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Research article

Orlicz estimates for parabolic Schrödinger operators with non-negative potentials on nilpotent Lie groups

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Abstract: In this paper, we study the Orlicz estimates for the parabolic Schrödinger operator

$$L = \partial_t - \Delta_X + V,$$

where the nonnegative potential V belongs to a reverse Hölder class on nilpotent Lie groups $\mathbb{G}$ and Δ_X is the sub-Laplace operator on $\mathbb{G}$. Under appropriate growth conditions of the Young function, we obtain the regularity estimates of the operator L in the Orlicz space by using the domain decomposition method. Our results generalize some existing ones of the L^p estimates.

Keywords: nilpotent Lie group; Orlicz space; parabolic Schrödinger operator; non-negative potential; domain decomposition method
Mathematics Subject Classification: 35J10, 46E30, 49N60

1. Introduction

Orlicz spaces have been widely studied since they were introduced by Orlicz [1], see for instance Orlicz space [2, 3], weighted Orlicz space [4] and fractional order Orlicz space [5]. The regularity estimates of operators play an important role in various fields of analysis. Several boundedness properties of integral operators are applied to the study of the regularity of elliptic and parabolic equations with discontinuous coefficients [6, 7]. Moreover, Abdalmonem and Scapellato [8] have considered some Morrey-Herz spaces with variable exponents and have examined some boundedness properties of intrinsic square functions and their commutators in this framework. Additionally, in [9], Deringoz et al. studied the Calderon-Zygmund operators and their commutators on generalized weighted Orlicz-Morrey spaces. Recently, Orlicz estimates for some operators have also been obtained [10–12].

The analytical study of nonlinear partial differential equations is one of the most interesting fields of research for many researchers in recent years [13–18]. In this paper, we are interested in the regularity estimates in Orlicz spaces for the parabolic Schrödinger operator with non-negative potentials on nilpotent Lie groups. In Euclidean space, many scholars have obtained many regularity estimates for elliptic and parabolic Schrödinger operators, such as L^p estimates [19–21] and Orlicz estimates [22–24]. Li [25] obtained the L^p estimates for the Schrödinger operators on nilpotent Lie groups, which generalizes the results in Euclidean space [20]. Liu, Huang and Dong [26] obtained the L^p estimates for the parabolic Schrödinger operators on nilpotent Lie groups. Yang and Li [27] established the boundedness and compactness of commutators related with Schrödinger operators on Heisenberg groups.

Acerbi and Mingione [28] proposed a new domain decomposition approach, and applied it to study the local Sobolev estimates for the degenerate parabolic p-Laplacian systems. Wang, Yao, Zhou and Jia [29] encoded and simplified the iteration-covering procedure used in [28] and extended it to the whole space, and obtained the Orlicz estimates for the Poisson equation. Moreover, Yao [22, 23] improved the method in [28, 29], and obtained the Orlicz estimates for the Schrödinger operators in Euclidean space. We will extend the method of [22, 23] to the nilpotent Lie groups, and apply it to obtain the Orlicz estimates for the parabolic Schrödinger operator with non-negative potentials on nilpotent Lie groups, which extend the L^p estimates in [26].

Let $\mathbb{G}$ be a simply connected nilpotent Lie group, and the corresponding Lie algebra is g. Assume $X = \{X_1, ..., X_m\}$ is a Hörmander system of left invariant vector fields on $\mathbb{G}$. In this paper, we consider the following parabolic Schrödinger operators on $\mathbb{G}$

$$Lu(z) = \partial_t u(z) - \Delta_X u(z) + V(x)u(z),\ z = (x, t) \in \mathbb{G} \times (0, +\infty), \tag{1.1}$$

where $\Delta_X = \sum_{i=1}^m X_i^2$ is the sub-Laplacian on $\mathbb{G}$, and the potential $V(x)$ belongs to the reverse Hölder class on $\mathbb{G}$: $V(x) \in RH_q$ $(1 < q < +\infty)$ if $V(x) \in L^q_{loc}(\mathbb{G})$, $V(x) > 0$ almost everywhere, and there is a positive constant C such that for all metric balls of $\mathbb{G}$,

$$\left(|B_r|^{-1} \int_{B_r} V^q(x)dx\right)^{1/q} \leqslant C\left(|B_r|^{-1} \int_{B_r} V(x)dx\right). \tag{1.2}$$

The smallest constant C that makes (1.2) true is called the RH_q constant of V. If $q = +\infty$, then the left-hand side of (1.2) is essential supremum of V on B_r, i.e.,

$$\sup_{B_r} V(x) \leqslant C\left(|B_r|^{-1} \int_{B_r} V(x)dx\right). \tag{1.3}$$

It's clear that $V \in RH_\infty$ implies that $V \in RH_q$ for $1 < q < +\infty$.

Inspired by [22, 23] and based on the L^p estimate in [26], we will study the Orlicz estimates for the operator (1.1) on nilpotent Lie groups. When $V \in RH_q$ $(1 < q < +\infty)$ according to the method in [22, 23], it is difficult to obtain a result similar to (3.7) in [22] or (2.24) in [23]. Therefore, we need to improve the domain decomposition method and the measure estimation of level sets in [22, 23].

The main results of this paper are as follows.

Theorem 1. *Assume $\phi \in \Delta_2 \cap \nabla_2$ and $V \in RH_q$, $q > \max\{D/2, \alpha_1\}$. Then for any $u \in C_0^\infty(\mathbb{G} \times (0, +\infty))$, we have $Vu \in L^\phi(\mathbb{G} \times (0, +\infty))$ and*

$$\int_{\mathbb{G}\times(0,+\infty)} \phi(|Vu|)dz \leqslant c \int_{\mathbb{G}\times(0,+\infty)} \phi(|Lu|)dz, \tag{1.4}$$

where the positive constant c is independent of V and u. See Section 2 for the young function ϕ, the global Δ_2 condition, the global ∇_2 condition, the constant α_1 and the dimension D.

The proof of Theorem 1 is based on the following local estimate.

