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Research article

Relation-theoretic almost ϕ-contractions with an application to elastic beam equations

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Abstract: In this article, we prove some results on existence and uniqueness of fixed points for an almost ϕ-contraction mapping defined on a metric space endowed with an amorphous relation. Our results generalize and improve several well known fixed point theorems of the existing literature. To substantiate the credibility of our results, we construct some examples. We also apply our results to determine a unique solution of a boundary value problem associated with nonlinear elastic beam equations.

Keywords: almost contractions; boundary value problems; fixed points; binary relations
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1. Introduction

Various generalizations of Banach contraction principle (abbreviated as: BCP) are established in the literature by improving the underlying contraction condition. In 1968, Browder [1] introduced the concept ϕ-contractions, whereas an appropriate mapping, say $\phi : [0,\infty) \to [0,\infty)$, is employed instead of the Lipschitzian constant $k \in [0, 1)$.

Definition 1.1. *[1] A function $\mathcal{J}$ from a metric space $(\mathbb{P}, \varrho)$ into itself is called ϕ-contraction if $\exists$ a function $\phi : [0,\infty) \to [0,\infty)$ verifying*

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq \phi(\varrho(p, q)), \quad \forall\ p, q \in \mathbb{P}. \tag{1.1}$$

Afterward, Boyd and Wong [2] and Matkowski [3] extended Browder fixed point theorem on improving the properties of auxiliary function ϕ.

Definition 1.2. *[4] A test function $\phi : [0,\infty) \to [0,\infty)$ is termed as (c)-comparison function if*

(i) *ϕ is monotonically increasing,*

(ii) $\sum\limits_{i=1}^{\infty} \phi^i(a) < \infty, \quad \forall\ a > 0.$

Remark 1.1. *[4] Any (c)-comparison function ϕ possesses the following properties:*

(i) $\phi(a) < a,\ \forall\ a > 0$,
(ii) $\phi(0) = 0$,
(iii) ϕ *remains right continuous at* 0.

In the following lines, we indicate a natural extension of BCP to ϕ-contractions, which is a particular case of Matkowski fixed point theorem [3].

Theorem 1.1. *Given a (c)-comparison function ϕ, the ϕ-contraction map on a complete metric space possesses a unique fixed point.*

The concept of almost contraction is an another noted variant of contraction mapping, which was introduced by Berinde [5] and developed by many researchers, e.g., [6–9].

Definition 1.3. *[5,6] A function $\mathcal{J}$ from a metric space $(\mathbb{P}, \varrho)$ into itself is termed as almost contraction if $\exists\ 0 < k < 1$ and $L \geq 0$ such that*

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq k\varrho(p, q) + L\varrho(p, \mathcal{J}q), \quad \forall\ p, q \in \mathbb{P}. \tag{1.2}$$

Berinde [5] also introduced a dual to the contractivity condition (1.2) as follows:

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq k\varrho(p, q) + L\varrho(q, \mathcal{J}p), \quad \forall\ p, q \in \mathbb{P}. \tag{1.3}$$

Owing to symmetry of the metric ϱ, both the contractivity conditions (1.2) and (1.3) are equivalent (c.f. [5]).

Theorem 1.2. *[5] Every almost contraction on a complete metric space possesses a fixed point.*

On making slight modification in Definition 1.3, Berinde [5] established the following corresponding uniqueness theorem.

Theorem 1.3. *[5] Assume that $(\mathbb{P}, \varrho)$ remains a complete metric space and $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ remains a map. If $\exists\ 0 < k < 1$ and $L \geq 0$ such that*

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq k\varrho(p, q) + L\varrho(p, \mathcal{J}p), \quad \forall\ p, q \in \mathbb{P}.$$

Then $\mathcal{J}$ admits a unique fixed point.

