^{n-2j+1}$. In this case, the Askey-Wilson polynomials make sense only when $0 \leq i \leq k$, where

$$k = \min\left\{j \,\middle|\, a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} = q^{n-2j+1} \text{ for } (\varepsilon_1,\varepsilon_2,\varepsilon_3) \in \mathfrak{S}, 1 \leq j \leq n\right\}.$$

Also, we see that $V_n(a,b,c)$ is not irreducible. Note that the actions of central elements α,β,γ are all fixed automatically, and we denote Δ_q by $\Delta_q(\alpha,\beta,\gamma)$ in this case. The structures of finite-dimensional $\Delta_q(\alpha,\beta,\gamma)$-module $V_n(a,b,c)$ will be discussed.

3. The structures of $V_n(a,b,c)$

In this section, we consider $(n+1)$-dimensional $\Delta_q(\alpha,\beta,\gamma)$-module. Firstly, we have the following result.

Theorem 3.1. *If the* **Condition A** *fails, then* $\Delta_q(\alpha,\beta,\gamma)$ *has at most four isomorphism classes of* $(n+1)$*-dimensional irreducible modules.*

Proof. First of all, assume that $abc \in \mathbf{T}_n$, that is, there is $j(1 \leq j \leq n)$ such that $abc = q^{n-2j+1}$.

Assume that $\Delta_q(\alpha,\beta,\gamma)$ has an $(n+1)$-dimensional irreducible module V, we can get a triple (a',b',c') such that $V \cong V_n(a',b',c')$ by Theorem 2.5. Obviously, $(a',b',c') \notin [a,b,c]$, otherwise, $V_n(a',b',c') \cong V_n(a,b,c)$ is reducible. Let

$$\begin{aligned}
f(x) = &-\left[\left(q^{3n-4j+8} + q^{n-4j+6}\right)a^2b^3\right]x^8 \\
&+\left[\left(q^{n-2j+6}a^4 + q^{n-2j+6}a^2\right)b^4 + \left(q^{3n-6j+8}a^2 + q^{3n-6j+8}\right)b^2\right]x^7 \\
&+\left[\left(q^{n-2j+6}a^4 + q^{n-2j+6}a^2\right)b^5 + \left(q^{n-2j+6}a^4 + \left(2q^{3n-4j+8} + 2q^{n-4j+6} + q^{n-2j+6}\right.\right.\right. \\
&\left.\left.\left.+q^{3n-6j+8} + 2q^{4-n-4j} + 2q^{5n+-4j+10}\right)a^2 + q^{3n-6j+8}\right)b^3 + \left(q^{3n-6j+8}a^2 + q^{3n-6j+8}\right)b\right]x^6 \\
&-\left[\left(q^{n+6} + q^{4-n}\right)a^4b^6 + \left(\left(q^{3n-4j+8} + q^{n-4j+6} + 3q^{n-2j+6} + q^{n+6} + q^{4-n} + 3q^{4-n-2j}\right.\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+3q^{3n-2j+8}\Big)a^4+\Big(4q^{3n-4j+8}+4q^{n-4j+6}+3q^{n-2j+6}+3q^{4-n-2j}+3q^{3n-2j+8}\Big)a^2\\
&+q^{3n-4j+8}+q^{n-4j+6}\Big)b^4+\Big(\Big(q^{3n-4j+8}+q^{n-4j+6}\Big)a^4+\Big(4q^{3n-4j+8}+4q^{n-4j+6}\\
&+3q^{3n-6j+8}+3q^{n-6j+6}+3q^{5n-6j+10}\Big)a^2+q^{3n-4j+8}+q^{n-4j+6}+3q^{3n-6j+8}\\
&+3q^{n-6j+6}+3q^{5n-6j+10}+q^{5n-8j+10}+q^{3n-8j+8}\Big)b^2+q^{5n-8j+10}+q^{3n-8j+8}\Big]x^5\\
&+\Big[\Big(\Big(3q^{n+6}+3q^{4-n}+4q^{n-2j+6}+q^{-n+4-2j}+q^{3n-2j+8}+q^{2-3n}+q^{3n+8}\Big)a^4\\
&+\Big(4q^{n-2j+6}+q^{3n-2j+8}+q^{4-n-2j}\Big)a^2\Big)b^5+\Big(\Big(3q^{n-4j+6}+3q^{3n-4j+8}+4q^{n-2j+6}\\
&+q^{4-n-4j}+q^{4-n-2j}+q^{3n-2j+8}+q^{5n-4j+10}\Big)a^4+\Big(4q^{3n-6j+8}-q^{7n-4j+12}-q^{2-3n-4j}\\
&+4q^{n-2j+6}+q^{n-6j+6}+q^{5n-6j+10}+q^{4-n-2j}+q^{3n-2j+8}+8q^{3n-4j+8}+8q^{n-4j+6}\\
&+3q^{4-n-4j}+3q^{5n-4j+10}\Big)a^2+3q^{3n-4j+8}+3q^{n-4j+6}+4q^{3n-6j+8}+q^{4-n-4j}+q^{n-6j+6}\\
&+q^{5n-4j+10}+q^{5n-6j+10}\Big)b^3+\Big(\Big(4q^{3n-6j+8}+q^{n-6j+6}+q^{5n-6j+10}\Big)a^2+4q^{3n-6j+8}\\
&+q^{n-6j+6}+q^{5n-6j+10}+3q^{5n-8j+10}+3q^{3n-8j+8}+q^{n-8j+6}+q^{7n-8j+12}\Big)b\Big]x^4\\
&-\Big[\Big(q^{n+6}+q^{4-n}\Big)a^4b^6+\Big(\Big(q^{3n-4j+8}+q^{n-4j+6}+3q^{n-2j+6}+q^{n+6}+q^{4-n}+3q^{4-n-2j}\\
&+3q^{3n-2j+8}\Big)a^4+\Big(4q^{3n-4j+8}+4q^{n-4j+6}+3q^{n-2j+6}+3q^{4-n-2j}+3q^{3n-2j+8}\Big)a^2\\
&+q^{3n-4j+8}+q^{n-4j+6}\Big)b^4+\Big(\Big(q^{3n-4j+8}+q^{n-4j+6}\Big)a^4+\Big(4q^{3n-4j+8}+4q^{n-4j+6}\\
&+3q^{3n-6j+8}+3q^{n-6j+6}+3q^{5n-6j+10}\Big)a^2+q^{3n-4j+8}+q^{n-4j+6}+3q^{3n-6j+8}\\
&-3q^{n-6j+6}+3q^{5n-6j+10}+q^{5n-8j+10}+q^{3n-8j+8}\Big)b^2+q^{5n-8j+10}+q^{3n-8j+8}\Big]x^3\\
&+\Big[\Big(q^{n-2j+6}a^4+q^{n-2j+6}a^2\Big)b^5+\Big(q^{n-2j+6}a^4+\Big(2q^{3n-4j+8}+2q^{n-4j+6}+2q^{5n-4j+10}\\
&+q^{n-2j+6}+q^{3n-6j+8}+2q^{4-n-4j}\Big)a^2+q^{3n-6j+8}\Big)b^3+\Big(q^{3n-6j+8}a^2+q^{3n-6j+8}\Big)b\Big]x^2\\
&+\Big[q^{n-2j+6}a^4+q^{n-2j+6}a^2b^4+\Big(q^{3n-6j+8}a^2+q^{3n-6j+8}\Big)b^2\Big]x\\
&-\Big[q^{3n-4j+8}+q^{n-4j+6}\Big]a^2b^3.
\end{aligned}
$$

By the direct calculation, we get that $f(x) = x^8 f(x^{-1})$. So, if $f(x) = 0$, then $f(x^{-1}) = 0$.