Theorem 2. *Suppose that $V \in RH_q$, $q > \frac{D}{2}$. If $h(x,t)$ satisfies $\partial_t h - \Delta_X h + Vh = 0$ in $Q_{4r}(z_0)$, then there is a constant $c > 0$ independent of $V,\ h,\ r,\ z_0$ such that*

$$\sup_{z\in Q_r(z_0)} |h(z)| \leqslant \frac{c}{r^2 V(B_{4r}(x_0))} \int_{Q_{4r}(z_0)} V\,|h|\,dz, \tag{1.5}$$

where $z_0 = (x_0,t_0) \in \mathbb{G}\times(0,+\infty)$, $Q_r(z_0)$ is the parabolic cylinders in $\mathbb{G}\times(0,+\infty)$, and $B_r(x_0)$ is the metric ball of center at x_0 and radius r in $\mathbb{G}$ (see Section 2), $V(B_{4r}(x_0)) = \int_{B_{4r}(x_0)} V(x)dx$.

Using Theorem 1 and the approximation method, it immediately gets the following corollary.

Corollary 3. *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and $V \in RH_q,\ q > \max\{D/2, \alpha_1\}$. Then for any $u \in W^{1,2}_{\phi,0}(\mathbb{G}\times(0,+\infty))$, there exists is a positive constant c independent of V and u such that*

$$\int_{\mathbb{G}\times(0,+\infty)} \phi(|Vu|)dxdt \leqslant c \int_{\mathbb{G}\times(0,+\infty)} \phi(|Lu|)dxdt. \tag{1.6}$$

See Definition 11 for the definition of $W^{1,2}_{\phi,0}(\mathbb{G}\times(0,+\infty))$.

Remark 4. *Theorem 1 generalizes the L^p estimate for the Schrödinger operator in [26] to the Orlicz estimate. In fact, due to $t^p \in \Delta_2\cap\nabla_2,\ t>0,\ p>1$, letting $\phi(t) = t^p,\ t>0,\ p>1$, we immediately obtain the L^p estimates for (1.1) which have been obtained in [26] by the pointwise estimate for the heat kernel of Schrödinger operators. Here we generalizes the condition $V \in RH_\infty$ to the condition $V \in RH_q$, which is different from [22, 23].*

This paper is organized as follows. In Section 2, we introduce the definitions and related conclusions of nilpotent Lie groups and Orlicz spaces. In Section 3, inspired by [22, 23], the level set $E_\lambda(1)$ (see (3.5)) is decomposed into a family of disjoint parabolic cylinders by using the covering lemma in homogeneous spaces (see Lemma 14). We prove Theorem 1, Theorem 2 and Corollary 3 in Section 4. In Section 5, we state the main conclusions of this paper.

2. Preliminaries

This subsection introduces the relevant results of nilpotent Lie groups, and the proofs and more properties and examples can be found in [30].

2.1. Background on nilpotent Lie groups

Let $\mathbb{G}$ be a simply connected nilpotent Lie group, and the corresponding Lie algebra is g. Assume $X = \{X_1,\ X_2, ..., X_m\}$ is a Hörmander system of left invariant vector fields on $\mathbb{G}$. $X_1,\ X_2, ..., X_m$ satisfy Hörmander's condition [31]. It can be seen from [32] that the left invariant vector field can induce the Carnot-Carathéodory distance d_c: for any $\delta > 0$, let $A(\delta)$ be the set of absolutely continuous curves $\gamma:\ [0,1] \to \mathbb{G}$ such that

$$\gamma'(t) = \sum_{i=1}^{m} a_i(t) X_i(\gamma(t)),\ \sum_{i=1}^{m} |a_i(t)| \leqslant \delta,\ a.e.\ t\in[0,1].$$

Then for any $\xi,\ \eta \in \mathbb{G}$,

$$d_c(\xi,\eta) = \inf\{\delta > 0:\ \exists\delta \in A(\delta),\ \gamma(0) = \xi,\ \gamma(1) = \eta\}.$$

Let

$$B(x,r) \equiv B_r(x) = \{y \in \mathbb{G} : d_c(y,x) < r\}$$

be the metric ball of center at x and radius r in $\mathbb{G}$. Let dx be the Haar measure on $\mathbb{G}$. For any measurable set $A \subseteq \mathbb{G}$, $|A|$ denotes the measure of A. Assume that e is the unit element of $\mathbb{G}$, then for any $x \in \mathbb{G}$ and $r > 0$, $|B(e,r)| = |B(x,r)|$. d and D denote the local dimension and the dimension at infinity of $\mathbb{G}$, respectively. Let $D \geqslant d \geqslant 2$. According to [30], there is a positive constant C_1 such that

$$C_1^{-1} r^d \leqslant |B(e,r)| \leqslant C_1 r^d, \quad \forall 0 \leqslant r \leqslant 1,$$

$$C_1^{-1} r^D \leqslant |B(e,r)| \leqslant C_1 r^D, \quad \forall 1 \leqslant r < +\infty.$$

Moreover, there exist positive constants $C_2 = C_2(C_1,d,D) > 0$ and C_3 such that

$$C_2^{-1}\Big(\frac{R}{r}\Big)^d \leqslant \frac{|B(e,R)|}{|B(e,r)|} \leqslant C_2\Big(\frac{R}{r}\Big)^D, \quad \forall 0 < r < R < +\infty,$$

$$|B(e,2r)| \leqslant C_3\,|B(e,r)|\,, \quad \forall 0 < r < +\infty. \tag{2.1}$$

The parabolic metric d_p in $\mathbb{G} \times (0,+\infty)$ is defined by

$$d_p(z,z_0) = \max\left\{d_c(x,x_0),\ |t-t_0|^{1/2}\right\},$$

where $z = (x,t),\ z_0 = (x_0,\ t_0) \in \mathbb{G} \times (0,+\infty)$. For any $z_0 = (x_0,t_0) \in \mathbb{G} \times (0,\infty)$ and $r > 0$, let

$$Q(z_0,r) = Q_r(z_0) = \left\{z = (x,t) \in \mathbb{G} \times (0,+\infty):\ d_c(x,x_0) < r,\ |t-t_0| < r^2\right\}$$

be the parabolic cylinders of center at z_0 and radius r in $\mathbb{G} \times (0,+\infty)$.

2.2. Orlicz spaces on nilpotent Lie groups

Here for the readers convenience, we give some definitions and related lemmas in Orlicz spaces, and more properties and proofs can be found in [23, 33–35].

We denote by Φ the function class that consists of all monotonically increasing convex functions $\phi:\ [0,+\infty) \to [0,+\infty)$.

