Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 18618–18630.
DOI: 10.3934/math.2023948
Received: 07 March 2023
Revised: 15 May 2023
Accepted: 21 May 2023
Published: 02 June 2023

Research article

Asymptotic analysis of high frequency modes for thin elastic plates

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Abstract: In this paper, we show that the high frequency modes of a thin clamped plate and the associated eigenfunctions converge, as the thickness of the plate goes to zero, to the eigenvalues and the eigenfunctions of a two-dimensional eigenvalue problem associated to the stretching displacements of the plate.

Keywords: linear elasticity; asymptotic analysis; thin plates; eigenvalue problem; high frequency modes; stretching vibrations
Mathematics Subject Classification: 35E20, 35C20, 74B05, 74K20, 74G10

1. Introduction

The purpose of this article is to study the asymptotic behavior of the high frequency modes of thin plates when the thickness of the plates goes to zero. The asymptotic methods were used to study a large variety of problems in thin elastic structures. Let us refer to Ciarlet and Destunder [1] for the justification of the two-dimensional linear plate model, Ciarlet et al. [2] for the junctions between three-dimensional and two-dimensional linear elastic structures, Le Dret [3,4] for modeling of a folded plate and [5] for modeling of the junction between two rods, and Trabucho and Viaño [6] for asymptotic analysis of linearized elastic beams.

The problem of modeling the vibrations of thin elastic structures using a rigourous asymptotic technique was first done by Ciarlet and Kessavan [7] in the case of a clamped plate. The techniques introduced in this article were adapted and used to study different spectral problems: Le Dret [8] for the vibrations of a folded plate, Bourquin and Ciarlet [9] and Lods [10] for a plate inserted in a three-dimensional body, Kerdid [11,12] for a single rod and junction between two rods. All these works are concerned with the convergence of low frequency modes of the three-dimensional linear elasticity, as the thickness of the body tends to zero. The limit problems obtained are the classical spectral problems associated with the flexural displacement of the structure. However, these techniques fail to obtain the limit problem for higher frequency modes.

We also refer to some interesting works related to this article, which deal with the asymptotic analysis of the eigenvalue problem in different thin linear elastic structures: Jumbo and Rodriguez Mulet [13] and Jumbo et al. [14] for thin elastic rod with non-uniform cross-section, Serpilli and Lenci [15] for laminated beams, Tambača [16] for curved rods and Qaudiello et al. [17] for a thin T-like shaped structure.

A nonstandard technique has been proposed in Irago et al. [18, 19] to study the behavior of the high frequency modes and their associated eigenfunctions in thin elastic rods. The limit problem obtained is the coupled one-dimensional spectral problem giving the classical equations for torsion and stretching vibrations.

In this work, we combine the techniques of [7, 19] to study the asymptotic behavior of high frequency modes in a thin clamped plate when the thickness of the plate goes to zero. Indeed, we will construct a suitable families of index $\{\ell_\varepsilon^m\}_{\varepsilon>0}$, which varies with ℓ and ε, and for which the high frequency eigenvalues $\eta_{\ell_\varepsilon^m}$ converge when the thickness of the plate approaches zero towards the eigenvalues η_m of a two-dimensional spectral problem. The limit problem is identified to be the standard eigenvalue problem associated with the classical equations for stretching vibrations of the plate. The limit eigenfunctions are determined by the couples $\left(\zeta_1^m, \zeta_2^m\right)$ of functions of the longitudinal variables of the plate that are the unique solution of the limit problem and correspond to the stretching displacements of the plate.

2. The three-dimensional problem

Let ω be an open bounded set of $\mathbb{R}^2$ and $\gamma = \partial\omega$ its boundary which is assumed to be sufficiently smooth. Given $\varepsilon \geq 0$ we define

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon), \qquad \Gamma_\varepsilon = \gamma \times (-\varepsilon, \varepsilon). \tag{2.1}$$

Ω^ε is assumed to be the reference configuration of the plate under consideration. The plate is clamped on its boundary Γ_ε. The material that constitute the plate is assumed to be homogeneous and isotropic with Young's modulus E and Poisson's ratio ν, all independent of ε.

We will also use the Lame's coefficients λ and μ related to E and ν by the formulas:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{2.2}$$

In the sequel, we shall use the repeated index convention, the Greek indices take their values in the set $\{1, 2\}$ and the Latin indices take their values in the set $\{1, 2, 3\}$.

