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Research article

Yudovich type solution for the two dimensional Euler-Boussinesq system with critical dissipation and general source term

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Abstract: The present article investigates the two-dimensional Euler-Boussinesq system with critical fractional dissipation and general source term. First, we show that this system admits a global solution of Yudovich type, and as a consequence, we treat the regular vortex patch issue.

Keywords: Boussinesq system; weak solutions; Besov spaces; fractional dissipation; regular vortex patch
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1. Introduction

The generalized $2d$ Euler-Boussinesq system with fractional dissipation is given by the following typical system:

$$\begin{cases} \partial_\tau v + v\cdot\nabla v + \nabla b = \mathbb{H}(\varrho) & \text{if } (\tau,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \partial_\tau \varrho + v\cdot\nabla\varrho + \kappa|\mathbf{D}|^\alpha\varrho = 0 & \text{if } (\tau,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \operatorname{div} v = 0, \\ (v,\varrho)_{|\tau=0} = (v_0,\varrho_0), \end{cases} \tag{1.1}$$

where $v = (v^1; v^2)^{\mathbf{T}}$ is the velocity of the fluid, ϱ represents the density/temperature and the pressure b is a scalar function. The action of the buoyancy forces are represented by the function $\mathbb{H}(\varrho) = (\mathbb{H}_1(\varrho), \mathbb{H}_2(\varrho))^{\mathbf{T}}$. The coefficient $\kappa \geq 0$ stands for the thermal diffusivity and the operator $|\mathbf{D}|^\alpha$ denotes the fractional Zygmund operator and it is defined formally as follows,

$$|\mathbf{D}|^\alpha f(x) \triangleq F^{-1}(|\cdot|^\alpha F(f))(x),$$

where F denotes the Fourier transform and F^{-1} is its inverse. In [5], Brenier introduced and studied the optimal transport and the magnetic relaxation concepts related to the Boussinesq system with general source term. The system (1.1) is a natural generalization for the $2d$ classical Boussinesq system, in fact, the system (1.1) coincides with the classical Boussinesq system when $\mathbb{H}(s) = (0, s)^{\mathbf{T}}$. Now, we list some results regarding the classical Boussinesq system i.e. ($\mathbb{H}(\varrho) = \varrho\vec{e}_2$). The existence of local solution of the inviscid system (with $\kappa = 0$) was proved by many authors in a different spaces, see for instance [7,8]. For the case $\kappa > 0$ and $\alpha = 2$ Chae proved in [6] the global existence for initial datum lies in H^s, with $s > 2$. More recently, Hmidi and Keraani [17] extended the result of [6] from the subcritical regularities to the critical regularities. The study of the classical Boussinesq system in the critical case ($\alpha = 1$) attracted the attention of many authors, where, Hmidi et al. [18] established a new technique to prove the global existence of solutions, this technique uses deeply the structure of the equation and they introduced the "coupled function" $\Gamma \triangleq w - \mathcal{R}\varrho$ with w stands for the vorticity and $\mathcal{R} = \partial_1|D|^{-1}$ is the Riesz transform. Thereafter, Ye [30] gave another approach to studying the classical Boussinesq system in the critical regime. The method used by [30] is based on a prestigious result obtained by Ye and Xue [28], (see Theorem 1.6 below). Concerning the existence/uniqueness problem for the Boussinesq system with general source term (1.1), Sulaiman showed in [25] that the system (1.1) is globally well-posed whenever $\alpha \in (1, 2)$. Recently, Wu and Zheng established in [26] that the system (1.1), with the presence of the vertical dissipation in the velocity and density equations, admits a unique global solution. For more results on this subject we may refer to the papers [1, 20, 22, 27].

The notion of vorticity ($w \triangleq \nabla \times v$) plays an interesting role in the study of the fluid dynamics. By performing the operator $\nabla\times$ to the Eq $(1.1)_1$, we get the following new system:

$$\begin{cases} \partial_\tau w + v \cdot \nabla w = \nabla \times (\mathbb{H}(\varrho)) & \text{if } (\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_\tau \varrho + v \cdot \nabla \varrho + \kappa|\mathbf{D}|^\alpha \varrho = 0 & \text{if } (\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} v = 0, \\ (w, \varrho)_{|\tau=0} = (w_0, \varrho_0). \end{cases} \tag{1.2}$$