Definition 5. *([23]) A function $\phi \in \Phi$ is called a Young function if*

$$\phi(0) = 0,\ \lim_{t\to+\infty} \phi(t) = +\infty,\ \lim_{t\to 0^+} \frac{\phi(t)}{t} = 0,\ \lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty.$$

Definition 6. *([23]) A Young function ϕ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a constant $K > 0$ such that for any $t > 0$,*

$$\phi(2t) \leqslant K\phi(t). \tag{2.2}$$

Definition 7. *([23]) A Young function ϕ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a constant $a > 1$ such that for any $t > 0$,*

$$\phi(at) \geqslant 2a\phi(t). \tag{2.3}$$

The following lemma can be easily obtained from (2.2) and (2.3). For example, see [35].

Lemma 8. *Let ϕ be a Young function. If $\phi \in \Delta_2 \cap \nabla_2$, then for any $0 < \theta_2 \leqslant 1 \leqslant \theta_1 < +\infty$,*

$$\phi(\theta_1 t) \leqslant K{\theta_1}^{\alpha_1}\phi(t), \tag{2.4}$$

$$\phi(\theta_2 t) \leqslant 2a{\theta_2}^{\alpha_2}\phi(t), \tag{2.5}$$

where $\alpha_1 = \log_2^K,\ \alpha_2 = \log_a^2 +1,\ \alpha_1 > \alpha_2 > 1.$

Definition 9. *(Orlicz spaces, [33, 34]) Let ϕ be a Young function. Then the Orlicz class $K^\phi\left(\mathbb{G} \times (0, +\infty)\right)$ is a set of all measurable functions $g : \mathbb{G} \times (0, +\infty) \to \mathbb{R}$ satisfying*

$$\|u\|_{L^\phi(\mathbb{G}\times(0,+\infty))} = \inf\left\{k > 0 : \int_{\mathbb{G}\times(0,+\infty)} \phi\left(\frac{|u|}{k}\right)dz \leqslant 1\right\}. \tag{2.6}$$

Definition 10. *(Orlicz-Sobolev spaces, [33,34]) The Orlicz-Sobolev space $W^{1,2}_\phi(\mathbb{G}\times(0, +\infty))$ is the set of all functions u satisfying $u_t,\ X^\alpha u \in L^\phi(\mathbb{G} \times (0, \infty)),\ 0 \leqslant |\alpha| \leqslant 2$ with the norm defined by*

$$\begin{aligned}\|u\|_{W^{1,2}_\phi(\mathbb{G}\times(0,+\infty))} = {} & \|u\|_{L^\phi(\mathbb{G}\times(0,+\infty))} + \|Xu\|_{L^\phi(\mathbb{G}\times(0,+\infty))}\\ & + \left\|X^2u\right\|_{L^\phi(\mathbb{G}\times(0,+\infty))} + \|u_t\|_{L^\phi(\mathbb{G}\times(0,+\infty))},\end{aligned}$$

where $Xu = (X_1u, ..., X_mu)$, $\|Xu\|_{L^\phi(\mathbb{G}\times(0,+\infty))} = \sum_{i=1}^m \|X_iu\|_{L^\phi(\mathbb{G}\times(0,+\infty))}$, $X^2u = \{X_iX_ju\}^m_{i,j=1}$, $\left\|X^2u\right\|_{L^\phi(\mathbb{G}\times(0,+\infty))} = \sum_{i,j=1}^m \left\|X_iX_ju\right\|_{L^\phi(\mathbb{G}\times(0,+\infty))}$.

Bramanti et al. [21] defined the function space $W^{2,p}_V(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|u\|_{W^{2,p}_V(\mathbb{R}^n)} = \|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)}.$$

Inspired by this, we introduce the following function space.

Definition 11. *The function space $W^{1,2}_{\phi,0}(\mathbb{G} \times (0, +\infty))$ is the closure of $C_0^\infty(\mathbb{G} \times (0, +\infty))$ in the norm*

$$\|u\|_{W^{1,2}_{\phi,V}(\mathbb{G}\times(0,+\infty))} = \|u\|_{W^{1,2}_\phi(\mathbb{G}\times(0,+\infty))} + \|Vu\|_{L^\phi(\mathbb{G}\times(0,+\infty))}.$$

Using the proof method of Lemma 2.1 in [23], it can be proved that if $g \in L^\phi\left(\mathbb{G} \times (0, +\infty)\right)$, then

$$\int_{\mathbb{G}\times(0,+\infty)} \phi\left(|g|\right)dz = \int_0^{+\infty} |\{x \in \mathbb{G} \times (0, +\infty) : |g| > t\}|d\left[\phi\left(t\right)\right]. \tag{2.7}$$

Similar to Byun and Ryu [36], it obtains the following lemma.

Lemma 12. *Let $\Omega \subset \mathbb{G} \times (0, +\infty)$ be a bounded domain. If $\phi \in \Delta_2 \cap \nabla_2$, then*

$$L^{\alpha_1}\left(\Omega\right) \subset L^\phi\left(\Omega\right) \subset L^{\alpha_2}\left(\Omega\right) \subset L^1\left(\Omega\right),$$

where α_1 and α_2 are the constant in Lemma 8.

3. Domain decomposition method

Denoting

$$p = (1 + \alpha_2)/2, \tag{3.1}$$

then we have $1 < p < \alpha_2$, where α_2 is the constant in Lemma 8. Assume that $u \in C_0^\infty(\mathbb{G} \times (0, +\infty))$. Some notations are given below for convenience. Denote

$$\lambda_0^p = \int_{\mathbb{G}\times(0,\,+\,\infty)} |Vu|^p dz + \frac{1}{\varepsilon^p} \int_{\mathbb{G}\times(0,\,+\,\infty)} |f|^p dz, \tag{3.2}$$

where $\varepsilon \in (0, 1)$ is a small enough constant which will be determined later, $f = Lu$, $dz = dxdt$. Let

$$u_\lambda = \frac{u}{\lambda_0 \lambda},\ f_\lambda = \frac{f}{\lambda_0 \lambda},\ \forall \lambda > 0. \tag{3.3}$$

Then it infers $Lu_\lambda = f_\lambda$. Additionally, for any parabolic cylinder Q in $\mathbb{G} \times (0, +\infty)$, we write

$$J_\lambda[Q] = \frac{1}{|Q|} \int_Q |Vu_\lambda|^p dz + \frac{1}{\varepsilon^p\,|Q|} \int_Q |f_\lambda|^p dz \tag{3.4}$$

and the level set as

$$E_\lambda(1) = \{z = (x, t) \in \mathbb{G} \times (0, +\infty) : |Vu_\lambda| > 1\}. \tag{3.5}$$

Lemma 13. *(Covering lemma, [37]) Let E be a bounded measurable set in the homogeneous space* (S, d, μ). *If* $\{B(x, \rho_x)\}$ *is any family of spheres with bounded radius covering E, then there exists at most countable disjoint subfamily* $\{B(x_i, \rho_i)\}$ *such that* $\{B(x_i, k_0\rho_i)\}$ *covering E with*

$$c\,|E| \leqslant \sum_i |B(x_i, \rho_i)|,$$

where the constants $k_0 \geqslant 1$ and $c > 0$ only depend on S.
Inspired by [22, 23], we decompose the level set $E_\lambda(1)$ into a family of disjoint parabolic cylinders.