The classical eigenvalue problem for the plate under consideration consists in finding pairs $(\eta^\varepsilon, u^\varepsilon)$ satisfying:

$$\begin{cases} -\partial_j \sigma_{ij}^\varepsilon &= \eta^\varepsilon u_i^\varepsilon & \text{in } \Omega^\varepsilon, \\ \sigma_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) &= \lambda e_{pp}^\varepsilon(\boldsymbol{u}^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon &= 0 & \text{on } \Gamma_\varepsilon, \\ \sigma^\varepsilon n^\varepsilon &= 0 & \text{on } \partial\Omega^\varepsilon \backslash \Gamma_\varepsilon, \end{cases} \tag{2.3}$$

where $\sigma^\varepsilon(\boldsymbol{u}^\varepsilon)$ is the stress tensor, n^ε is the outer unit normal vector to $\partial\Omega^\varepsilon$, and $e^\varepsilon(\boldsymbol{u}^\varepsilon)$ is the linearized strain tensor corresponding to the displacement $\boldsymbol{u}^\varepsilon$:

$$e^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon) = \frac{1}{2}\left(\frac{\partial u^\varepsilon_j}{\partial x^\varepsilon_i} + \frac{\partial u^\varepsilon_i}{\partial x^\varepsilon_j}\right). \tag{2.4}$$

In order to put the above problem in variational form, we introduce the space

$$V^\varepsilon = \{v^\varepsilon = (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3, \;\; v^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon\}. \tag{2.5}$$

So, problem (2.3) is equivalent to finding $(\eta^\varepsilon, u^\varepsilon) \in \mathbb{R} \times V^\varepsilon$ satisfying

$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{ij}(\boldsymbol{v}^\varepsilon) dx^\varepsilon = \eta^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon_i v^\varepsilon_i dx^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon. \tag{2.6}$$

Thanks to Korn inequality, the bilinear form

$$(u^\varepsilon, v^\varepsilon) \in V^\varepsilon \times V^\varepsilon \longmapsto \int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{ij}(v^\varepsilon) dx^\varepsilon \tag{2.7}$$

is V^ε-elliptic. From spectral theory, it is known that problem (2.6) has a sequence of eigenvalues $(\eta^\varepsilon_m)_{m\geq 1}$ satisfying

$$0 < \eta^\varepsilon_1 \leq \eta^\varepsilon_2 \leq \eta^\varepsilon_3 \leq ... \leq \eta^\varepsilon_m \leq ... \tag{2.8}$$

with

$$\lim_{m\longrightarrow\infty} \eta^\varepsilon_m = +\infty, \tag{2.9}$$

associated with a family of eigenfunctions $(u^{\varepsilon,m})_{m\geq 1}$, that is

$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij}(\boldsymbol{u}^{\varepsilon,m}) e^\varepsilon_{ij}(\boldsymbol{v}^\varepsilon) dx^\varepsilon = \eta^\varepsilon_m \int_{\Omega^\varepsilon} u^{\varepsilon,m}_i v^\varepsilon_i dx^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon, \tag{2.10}$$

which can be orthonormalized as

$$\int_{\Omega^\varepsilon} u^{\varepsilon,m}_i u^{\varepsilon,n}_i dx^\varepsilon = \delta_{mn} \quad \forall m, n \geq 1, \tag{2.11}$$

and which make a basis in both Hilbert spaces V^ε and $[L^2(\Omega^\varepsilon)]^3$.

3. The scaled problem

In order to define a problem equivalent to problem (2.6) but posed over a domain which does not depend on ε, we let

$$\Omega = \omega \times (-1, 1), \qquad \Gamma = \gamma \times (-1, 1), \tag{3.1}$$

and

$$V = H^1_\Gamma(\Omega; \mathbb{R}^3). \tag{3.2}$$

We introduce the following mapping:

$$\phi^\varepsilon \; : \; \Omega \;\; \rightarrow \;\; \Omega^\varepsilon,$$

$$x \quad \mapsto \quad x^{\varepsilon} = (x_1, x_2, \varepsilon x_3),$$

and the scaling functions $v(\varepsilon) \in V$ defined as:

$$v_\alpha(\varepsilon)(x) = \varepsilon^{-1} v_\alpha^{\varepsilon}(x^{\varepsilon}); \qquad v_3(\varepsilon)(x) = v_3^{\varepsilon}(x^{\varepsilon}). \tag{3.3}$$

Due to scaling, we have

$$\left\{ \begin{array}{lcl} e_{\alpha\beta}^{\varepsilon}(v^{\varepsilon}) & = & e_{\alpha\beta}(v(\varepsilon)), \\ e_{\alpha 3}^{\varepsilon}(v^{\varepsilon}) & = & \varepsilon^{-1} e_{\alpha 3}(v(\varepsilon)), \\ e_{33}^{\varepsilon}(v^{\varepsilon}) & = & \varepsilon^{-2} e_{33}(v(\varepsilon)). \end{array} \right. \tag{3.4}$$

Substituting (3.4) in (2.6), we obtain the following scaled variational formulation:

Find $(\eta_m(\varepsilon), u^m(\varepsilon)) \in \mathbb{R} \times V$ such that for, all $v \in V$

$$\int_\Omega b_\varepsilon(u^m(\varepsilon), v) dx = \eta_m(\varepsilon) \left\{ \int_\Omega u_\alpha^m(\varepsilon) v_\alpha dx + \varepsilon^{-2} \int_\Omega u_3^m(\varepsilon) v_3 dx \right\}, \tag{3.5}$$

where

$$\eta_m(\varepsilon) = \eta_m^{\varepsilon}, \tag{3.6}$$

and

$$\begin{array}{rcl} b_\varepsilon(u, v) & = & 2\mu e_{\alpha\beta}(u) e_{\alpha\beta}(v) + \lambda e_{\alpha\alpha}(u) e_{\beta\beta}(v) \\ & + & \varepsilon^{-2} \left[4\mu e_{\alpha 3}(u) e_{\alpha 3}(v) + \lambda (e_{\alpha\alpha}(u) e_{33}(v) + e_{33}(u) e_{\alpha\alpha}(v)) \right] \\ & + & \varepsilon^{-4} (\lambda + 2\mu) e_{33}(u) e_{33}(v), \end{array} \tag{3.7}$$

with the normalization condition

$$\int_\Omega u_\alpha^m(\varepsilon) u_\alpha^n(\varepsilon) dx + \varepsilon^{-2} \int_\Omega u_3^m(\varepsilon) u_3^n(\varepsilon) dx = \delta_{mn} \qquad \forall m, n \geq 1. \tag{3.8}$$

We define the space of Kirchhoff-Love on Ω as

$$V_{KL} = \{ v \in V, \ \ e_{i3}(v) = 0 \}. \tag{3.9}$$

Elements of this space are characterized by

$$\left\{ \begin{array}{ll} v_\alpha(0)(x) & = \zeta_\alpha(x_1, x_2) - x_3 \partial_\alpha \zeta_3(x_1, x_2), \\ v_3(0)(x) & = \zeta_3(x_1, x_2), \end{array} \right. \tag{3.10}$$

where $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$.

4. Convergence of low frequency modes

A first convergence analysis of the low frequencies of the three-dimensional linearized elasticity system in a thin plate, when the thickness of the plate approaches zero, was done in [7]. It has been shown that the standard biharmonic two-dimensional eigenvalue problem associated with the flexural displacements of the plate can be derived mathematically from the standard three-dimensional eigenvalue problem of linear elasticity through a rigourous convergence analysis as the thickness of the plate tends to zero.

More precisely, it has been proven that for each integer $m \geq 1$,

$$\lambda_m(\varepsilon) = \varepsilon^{-2}\eta_m(\varepsilon) \to \lambda_m(0) \tag{4.1}$$

and

$$u^m(\varepsilon) \to u^m(0) \quad \text{strongly in } \; V \tag{4.2}$$

with

$$u^m(0) = (-x_3\partial_1 u_3^m(x_1, x_2), -x_3\partial_2 u_3^m(x_1, x_2), u_3^m(x_1, x_2)),$$

where $(\lambda_m(0), u_3^m) \in \mathbb{R} \times H_0^2(\omega)$ are eigensolutions of the limit spectral problem:

Find $(\lambda, u) \in \mathbb{R} \times H_0^2(\omega)$ such that, for all $v \in H_0^2(\omega)$

$$\frac{E}{3(1-\nu^2)} \int_\omega \Delta u \Delta v dx_1 dx_2 = \lambda \int_\omega uv dx_1 dx_2. \tag{4.3}$$

The pairs $(\lambda^m(0), u_3^m)$ are solutions of the eigenvalue problem for the biharmonic operator Δ^2

$$\frac{E}{3(1-\nu^2)}\Delta^2 u = \lambda u, \tag{4.4}$$

corresponding to the classical equations for the flexural vibrations.

5. Convergence of high frequency modes

Our objective in this work is to characterize the limit problem associated to high frequency modes when the thickness of the plate goes to zero. Unfortunately, the techniques used in [7] are not adapted for the asymptotic analysis of higher frequency modes. So, we will be inspired by the idea proposed in [18, 19] for the convergence analysis of high frequencies in a thin rod in order to characterize the limit problem for high frequency modes in a thin plate.