By the actions of α, β, γ as (2.7), we have

$$
\begin{aligned}
\omega(a', b', c') &= \omega(a, b, c),\\
\omega(b', c', a') &= \omega(b, c, a),\\
\omega(c', a', b') &= \omega(c, a, b).
\end{aligned}
\tag{3.1}
$$

Eliminating b' and c' from the Eq (3.1) implies that a' is the root of $f(x) = 0$. Since $\mathbb{F}$ is an algebraically closed field, there are 8 roots a' that satisfy the above equation.

Since $U(q; a', b', c') = U(q; a', b', c'^{-1})$ and $\omega(q; a', b', c') = \omega(q; a', b', c'^{-1})$, we can get that $V_n(a', b', c') \cong V_n(a', b', c'^{-1})$ by (2.6)–(2.7). Also $V_n(a', b', c') \cong V_n(a'^{-1}, b', c'^{-1})$ by Lemma 2.3, so we have $V_n(a'^{-1}, b', c') \cong V_n(a', b', c')$.

It follows that $\Delta_q(\alpha, \beta, \gamma)$ has at most four $(n+1)$-dimensional irreducible module up to isomorphism. Other cases for $a^{-1}bc$, $ab^{-1}c$, abc^{-1} belonging to $\mathbf{T}_n$ can be similarly to discuss.

Therefore, there is at most four $(n+1)$-dimensional irreducible $\Delta_q(\alpha,\beta,\gamma)$-module up to isomorphism if the **Condition A** fails. □

In the following, we discuss the structures of $V_n(a,b,c)$ when $a, b, c \in \mathbb{F}$ are fixed and **Condition A** fails. For this purpose, we first give some notations.

For $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$, $(\varepsilon_1',\varepsilon_2',\varepsilon_3')$, $(\varepsilon_1'',\varepsilon_2'',\varepsilon_3'') \in \mathfrak{S}$, denote

$$V_1 = \sum_{i=j}^{n} \mathbb{F}v_i^{(\varepsilon_1\varepsilon_2,1)}, \quad V_2 = \sum_{i=k}^{n} \mathbb{F}v_i^{(\varepsilon_1'\varepsilon_2',1)}, \quad V_3 = \sum_{i=l}^{n} \mathbb{F}v_i^{(\varepsilon_1''\varepsilon_2'',1)}, \tag{3.2}$$

we have the following results.

Theorem 3.2. *Assume that $a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} = q^{n-2j+1}$ for some $j(1 \le j \le n)$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3) \in \mathfrak{S}$. Then*

1) *V_1 is a submodule of $V_n(a,b,c)$ and $V_1 \cong V_{n-j}(q^j a^{\varepsilon_1\varepsilon_2}, q^j b, q^j c^{\varepsilon_1\varepsilon_2})$;*
2) *$V_n(a,b,c)/V_1 \cong V_{j-1}(q^{j-n-1}a^{\varepsilon_1\varepsilon_2}, q^{j-n-1}b, q^{j-n-1}c^{\varepsilon_1\varepsilon_2})$;*
3) *the equivalent conditions of irreducibility of V_1 are described as follows:*

$(\varepsilon_1,\varepsilon_2,\varepsilon_3)$	*V_1 is irreducible*
$(1,1,1)$	$a^2,b^2,c^2 \notin \left\{q^{2i} \mid 1 \le i \le n-j\right\}$
$(-1,1,1)$	$a^2,b^{-2} \notin \left\{q^{2j-2i} \mid 1 \le i \le n-j\right\}$ *and* $c^2 \notin \left\{q^{2i+2j} \mid 1 \le i \le n-j\right\}$
$(1,-1,1)$	$a^2,b^{-2},c^2 \notin \left\{q^{2i-2j} \mid 1 \le i \le n-j\right\}$ *and* $j > \frac{n}{3}$
$(1,1,-1)$	$a^2,b^2 \notin \left\{q^{2i-2j} \mid 1 \le i \le n-j\right\}$ *and* $c^2 \notin \left\{q^{2i-2n-2} \mid 1 \le i \le n-j\right\}$

;

4) *the equivalent conditions of maximality of V_1 are described as follows:*

$(\varepsilon_1,\varepsilon_2,\varepsilon_3)$	*V_1 is maximal*
$(1,1,1)$	$a^2,b^2,c^2 \notin \left\{q^{2i-2j} \mid 1 \le i \le j-1\right\}$
$(-1,1,1)$	$a^2,b^{-2} \notin \left\{q^{2i+2j-2n-2} \mid 1 \le i \le j-1\right\}$ *and* $c^2 \notin \left\{q^{2i-2n-2} \mid 1 \le i \le j-1\right\}$
$(1,-1,1)$	$a^2,b^{-2},c^2 \notin \left\{q^{2n-2i-2j+2} \mid 1 \le i \le j-1\right\}$ *with* $j < \frac{2}{3}n+1$
$(1,1,-1)$	$a^2,b^2 \notin \left\{q^{2n-2i-2j+2} \mid 1 \le i \le j-1\right\}$ *and* $c^2 \notin \left\{q^{2n-2i+2} \mid 1 \le i \le j-1\right\}$

.

Proof. We only prove the case when $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (1,1,1)$, the others are similar. At this time, by (3.2), we have

$$V_1 = \sum_{i=j}^{n} \mathbb{F}v_i^{(1,1)}.$$

For convenience, we denote

$$(a',b',c') = (q^j a, q^j b, q^j c), \quad (a'',b'',c'') = (q^{j-n-1}a, q^{j-n-1}b, q^{j-n-1}c).$$

1) Firstly, let us show that V_1 is a submodule of $V_n(a,b,c)$.

By Theorem 2.2 and Lemma 2.3, the actions of A, B, γ on this basis $\{v_i^{(1,1)} | 0 \le i \le n\}$ of $V_n(a,b,c)$ are as follows:

$$Av_i^{(1,1)} = \theta_i(q;a)v_i^{(1,1)} + v_{i+1}^{(1,1)}, \tag{3.3}$$
$$Bv_i^{(1,1)} = \theta_i(q;b)v_i^{(1,1)} + \phi_i(q;a,b,c)v_{i-1}^{(1,1)}, \tag{3.4}$$
$$\gamma v_i^{(1,1)} = \omega(q;a,b,c)v_i^{(1,1)}. \tag{3.5}$$

By (3.3) and (3.5), $V_1 = \sum_{k=j}^{n} \mathbb{F}v_k^{(1,1)}$ is an A- and γ-invariant subspace of $V_n(a,b,c)$, it is enough to show that V_1 is B-invariant.

Since $abc = q^{n-2j+1}$, we have $\phi_j(q;a,b,c) = 0$ by direct calculation.

By (3.4), we know

$$Bv_j^{(1,1)} = \theta_j(q;b)v_j^{(1,1)}.$$

Therefore V_1 is B-invariant and V_1 is a $\Delta_q(\alpha,\beta,\gamma)$-submodule of $V_n(a,b,c)$.

Now we show that $V_1 \cong V_{n-j}(a',b',c')$.

Let $\{v_i'^{(1,1)} | 0 \le i \le n-j\}$ be a canonical basis of $V_{n-j}(a',b',c')$.

Construct a linear map $\varphi : V_1 \to V_{n-j}(a',b',c')$ sending $v_{j+k}^{(1,1)}$ to $v_k'^{(1,1)}$. Obviously, φ is bijective since $\dim V_1 = \dim V_{n-j} = n-j+1$. So it is enough to show that φ is a $\Delta_q(\alpha,\beta,\gamma)$-modules homomorphism.