Lemma 14. *For any* $\lambda > 0$, *there is a family of disjoint parabolic cylinders* $\{Q(z_i, \rho_i)\}_{i\geqslant 1}$ *with* $z_i = (x_i, t_i) \in E_\lambda(1)$, $\rho_i = \rho_{z_i}(\lambda) > 0$ *such that*

$$J_\lambda[Q(z_i, \rho_i)] = 1,\ J_\lambda[Q(z_i, \rho)] < 1,\ \forall \rho > \rho_i \tag{3.6}$$

and

$$E_\lambda(1) \subset \bigcup_{i\geqslant 1} Q(z_i, k_0\rho_i) \bigcup \text{negligible set}, \tag{3.7}$$

where $k_0 \geqslant 1$ is a constant. Moreover, we have

$$\begin{aligned} \left|Q_{\rho_i}(z_i)\right| &\leqslant \frac{3^p}{3^p - 2} \\ &\cdot \left(\int_{\left\{z\in Q_{\rho_i}(z_i):|Vu_\lambda|>\frac{1}{3}\right\}} |Vu_\lambda|^p dz + \frac{1}{\varepsilon^p} \int_{\left\{z\in Q_{\rho_i}(z_i):|f_\lambda|>\frac{\varepsilon}{3}\right\}} |f_\lambda|^p dz \right). \end{aligned} \tag{3.8}$$

Proof of Lemma 14: For any fixed $z = (x, t) \in \mathbb{G} \times (0, +\infty)$ and $\rho(\lambda) \geqslant \rho_0(\lambda) > 0$ with $\lambda^p \left|Q_{\rho_0}(z)\right| = 1$, it follows from [33–35] that

$$\begin{aligned}
& J_\lambda \left[Q_\rho(z)\right] \\
& \leqslant \frac{1}{\left|Q_\rho(z)\right|} \left(\int_{\mathbb{G}\times(0,\infty)} |V(y)u_\lambda(y,s)|^p dyds + \frac{1}{\varepsilon^p} \int_{\mathbb{G}\times(0,\infty)} |f_\lambda(y,s)|^p dyds \right) \\
& = \frac{1}{\lambda^p \left|Q_\rho(z)\right|} \\
& \leqslant 1.
\end{aligned}$$

Thus it infers

$$\sup_{z\in\mathbb{G}\times(0,+\infty)} \sup_{\rho\geqslant\rho_0} J_\lambda \left[Q_\rho(z)\right] \leqslant 1. \tag{3.9}$$

Then for a.e. $z = (x, t) \in E_\lambda(1)$, by (3.4) and Lebesgue's differential theorem we get

$$\begin{aligned}
\lim_{\rho\to 0} J_\lambda \left[Q_\rho(z)\right] &\geqslant \lim_{\rho\to 0} \frac{1}{\left|Q_\rho(z)\right|} \int_{B_\rho(z)} |V(y)u_\lambda(y,s)|^p dyds \\
&= |V(x)u_\lambda(x,t)|^p \\
&> 1.
\end{aligned} \tag{3.10}$$

From (3.10) we know that there is $\rho(\lambda) > 0$ such that

$$J_\lambda \left[Q_\rho(z)\right] > 1.$$

By using the above formula and (3.9), it can be seen that there is $\rho_z \in (0, \rho_0]$ such that

$$\rho_z = \max\left\{\rho \in (0, \rho_0] : J_\lambda \left[Q_\rho(z)\right] = 1\right\}.$$

Thus,

$$J_\lambda \left[Q_{\rho_z}(z)\right] = 1,\ J_\lambda \left[Q_\rho(z)\right] < 1,\ \forall \rho > \rho_z.$$

To sum up, for a.e. $z \in E_\lambda(1)$, there is a family of parabolic cylinders $Q_{\rho_z}(z)$ constructed as above. $\left(\mathbb{G} \times (0, +\infty), d_p, dxdt\right)$ is a homogeneous space, therefore, according to Lemma 13, there is a family of countable disjoint parabolic cylinders $\left\{Q_{\rho_i}(z_i)\right\}_{i\geqslant 1}$ such that (3.6) and (3.7) hold.
Moreover, from (3.6) we obtain

$$\left|Q_{\rho_i}(z_i)\right| = \int_{Q_{\rho_i}(z_i)} |Vu_\lambda|^p dz + \frac{1}{\varepsilon^p} \int_{Q_{\rho_i}(z_i)} |f_\lambda|^p dz.$$

It follows from the above formula that

$$\begin{aligned}
\left|Q_{\rho_i}(z_i)\right| \leqslant & \int_{\left\{z\in Q_{\rho_i}(z_i): |Vu_\lambda| > \frac{1}{3}\right\}} |Vu_\lambda|^p dz + \frac{1}{3^p} \left|Q_{\rho_i}(z_i)\right| \\
& + \frac{1}{\varepsilon^p} \int_{\left\{z\in Q_{\rho_i}(z_i): |f_\lambda| > \frac{\varepsilon}{3}\right\}} |f_\lambda|^p dz + \frac{1}{3^p} \left|Q_{\rho_i}(z_i)\right|.
\end{aligned}$$

Then (3.8) is immediately obtained. This completes our proof.

4. Proof of the main results

4.1. Proof of Theorem 2

To prove Theorem 2, three useful lemmas are first given.

Lemma 15. *([38, Chapter I]) If $V \in RH_q$, $q > 1$, then there exist $1 \leqslant p_0 < +\infty$ and a constant $c > 0$ such that for any nonnegative function g and all parabolic cylinders Q_r,*

$$\left(\frac{1}{|Q_r|}\int_{Q_r} gdz\right)^{p_0} \leqslant \frac{c}{r^2 V(B_r)}\int_{Q_r} g^{p_0} V dz,$$

where $V(B_r) = \int_{B_r} V dx$.