Let us start with the following lemma:

Lemma 5.1. *There exists an increasing sequence of constants $K_m > 0, m \geq 1$, independent of ε such that*

$$\eta_m(\varepsilon) \leq K_m \varepsilon^2. \tag{5.1}$$

Proof. From [7] Lemma 1 we have, for each integer $m \geq 1, \varepsilon^{-2}\eta_m(\varepsilon) \leq K_m$ where K_m is a constant independent of ε, which gives (5.1). □

So, if we fix the index m and we make ε tend to zero, all the sequence $\eta_m(\varepsilon)$ goes to zero. indeed, the high frequency modes are concentrated at infinity when ε approaches zero and cannot be obtained using such a passage to the limit. So, the idea in order to characterize this kind of frequencies, consists in associating to each integer $m \geq 1$, a family of index $\{\ell_\varepsilon^m\}_{\varepsilon>0}$ that depend on ε and such that

$$\lim_{\varepsilon \to 0} \ell_\varepsilon^m = +\infty, \tag{5.2}$$

and

$$\eta_{\ell_\varepsilon^m}(\varepsilon) < K_m. \tag{5.3}$$

This family of index can be defined by

$$\ell_\varepsilon^m = \max\{j \in \mathbb{N}^* : \eta_j(\varepsilon) \le K_m\}. \tag{5.4}$$

It is clear that (5.4) satisfies (5.2) and (5.3).

The family of index $\{\ell_\varepsilon^m\}$ varies with m and ε, and for each $\varepsilon > 0$, $\{\ell_\varepsilon^m\}_{m\ge 1}$ is an increasing subsequence of positive integers satisfying $\ell_\varepsilon^m \ge m, \forall m \ge 1$. It contains the indices of the stretching modes among all the modes $\{\eta_m(\varepsilon)\}_{m\ge1}$ of the plate.

To illustrate this idea and show the layout of the stretching modes $\{\eta_{\ell_\varepsilon^m}(\varepsilon)\}$ when m and ε vary, let us represent the family $\{\eta_{\ell_\varepsilon^m}(\varepsilon)\}$ in a double-entry table. Consider a decreasing sequence $\{\varepsilon_n\}_{n\ge1}$ converging to 0, the elements of the sequences $\{\eta_m(\varepsilon_n)\}_{m\ge1}$ are arranged in rows while the elements of the sequences $\{\eta_m(\varepsilon_n)\}_{n\ge1}$ are arranged in columns.

$$\begin{array}{ccccccccc}
\eta_1(\varepsilon_1) & \cdots & \eta_{\ell^1_{\varepsilon_1}}(\varepsilon_1) & \cdots & \eta_{\ell^m_{\varepsilon_1}}(\varepsilon_1) & \cdots & & & \\
\eta_1(\varepsilon_2) & \cdots & & \eta_{\ell^1_{\varepsilon_2}}(\varepsilon_2) & \cdots & \eta_{\ell^m_{\varepsilon_2}}(\varepsilon_2) & \cdots & & \\
\vdots & & & & \ddots & & \ddots & & \\
\eta_1(\varepsilon_n) & \cdots & & & \eta_{\ell^1_{\varepsilon_n}}(\varepsilon_n) & \cdots & \eta_{\ell^m_{\varepsilon_n}}(\varepsilon_n) & \cdots & \\
\downarrow & & & & & \searrow & & \searrow & \\
0 & \cdots & & & & \eta_1(0) & \cdots & \eta_m(0) & \cdots
\end{array}$$

Since for each $m \ge 1$ the family $\{\ell^m_{\varepsilon_n}\}_{n\ge1}$ is increasing, that is $\ell^m_{\varepsilon_n} \ge \ell^m_{\varepsilon_{n'}}$ for $n > n'$, the elements of the sequence $\{\eta_{\ell^m_{\varepsilon_n}}(\varepsilon_n)\}_{n\ge1}$ corresponding to the modes associated to the stretching vibrations of the plate are arranged diagonally. As the stretching vibrations are high frequency modes and are concentrated at infinity when ε approaches zero, they can only be reached through such a family of indices.

The following theorem summarize the results obtained when passing to the limit on these families of sequences and constitute the main result of this paper.

Theorem 5.2. *For each integer $m \ge 1$, there exists a sequence $\{\ell_\varepsilon^m\}$ such that*

$$\eta_{\ell_\varepsilon^m}(\varepsilon) \to \eta_m(0) \tag{5.5}$$

where $\eta_m(0)$ is an eigenvalue of the limit spectral problem:

Find $(\eta, \zeta) \in \mathbb{R} \times [H_0^1(\omega)]^2$ such that, for all $\xi \in [H_0^1(\omega)]^2$,

$$\frac{E}{1+\nu}\int_\omega e_{\alpha\beta}(\zeta)e_{\alpha\beta}(\xi)dx_1dx_2 \quad + \quad \frac{E\nu}{1-\nu^2}\int_\omega e_{\alpha\alpha}(\zeta)e_{\beta\beta}(\xi)dx_1dx_2 = \eta(0)\int_\omega \zeta_\alpha\xi_\alpha dx_1dx_2. \tag{5.6}$$