Since $a' = q^j a$, we have

$$\begin{aligned}
A\varphi(v_{j+k}^{(1,1)}) &= Av_k'^{(1,1)} \\
&= \theta_k(q^{n-j},q;a')v_k'^{(1,1)} + v_{k+1}'^{(1,1)} \\
&= \theta_{j+k}(q^n,q;a)v_k'^{(1,1)} + v_{k+1}'^{(1,1)} \\
&= \varphi(\theta_{j+k}(q^n,q;a)v_{j+k}^{(1,1)} + v_{j+k+1}^{(1,1)}) \\
&= \varphi(Av_{j+k}^{(1,1)}).
\end{aligned}$$

Similarly, $B\varphi(v_{j+k}^{(1,1)}) = \varphi(Bv_{j+k}^{(1,1)})$ and $\gamma\varphi(v_{j+k}^{(1,1)}) = \varphi(\gamma v_{j+k}^{(1,1)})$.

Therefore, $V_1 \cong V_{n-j}(q^j a, q^j b, q^j c)$ is a submodule of $V_n(a,b,c)$.

2) $V_n(a,b,c)/V_1$ has a basis $\left\{v_i^{(1,1)} + V_1 | 0 \le i \le j-1\right\}$ and let $\left\{v_i''^{(1,1)} | 0 \le i \le j-1\right\}$ be a basis of $V_{j-1}(a'',b'',c'')$.

Define a map $\psi : V_n(a,b,c)/V_1 \to V_{j-1}(a'',b'',c'')$ that sends $v_i^{(1,1)} + V_1$ to $v_i''^{(1,1)}$. Obviously, ψ is bijective since $\dim V_n(a,b,c)/V_1 = \dim V_{j-1}(a'',b'',c'') = j$. So it is enough to show that ψ is a $\Delta_q(\alpha,\beta,\gamma)$-modules homomorphism.

Since $a'' = q^{j-n-1}a$, we have

$$\begin{aligned}
A\psi(v_i^{(1,1)} + V_1) &= Av_i''^{(1,1)} \\
&= \theta_i(q^{j-1},q;a'')v_i''^{(1,1)} + v_{i+1}''^{(1,1)} \\
&= \theta_i(q^n,q;a)v_i''^{(1,1)} + v_{i+1}''^{(1,1)}
\end{aligned}$$

$$= \psi(\theta_i(q^n, q; a)(v_i^{(1,1)} + V_1) + (v_{i+1}^{(1,1)} + V_1))$$
$$= \psi(A(v_i^{(1,1)} + V_1)).$$

Similarly, $B\psi(v_i^{(1,1)} + V_1) = \psi(B(v_i^{(1,1)} + V_1))$ and $\gamma\psi(v_i^{(1,1)} + V_1) = \psi(\gamma(v_i^{(1,1)} + V_1))$.

It follows that $V_n(a, b, c)/V_1$ is isomorphic to $V_{j-1}(a'', b'', c'')$.

3) By 1), it is sufficient to describe the equivalent conditions for $V_{n-j}(a', b', c')$ is irreducible or equivalently $a'b'c'$, $a'^{-1}b'c'$, $a'b'^{-1}c'$, $a'b'c'^{-1} \notin \mathbf{T}_{n-j}$ by Theorem 2.4. Indeed

$$a'b'c', a'^{-1}b'c', a'b'^{-1}c', a'b'c'^{-1} \notin \mathbf{T}_{n-j} \Longleftrightarrow a^2, b^2, c^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}.$$

($\Longrightarrow$) Since $abc = q^{n-2j+1}$ and $a'^{-1}b'c' = q^j a^{-1}bc = q^{n-j+1}a^{-2} \notin \mathbf{T}_{n-j}$, we have $a^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$.

Similarly, we can obtain that b^2, $c^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$.

($\Longleftarrow$) Assume that $abc = q^{n-2j+1}$ and a^2, b^2, $c^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$. Obviously, $a'b'c' = q^{3j}abc = q^{n+j+1} \notin \mathbf{T}_{n-j}$.

Since $a^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$, we have $a'^{-1}b'c' = q^j a^{-1}bc = q^{n-j+1}a^{-2} \notin \mathbf{T}_{n-j}$.

Since $b^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$, we have $a'b'^{-1}c' = q^j ab^{-1}c = q^{n-j+1}b^{-2} \notin \mathbf{T}_{n-j}$.

Since $c^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$, we have $a'b'c'^{-1} = q^j abc^{-1} = q^{n-j+1}c^{-2} \notin \mathbf{T}_{n-j}$.

In summary, V_1 is irreducible if and only if $a^2, b^2, c^2 \notin \left\{q^{2i}|1 \leq i \leq n-j\right\}$.

4) By 2), it is sufficient to describe the equivalent conditions for $V_n(a, b, c)/V_1 \cong V_{j-1}(a'', b'', c'')$ is irreducible or equivalently $a''b''c''$, $a''^{-1}b''c''$, $a''b''^{-1}c''$, $a''b''c''^{-1} \notin \mathbf{T}_{j-1}$ by Theorem 2.4. Indeed

$$a''b''c'', a''^{-1}b''c'', a''b''^{-1}c'', a''b''c''^{-1} \notin \mathbf{T}_{j-1} \Longleftrightarrow a^2, b^2, c^2 \notin \left\{q^{2i-2j}|1 \leq i \leq j-1\right\}.$$

($\Longrightarrow$) Since $abc = q^{n-2j+1}$, and $a''^{-1}b''c'' = q^{j-n-1}a^{-1}bc = q^{-j}a^{-2} \notin \mathbf{T}_{j-1}$, we have $a^2 \notin \left\{q^{2i-2j}|1 \leq i \leq j-1\right\}$.

Similarly, we can obtain that b^2, $c^2 \notin \left\{q^{-2i}|1 \leq i \leq j-1\right\}$.

($\Longleftarrow$) Assume that $abc = q^{n-2j+1}$ and a^2, b^2, $c^2 \notin \left\{q^{2i-2j}|1 \leq i \leq j-1\right\}$. Obviously, $a''b''c'' = q^{3j-3n-3}abc = q^{j-2-2n} = q^{j-2(n+1)} \notin \mathbf{T}_{j-1}$.

Since $a^2 \notin \left\{q^{2i-2j}|1 \leq i \leq j-1\right\}$, we have $a''^{-1}b''c'' = q^{j-n-1}a^{-1}bc = q^{-j}a^{-2} \notin \mathbf{T}_{j-1}$.

Since $b^2 \notin \left\{q^{2i-2j}|1 \leq i \leq n-j\right\}$, we have $a''b''^{-1}c'' = q^{j-n-1}ab^{-1}c = q^{-j}b^{-2} \notin \mathbf{T}_{j-1}$.

Since $c^2 \notin \left\{q^{2i-2j}|1 \leq i \leq n-j\right\}$, we have $a''b''c''^{-1} = q^{j-n-1}abc^{-1} = q^{-j}c^{-2} \notin \mathbf{T}_{j-1}$.

In summary, V_1 is maximal if and only if a^2, b^2, $c^2 \notin \left\{q^{2i-2j}|1 \leq i \leq j-1\right\}$.