In the recent past, various fixed point results have been investigated for relation-theoretic contractions, which are indeed desirable to hold for those elements that remain related via the underlying relation defined on the metric space. This idea is initiated by Alam and Imdad [10] in 2015 and generalized by various authors, e.g., [11–16] and references therein. Such results are utilized to solve ordinary differential equations, matrix equations, integral equations, nonlinear elliptic problems, elastic beam equations and fractional differential equations satisfying certain auxiliary conditions, e.g., [17–25]. Alam and Imdad [11] and Arif et al. [12], respectively, established the relational analogues of Boyd-Wong fixed point theorem [2] and Matkowski fixed point theorem [3], wherein authors [11, 12] employed a class of transitive binary relation. Very recently, Algehyne et al. [14] generalized Theorem 1.1 in the framework of relational metric space. On the other hand, Khan [15] proved some metrical fixed point results under almost contractions employing an amorphous relation.

Fourth-order two-point BVP (abbreviation of "boundary value problems") have the importance in engineering and science. Such BVP describe the deflection of an elastic beam in an equilibrium state and have applications in the fields of physics, material mechanics, micro-electromechanical systems, aircraft design, chemical sensors and medical diagnostics. To describe the existence of positive solutions of the nonlinear elastic beam equations, many authors employed the monotone iterative methods. In this paper, one considers a special case of the BVP associated with elastic beam equation:

$$\begin{cases} \vartheta''''(s) = h(s, \vartheta(s)), \quad 0 \leq s \leq 1, \\ \vartheta(0) = \vartheta'(0) = \vartheta''(1) = \vartheta'''(1) = 0 \end{cases} \tag{1.4}$$

where $h : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function. Equation (1.4) characterizes the bending equilibria of an elastic beam model of length 1, whose both ends are rigidly fixed. Such a beam is called cantilever beam in material mechanics.

A more general elastic beam equation can be represented by

$$\begin{cases} \vartheta''''(s) = h(s, \vartheta(s), \vartheta'(s), \vartheta''(s), \vartheta'''(s)), \quad 0 \leq s \leq 1, \\ \vartheta(0) = \vartheta'(0) = \vartheta''(1) = \vartheta'''(1) = 0 \end{cases} \tag{1.5}$$

where $h : [0, 1] \times [0, \infty)^3 \to [0, \infty)$ is a continuous function. In this equation, ϑ represents the load density stiffness, $\vartheta'''(s)$ represents the shear force stiffness, $\vartheta''(s)$ represents the bending moment stiffness and $\vartheta'(s)$ is the slope.

On the other hand, an elastic beam equation along with nonlinear boundary conditions can be described by

$$\begin{cases} \vartheta''''(s) = h(s, \vartheta(s), \vartheta'(s), \vartheta''(s), \vartheta'''(s)), \quad 0 \leq s \leq 1, \\ \vartheta(0) = \vartheta'(0) = \vartheta''(1) = \vartheta'''(1) = g(\vartheta(1)). \end{cases} \tag{1.6}$$

In such a beam the left end is fixed while the right end is attached to an elastic bearing device, represented by the function $g : [0, \infty) \to [0, \infty)$.

In this paper, we prove a new metrical fixed point results under almost ϕ-contractions (whereas ϕ remains a (c)-comparison function) employing an arbitrary relation. The idea utilized in our main results combines those involved in [14, 15]. In support of our results, we furnish two illustrative examples. Using our results, we discuss the existence of a unique solution for the BVP represented by Eq (1.4).

2. Preliminaries

As usual, $\mathbb{N}$ and $\mathbb{R}$ will stand respectively for the set of natural numbers and that of real numbers. A binary relation (or simply, a relation) $\mathfrak{S}$ on a given set $\mathbb{P}$ is a subset of cartesian product $\mathbb{P}^2$. In the sequel, $\mathbb{P}$ remains a set, ϱ is a metric on $\mathbb{P}$, $\mathfrak{S}$ remains a relation on $\mathbb{P}$ and $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ is a mapping.