In addition, there exists a subsequence (still denoted ε) and $u^m(0) \in V$ such that

$$u^{\ell_\varepsilon^m}(\varepsilon) \rightharpoonup u^m(0) \qquad \textit{weakly in } V, \tag{5.7}$$

where

$$u_\alpha^m(0) = \zeta_\alpha^m(x_1, x_2) \tag{5.8}$$

and

$$u_3^m(0) = 0, \tag{5.9}$$

with

$$(\zeta_1^m, \zeta_2^m) \quad \in [H_0^1(\omega)]^2.$$

If $(\zeta_1^m, \zeta_2^m) \neq (0,0)$ *then it is an eigenfunction associated to* $\eta_m(0)$*.*

To prove this theorem we combine the techniques in [7, 19]. First, we start by establishing an appropriate bound for the eigenfunctions.

Lemma 5.3. *For each* $m \geq 1$*, there exists a constant* $C_m > 0$ *independent of* ε*, such that*

$$\| u^{\ell_\varepsilon^m}(\varepsilon) \|_{H^1(\Omega;\mathbb{R}^3)} \leq C_m. \tag{5.10}$$

Proof. Let us define the scaled strain tensors

$$\begin{cases} \kappa_{\alpha\beta}^{\ell_\varepsilon^m}(\varepsilon) &= e_{\alpha\beta}(u^{\ell_\varepsilon^m}(\varepsilon)), \\ \kappa_{\alpha 3}^{\ell_\varepsilon^m}(\varepsilon) &= \varepsilon^{-1} e_{\alpha 3}(u^{\ell_\varepsilon^m}(\varepsilon)), \\ \kappa_{33}^{\ell_\varepsilon^m}(\varepsilon) &= \varepsilon^{-2} e_{33}(u^{\ell_\varepsilon^m}(\varepsilon)). \end{cases} \tag{5.11}$$

Taking $v = u^{\ell_\varepsilon^m}(\varepsilon)$ in (3.5) and using (3.7) and (3.8), we obtain

$$2\mu \|\kappa^{\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega;M^3)} \quad \leq \quad \int_\Omega b_\varepsilon(u^{\ell_\varepsilon^m}(\varepsilon), u^{\ell_\varepsilon^m}(\varepsilon)) dx_1 dx_2 = \eta_{\ell_\varepsilon^m}(\varepsilon) \leq K_m.$$

So, we have

$$\|\kappa_{ij}^{\ell_\varepsilon^m}(\varepsilon)\|_{L^2(\Omega)} \leq C_m, \tag{5.12}$$

and consequently, since $0 < \varepsilon \leq 1$,

$$\begin{aligned} &\|e_{\alpha\beta}(u^{\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega)} \leq C_m, \\ &\|e_{\alpha 3}(u^{\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega)} \leq C_m \varepsilon \leq C_m, \\ &\|e_{33}(u^{\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega)} \leq C_m \varepsilon^2 \leq C_m. \end{aligned} \tag{5.13}$$

Therefor, (5.10) is obtained using Korn inequality in $H^1_\Gamma(\Omega; \mathbb{R}^3)$. □

Lemma 5.4. *For each* $m \geq 1$*, there exists a subsequence (still denoted* ε*) such that*

$$\eta_{\ell_\varepsilon^m}(\varepsilon) \to \eta_m(0) \tag{5.14}$$

and

$$u^{\ell_\varepsilon^m}(\varepsilon) \rightharpoonup u^m(0) \qquad \text{weakly in } V \tag{5.15}$$

where

$$u_\alpha^m(0)(x) = \zeta_\alpha^m(x_1, x_2) - x_3 \partial_\alpha \zeta_3^m(x_1, x_2), \tag{5.16}$$

$$u_3^m(0)(x) = \zeta_3^m(x_1, x_2), \tag{5.17}$$

with $\zeta_\alpha \in H_0^1(\omega)$ *and* $\zeta_3 \in H_0^2(\omega)$*.*

Proof. Convergences (5.14) and (5.15) come from (5.3) and (5.10). Now using (5.13) we have

$$e_{i3}(u^{\ell_\varepsilon^m}(\varepsilon)) \to 0 \quad \text{strongly} \;\; \text{in } L^2(\Omega),$$

and since

$$u_i^{\ell_\varepsilon^m}(\varepsilon) \rightharpoonup u_i^m(0) \quad \text{weakly} \;\; \text{in } H^1(\Omega),$$

then

$$e_{i3}(u^{\ell_\varepsilon^m}(\varepsilon)) \rightharpoonup e_{i3}(u^m(0)) \;\; \text{weakly} \;\; \text{in } L^2(\Omega).$$

Thus,

$$e_{i3}(u^m(0)) = 0.$$

Therefore, $u^m(0) \in V_{KL}$ and consequently, we deduce (5.16) and (5.17) from (3.10). □