This proof is finished. □

Theorem 3.3. *Assume that* $a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} = q^{n-2j+1}$ *and* $a^{\varepsilon'_1}b^{\varepsilon'_2}c^{\varepsilon'_3} = q^{n-2k+1}$ *for some* $j, k(1 \leq j, k \leq n)$ *and* $(\varepsilon_1, \varepsilon_2, \varepsilon_3), (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) \in \mathfrak{S}$. *Then*

1) V_1 *and* V_2 *are* $\Delta_q(\alpha, \beta, \gamma)$*-submodules of* $V_n(a, b, c)$ *and*

$$V_1 \cong V_{n-j}(q^j a^{\varepsilon_1\varepsilon_2}, q^j b, q^j c^{\varepsilon_1\varepsilon_2}), \quad V_2 \cong V_{n-k}(q^k a^{\varepsilon'_1\varepsilon'_2}, q^k b, q^k c^{\varepsilon'_1\varepsilon'_2});$$

2) if $\varepsilon_1\varepsilon_2 = \varepsilon'_1\varepsilon'_2$, then V_2 is reducible and V_1 is a submodule of V_2 when $j > k$; V_1 is reducible and V_2 is a submodule of V_1 when $j < k$;

3) the equivalent conditions of the irreducibility of V_1 and V_2 are described as follows:

$(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$	V_1 *is irreducible*	V_2 *is irreducible*
$(1, 1, 1)$ $(-1, 1, 1)$	$j > k - 1$ *and* $b^2 \notin \left\{q^{2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}$	$k > n - j$ *and* $b^2 \notin \left\{q^{2i-2k}, q^{2i-2k-2j} \mid 1 \le i \le n - k\right\}$
$(1, 1, 1)$ $(1, -1, 1)$	$j > k - 1$ *and* $a^2 \notin \left\{q^{2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}$	$k > n - j$ *and* $a^2 \notin \left\{q^{2i-2k}, q^{2i+2k-2j} \mid 1 \le i \le n - k\right\}$
$(1, 1, 1)$ $(1, 1, -1)$	$j > k$ *and* $a^2 \notin \left\{q^{2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}$	$j < k$ *and* $a^2 \notin \left\{q^{2i+2k-2j}, q^{2i-2k} \mid 1 \le i \le n - k\right\}$
$(-1, 1, 1)$ $(1, -1, 1)$	$j > \max\left\{k, \frac{n}{2} - \frac{k}{2}\right\}$ *and* $a^2 \notin \left\{q^{2j-2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}$	$k > \max\left\{j, \frac{n}{3}\right\}$ *and* $a^2 \notin \left\{q^{2j-2i}, q^{2i-2k} \mid 1 \le i \le n - k\right\}$
$(-1, 1, 1)$ $(1, 1, -1)$	$j > k - 1$ *and* $a^2 \notin \left\{q^{2j-2i}, q^{2i-2k+4j} \mid 1 \le i \le n - j\right\}$	$k > j - 1$ *and* $a^2 \notin \left\{q^{2i-2k}, q^{2j-2i-4k} \mid 1 \le i \le n - k\right\}$
$(1, -1, 1)$ $(1, 1, -1)$	$j > \max\left\{k - 1, \frac{n}{3}\right\}$ *and* $b^2 \notin \left\{q^{2j-2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}$	$k > j - 1$ *and* $b^2 \notin \left\{q^{2i-2k}, q^{2j-2i-4k} \mid 1 \le i \le n - k\right\}$

Proof. 1) It can be implied directly by Theorem 3.2.

2) If $\varepsilon_1\varepsilon_2 = \varepsilon'_1\varepsilon'_2$, it is obvious by (3.2).

3) We only prove the case when $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 1)$ and $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) = (-1, 1, 1)$, the others are similar. At this time, by (3.2), we have

$$V_1 = \sum_{i=j}^{n} \mathbb{F}v_i^{(1,1)} \cong V_{n-j}(q^j a, q^j b, q^j c), \tag{3.6}$$

$$V_2 = \sum_{i=k}^{n} \mathbb{F}v_i^{(1,1)} \cong V_{n-k}(q^k a^{-1}, q^k b, q^k c^{-1}). \tag{3.7}$$

For convenience, we denote

$$(a_1, b_1, c_1) = (q^j a, q^j b, q^j c), \qquad (a_2, b_2, c_2) = (q^k a^{-1}, q^k b, q^k c^{-1}).$$

Firstly, we discuss the equivalent conditions of the irreducibility of V_1.

By (3.6), it is sufficient to describe the equivalent conditions for $V_{n-j}(a_1, b_1, c_1)$ is irreducible or equivalently $a_1b_1c_1, a_1^{-1}b_1c_1, a_1b_1^{-1}c_1, a_1b_1c_1^{-1} \notin \mathbf{T}_{n-j}$ by Theorem 2.4. Indeed

$$a_1b_1c_1, a_1^{-1}b_1c_1, a_1b_1^{-1}c_1, a_1b_1c_1^{-1} \notin \mathbf{T}_{n-j} \Longleftrightarrow b^2 \notin \left\{q^{2i}, q^{2i-2k} \mid 1 \le i \le n - j\right\}, j > k - 1.$$

($\Longrightarrow$) Since $abc = q^{n-2j+1}$, we have $a_1b_1^{-1}c_1 = q^j ab^{-1}c = q^{n-j+1}b^{-2} \notin \mathbf{T}_{n-j}$ and $b^2 \notin \left\{q^{2i} \mid 1 \le i \le n - j\right\}$.

Since $a^{-1}bc = q^{n-2k+1}$, we have $a_1^{-1}b_1c_1 = q^j a^{-1}bc = q^{j+n-2k+1} \notin \mathbf{T}_{n-j}$, so $j > k - 1$. And $a_1b_1c_1^{-1} = q^j abc^{-1} = q^{j-n+2k-1}b^2 \notin \mathbf{T}_{n-j}$, we have $b^2 \notin \left\{q^{2i-2k} \mid 1 \le i \le n - j\right\}$.

Therefore, $b^2 \notin \left\{q^{2i}, q^{2i-2k} | 1 \le i \le n-j\right\}$ and $j > k-1$.

($\Longleftarrow$) Assume that $b^2 \notin \left\{q^{2i}, q^{2i-2k} | 1 \le i \le n-j\right\}$ and $j > k-1$.

Since $abc = q^{n-2j+1}$, we have $a_1b_1c_1 = q^{3j}abc = q^{n+j+1} \notin \mathbf{T}_{n-j}$ and $a_1b_1^{-1}c_1 = q^jab^{-1}c = q^{n-j+1}b^{-2} \notin \mathbf{T}_{n-j}$.

Since $a^{-1}bc = q^{n-2k+1}$, we have $a_1^{-1}b_1c_1 = q^ja^{-1}bc = q^{j+n-2k+1} \notin \mathbf{T}_{n-j}$ and $a_1b_1c_1^{-1} = q^{j-n+2k-1}b^2 \notin \mathbf{T}_{n-j}$.

Therefore V_1 is irreducible if and only if $b^2 \notin \left\{q^{2i}, q^{2i-2k} | 1 \le i \le n-j\right\}$ with $j > k-1$.

Now we discuss the equivalent conditions of the irreducibility of V_2.

Similarly, it is enough to show that

$$a_2b_2c_2, a_2^{-1}b_2c_2, a_2b_2^{-1}c_2, a_2b_2c_2^{-1} \notin \mathbf{T}_{n-k} \Longleftrightarrow b^2 \notin \left\{q^{2i-2k}, q^{2i-2k-2j} | 1 \le i \le n-k\right\}, k > n-j.$$

($\Longrightarrow$) Since $abc = q^{n-2j+1}$, we have $a_2b_2c_2 = q^{3k}a^{-1}bc^{-1} = q^{2k-n+2j-1}b^2 \notin \mathbf{T}_{n-k}$, so $b^2 \notin \left\{q^{2i-2k-2j} | 1 \le i \le n-k\right\}$. And $a_2b_2^{-1}c_2 = q^ka^{-1}b^{-1}c^{-1} = q^{k-n+2j-1} \notin \mathbf{T}_{n-k}$, so $k > n-j$.

Since $a^{-1}bc = q^{n-2k+1}$, we have $a_2^{-1}b_2c_2 = q^kabc^{-1} = q^{3k-n-1}b^2 \notin \mathbf{T}_{n-k}$, so $b^2 \notin \left\{q^{2i-2k} | 1 \le i \le n-k\right\}$.

Therefore, $b^2 \notin \left\{q^{2i-2k}, q^{2i-2k-2j} | 1 \le i \le n-k\right\}$ and $k > n-j$.

($\Longleftarrow$) Assume that $b^2 \notin \left\{q^{2i-2k}, q^{2i-2k-2j} | 1 \le i \le n-k\right\}$ and $k > n-j$.