Definition 2.1. *[10] A pair $p, q \in \mathbb{P}$ is said to be $\mathfrak{S}$-comparative (offten denoted by $[p, q] \in \mathfrak{S}$) if*

$$(p, q) \in \mathfrak{S} \quad or \quad (q, p) \in \mathfrak{S}.$$

Definition 2.2. *[26] $\mathfrak{S}^{-1} := \{(p, q) \in \mathbb{P}^2 : (q, p) \in \mathfrak{S}\}$ is said to be transpose of $\mathfrak{S}$.*

Definition 2.3. *[26] The symmetric closure of $\mathfrak{S}$ is a relation $\mathfrak{S}^s := \mathfrak{S} \cup \mathfrak{S}^{-1}$.*

Proposition 2.1. *[10] $(p, q) \in \mathfrak{S}^s \Longleftrightarrow [p, q] \in \mathfrak{S}$.*

Definition 2.4. *[27] On a subset $\mathbb{Q} \subseteq \mathbb{P}$, $\mathfrak{S}$ induces a relation:*

$$\mathfrak{S}|_{\mathbb{Q}} := \mathfrak{S} \cap \mathbb{Q}^2,$$

which is known as the restriction of $\mathfrak{S}$ on $\mathbb{Q}$.

Definition 2.5. *[10] $\mathfrak{S}$ is called $\mathcal{J}$-closed if it satisfies*

$$(\mathcal{J}p, \mathcal{J}q) \in \mathfrak{S},$$

$\forall\ p, q \in \mathbb{P}$ verifying $(p, q) \in \mathfrak{S}$.

Proposition 2.2. *[12] $\mathfrak{S}$ is $\mathcal{J}^n$-closed provided $\mathfrak{S}$ remains $\mathcal{J}$-closed.*

Definition 2.6. *[10] A sequence $\{p_n\} \subset \mathbb{P}$ verifying $(p_n, p_{n+1}) \in \mathfrak{S}$, $\forall\ n \in \mathbb{N}$, is said to be $\mathfrak{S}$-preserving.*

Definition 2.7. *[11] One says that $(\mathbb{P}, \varrho)$ is $\mathfrak{S}$-complete if every $\mathfrak{S}$-preserving Cauchy sequence in $\mathbb{P}$ remains convergent.*

Definition 2.8. *[11] $\mathcal{J}$ is termed as $\mathfrak{S}$-continuous at $p \in \mathbb{P}$ if whenever $\{p_n\} \subset \mathbb{P}$ is a $\mathfrak{S}$-preserving sequence satisfying $p_n \xrightarrow{\varrho} p$, then one has*

$$\mathcal{J}(p_n) \xrightarrow{\varrho} \mathcal{J}(p).$$

A function, which is $\mathfrak{S}$-continuous at every point of $\mathbb{P}$, is termed as $\mathfrak{S}$-continuous.

Definition 2.9. *[10] $\mathfrak{S}$ is referred as ϱ-self-closed if for every $\mathfrak{S}$-preserving convergent sequence in $\mathbb{P}$, $\exists$ a subsequence each term of which is $\mathfrak{S}$-comparative with the convergence limit.*

In our subsequent discussions, we will adopt the following notations.

- $F(\mathcal{J})$:=the set of fixed points of $\mathcal{J}$;
- $\mathbb{P}(\mathcal{J}, \mathfrak{S}) := \{p \in \mathbb{P} : (p, \mathcal{J}p) \in \mathfrak{S}\}$.

Proposition 2.3. *Given a (c)-comparison function ϕ and $L \geq 0$, then the following assumptions are equivalent:*

(I) $\varrho(\mathcal{J}p, \mathcal{J}q) \leq \phi(\varrho(p, q)) + L\varrho(q, \mathcal{J}p),\ \forall\ p, q \in \mathbb{P}$ *with* $(p, q) \in \mathfrak{S}$,
(II) $\varrho(\mathcal{J}p, \mathcal{J}q) \leq \phi(\varrho(p, q)) + L\varrho(q, \mathcal{J}p),\ \forall\ p, q \in \mathbb{P}$ *with* $[p, q] \in \mathfrak{S}$.

Above result follows proceeding the lines of Proposition 3 [14].

3. Main results

We present the following result on existence of fixed point under almost ϕ-contraction.