Lemma 5.5. *For each* $m \geq 1$*, if* $\eta_m(0) \neq 0$ *then*

$$\zeta_3^m = 0. \tag{5.18}$$

Proof. Let $v = (0, 0, v_3)$, $v_3 \in H_0^1(\omega)$ we have

$$e_{\alpha\beta}(v) = 0, \;\; e_{\alpha 3}(v) = \frac{1}{2}\partial_\alpha v_3, \;\text{and}\; e_{33}(v) = 0.$$

Substituting in (3.5) and multiplying the equation by ε^2 we have

$$\varepsilon \int_\Omega 2\mu \kappa_{\alpha 3}^{\ell_\varepsilon^m}(\varepsilon)\partial_\alpha v_3 dx = \eta_m(\varepsilon) \int_\Omega u_3^{\ell_\varepsilon^m}(\varepsilon) v_3 dx.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_\omega \zeta_3^m v_3 dx_1 dx_2 = 0 \qquad \forall v_3 \in H_0^1(\omega).$$

Therefore,

$$\zeta_3^m = 0.$$

□

Lemma 5.6. *For each* $m \geq 1$*, there exists a subsequence (still denoted* ε*) such that*

$$\kappa_{\alpha i}^{\ell_\varepsilon^m}(\varepsilon) \rightharpoonup \kappa_{\alpha i}^m(0) \qquad \textit{weakly in } L^2(\Omega), \tag{5.19}$$

with

$$\kappa_{\alpha\beta}^m(0) = e_{\alpha\beta}(\zeta^m), \tag{5.20}$$

and

$$\kappa_{\alpha 3}^m(0) = 0. \tag{5.21}$$

Proof. Convergence (5.19) comes from (5.12) and from (5.15) we have

$$\kappa^{\ell^m_\varepsilon}_{\alpha\beta}(\varepsilon) \rightharpoonup e_{\alpha\beta}(u^m(0)) \qquad \text{weakly in } L^2(\Omega). \tag{5.22}$$

Using (5.16) and (5.18) we obtain

$$u^m_\alpha(0) = \zeta^m_\alpha(x_1, x_2),$$

and then

$$\kappa^m_{\alpha\beta}(0) = e_{\alpha\beta}(\zeta^m).$$

Taking now $v = (v_1, v_2, 0)$ in (3.5), with $v_\alpha \in H^1_\Gamma(\Omega)$, and multiplying the equation by ε, we obtain

$$\begin{aligned}&\varepsilon \int_\Omega \left[4\mu\kappa^{\ell^m_\varepsilon}_{\alpha\beta}(\varepsilon)e_{\alpha\beta}(v) + \kappa^{\ell^m_\varepsilon}_{\alpha\alpha}(\varepsilon)e_{\beta\beta}(v)\right]dx + 4\mu \int_\Omega \kappa^{\ell^m_\varepsilon}_{\alpha3}(\varepsilon)e_{\alpha3}(v)dx + \varepsilon\lambda \int_\Omega \kappa^{\ell^m_\varepsilon}_{33}(\varepsilon)e_{\alpha\alpha}(v)dx \\ &= \varepsilon\eta_m(\varepsilon) \int_\Omega u^{\ell^m_\varepsilon}_\alpha(\varepsilon)v_\alpha dx.\end{aligned} \tag{5.23}$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$4\mu \int_\Omega \kappa^m_{\alpha3}(0)\partial_3 v_\alpha dx = 0, \qquad \forall v_\alpha \in H^1_\Gamma(\Omega),$$

which has as unique solution (see [8])

$$\kappa^m_{\alpha3}(0) = 0.$$

□

Lemma 5.7. *For each $m \geq 1$, there exists a subsequence, still denoted ε, such that*

$$\kappa^{\ell^m_\varepsilon}_{33}(\varepsilon) \rightharpoonup \kappa^m_{33}(0) \quad \textit{weakly in } L^2(\Omega) \tag{5.24}$$

where

$$\kappa^m_{33}(0) = \frac{-\lambda}{\lambda + 2\mu}\kappa^m_{\alpha\alpha}(0). \tag{5.25}$$

Proof. Let $v = (0, 0, v_3)$, $v_3 \in H^1_\Gamma(\Omega)$, we have

$$e_{\alpha\beta}(v) = 0 \quad \text{and} \quad e_{\alpha3}(v) = \frac{1}{2}\partial_\alpha v_3.$$