Lemma 16. *([39, Lemma 1.4]) Let E be an open subset of the homogeneous space (S, d, μ). $\mathcal{F}(E)$ is the set of all metric spheres in E. If for $0 < q_1 < p$ and $0 \leqslant f \in L^p_{loc}(\mu)$, there exist the constants $A > 1,\ 1 < \sigma_0 \leqslant \sigma_0{}'$ such that*

$$\left(\frac{1}{|B|}\int_B f^p d\mu\right)^{1/p} \leqslant A\left(\frac{1}{|\sigma_0 B|}\int_{\sigma_0 B} f^{q_1} d\mu\right)^{1/q_1}, \quad \forall B:\ \sigma_0' B \in \mathcal{F}(E),$$

then for $0 < r < q_1$ and $1 < \sigma \leqslant \sigma' \leqslant \sigma_0{}'$, there exists a constant $A' > 1$ such that

$$\left(\frac{1}{|B|}\int_B f^p d\mu\right)^{1/p} \leqslant A'\left(\frac{1}{|\sigma B|}\int_{\sigma B} f^{r} d\mu\right)^{1/r}, \quad \forall B:\ \sigma' B \in \mathcal{F}(E).$$

For $z_0 = (x_0, t_0) \in \mathbb{G} \times (0, +\infty)$ and $r > 0$, we write

$$\tilde{Q}(z_0, r) = \left\{z = (x, t) \in \mathbb{G} \times (0, +\infty):\ d_c(x, x_0) < r, t_0 - r^2 < t < t_0\right\}.$$

Lemma 17. *([26, Lemma 2.9]) Assume that $V \in RH_q, q > \frac{D}{2}$. If $h(x,t)$ satisfies the homogeneous equation*

$$\partial_t h - \Delta_X h + Vh = 0 \text{ in } \tilde{Q}(z_0, 4r),$$

then there exists a positive constant c such that

$$\sup_{z \in \tilde{Q}(z_0, r)} |h| \leqslant c\left(\frac{1}{\left|\tilde{Q}(z_0, 2r)\right|}\int_{\tilde{Q}(z_0, 2r)} |h|^2 dz\right)^{1/2}.$$

Now we begin to prove Theorem 2.

Proof of Theorem 2: Denoting $Q_1 = Q((x_0, t_0 + r^2), \sqrt{2}r)$, $\tilde{Q}_1 = \tilde{Q}((x_0, t_0 + r^2), \sqrt{2}r)$, $2Q_1 = Q((x_0, t_0 + r^2), 2\sqrt{2}r)$, $2\tilde{Q}_1 = \tilde{Q}((x_0, t_0 + r^2), 2\sqrt{2}r)$, then

$$Q(z_0, r) \subset \tilde{Q}_1 \subset 2\tilde{Q}_1 \subset 2Q_1 \subset Q(z_0, 3r).$$

It follows from Lemma 17 that

$$
\begin{aligned}
\sup_{Q(z_0,r)}|h| \leqslant \sup_{\tilde{Q}_1}|h| &\leqslant c\left(\frac{1}{|2\tilde{Q}_1|}\int_{2\tilde{Q}_1}|h|^2dz\right)^{1/2} \\
&\leqslant c\left(\frac{1}{|Q(z_0,r)|}\int_{Q(z_0,4r)}|h|^2dz\right)^{1/2} \\
&\leqslant c\left(\frac{1}{|Q(z_0,4r)|}\int_{Q(z_0,4r)}|h|^2dz\right)^{1/2}.
\end{aligned}
\tag{4.1}
$$

Using (4.1) and Lemma 16, we obtain

$$
\sup_{Q(z_0,r)}|h| \leqslant c\left(\frac{1}{|Q(z_0,4r)|}\int_{Q(z_0,4r)}|h|^l dz\right)^{1/l}, \quad \forall 0<l<2.
$$

When $l>2$, by (4.1) and Hölder's inequality, it infers

$$
\sup_{Q(z_0,r)}|h| \leqslant c\left(\frac{1}{|Q(z_0,4r)|}\int_{Q(z_0,4r)}|h|^l dz\right)^{1/l}, \quad \forall l>2.
$$

Thus,

$$
\sup_{Q(z_0,r)}|h| \leqslant c\left(\frac{1}{|Q(z_0,4r)|}\int_{Q(z_0,4r)}|h|^l dz\right)^{1/l}, \quad \forall l>0. \tag{4.2}
$$

Letting $\frac{1}{l}=p_0$, from (4.2) and Lemma 15 we get

$$
\begin{aligned}
\sup_{Q(z_0,r)}|h| &\leqslant c\left(\frac{1}{|Q(z_0,4r)|}\int_{Q(z_0,4r)}|h|^{\frac{1}{p_0}}dz\right)^{p_0} \\
&\leqslant \frac{C}{r^2V(B_{4r}(x_0))}\int_{Q(z_0,4r)}V\,|h|\,dz.
\end{aligned}
$$

This completes our proof.

4.2. *Proof of Theorem 1*

In this subsection, based on Theorem 2 and the domain decomposition method, we give the proof of Theorem 1.

Proof of Theorem 1: For $u\in C_0^\infty(\mathbb{G}\times(0,+\infty))$, there exists a parabolic cylinder Q_{R_0} such that $\operatorname{spt}(u)\subset Q_{R_0}$. Combining $V\in RH_q$, $q>\max\{D/2,\alpha_1\}$, (2.2) and (2.3), we obtain

$$
\begin{aligned}
&\int_{\mathbb{G}\times(0,+\infty)}\phi(|Vu|)dz \\
&= \int_{\{z\in\mathbb{G}\times(0,+\infty):|Vu|\geqslant 1\}}\phi(|Vu|)\,dz + \int_{\{z\in\mathbb{G}\times(0,+\infty):|Vu|<1\}}\phi(|Vu|)\,dz \\
&\leqslant K\phi(1)\int_{\mathbb{G}\times(0,+\infty)}|Vu|^{\alpha_1}dz + 2a\phi(1)\int_{\mathbb{G}\times(0,+\infty)}|Vu|^{\alpha_2}dz
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c\left(\sup_{Q_{R_0}}|u|^{\alpha_1}+\sup_{Q_{R_0}}|u|^{\alpha_2}\right)\left(\int_{Q_{R_0}}|V|^{\alpha_1}dz+\int_{Q_{R_0}}|V|^{\alpha_2}dz\right)\\
&<+\infty,
\end{aligned}
$$