Since $abc = q^{n-2j+1}$, we have $a_2b_2^{-1}c_2 = q^ka^{-1}b^{-1}c^{-1} = q^{k-n+2j-1} \notin \mathbf{T}_{n-k}$ and $a_2b_2c_2 = q^{3k}a^{-1}bc^{-1} = q^{2k-n+2j-1}b^2 \notin \mathbf{T}_{n-k}$.

Since $a^{-1}bc = q^{n-2k+1}$, we have $a_2b_2c_2^{-1} = q^kabc^{-1} = q^{n-k+1} \notin \mathbf{T}_{n-k}$ and $a_2^{-1}b_2c_2 = q^kabc^{-1} = q^{3k-n-1}b^2 \notin \mathbf{T}_{n-k}$.

Therefore V_2 is irreducible if and only if $b^2 \notin \left\{q^{2i-2k}, q^{2i-2k-2j} | 1 \le i \le n-k\right\}$ with $k > n-j$.

This proof is finished. □

Theorem 3.4. *Assume that* $a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} = q^{n-2j+1}$, $a^{\varepsilon_1'}b^{\varepsilon_2'}c^{\varepsilon_3'} = q^{n-2k+1}$, $a^{\varepsilon_1''}b^{\varepsilon_2''}c^{\varepsilon_3''} = q^{n-2l+1}$, *for some* $j, k, l (1 \le j, k, l \le n)$ *and* $(\varepsilon_1, \varepsilon_2, \varepsilon_3), (\varepsilon_1', \varepsilon_2', \varepsilon_3'), (\varepsilon_1'', \varepsilon_2'', \varepsilon_3'') \in \mathfrak{S}$. *If* $\varepsilon_1\varepsilon_2 = \varepsilon_1'\varepsilon_2'$, *then*

1) V_1, V_2 *and* V_3 *are submodules of* $V_n(a, b, c)$ *and*

$$V_1 \cong V_{n-j}(q^ja^{\varepsilon_1\varepsilon_2}, q^jb, q^jc^{\varepsilon_1\varepsilon_2}), \quad V_2 \cong V_{n-k}(q^ka^{\varepsilon_1'\varepsilon_2'}, q^kb, q^kc^{\varepsilon_1'\varepsilon_2'}), \quad V_3 \cong V_{n-l}(q^la^{\varepsilon_1''\varepsilon_2''}, q^lb, q^lc^{\varepsilon_1''\varepsilon_2''});$$

2) *if* $j > k$, V_2 *is reducible and* V_1 *is a submodule of* V_2; *if* $j < k$, V_1 *is reducible and* V_2 *is a submodule of* V_1;

3) *the equivalent conditions of the irreducibility of* V_1 *and* V_2 *are described as follows*:

$(\varepsilon_1'', \varepsilon_2'', \varepsilon_3'')$	V_1 *is irreducible*	V_2 *is irreducible*
$(1, 1, 1)$	$j > \max\left\{k, n-l, \frac{n}{2} - \frac{k}{2}\right\}$ *and* $c^2 \notin \left\{q^{2i+2j-2l} | 1 \le i \le n-j\right\}$	$k > \max\left\{j, \frac{n}{3}, n-l\right\}$ *and* $c^2 \notin \left\{q^{2i+2k-2l} | 1 \le i \le n-k\right\}$
$(-1, 1, 1)$	$j > \max\{k, l-1\}$ *and* $b^2 \notin \left\{q^{2i} | 1 \le i \le n-j\right\}$	$k > \max\{j, l-1\}$ *and* $b^2 \notin \left\{q^{2i+2k-2j} | 1 \le i \le n-k\right\}$
$(1, -1, 1)$	$j > \max\{k, l-1\}$ *and* $a^2 \notin \left\{q^{2i} | 1 \le i \le n-j\right\}$	$k > \max\{j, l-1\}$ *and* $a^2 \notin \left\{q^{2i+2k-2j} | 1 \le i \le n-k\right\}$
$(1, 1, -1)$	$j > \max\left\{k, l-1, \frac{n}{2} - \frac{k}{2}\right\}$ *and* $a^2 \notin \left\{q^{2j-2i} | 1 \le i \le n-j\right\}$	$k > \max\left\{j, \frac{n}{3}, l-1\right\}$ *and* $b^2 \notin \left\{q^{2k-2i} | 1 \le i \le n-k\right\}$

;

4) the equivalent conditions of the irreducibility of V_3 are described as follows:

$(\varepsilon_1'', \varepsilon_2'', \varepsilon_3'')$	V_3 *is irreducible*
$(1,1,1)$	$l > \max\{j-1, k-1\}$ *and* $c^2 \notin \left\{q^{2i} \mid 1 \le i \le n-l\right\}$
$(-1,1,1)$	$l > \max\{k-1, n-j\}$ *and* $b^2 \notin \left\{q^{2i-2j-2l} \mid 1 \le i \le n-l\right\}$
$(1,-1,1)$	$l > \max\left\{\frac{n}{3}, k-1, n-j\right\}$ *and* $a^2 \notin \left\{q^{2i+2l-2j} \mid 1 \le i \le n-l\right\}$
$(1,1,-1)$	$l > \max\{j-1, k-1\}$ *and* $c^2 \notin \left\{q^{-2i-2l} \mid 1 \le i \le n-l\right\}$

.

Proof. 1) It is obvious by Theorem 3.3.

2) If $\varepsilon_1\varepsilon_2 = \varepsilon_1'\varepsilon_2'$, then $V_1 = \sum_{i=j}^{n} v_i^{(\varepsilon_1\varepsilon_2,1)}$ and $V_2 = \sum_{i=k}^{n} v_i^{(\varepsilon_1\varepsilon_2,1)}$ by (3.2). Therefore, it is obvious.

We only prove the case of $(\varepsilon_1'', \varepsilon_2'', \varepsilon_3'') = (1,1,1)$, the others are similar. In this case, $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,1,1)$ and $(\varepsilon_1', \varepsilon_2', \varepsilon_3') = (1,-1,1)$. By (3.2), we get that

$$V_1 = \sum_{i=j}^{n} v_i^{(-1,1)} \cong V_{n-j}(q^k a^{-1}, q^k b, q^k c^{-1}), \tag{3.8}$$

$$V_2 = \sum_{i=k}^{n} v_i^{(-1,1)} \cong V_{n-k}(q^j a^{-1}, q^j b, q^j c^{-1}), \tag{3.9}$$

$$V_3 = \sum_{i=l}^{n} v_i^{(1,1)} \cong V_{n-l}(q^l a, q^l b, q^l c). \tag{3.10}$$

For convenience, denote

$$(a_1, b_1, c_1) = (q^k a^{-1}, q^k b, q^k c^{-1}),$$
$$(a_2, b_2, c_2) = (q^j a^{-1}, q^j b, q^j c^{-1}),$$
$$(a_3, b_3, c_3) = (q^l a, q^l b, q^l c).$$

3) Firstly, we show the equivalent conditions of the irreducibility of V_1.

By 2), the equivalent condition of irreducibility of V_1 should be discussed in the case of $j > k$. By (3.8), it is sufficient to describe the equivalent conditions for $V_{n-j}(a_1, b_1, c_1)$ is irreducible or equivalently $a_1b_1c_1, a_1^{-1}b_1c_1, a_1b_1^{-1}c_1, a_1b_1c_1^{-1} \notin \mathbf{T}_{n-j}$ by Theorem 2.4. Indeed

$$a_1b_1c_1, a_1^{-1}b_1c_1, a_1b_1^{-1}c_1, a_1b_1c_1^{-1} \notin \mathbf{T}_{n-j} \Longleftrightarrow c^2 \notin \left\{q^{2i+2j-2l} \mid 1 \le i \le n-j\right\}, j > \max\left\{n-l, \frac{n}{2} - \frac{k}{2}\right\}.$$

($\Longrightarrow$) Since $ab^{-1}c = q^{n-2k+1}$, we have $a_1b_1c_1 = q^{3j}a^{-1}bc^{-1} = q^{3j-n+2k-1} \notin \mathbf{T}_{n-j}$, so $j > \frac{n}{2} - \frac{k}{2}$.