We say that initial data is of Yudovich type if the initial vorticity w_0 is bounded, where Yudovich in his famous paper [31] proved the global existence and uniqueness for Euler equation. Recently, Danchin and Paicu [12] proved that the system (1.1) admits unique solution of Yudovich type when $\alpha = 2$. Recently, Xu and Xue [29] extended the results of [12] in the supercritical/critical regime $\alpha \in (0, 1]$, but in the supercritical case, they have imposed a smallness condition on the initial density ϱ_0. A particular sub-class of Yudovich type data is the vortex patch, The initial vorticity w_0 is said to be a regular vortex patch, if the initial vorticity w_0 is the characteristic function of a regular domain D_0. In [9] Chemin showed that, if the initial boundary ∂D_0 is $C^{1+\eta}$ for $\eta \in (0, 1)$, then ∂D_τ is still $C^{1+\eta}$, where $D_\tau \triangleq \Phi(\tau, D_0)$ is the flow image of the initial domain and Φ is the flow map defined as follows

$$\begin{cases} \partial_\tau \Phi(\tau, x) = v(\tau, \Phi(\tau, x)) & \text{if } (\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \Phi_{|\tau=0}(x) = x. \end{cases}$$

The approach of Chemin consists to control the L^∞ norm of the gradient of the velocity with respect to the co-normal regularity of the vorticity by introducing the logarithmic estimate (see also Serfati [24] for another proof). In this context, for the case $\kappa > 0$, and $\alpha = 2$, Hmidi and Zerguine [19] proved that for a smooth initial boundary then the velocity remains Lipschitz for any positive time. More recently, the reference [32] investigates the vortex patch issue for the classical Euler-Boussinesq system with critical dissipation. Hassainia and Hmidi [14] established the local existence for the regular/singular patch for the inviscid Euler-Boussinesq system, this result was improved later by Hmidi et al. in [16]. Also, the first author and Zerguine [21] established a similar results to [14] for the system $(1.1)_{\kappa=0}$. For more information and details we refer the reader to [4, 11, 13, 15, 23] and the references therein. Inspired by the method used by [30] and motivated by the above works, we investigate the generalized Boussinesq system (1.1) with critical dissipation corresponding to the case $\alpha = 1$. The first theorem of this paper deals with the Yudovich type solutions.

Theorem 1.1. *Let $\alpha = 1$ and $\mathbb{H} \in C^1(\mathbb{R};\mathbb{R}^2)$, such that $\mathbb{H}(0) = \mathbf{0}$. We assume that the initial datum $(v_0, \varrho_0) \in L^2(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, and $w_0 \in L^3(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then the system $(1.1)_{\alpha=1}$ has a global unique output (w, ϱ) such that*

$$w \in L^{\infty,loc}(\mathbb{R}_+, L^3(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)), \varrho \in L^{\infty,loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)), |\mathbf{D}|^{3/2}\varrho \in L^{2,loc}(\mathbb{R}_+, L^2(\mathbb{R}^2)).$$

Remark 1.2. *In what follows, κ is a non-negative parameter ($\kappa > 0$), and for the sake of simplicity, we assume that $\kappa = 1$.*

Remark 1.3. *Sulaiman [25] assumed that the function $\mathbb{H}$ is $C^5(\mathbb{R};\mathbb{R}^2)$ function, this assumption is rather strong. In the reference [26], Wu and Zheng supposed that the function $\mathbb{H}$ belongs to $C^2(\mathbb{R};\mathbb{R}^2)$. Compared with these results, in this theorem, we have weakened the assumption imposed on the general source term $\mathbb{H}(\varrho)$, where we have assumed that $\mathbb{H} \in C^1(\mathbb{R};\mathbb{R}^2)$, which is quite natural because we need to deal with the vorticity equation.*

The second result of this manuscript is devoted to studying the regular vortex patch issue, and we have the following theorem:

Theorem 1.4. *Let $\mathbb{H} \in C^3(\mathbb{R};\mathbb{R}^2)$, such that $\mathbb{H}(0) = \mathbf{0}$ and $\eta \in (0,1)$ and D_0 be a bounded smooth domain of $\mathbb{R}^2$ with $\partial D_0 \in C^{1+\eta}(\mathbb{R}^2)$. Suppose the initial vorticity is of the form $w_0 = \mathbf{1}_{D_0}$ such that $w_0 \in C^\eta(X_0)$ and $\varrho_0 \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. Then the system $(1.1)_{\alpha=1}$ has a unique global output*

$$(v, \rho) \in L^{\infty,loc}(\mathbb{R}_+; \dot{W}^{1,\infty}(\mathbb{R}^2)) \times L^{\infty,loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)).$$

Moreover, the boundary of the advected domain $D_\tau \in C^{\eta+1}(\mathbb{R}^2)$, where $D_\tau = \Phi(\tau, D_0)$.