Theorem 3.1. *Assume that $\mathbb{P}$ remains a set endowed with a relation $\mathfrak{S}$ and a metric ϱ while $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ remains a function. Also,*

(i) $(\mathbb{P}, \varrho)$ *remains $\mathfrak{S}$-complete metric space,*
(ii) $\mathbb{P}(\mathcal{J}, \mathfrak{S}) \neq \emptyset$,
(iii) $\mathfrak{S}$ *is $\mathcal{J}$-closed,*
(iv) $\mathcal{J}$ *remains $\mathfrak{S}$-continuous or $\mathfrak{S}$ remains ϱ-self-closed,*
(v) $\exists$ *a (c)-comparison function ϕ and a constant $L \geq 0$ satisfying*

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq \phi(\varrho(p, q)) + L\varrho(q, \mathcal{J}p), \ \forall\ p, q \in \mathbb{P} \text{ with } (p, q) \in \mathfrak{S}.$$

Then, $\mathcal{J}$ possesses a fixed point.

Proof. Using assumption (ii), take $p_0 \in \mathbb{P}(\mathcal{J}, \mathfrak{S})$ and construct the sequence $\{p_n\} \subset \mathbb{P}$ verifying

$$p_n = \mathcal{J}^n(p_0) = \mathcal{J}(p_{n-1}), \ \forall\ n \in \mathbb{N}. \tag{3.1}$$

As $(p_0, \mathcal{J}p_0) \in \mathfrak{S}$, by $\mathcal{J}$-closedness of $\mathfrak{S}$ (along with Proposition 2.2), one finds

$$(\mathcal{J}^n r_0, \mathcal{J}^{n+1} p_0) \in \mathfrak{S}$$

which making use of (3.1) becomes

$$(p_n, p_{n+1}) \in \mathfrak{S}, \quad \forall\ n \in \mathbb{N}. \tag{3.2}$$

Therefore, $\{p_n\}$ remains an $\mathfrak{S}$-preserving sequence.
Denote $\varrho_n := \varrho(p_n, p_{n+1})$. Making use of assumption (v) along with (3.1) and (3.2), we get

$$\begin{aligned} \varrho_n &= \varrho(p_n, p_{n+1}) = \varrho(\mathcal{J}p_{n-1}, \mathcal{J}p_n) \\ &\leq \phi(\varrho(p_{n-1}, p_n)) + L\varrho(p_n, \mathcal{J}p_{n-1}), \quad n \in \mathbb{N}_0 \end{aligned}$$

which in view of (3.1) reduces to

$$\varrho_n \leq \phi(\varrho_{n-1}). \tag{3.3}$$

By induction, (3.3) reduces to

$$\varrho_n \leq \phi(\varrho_{n-1}) \leq \phi^2(\varrho_{n-2}) \leq \cdots \leq \phi^n(\varrho_0)$$

so that

$$\varrho_n \leq \phi^n(\varrho_0), \quad \forall\ n \in \mathbb{N}. \tag{3.4}$$

Making use of (3.4), $\forall\, m, n \in \mathbb{N}$ with $m < n$, one gets

$$\begin{aligned}\varrho(p_m, p_n) &\leq \varrho(p_m, p_{m+1}) + \varrho(p_{m+1}, p_{m+2}) + \cdots + \varrho(p_{n-1}, p_n)\\ &\leq \phi^m(\varrho_0) + \phi^{m+1}(\varrho_0) + \cdots + \phi^{n-1}(\varrho_0)\\ &= \sum_{i=m}^{n-1} \phi^i(\varrho_0) \leq \sum_{i\geq m} \phi^i(\varrho_0)\\ &\to 0 \text{ as } m \text{ (and hence } n) \to \infty.\end{aligned}$$

This implies that $\{p_n\}$ is Cauchy. Therefore, $\{p_n\}$ remains an $\mathfrak{S}$-preserving Cauchy sequence and hence the $\mathfrak{S}$-completeness of $\mathbb{P}$ guarantees the existence of $p \in \mathbb{P}$ satisfying $p_n \xrightarrow{\varrho} p$.