Substituting in (3.5) and multiplying the equation by ε^2, we obtain

$$\varepsilon \int_\Omega 2\mu\kappa^{\ell^m_\varepsilon}_{\alpha3}(\varepsilon)\partial_\alpha v_3 dx + \lambda \int_\Omega \kappa^{\ell^m_\varepsilon}_{\alpha\alpha}(\varepsilon)\partial_3 v_3 dx \quad + \quad (\lambda + 2\mu) \int_\Omega \kappa^{\ell^m_\varepsilon}_{33}(\varepsilon)\partial_3 v_3 dx = \eta_m(\varepsilon) \int_\Omega u^{\ell^m_\varepsilon}_3(\varepsilon)v_3 dx.$$

Which gives by passing to the limit as $\varepsilon \to 0$

$$\int_\Omega \left[\lambda\kappa^m_{\alpha\alpha}(0) + (\lambda + 2\mu)\kappa^m_{33}(0)\right]\partial_3 v_3 dx = 0, \quad \forall v_3 \in H^1_\Gamma(\Omega),$$

and consequently,

$$\lambda\kappa^m_{\alpha\alpha}(0) + (\lambda + 2\mu)\kappa^m_{33}(0) = 0.$$

□

Lemma 5.8. *The stretching displacements* $(\zeta_1^m, \zeta_2^m) \in [H_0^1(\omega)]^2$ *satisfy for all* $(\xi_1, \xi_2) \in [H_0^1(\omega)]^2$,

$$\begin{aligned} & \frac{E}{1+\nu}\int_\omega e_{\alpha\beta}(\zeta^m)e_{\alpha\beta}(\xi)dx_1dx_2 + \frac{E\nu}{1-\nu^2}\int_\omega e_{\alpha\alpha}(\zeta^m)e_{\beta\beta}(\xi)dx_1dx_2 \\ = \ & \eta_m(0)\int_\omega \zeta_\alpha^m \xi_\alpha dx_1dx_2. \end{aligned} \tag{5.26}$$

Proof. Taking $v = (\xi_1, \xi_2, 0)$ in (3.5) with $(\xi_1, \xi_2) \in [H_0^1(\omega)]^2$, we have

$$\int_\Omega \left[2\mu\kappa_{\alpha\beta}^{\ell_\varepsilon^m}(\varepsilon)e_{\alpha\beta}(v) + \lambda\kappa_{\alpha\alpha}^{\ell_\varepsilon^m}(\varepsilon)e_{\beta\beta}(v)\right]dx \quad + \quad \int_\Omega \lambda\kappa_{33}^{\ell_\varepsilon^m}(\varepsilon)e_{\beta\beta}(v)dx = \eta_m(\varepsilon)\int_\Omega u_\alpha^{\ell_\varepsilon^m}(\varepsilon)v_\alpha dx.$$

Passing to the limit when $\varepsilon \to 0$, we obtain

$$\begin{aligned} 2\mu\int_\omega \kappa_{\alpha\beta}^m(0)e_{\alpha\beta}(\xi)dx_1dx_2 + \lambda\int_\omega \kappa_{\alpha\alpha}^m(0)e_{\beta\beta}(\xi)dx_1dx_2 \\ +\lambda\int_\Omega \kappa_{33}^m(0)e_{\beta\beta}(\xi)dx_1dx_2 = \eta_m(0)\int_\omega u_\alpha^m(0)\xi_\alpha dx_1dx_2, \end{aligned}$$

which can be written, using (5.25)

$$2\mu\int_\omega \kappa_{\alpha\beta}^m(0)e_{\alpha\beta}(\xi)dx_1dx_2 + \frac{2\mu\lambda}{\lambda+2\mu}\int_\omega \kappa_{\alpha\alpha}^m(0)e_{\beta\beta}(\xi)dx_1dx_2 = \eta_m(0)\int_\omega u_\alpha^m(0)\xi_\alpha dx_1dx_2.$$

Replacing $\kappa_{\alpha\beta}^m(0)$ by their expressions (5.20), we obtain

$$2\mu\int_\omega e_{\alpha\beta}(\zeta^m)e_{\alpha\beta}(\xi)dx_1dx_2 + \frac{2\mu\lambda}{\lambda+2\mu}\int_\omega e_{\alpha\alpha}(\zeta^m)e_{\beta\beta}(\xi)dx_1dx_2 = \eta_m(0)\int_\omega \zeta_\alpha^m\xi_\alpha dx_1dx_2.$$

(5.26) comes using relations (2.2). □

Lemma 5.9. *For each* $m \geq 1$*, the whole family* $(\eta_{\ell_\varepsilon^m}(\varepsilon))_{\varepsilon>0}$ *converges as* $\varepsilon \to 0$*. In addition, if* $\eta_m(0)$ *is a simple eigenvalue of* (5.26)*, then* $\eta_{\ell_\varepsilon^m}(\varepsilon)$ *is also a simple eigenvalue of* (3.5) *for* $\varepsilon < \varepsilon_0$ *small enough.*