Since $abc = q^{n-2l+1}$, we have $a_1^{-1}b_1c_1 = q^j abc^{-1} = q^{j+n-2l+1}c^{-2} \notin \mathbf{T}_{n-j}$, so $c^2 \notin q^{2i+2j-2l}$. And $a_1b_1^{-1}c_1 = q^j a^{-1}b{-}1c^{-1} = q^{j-n+2l-1} \notin \mathbf{T}_{n-j}$, so $j > n-l$.

Therefore, $c^2 \notin \left\{q^{2i+2j-2l} \mid 1 \le i \le n-j\right\}$ with $j > \max\left\{n-l, \frac{n}{2} - \frac{k}{2}\right\}$.

($\Longleftarrow$) Assume that $c^2 \notin \left\{q^{2i+2j-2l} \mid 1 \le i \le n-j\right\}$ and $j > \max\left\{k, n-l, \frac{n}{2} - \frac{k}{2}\right\}$.

Since $ab^{-1}c = q^{n-2k+1}$, we have $a_1b_1c_1 = q^{3j}a^{-1}bc^{-1} = q^{3j-n+2k-1} \notin \mathbf{T}_{n-j}$.

Since $abc = q^{n-2l+1}$, we have $a_1^{-1}b_1c_1 = q^j abc^{-1} = q^{j+n-2l+1}c^{-2} \notin \mathbf{T}_{n-j}$ and $a_1b_1^{-1}c_1 = q^j a^{-1}b^{-1}c^{-1} = q^{j-n+2l-1} \notin \mathbf{T}_{n-j}$.

Since $a^{-1}bc = q^{n-2j+1}$, we have $a_1b_1c_1^{-1} = q^j a^{-1}bc = q^{n-j+1} \notin \mathbf{T}_{n-j}$.

Therefore, V_1 is irreducible if and only if $c^2 \notin \left\{q^{2i+2j-2l} | 1 \le i \le n-j\right\}$ with $j > \max\left\{k, n-l, \frac{n}{2} - \frac{k}{2}\right\}$.

Now we show the equivalent conditions of the irreducibility of V_2.

By 2), the equivalent condition of irreducibility of V_2 should be discussed in the case of $j < k$. In this case, it is enough to show that

$$a_2b_2c_2, a_2^{-1}b_2c_2, a_2b_2^{-1}c_2, a_2b_2c_2^{-1} \notin \mathbf{T}_{n-k} \Longleftrightarrow c^2 \notin \left\{q^{2i+2k-2l} | 1 \le i \le n-k\right\}, k > \max\left\{\frac{n}{3}, n-l\right\}.$$

The proof is analogous to the above process omitted here.

$$a_3b_3c_3, a_3^{-1}b_3c_3, a_3b_3^{-1}c_3, a_3b_3c_3^{-1} \notin \mathbf{T}_{n-l} \Longleftrightarrow c^2 \notin \left\{q^{2i} | 1 \le i \le n-l\right\}, l > \max\{j-1, k-1\}.$$

($\Longrightarrow$) Since $a^{-1}bc = q^{n-2j+1}$, we have $a_3^{-1}b_3c_3 = q^l a^{-1}bc = q^{l+n-2j+1} \notin \mathbf{T}_{n-l}$, so $l > j-1$.

Since $ab^{-1}c = q^{n-2k+1}$, we have $a_3b_3^{-1}c_3 = q^l ab^{-1}c = q^{l+n-2k+1} \notin \mathbf{T}_{n-l}$, so $l > k-1$.

Since $abc = q^{n-2l+1}$, $a_3b_3c_3^{-1} = q^l abc^{-1} = q^{n-l+1}c^{-2} \notin \mathbf{T}_{n-l}$, so $c^2 \notin \left\{q^{2i} | 1 \le i \le n-l\right\}$.

Therefore, $c^2 \notin \left\{q^{2i} | 1 \le i \le n-l\right\}$ and $l > \max\{j-1, k-1\}$.

($\Longleftarrow$) Assume that $c^2 \notin \left\{q^{2i} | 1 \le i \le n-l\right\}$ and $l > \max\{j-1, k-1\}$.

Since $abc = q^{n-2l+1}$, we have $a_3b_3c_3 = q^{3l}abc = q^{n+l+1} \notin \mathbf{T}_{n-l}$ and $a_3b_3c_3^{-1} = q^l abc^{-1} = q^{n-l+1}c^{-2} \notin \mathbf{T}_{n-l}$.

Since $a^{-1}bc = q^{n-2j+1}$, we have $a_3^{-1}b_3c_3 = q^l a^{-1}bc = q^{l+n-2j+1} \notin \mathbf{T}_{n-l}$.

Since $ab^{-1}c = q^{n-2k+1}$, we have $a_3b_3^{-1}c_3 = q^l ab^{-1}c = q^{l+n-2k+1} \notin \mathbf{T}_{n-l}$.

In summary, V_3 is irreducible if and only if $c^2 \notin \left\{q^{2i} | 1 \le i \le n-l\right\}$ with $l > \max\{j-1, k-1\}$. □

Denote

$$V_1 = \sum_{i=j}^{n} \mathbb{F}v_i^{(1,1)}, \quad V_2 = \sum_{i=k}^{n} \mathbb{F}v_i^{(-1,1)}, \quad V_3 = \sum_{i=l}^{n} \mathbb{F}v_i^{(-1,1)}, \quad V_4 = \sum_{i=m}^{n} \mathbb{F}v_i^{(1,1)}. \tag{3.11}$$

Theorem 3.5. *Assume that* $abc = q^{n-2j+1}$, $a^{-1}bc = q^{n-2k+1}$, $ab^{-1}c = q^{n-2l+1}$ *and* $abc^{-1} = q^{n-2m+1}$ *for some* $j, k, l, m (1 \le j, k, l, m \le n)$. *Then*

1) V_1, V_2, V_3 *and* V_4 *are submodules of* $V_n(a, b, c)$ *and*

$$V_1 \cong V_{n-j}(q^j a, q^j b, q^j c), \qquad V_2 \cong V_{n-k}(q^k a^{-1}, q^k b, q^k c^{-1}),$$
$$V_3 \cong V_{n-l}(q^l a^{-1}, q^l b, q^l c^{-1}), \qquad V_4 \cong V_{n-l}(q^m a, q^m b, q^m c);$$

2) we have the following statements:

$j > m$	V_1 *is irreduciible* $\Longleftrightarrow$ $j > \max\{k-1, l-1\}$	V_4 *is reducible* V_1 *is a submodule of* V_4
$j < m$	V_1 *is reducible* V_4 *is a submodule of* V_1	V_4 *is irreduciible* $\Longleftrightarrow$ $m > \{l-1, k-1\}$
$k > l$	V_2 *is irreduciible* $\Longleftrightarrow$ $k > \max\left\{\frac{n-l}{2}, m-1, n-j\right\}$	V_3 *is reducible* V_2 *is a submodule of* V_3
$k < l$	V_2 *is reducible* V_3 *is a submodule of* V_2	V_3 *is irreduciible* $\Longleftrightarrow$ $l > \max\left\{\frac{n}{3}, m-1, n-j\right\}$

.

Proof. 1) It is obvious by Theorem 3.2.

For convenience, denote

$$(a_1, b_1, c_1) = (q^k a, q^k b, q^k c), \qquad (a_2, b_2, c_2) = (q^j a^{-1}, q^j b, q^j c^{-1}),$$
$$(a_3, b_3, c_3) = (q^l a^{-1}, q^l b, q^l c^{-1}), \qquad (a_4, b_4, c_4) = (q^m a, q^m b, q^m c).$$

2) Firstly, let us consider $j > m$.