Now, we shall use assumption (iv) to show that $p \in F(\mathcal{J})$. If $\mathcal{J}$ is $\mathfrak{S}$-continuous, then one has $p_{n+1} = \mathcal{J}(p_n) \xrightarrow{\varrho} \mathcal{J}(p)$. Using uniqueness of limit, one obtains $\mathcal{J}(p) = p$. Alternately, when $\mathfrak{S}$ is ϱ-self-closed, $\exists$ a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with $[p_{n_k}, p] \in \mathfrak{S}, \quad \forall\, k \in \mathbb{N}$. Making use of Proposition 2.3, $[p_{n_k}, p] \in \mathfrak{S}$ and condition (v), we get

$$\begin{aligned}\varrho(p_{n_k+1}, \mathcal{J}p) &= \varrho(\mathcal{J}p_{n_k}, \mathcal{J}p)\\ &\leq \phi(\varrho(p_{n_k}, p)) + L\varrho(p, \mathcal{J}\vartheta_{n_k})\\ &= \phi(\varrho(p_{n_k}, p)) + L\varrho(p, p_{n_k+1}).\end{aligned}$$

Employing (i) and (ii) of Remark 1.1 and $p_{n_k} \xrightarrow{\varrho} p$, we get

$$\begin{aligned}\varrho(p_{n_k+1}, \mathcal{J}p) &\leq \varrho(p_{n_k}, p) + L\varrho(p, p_{n_k+1})\\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

implying thereby $p_{n_k+1} \xrightarrow{\varrho} \mathcal{J}(p)$. Using uniqueness of limit again, one obtains $\mathcal{J}(p) = p$. Therefore, in each of the cases, p remains a fixed point of $\mathcal{J}$. □

On making a slight modification in contraction condition together with an additional hypothesis, the following uniqueness result can be presented.

Theorem 3.2. *Along with the assumptions* (i)-(iv) *of Theorem 3.1, if the following conditions hold:*

(vi) $\exists$ *a (c)-comparison function* ϕ *and a constant* $L \geq 0$ *satisfying*

$$\varrho(\mathcal{J}p, \mathcal{J}q) \leq \phi(\varrho(p, q)) + L\varrho(p, \mathcal{J}p),\ \forall\, p, q \in \mathbb{P} \textit{ with } (p, q) \in \mathfrak{S}$$

(vii) $\mathfrak{S}|_{\mathcal{J}(\mathbb{P})}$ *remains complete,*

then $\mathcal{J}$ *admits a unique fixed point.*

Proof. Similar to Theorem 3.1, one can find that $F(\mathcal{J}) \neq \emptyset$. Take $p, q \in F(\mathcal{J})$, i.e.,

$$\mathcal{J}(p) = p \text{ and } \mathcal{J}(q) = q. \tag{3.5}$$

On contrary, assume that $p \neq q$. Clearly $p, q \in \mathcal{J}(\mathbb{P})$. Therefore, by assumption (vii), one gets

$$[p, q] \in \mathfrak{S}. \tag{3.6}$$

Making use of (3.5), (3.6) and assumption (vi), one gets

$$\begin{aligned} \varrho(p,q) &= \varrho(\mathcal{J}p,\mathcal{J}q) \\ &\leq \phi(\varrho(p,q)) + L\varrho(p,\mathcal{J}p) = \phi(\varrho(p,q)) \end{aligned}$$

which by using item (i) of Remark 1.1 gives rise to

$$\varrho(p,q) \leq \phi(\varrho(p,q)) < \varrho(p,q)$$

which remains a contraction. Hence, $p = q$. □

Notice that Theorems 3.1 and 3.2 generalized, unified, sharpened and improved several existing results in the following respects:

- On setting, $\phi(a) = ka$, $0 \leq k < 1$, one deduces the results of Khan [15].
- By putting $L = 0$, one deduces the results of Algehyne et al. [14].
- Particularly for $\phi(a) = ka$, $0 \leq k < 1$ and $L = 0$, our results reduce to relation-theoretic contraction principle of Alam and Imdad [10].
- Under universal relation $\mathfrak{S} = \mathbb{P}^2$ and for $L = 0$, our results reduce to Theorem 1.1.
- Under universal relation $\mathfrak{S} = \mathbb{P}^2$ and for $\phi(a) = ka$, $0 \leq k < 1$, Theorems 3.1 and 3.2 respectively reduce to Theorems 1.2 and 1.3.

4. Examples

This section is comprised of two illustrative examples, which demonstrate the importance of Theorems 3.1 and 3.2.