that is, $Vu \in L^{\phi}(\mathbb{G}\times(0,+\infty))$. It follows from (2.7) that

$$
\begin{aligned}
&\int_{\mathbb{G}\times(0,+\infty)}\phi(|Vu|)dx\\
=&\int_0^{+\infty}|\{z\in\mathbb{G}\times(0,+\infty):|Vu|>2N_0\lambda_0\lambda\}|d\left[\phi(\lambda_0\lambda)\right],
\end{aligned}
\tag{4.3}
$$

where N_0 is taken as

$$
N_0=\left(\frac{1}{\varepsilon}\right)^{\frac{p-1}{\alpha_1-p}}>1, \tag{4.4}
$$

and the constants p and α_1 are the same as those in (3.1) and Lemma 8. In fact, from the proof of the following Theorem 1, we only need to take $N_0>1$ to satisfy $\lim\limits_{\varepsilon\to 0}\varepsilon^p N_0^{\alpha_1-p}=0$.
Now we begin to estimate $|\{z\in\mathbb{G}\times(0,+\infty):|Vu|>2N_0\lambda_0\lambda\}|$.
For any $i\geqslant 1$, from Lemma 14 and (3.4) we deduce that

$$
\frac{1}{\left|Q_{4k_0\rho_i}(z_i)\right|}\int_{Q_{4k_0\rho_i}(z_i)}|Vu_\lambda|^p dz\leqslant 1 \tag{4.5}
$$

and

$$
\frac{1}{\left|Q_{4k_0\rho_i}(z_i)\right|}\int_{Q_{4k_0\rho_i}(z_i)}|f_\lambda|^p dz\leqslant \varepsilon^p, \tag{4.6}
$$

where k_0 is the constant in Lemma 14. Let w satisfy

$$
Lw=\overline{Lu_\lambda}=\begin{cases}Lu_\lambda, & z\in\ Q_{4k_0\rho_i}(z_i);\\ 0, & z\in(\mathbb{G}\times\mathbb{R})\backslash Q_{4k_0\rho_i}(z_i).\end{cases} \tag{4.7}
$$

By [26, Theorem 3.1], we have

$$
\int_{\mathbb{G}\times\mathbb{R}}|Vw|^p dz\leqslant c\int_{\mathbb{G}\times\mathbb{R}}\left|\overline{Lu_\lambda}\right|^p dz.
$$

Combined with (4.6) and (4.7),

$$
\begin{aligned}
\int_{Q_{4k_0\rho_i}(z_i)}|Vw|^p dz&\leqslant\int_{\mathbb{G}\times\mathbb{R}}|Vw|^p dz\\
&\leqslant c\int_{Q_{4k_0\rho_i}(z_i)}|Lu_\lambda|^p dz\\
&\leqslant c\varepsilon^p\left|Q_{4k_0\rho_i}(z_i)\right|.
\end{aligned}
\tag{4.8}
$$

Let $h=u_\lambda-w$. Then h satisfies

$$
\partial_t h-\Delta_X h+Vh=0 \text{ in } Q_{4k_0\rho_i}(z_i).
$$

By virtue of (4.5) and (4.8), we get

$$\begin{aligned}\int_{Q_{4k_0\rho_i}(z_i)} |Vh|^p dz &\leqslant 2^{p-1}\left(\int_{Q_{4k_0\rho_i}(z_i)} |Vw|^p dz + \int_{Q_{4k_0\rho_i}(z_i)} |Vu_\lambda|^p dz\right)\\ &\leqslant c\left(\varepsilon^p \left|Q_{4k_0\rho_i}(z_i)\right| + \left|Q_{4k_0\rho_i}(z_i)\right|\right)\\ &\leqslant c\left|Q_{4k_0\rho_i}(z_i)\right|.\end{aligned} \tag{4.9}$$

Denoting $\mu = \lambda_0\lambda$, we deduce that

$$\begin{aligned}&\left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vu| > 2N_0\mu\right\}\right|\\ =&\left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vu_\lambda| > 2N_0\right\}\right|\\ \leqslant&\left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vh| > N_0\right\}\right| + \left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vw| > N_0\right\}\right|\\ \equiv& I_1 + I_2.\end{aligned}$$

Next we estimate I_1 and I_2. From (4.8) we find that

$$\begin{aligned}I_2 &= \left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vw| > N_0\right\}\right|\\ &\leqslant \frac{1}{N_0^p}\int_{Q_{k_0\rho_i}(z_i)} |Vw|^p dx\\ &\leqslant c\frac{\varepsilon^p}{N_0^p}\left|Q_{4k_0\rho_i}(z_i)\right|.\end{aligned}$$

Using (1.2), Theorem 2, Hölder's inequality and (4.9), we conclude that

$$\begin{aligned}I_1 &= \left|\left\{x \in Q_{k_0\rho_i}(z_i) : |Vh| > N_0\right\}\right|\\ &\leqslant \frac{1}{N_0^q}\int_{Q_{k_0\rho_i}(z_i)} V^q |h|^q dx\\ &\leqslant \frac{1}{N_0^q} 2(k_0\rho_i)^2 \int_{B_{k_0\rho_i}(x_i)} V^q dx \left(\sup_{Q_{k_0\rho_i}(z_i)} |h|\right)^q\\ &\leqslant c\frac{1}{N_0^q}(k_0\rho_i)^2 \left|B_{k_0\rho_i}(x_i)\right|^{1-q}\left(\int_{B_{k_0\rho_i}(x_i)} V dx\right)^q \left(\sup_{Q_{k_0\rho_i}(z_i)} |h|\right)^q\\ &\leqslant c\frac{1}{N_0^q}\left|Q_{k_0\rho_i}(z_i)\right|^{1-q}\left(\int_{Q_{4k_0\rho_i}(z_i)} |Vh| dz\right)^q\\ &\leqslant c\frac{1}{N_0^q}\left|Q_{4k_0\rho_i}(z_i)\right|^{1-q/p}\left(\int_{B_{4k_0\rho_i}(x_i)} |Vh|^p dz\right)^{q/p}\\ &\leqslant c\frac{1}{N_0^q}\left|Q_{4k_0\rho_i}(z_i)\right|.\end{aligned}$$

Therefore, we have

$$\left|\left\{z \in Q_{k_0\rho_i}(z_i) : |Vu| > 2N_0\mu\right\}\right| \leqslant c\left(\frac{1}{N_0^q} + \frac{\varepsilon^p}{N_0^p}\right)\left|Q_{4k_0\rho_i}(z_i)\right|. \tag{4.10}$$