Proof. See [7]. □

Proposition 5.10. *For each* $m \geq 1$*, the limit eigensolutions* $(\eta_m(0), \zeta_1^m, \zeta_2^m)$ *verify the classical equations of stretching vibrations:*

$$\begin{cases} \dfrac{E}{2(1-\nu^2)}\left[2\dfrac{\partial^2\zeta_1^m}{\partial x_1^2} + (1-\nu)\dfrac{\partial^2\zeta_1^m}{\partial x_2^2} + (1+\nu)\dfrac{\partial^2\zeta_2^m}{\partial x_1\partial x_2}\right] = \eta_m(0)\zeta_1^m, \\ \dfrac{E}{2(1-\nu^2)}\left[2\dfrac{\partial^2\zeta_2^m}{\partial x_2^2} + (1-\nu)\dfrac{\partial^2\zeta_2^m}{\partial x_1^2} + (1+\nu)\dfrac{\partial^2\zeta_1}{\partial x_1\partial x_2}\right] = \eta_m(0)\zeta_2^m, \end{cases} \tag{5.27}$$

with

$$\zeta_1^m = \zeta_2^m = 0 \quad on\ \gamma. \tag{5.28}$$

Proof. Performing an integrating by part in the left-hand side of Eq (5.26) we obtain, for all $(\xi_1, \xi_2) \in [H_0^1(\omega)]^2$

$$
\begin{aligned}
&\frac{E}{1+\nu}\int_\omega\left[\frac{\partial^2\zeta_1^m}{\partial x_1^2}+\frac{1}{2}\frac{\partial^2\zeta_1^m}{\partial x_2^2}+\frac{1}{2}\frac{\partial^2\zeta_2^m}{\partial x_1\partial x_2}\right]\xi_1 dx_1dx_2\\
&+\frac{E}{1+\nu}\int_\omega\left[\frac{\partial^2\zeta_2^m}{\partial x_2^2}+\frac{1}{2}\frac{\partial^2\zeta_2^m}{\partial x_1^2}+\frac{1}{2}\frac{\partial^2\zeta_1^m}{\partial x_1\partial x_2}\right]\xi_2 dx_1dx_2\\
&+\frac{E\nu}{1-\nu^2}\int_\omega\left[\frac{\partial^2\zeta_1^m}{\partial x_1^2}+\frac{\partial^2\zeta_2^m}{\partial x_1\partial x_2}\right]\xi_1 dx_1dx_2\\
&+\frac{E\nu}{1-\nu^2}\int_\omega\left[\frac{\partial^2\zeta_2^m}{\partial x_2^2}+\frac{\partial^2\zeta_1^m}{\partial x_1\partial x_2}\right]\xi_2 dx_1dx_2\\
&=\eta_m(0)\int_\omega \zeta_\alpha^m\xi_\alpha dx_1dx_2
\end{aligned}
$$

which gives, for all $(\xi_1, \xi_2) \in [H_0^1(\omega)]^2$

$$
\begin{aligned}
&\frac{E}{2(1-\nu^2)}\int_\omega\left[2\frac{\partial^2\zeta_1^m}{\partial x_1^2}+(1-\nu)\frac{\partial^2\zeta_1^m}{\partial x_2^2}+(1+\nu)\frac{\partial^2\zeta_2^m}{\partial x_1\partial x_2}\right]\xi_1\, dx_1dx_2\\
&+\frac{E}{2(1-\nu^2)}\int_\omega\left[2\frac{\partial^2\zeta_2^m}{\partial x_2^2}+(1-\nu)\frac{\partial^2\zeta_2^m}{\partial x_1^2}+(1+\nu)\frac{\partial^2\zeta_1^m}{\partial x_1\partial x_2}\right]\xi_2\, dx_1dx_2\\
&=\eta_m(0)\int_\omega \zeta_1^m\xi_1\, dx_1dx_2+\eta_m(0)\int_\omega \zeta_2^m\xi_2\, dx_1dx_2.
\end{aligned}
$$

(5.27) is obtained by taking respectively $\xi_2 = 0$ and $\xi_1 = 0$. □

6. Conclusions

In this work, we have proved that the stretching frequencies of an elastic thin plate is the limit of a family of high frequencies of the three-dimensional elastic model of the plate, as the thickness approaches zero. We have also shown, that the standard spectral problem associated to stretching modes in linear elastic plates can be derived mathematically from the standard three-dimensional eigenvalue problem of linear elasticity through a non-standard asymptotic analysis technique. This technique can be used to study a wide variety of problems of modeling vibrations for thin structure like folded plate and junction between different thin structures.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author gratefully acknowledges the financial support of the Deanship of research of Imam Mohammad Ibn Saud Islamic University (IMSIU).

Conflict of interest

The author declares no conflict of interest.

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