If $j > m$, it is obvious that V_4 is reducible, V_1 is a submodule of V_4 by (3.11). Also, V_1 is irreducible if and only if $V_{n-1}(a_1, b_1, c_1)$ is irreducible by 1), or equivalently $a_1 b_1 c_1$, $a_1^{-1} b_1 c_1$, $a_1 b_1^{-1} c_1$, $a_1 b_1 c_1^{-1} \notin \mathbf{T}_{n-j}$ by Theorem 2.4. Indeed

$$a_1 b_1 c_1, a_1^{-1} b_1 c_1, a_1 b_1^{-1} c_1, a_1 b_1 c_1^{-1} \notin \mathbf{T}_{n-j} \Longleftrightarrow j > \max\{k-1, l-1\}.$$

($\Longrightarrow$) Since $a^{-1}bc = q^{n-2k+1}$, we have $a_1^{-1} b_1 c_1 = q^j a^{-1} bc = q^{j+n-2k+1} \notin \mathbf{T}_{n-j}$, so $j > k-1$.

Since $ab^{-1}c = q^{n-2l+1}$, we have $a_1 b_1^{-1} c_1 = q^j ab^{-1} c = q^{j+n-2l+1} \notin \mathbf{T}_{n-j}$, so $j > l-1$.

Therefore, $j > \max\{k-1, l-1\}$.

($\Longleftarrow$) Assume that $j > \max\{k-1, l-1\}$.

Since $abc = q^{n-2j+1}$, we have $a_1 b_1 c_1 = q^{3j} abc = q^{n+j+1} \notin \mathbf{T}_{n-j}$.

Since $a^{-1}bc = q^{n-2k+1}$, we have $a_1^{-1} b_1 c_1 = q^j a^{-1} bc = q^{j+n-2k+1} \notin \mathbf{T}_{n-j}$.

Since $ab^{-1}c = q^{n-2l+1}$, we have $a_1 b_1^{-1} c_1 = q^j ab^{-1} c = q^{j+n-2l+1} \notin \mathbf{T}_{n-j}$.

Since $abc^{-1} = q^{n-2k+1}$, we have $a_1 b_1 c_1^{-1} = q^k a^{-1} bc = q^{n-k+1} \notin \mathbf{T}_{n-j}$.

Therefore V_1 is irreducible if and only if $j > \max\{k-1, l-1\}$.

Similarly, if $j < m$, we can get that V_1 is reducible, V_4 is a submodule of V_1 by (3.11). V_4 is irreducible if and only if $m > \max\{k-1, l-1\}$ by 1).

The remain proof is similar to the above. □

4. Some examples

Keeping all notations as the previous sections. Fixing $a, b, c \in \mathbb{F}$ and assuming that the **Condition A** fail or not, we give some examples to explain various structures of $\Delta_q(\alpha, \beta, \gamma)$-modules.

Example 4.1. *Assume that $V_1(a, b, c)$ has the basis $\left\{v_0^{(1,1)}, v_1^{(1,1)}\right\}$.*

(1) *If we choose a, b, c such that $abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \mathbf{T}_1 = \{1\}$, then $V_1(a, b, c)$ is an irreducible $\Delta_q(\alpha, \beta, \gamma)$-module by Theorem 2.4.*

(2) *If $a = q$, $b = q^{-\frac{1}{2}}$, $c = q^{-\frac{1}{2}}$, then*

$$\begin{aligned}
\omega(a, b, c) &= q^{\frac{5}{2}} + 2q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + 2q^{-\frac{3}{2}} + q^{-\frac{5}{2}},\\
\omega(b, c, a) &= q^3 + 2q + 2 + 2q^{-1} + q^{-3},\\
\omega(c, a, b) &= q^{\frac{5}{2}} + 2q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + 2q^{-\frac{3}{2}} + q^{-\frac{5}{2}}.
\end{aligned}$$

By the tedious calculation, it concludes that $\Delta_q(q^3 + 2q + 2 + 2q^{-1} + q^{-3},\ q^{\frac{5}{2}} + 2q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + 2q^{-\frac{3}{2}} + q^{-\frac{5}{2}},\ q^{\frac{5}{2}} + 2q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + 2q^{-\frac{3}{2}} + q^{-\frac{5}{2}})$ has three 2-dimensional irreducible modules up to isomorphism. We also have $abc = 1 \in \mathbf{T}_1$, $a^{-1}bc = q^{-2} \notin \mathbf{T}_1$, $ab^{-1}c = q \notin \mathbf{T}_1$, $abc^{-1} = q \notin \mathbf{T}_1$. By Theorem 3.2, we have $V_1(q, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$ is reducible and for $V = \mathbb{F}v_1^{(1,1)}$,

(a) V *is a submodule of* $V_1(q, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$ *and* $V \cong V_0(q^2, q^{\frac{1}{2}}, q^{\frac{1}{2}})$;
(b) $V_1(q, q^{-\frac{1}{2}}, q^{-\frac{1}{2}})/V \cong V_0(1, q^{-\frac{3}{2}}, q^{-\frac{3}{2}})$;
(c) V *is irreducible and maximal.*

Example 4.2. *Assume that* $V_3(a, b, c)$ *has two bases* $\left\{v_0^{(1,1)}, v_1^{(1,1)}, v_2^{(1,1)}, v_3^{(1,1)}\right\}$ *and* $\left\{v_0^{(-1,1)}, v_1^{(-1,1)}, v_2^{(-1,1)}, v_3^{(-1,1)}\right\}$.
(1) *If we choose* a, b, c *such that* $abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \mathbf{T}_3 = \left\{q^2, 1, q^{-2}\right\}$*, then* $V_3(a, b, c)$ *is an irreducible* $\Delta_q(\alpha, \beta, \gamma)$*-module by Theorem 2.4.*
(2) *If* $a = q^{\frac{3}{2}}$, $b = q^{-\frac{1}{2}}$, $c = q$*, then*

$$\begin{aligned}\omega(a, b, c) &= q^5 + q^3 + q^2 + q + q^{-1} + q^{-2} + q^{-3} + q^{-5},\\ \omega(b, c, a) &= q^{\frac{11}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{11}{2}},\\ \omega(c, a, b) &= q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}}.\end{aligned}$$

By the tedious calculation, it concludes that $\Delta_q(q^{\frac{11}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{11}{2}},\ q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}},\ q^5 + q^3 + q^2 + q + q^{-1} + q^{-2} + q^{-3} + q^{-5})$ *has four* 4*-dimensional irreducible modules up to isomorphism. We also have* $abc = q^2 \in \mathbf{T}_3$, $a^{-1}bc = q^{-1} \notin \mathbf{T}_3$, $ab^{-1}c = q^3 \notin \mathbf{T}_3$, $abc^{-1} = 1 \in \mathbf{T}_3$*. By Theorem 3.3, we have* $V_3(q^{\frac{3}{2}}, q^{-\frac{1}{2}}, q)$ *is reducible and*

(a) V_1, V_2 *are submodules of* $V_3(q^{\frac{3}{2}}, q^{-\frac{1}{2}}, q)$ *and*

$$V_1 \cong V_2(q^{\frac{5}{2}}, q^{\frac{1}{2}}, q^2), \qquad V_2 \cong V_1(q^{\frac{7}{2}}, q^{\frac{3}{2}}, q^3);$$

(b) V_1 *is reducible and* V_2 *is a submodule of* V_1;
(c) V_2 *is an irreducible module,*

where $V_1 = \mathbb{F}v_1^{(1,1)} + \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)}$ *and* $V_2 = \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)}$.
(3) *If* $a = q$, $b = q^{\frac{1}{2}}$, $c = q^{\frac{1}{2}}$*, then*

$$\begin{aligned}\omega(a, b, c) &= q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}},\\ \omega(b, c, a) &= q^5 + q^3 + q + 2 + q^{-1} + q^{-3} + q^{-5},\\ \omega(c, a, b) &= q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}}.\end{aligned}$$