It follows from (3.7), (4.10) and (3.8) that

$$
\begin{aligned}
&|\{z\in\mathbb{G}\times(0,+\infty):|Vu|>2N_0\mu\}|\\
\leqslant&\sum_i\left|\left\{z\in Q_{k_0\rho_i}(z_i):|Vu|>2N_0\mu\right\}\right|\\
\leqslant&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)\sum_i\left|Q_{4k_0\rho_i}(z_i)\right|\\
\leqslant&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)\sum_i\left|Q_{k_0\rho_i}(z_i)\right|\\
\leqslant&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)\left(\frac{1}{\mu^p}\sum_i[\int_{\left\{z\in Q_{k_0\rho_i}(z_i):|Vu|>\frac{\mu}{3}\right\}}|Vu|^pdz\right.\\
&\left.+\frac{1}{\varepsilon^p\mu^p}\int_{\left\{z\in Q_{k_0\rho_i}(z_i):|f|>\frac{\varepsilon\mu}{3}\right\}}|f|^pdz]\right)\\
\leqslant&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)\left(\frac{1}{\mu^p}\int_{\left\{z\in\mathbb{G}\times(0,+\infty):|Vu|>\frac{\mu}{3}\right\}}|Vu|^pdz\right.\\
&\left.+\frac{1}{\varepsilon^p\mu^p}\int_{\left\{z\in\mathbb{G}\times(0,+\infty):|f|>\frac{\varepsilon\mu}{3}\right\}}|f|^pdz\right).
\end{aligned}
$$

Furthermore, by the above inequality and (4.3) we have

$$
\begin{aligned}
&\int_{\mathbb{G}\times(0,+\infty)}\phi(|Vu|)dz\\
=&\int_0^{+\infty}|\{z\in\mathbb{G}\times(0,+\infty):|Vu|>2N_0\mu\}|d\left[\phi(2N_0\mu)\right]\\
\leqslant&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)\int_0^{+\infty}\frac{1}{\mu^p}\left\{\int_{\left\{z\in\mathbb{G}\times(0,+\infty):|Vu|>\frac{\mu}{3}\right\}}|Vu|^pdz\right\}d\left[\phi(2N_0\mu)\right]\\
&+c\left(\frac{1}{\varepsilon^pN_0^q}+\frac{1}{N_0^p}\right)\int_0^{+\infty}\frac{1}{\mu^p}\left\{\int_{\left\{z\in\mathbb{G}\times(0,+\infty):|f|>\frac{\varepsilon\mu}{3}\right\}}|f|^pdz\right\}d\left[\phi(2N_0\mu)\right]\\
\equiv&\, c\left(\frac{1}{N_0^q}+\frac{\varepsilon^p}{N_0^p}\right)I_3+c\left(\frac{1}{\varepsilon^pN_0^q}+\frac{1}{N_0^p}\right)I_4,
\end{aligned}
$$

where the constant c is independent of ε, N_0. Now we estimate I_3 and I_4. According to Fubini's theorem and integration by parts, we deduce that

$$\begin{aligned}
I_3 &= \int_0^{+\infty} \frac{1}{\mu^p} \left\{ \int_{\left\{x \in \mathbb{G}: |Vu| > \frac{\mu}{3}\right\}} |Vu|^p dz \right\} d\left[\phi\left(2N_0\mu\right)\right] \\
&= \int_{\mathbb{G}\times(0,+\infty)} |Vu|^p \left(\int_0^{3|Vu|} \frac{d[\phi\left(2N_0\mu\right)]}{\mu^p} \right) dz \\
&= 3^{-p} \int_{\mathbb{G}\times(0,+\infty)} \phi(6N_0\left|Vu\right|) dz + \int_{\mathbb{G}\times(0,\,+\,\infty)} |Vu|^p dz \times \lim_{\mu\to 0} \frac{\phi\left(2N_0\mu\right)}{\mu^p} \\
&\quad + p \int_{\mathbb{G}\times(0,\,+\,\infty)} |Vu|^p \left(\int_0^{3|Vu|} \frac{\phi\left(2N_0\mu\right)}{\mu^{p+1}} d\mu \right) dz.
\end{aligned}$$

By using (2.5) and $1 < p < \alpha_2$, it infers

$$0 \leqslant \lim_{\mu\to 0} \frac{\phi\left(2N_0\mu\right)}{\mu^p} \leqslant 2a\phi\left(2N_0\right) \lim_{\mu\to 0} \mu^{\alpha_2 - p} = 0.$$

Moreover, using Lemma 8, we have

$$\begin{aligned}
&\int_0^{3|Vu|} \phi\left(2N_0\mu\right) / \mu^{p+1} d\mu \\
&= \int_0^{3|Vu|} \phi\left(\frac{\mu}{3\left|Vu\right|} \cdot 6N_0\left|Vu\right|\right) / \mu^{p+1} d\mu \\
&\leqslant c \frac{1}{|Vu|^{\alpha_2}} \phi\left(6N_0\left|Vu\right|\right) \int_0^{3|Vu|} \mu^{\alpha_2} / \mu^{p+1} d\mu \\
&\leqslant c \frac{1}{|Vu|^p} \phi\left(6N_0\left|Vu\right|\right).
\end{aligned}$$

Note that $1 < p < \alpha_2$ must also be required in the above integral calculation process. Therefore, from the above analysis and (2.4), we observe that

$$\begin{aligned}
I_3 &\leqslant c \int_{\mathbb{G}\times(0,+\infty)} \phi\left(6N_0\left|Vu\right|\right) dz \\
&\leqslant c N_0^{\alpha_1} \int_{\mathbb{G}\times(0,+\infty)} \phi\left(\left|Vu\right|\right) dz,
\end{aligned} \tag{4.11}$$

where the constant c is independent of $\varepsilon,\ N_0$.
By applying Fubini's theorem, integration by parts and (2.5), I_4 becomes

$$\begin{aligned}
I_4 &= \int_0^{+\infty} \frac{1}{\mu^p} \left\{ \int_{\{z \in \mathbb{G}\times(0,+\infty):\ |f| > \frac{\varepsilon\mu}{3}\}} |f|^p dz \right\} d\left[\phi(2N_0\mu)\right] \\
&= \int_{\mathbb{G}\times(0,+\infty)} |f|^p \left(\int_0^{\frac{3|f|}{\varepsilon}} \frac{d\left[\phi(2N_0\mu)\right]}{\mu^p} \right) dz \\
&= \frac{\varepsilon^p}{3^p} \int_{\mathbb{G}\times(0,+\infty)} \phi\left(\frac{6N_0\left|f\right|}{\varepsilon}\right) dz + p \int_{\mathbb{G}\times(0,+\infty)} |f|^p \left(\int_0^{\frac{3|f|}{\varepsilon}} \frac{\phi(2N_0\mu)}{\mu^p} d\mu \right) dz.
\end{aligned}$$