Example 4.1. *Let $\mathbb{P} = [0,1]$ with the metric $\varrho(p,q) = |p-q|$ and the natural ordering $\mathfrak{S} =\geq$ as a relation on it. Then, $(\mathbb{P},\varrho)$ remains $\mathfrak{S}$-complete metric space. Define a function $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ by*

$$\mathcal{J}(p) = \begin{cases} 0, & \text{if } p = 1 \\ 1/3, & \text{otherwise.} \end{cases}$$

Then, $\mathcal{J}$ remains $\mathfrak{S}$-continuous and $\mathfrak{S}$ is $\mathcal{J}$-closed. Define a (c)-comparison function $\phi(a) = a/3$. Then for any $L \geq 1/3$ and $\forall\ p,q \in \mathbb{P}$ satisfying $(p,q) \in \mathfrak{S}$, one has

$$\varrho(\mathcal{J}p,\mathcal{J}q) \leq \phi(\varrho(p,q)) + L\varrho(q,\mathcal{J}p),$$

so that contraction condition (v) *of Theorem 3.1 is verified. Therefore, all the hypotheses of Theorem 3.1 are satisfied. Consequently, $\mathcal{J}$ admits a fixed point in $\mathbb{P}$. Moreover, all the hypotheses of Theorem 3.2 also hold. Here, $\mathcal{J}$ possesses a unique fixed point, namely: $p = 1/3$.*

Example 4.2. *Consider $\mathbb{P} = (0,1]$ with the metric $\varrho(p,q) = |p-q|$. On $\mathbb{P}$, take a relation $\mathfrak{S} = \{(p,q) : 1/4 \leq p \leq q \leq 1/3 \text{ or } 1/2 \leq p \leq q \leq 1\}$. Then $(\mathbb{P},\varrho)$ is $\mathfrak{S}$-complete metric space. Define a function $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ by*

$$\mathcal{J}(p) = \begin{cases} 1/4, & \text{if } 0 \leq p < 1/2 \\ 1, & \text{if } 1/2 \leq p \leq 1. \end{cases}$$

Then, $\mathcal{J}$ remains $\mathfrak{S}$-continuous and $\mathfrak{S}$ is $\mathcal{J}$-closed. The contraction condition (v) *of Theorem 3.1 is satisfied for any arbitrary (c)-comparison function ϕ and for an arbitrary constant $L \geq 0$. Thus, in view of Theorem 3.1, $\mathcal{J}$ admits a fixed point. As the elements* $1/4$ *and* 1 *are not $\mathfrak{S}$-comparative, $\mathfrak{S}|_{\mathcal{J}(\mathbb{P})}$ is not complete. Consequently, Theorem 3.2 cannot be applied for the present example. Here, $p = 1/4$ and $p = 1$ remain two fixed points of $\mathcal{J}$.*

5. Solutions of nonlinear elastic beam equations

In what follows, C[0,1] will denote the class of continuous real functions on $[0, 1]$. In the following lines, one shall prove the existence and uniqueness theorem to find a solution of the BVP (1.4).

Theorem 5.1. *Along with the problem* (1.4)*, suppose that $\exists$ a (c)-comparison function ϕ satisfying*

$$0 \leq h(s, \alpha) - h(s, \beta) \leq \phi(\alpha - \beta), \tag{5.1}$$

$\forall\ s \in [0, 1]$ *and* $\forall\ \alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$. *Also, assume that* $\exists\ \lambda \in C[0, 1]$ *verifying*

$$\lambda(s) \geq \int_0^1 \Omega(s, \xi) h(\xi, \lambda(\xi)) d\xi, \quad \forall\ s \in [0, 1], \tag{5.2}$$

where Green function $\Omega(s, \xi)$ *is defined as*

$$\Omega(s, \xi) = \frac{1}{6} \begin{cases} \xi^2(3s - \xi), & 0 \leq \xi \leq s \leq 1, \\ s^2(3\xi - s), & 0 \leq s \leq \xi \leq 1. \end{cases} \tag{5.3}$$

Then the BVP (1.4) *admits a unique solution.*

Proof. BVP (1.4) can be transformed to the integral equation:

$$\vartheta(s) = \int_0^1 \Omega(s, \xi) h(\xi, \vartheta(\xi)) d\xi, \quad \forall\ s \in [0, 1], \tag{5.4}$$