By the tedious calculation, it concludes that $\Delta_q(q^5 + q^3 + q + 2 + q^{-1} + q^{-3} + q^{-5},\ q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}},\ q^{\frac{9}{2}} + q^{\frac{7}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}} + q^{-\frac{5}{2}} + q^{-\frac{7}{2}} + q^{-\frac{9}{2}})$ *has three* 4*-dimensional irreducible modules up to isomorphism. We also have* $abc = q^2 \in \mathbf{T}_3$, $a^{-1}bc = 1 \in \mathbf{T}_3$, $ab^{-1}c = q \notin \mathbf{T}_3$ *and* $abc^{-1} = q \notin \mathbf{T}_3$*. By Theorem 3.3, we have* $V_3(q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$ *is reducible and*

(a) V_1, V_2 *are submodules of* $V_3(q, q^{\frac{1}{2}}, q^{\frac{1}{2}})$ *and*

$$V_1 \cong V_2(q^2, q^{\frac{3}{2}}, q^{\frac{3}{2}}), \qquad V_2 \cong V_1(q, q^{\frac{5}{2}}, q^{\frac{3}{2}});$$

(b) V_1 *and* V_2 *are reducible,*

where $V_1 = \mathbb{F}v_1^{(1,1)} + \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)}$ *and* $V_2 = \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)}$.

Example 4.3. *Assume that* $V_4(a,b,c)$ *has two bases* $\left\{v_0^{(1,1)}, v_1^{(1,1)}, v_2^{(1,1)}, v_3^{(1,1)}, v_4^{(1,1)}\right\}$ *and* $\left\{v_0^{(-1,1)}, v_1^{(-1,1)}, v_2^{(-1,1)}, v_3^{(-1,1)}, v_4^{(-1,1)}\right\}$.

(1) *If we choose* a,b,c *such that* $abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \mathbf{T}_4 = \left\{q^3, q, q^{-1}, q^{-3}\right\}$, *then* $V_4(a,b,c)$ *is an irreducible* $\Delta_q(\alpha,\beta,\gamma)$*-module by Theorem 2.4.*

(2) *If* $a = q^2$, $b = q^{-2}$, $c = q$, *then*

$$\begin{aligned}\omega(a,b,c) &= q^6 + 2q^4 + 2 + 2q^{-4} + q^{-6},\\ \omega(b,c,a) &= q^7 + 2q^3 + q + q^{-1} + 2q^{-3} + q^{-7},\\ \omega(c,a,b) &= q^7 + 2q^3 + q + q^{-1} + 2q^{-3} + q^{-7}.\end{aligned}$$

By tedious calculation, it concludes that $\Delta_q(q^7+2q^3+q+q^{-1}+2q^{-3}+q^{-7},\ q^7+2q^3+q+q^{-1}+2q^{-3}+q^{-7},\ q^6+2q^4+2+2q^{-4}+q^{-6})$ *has three* 5*-dimensional irreducible module up to isomorphism. We also have* $abc = q \in \mathbf{T}_4$, $a^{-1}bc = q^{-3} \in \mathbf{T}_4$, $ab^{-1}c = q^5 \notin \mathbf{T}_4$, $abc^{-1} = q^{-1} \in \mathbf{T}_4$. *By Theorem 3.4, we have* $V_4(q^2, q^{-2}, q)$ *is reducible and*

(a) V_1, V_2, V_3 *are submodules of* $V_4(q^2, q^{-2}, q)$ *and*

$$V_1 \cong V_2(q^4, 1, q^3), \qquad V_2 \cong V_1(q^5, q, q^4), \qquad V_3 \cong V_0(q^2, q^2, q^3);$$

(b) V_1 *and* V_2 *are reducible,* V_2 *is a submodule of* V_1;

(c) V_3 *is irreducible,*

where $V_1 = \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)} + \mathbb{F}v_4^{(1,1)}$, $V_2 = \mathbb{F}v_3^{(1,1)} + \mathbb{F}v_4^{(1,1)}$ *and* $V_3 = \mathbb{F}v_4^{(-1,1)}$.

Example 4.4. *Assume that* $V_5(a,b,c)$ *has two bases* $\left\{v_0^{(1,1)}, v_1^{(1,1)}, v_2^{(1,1)}, v_3^{(1,1)}, v_4^{(1,1)}, v_5^{(1,1)}\right\}$ *and* $\left\{v_0^{(-1,1)}, v_1^{(-1,1)}, v_2^{(-1,1)}, v_3^{(-1,1)}, v_4^{(-1,1)}, v_5^{(-1,1)}\right\}$.

(1) *If we choose* a,b,c *such that* $abc, a^{-1}bc, ab^{-1}c, abc^{-1} \notin \mathbf{T}_5 = \left\{q^4, q^2, 1, q^{-2}, q^{-4}\right\}$, *then* $V_5(a,b,c)$ *is an irreducible* $\Delta_q(\alpha,\beta,\gamma)$*-module by Theorem 2.4.*

(2) *If* $a = q^2$, $b = q$, $c = q^{-1}$, *then*

$$\begin{aligned}\omega(a,b,c) &= q^7 + q^5 + q^3 + q + q^{-1} + q^{-3} + q^{-5} + q^{-7},\\ \omega(b,c,a) &= q^8 + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-8},\\ \omega(c,a,b) &= q^7 + q^5 + q^3 + q + q^{-1} + q^{-3} + q^{-5} + q^{-7}.\end{aligned}$$

By tedious calculation, it concludes that $\Delta_q(q^8+q^4+q^2+2+q^{-2}+q^{-4}+q^{-8},\ q^7+q^5+q^3+q+q^{-1}+q^{-3}+q^{-5}+q^{-7},\ q^7+q^5+q^3+q+q^{-1}+q^{-3}+q^{-5}+q^{-7})$ *has three* 6*-dimensional irreducible modules up to isomorphism. We also have* $abc = q^2 \in \mathbf{T}_5$, $a^{-1}bc = q^{-2} \in \mathbf{T}_5$, $ab^{-1}c = 1 \in \mathbf{T}_5$, $abc^{-1} = q^4 \in \mathbf{T}_5$. *By Theorem 3.5, we have* $V_5(q^2, q, q^{-1})$ *is reducible and*

(a) V_1, V_2, V_3 *and* V_4 *are submodules of* $V_5(q^2, q, q^{-1})$ *and*

$$\begin{aligned}V_1 &\cong V_3(q^4, q^3, q), & V_2 &\cong V_1(q^2, q^5, q^5),\\ V_3 &\cong V_2(q, q^4, q^4), & V_4 &\cong V_4(q^3, q^2, 1);\end{aligned}$$

(b) V_1 *and* V_4 *are reducible,* V_1 *is a submodule of* V_4;

(c) V_3 *is reducible,* V_2 *is a submodule of* V_3 *and* V_2 *is irreducible,*

where $V_1 = \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)} + \mathbb{F}v_4^{(1,1)} + \mathbb{F}v_5^{(1,1)}$, $V_2 = \mathbb{F}v_4^{(-1,1)} + \mathbb{F}v_5^{(-1,1)}$, $V_3 = \mathbb{F}v_3^{(-1,1)} + \mathbb{F}v_4^{(-1,1)} + \mathbb{F}v_5^{(-1,1)}$ *and* $V_4 = \mathbb{F}v_1^{(1,1)} + \mathbb{F}v_2^{(1,1)} + \mathbb{F}v_3^{(1,1)} + \mathbb{F}v_4^{(1,1)} + \mathbb{F}v_5^{(1,1)}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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