Setting $\theta_2 = \frac{\varepsilon\mu}{3|f|}$, then $0 < \theta_2 < 1$. It follows from (2.5) that

$$\begin{aligned}\phi\left(2N_0\mu\right) &= \phi\left(\frac{\varepsilon\mu}{3\left|f\right|}\cdot\frac{6N_0\left|f\right|}{\varepsilon}\right)\\ &\leqslant \frac{2a\varepsilon^{\alpha_2}}{3^{\alpha_2}}\frac{1}{|f|^{\alpha_2}}\phi\left(\frac{6N_0\left|f\right|}{\varepsilon}\right).\end{aligned}$$

By the above inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{G}\times(0,+\infty)}|f|^p\left(\int_0^{\frac{3|f|}{\varepsilon}}\frac{\phi(2N_0\mu)}{\mu^p}d\mu\right)dz\\ \leqslant\ &\frac{2a\varepsilon^{\alpha_2}}{3^{\alpha_2}}\int_{\mathbb{G}\times(0,+\infty)}|f|^{p-\alpha_2}\phi\left(\frac{6N_0\left|f\right|}{\varepsilon}\right)\left(\int_0^{\frac{3|f|}{\varepsilon}}\frac{\mu^{\alpha_2}}{\mu^{p+1}}d\mu\right)dz\\ =\ &\frac{2a\varepsilon^p}{3^p(\alpha_2-p)}\int_{\mathbb{G}\times(0,+\infty)}\phi\left(\frac{6N_0\left|f\right|}{\varepsilon}\right)dz.\end{aligned}$$

Additionally, using (2.4), we get

$$\begin{aligned}I_4 &\leqslant c\varepsilon^p\int_{\mathbb{G}\times(0,+\infty)}\phi\left(\frac{6N_0\left|f\right|}{\varepsilon}\right)dz\\ &\leqslant c\varepsilon^p\int_{\mathbb{G}\times(0,+\infty)}K\left(\frac{6N_0}{\varepsilon}\right)^{\alpha_1}\phi\left(|f|\right)dz\\ &\leqslant c\frac{1}{\varepsilon^{\alpha_1-p}}N_0^{\alpha_1}\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|f|\right)dz,\end{aligned}\tag{4.12}$$

where the constant c is independent of $\varepsilon,\ N_0$.
Therefore, from (4.11) and (4.12) we obtain

$$\begin{aligned}\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|Vu|\right)dz &\leqslant c\left(\frac{1}{N_0^{q-\alpha_1}}+\varepsilon^pN_0^{\alpha_1-p}\right)\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|Vu|\right)dz\\ &\quad+c\frac{1}{\varepsilon^{\alpha_1}}\left(\frac{1}{N_0^{q-\alpha_1}}+\varepsilon^pN_0^{\alpha_1-p}\right)\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|f|\right)dz\\ &\leqslant c\left(\varepsilon^{\frac{(q-\alpha_1)(p-1)}{(\alpha_1-p)}}+\varepsilon\right)\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|Vu|\right)dz\\ &\quad+c\frac{1}{\varepsilon^{\alpha_1}}\left(\varepsilon^{\frac{(q-\alpha_1)(p-1)}{(\alpha_1-p)}}+\varepsilon\right)\int_{\mathbb{G}\times(0,+\infty)}\phi\left(|f|\right)dz,\end{aligned}$$

where the constant $c > 0$ is independent of ε. Finally, using $1 < p < \alpha_2 < \alpha_1 < q$ and choosing a suitable $\varepsilon > 0$ such that

$$c\left(\varepsilon^{\frac{(q-\alpha_1)(p-1)}{(\alpha_1-p)}}+\varepsilon\right)\leqslant\frac{1}{2},$$

we obtain (1.4). This completes our proof.

4.3. Proof of Corollary 3

Proof of Corollary 3: For $u \in W^{1,2}_{\phi,0}(\mathbb{G} \times (0, \infty))$, according to Definition 11, there exists a sequence of $\{u_k\}$ of functions in $C_0^\infty(\mathbb{G} \times (0, \infty))$ such that

$$\|u_k - u\|_{W^{1,2}_\phi(\mathbb{G}\times(0,\infty))} + \|V(u_k - u)\|_{L^\phi(\mathbb{G}\times(0,\infty))} \to 0, \quad k \to \infty.$$

Therefore, from the above formula, Theorem 1, the convexity and monotonicity of φ, and (2.4) and (2.5), Corollary 3 is immediately proved. We complete the proof of Corollary 3.

5. Conclusions

In this paper, we study the regularity estimates in Orlicz space for the parabolic Schrödinger operator $L = \partial_t - \Delta_X + V$ on nilpotent Lie groups. There are many essential differences between partial differential operators on nilpotent Lie groups and partial differential operators on Euclidean space. For example, the sub-Laplace operator $\Delta_X = \sum_{i=1}^{m} X_i^2$ on a nilpotent Lie group is a degenerate elliptic operator, while the Laplace operator $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$ in Euclidean space is a uniformly elliptic operator. Acerbi and Mingione [28] invented a new domain decomposition approach, which is completely free from harmonic analysis. Wang et al. [22, 23, 29] simplified and improved this approach, and obtained Orlicz estimates for some operators in Euclidean space. We extend the method of [22, 23] to nilpotent Lie groups, and in order to generalize the condition $V \in RH_\infty$ to the condition $V \in RH_q$, we have appropriately improved the domain decomposition method and the measure estimation of level sets in [22, 23]. By using this approach, we obtain the Orlicz estimates for the parabolic Schrödinger operator with non-negative potentials on a nilpotent Lie groups, which extend the L^p estimates in [26]. Because this method needs to rely on the estimate of the metric sphere measure, this method is applicable to the Heisenberg group, Carnot group, etc, but it may not be suitable for the Hörmander's vector fields.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflicts of interest regarding the publication of this paper.

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