It is straightforward to verify

$$0 \leq \Omega(s, \xi) \leq \frac{1}{2} s^2 \xi, \quad \forall\ s, \xi \in [0, 1]. \tag{5.5}$$

Take a metric ϱ on $\mathbb{P}$:=C[0,1] as

$$\varrho(\mu, \upsilon) = \max_{s \in [0,1]} |\mu(s) - \upsilon(s)|, \quad \forall\ \mu, \upsilon \in \mathbb{P},$$

Also, consider a relation $\mathfrak{S}$ on $\mathbb{P}$:=C[0,1] as

$$(\mu, \upsilon) \in \mathfrak{S} \Leftrightarrow \mu(s) \geq \upsilon(s), \quad \forall\ \mu, \upsilon \in \mathbb{P}, \ \forall\ s \in [0, 1].$$

It can be easily proved that $(\mathbb{P}, \varrho)$ is $\mathfrak{S}$-complete and $\mathfrak{S}$ is ϱ-self-closed.

Define a function $\mathcal{J} : \mathbb{P} \to \mathbb{P}$ by

$$\mathcal{J}(\mu)(s) = \int_0^1 \Omega(s,\xi)h(\xi,\mu(\xi))d\xi, \ \ \forall\, s \in [0,1], \ \forall\, \mu \in \mathbb{P}.$$

With the help of (5.1), it can be verified that $\mathcal{J}$ remains $\mathfrak{S}$-closed and $\mathfrak{S}|_{\mathcal{J}(\mathbb{P})}$ remains complete.

Owing to (5.2), one finds $\lambda(s) \geq \mathcal{J}(\lambda)(s)$ so that $\lambda \in \mathbb{P}(\mathcal{J},\mathfrak{S})$. Using (5.1), $\forall\, s \in [0,1]$ and $\forall\, \mu, \upsilon \in \mathbb{P}$ verifying $(\mu,\upsilon) \in \mathfrak{S}$, one has

$$\begin{aligned}
|\mathcal{J}(\mu)(s) - \mathcal{J}(\upsilon)(s)| &= \int_0^1 \Omega(s,\xi)(h(\xi,\mu(\xi)) - h(\xi,\upsilon(\xi)))d\xi \\
&\leq \int_0^1 \Omega(s,\xi)\phi(\mu(\xi) - \upsilon(\xi))d\xi \\
&\leq \left(\int_0^1 \Omega(s,\xi)d\xi\right)(\phi(\varrho(\mu,\upsilon)) \qquad \text{(as } \phi \text{ is increasing)} \\
&\leq \frac{\phi(\varrho(\mu,\upsilon))}{4} \qquad \text{(using (5.5))} \\
&\leq \phi(\varrho(\mu,\upsilon)) \\
&\leq \phi(\varrho(\mu,\upsilon)) + L\varrho(\mu,\mathcal{J}\mu) \qquad \text{(where } L \geq 0 \text{ is arbitrary)}
\end{aligned}$$

so that

$$\varrho(\mathcal{J}\mu,\mathcal{J}\upsilon) \leq \phi(\varrho(\mu,\upsilon)) + L\varrho(\mu,\mathcal{J}\mu).$$

□

Hence, all the conditions of Theorem 3.2 have been verified. Consequently, $\exists$ a unique $\overline{\vartheta} \in C([0,1])$ verifying $\mathcal{J}(\overline{\vartheta}) = \overline{\vartheta}$, which remains the unique solution of (5.4) and hence a solution of (1.4).

6. Conclusions

The results proved in this manuscript involved a weaker contraction condition, which holds for the comparative elements only. An application of our results is given to nonlinear cantilever beam incorporating some Euler-Bernoulli hypotheses, which shows the physical significance of our results. Some other physical situations can be modeled by special types of delayed hematopoiesis models, fractional differential equations and nonlinear elliptic problems. Thus far the results proved herein and the similar other results in future works are utilized in such physical phenomenon. As future works, our results can be extended to almost ϕ-contractions in the senses of Boyd and Wong [2] and Matkowski [3].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest.

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