Remark 1.5. *This result can be seen as a generalization of the results obtained in [25, 32]:*

- *Zerguine [32] studied the vortex patch problem for the classical Euler-Boussinesq system (1.1) with critical dissipation i.e. ($\mathbb{H}(\varrho) = \varrho\vec{e}_2$ and $\alpha = 1$), where he have used the technique introduced by [18], that dealing with the coupled function $\Gamma = w - \mathcal{R}\varrho$. Unfortunately, this method can not be adopted in our case, because the presence of the general source term $\mathbb{H}(\varrho)$ spoils the principle of the coupled function used in [18].*
- *Sulaiman [25] established the global well posedness for generalized Euler-Boussinesq system (1.1) with subcritical dissipation $\alpha \in (1, 2)$. This paper treats the critical dissipation $\alpha = 1$.*

The proof of the above theorems heavily lies on the prestigious theorem proved by [28], which is given as follows:

Theorem 1.6. *(Differentiability of the drift-diffusion equation) Consider the advection fractional-diffusion equation with critical fractional dissipation:*

$$\begin{cases} \partial_\tau \varrho + v\cdot\nabla\varrho + \kappa|\mathbf{D}|\varrho = 0 \quad \text{if } (\tau,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \mathrm{div} v = 0. \end{cases} \tag{1.3}$$

Let $\varrho_0 \in L^2(\mathbb{R}^2)$*. Suppose that* $v \in L^\infty_{loc}(\mathbb{R}_+; C^\delta(\mathbb{R}^2))$ *for some positive number* δ*. Then there exists some constant* η *belongs to* $(0,\delta)$ *such that* $\varrho \in L^{\infty,loc}(\mathbb{R}_+; C^{1+\eta}(\mathbb{R}^2))$.

Based on this theorem, we prove that v is a δ-Hölder function for some positive constant $\delta \in (0,\frac{1}{3})$, thus we obtain a bound for $\|w\|_{L^\infty_\tau L^\infty}$. Then we get Yudovich's type solution. The main tool of the proof of Theorem 1.4 is to control the Lipschitz norm of the velocity by using the logarithmic estimate to do so we shall control the co-normal regularity $\partial_{X_\tau} w$ in Hölder space of negative index $C^{\eta-1}$, (see Definitions B.7 and B.8 in Appendix B for the directional derivative ∂_{X_τ}), in the way, we need to deal with the term $\|\mathbb{H}(\varrho)\|_{C^{1+\eta}}$, to get rid of the nonlinear function $\mathbb{H}$, we use the paralinearization theorem (see Theorem A.5), which leads us to assume that $\mathbb{H}$ is a C^3 function, and by using results from Theorem 1.1, we bound the co-normal derivative of the vorticity. The rest of this paper is arranged as follows: The next section is devoted to prove Theorem 1.1. In Section 3, we investigate the vortex patch issue for the system (1.1). We finish this paper with two appendices, the first one gives some basic useful tools and definitions and the second one is devoted to giving some notions and results regarding the vortex patch.

Notations: (1) We denote by C any positive constant that changes from line to line, we shall use the notation $X \lesssim Y$ instead of the notation $\exists C_0 > 0$ such that $X \le C_0 Y$ and C_0 is a positive constant depending on the initial data.

(2) For every $p\in[1,\infty]$, $\|.\|_{L^p}$ denotes the L^p norms.

(3) The space C^η denotes the Hölder space. For $\eta\in\mathbb{R}_+\backslash\mathbb{N}$ the Hölder space C^η coincides with $B^\eta_{\infty,\infty}$.

(4) $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_{ij} = \frac{\partial^2}{\partial x_i x_j}$, where $x=(x_1,x_2)$ and $dx = dx_1 dx_2$.

(5) s on the involved norms of the initial data and the time.

2. Yudovich type solution

In this part, we perform only the classical A priori estimates. For the existence of solutions can be done by the classical arguments.

Proposition 2.1. *Let* (v,ϱ) *be a smooth solution for* (1.1) *such that* $\mathrm{div} v_0 = 0$*. Then we have:*

(i) $\|\varrho(\tau)\|_{L^\eta} \le \|\varrho_0\|_{L^\eta}, \quad \forall\eta\in[2,\infty]$,

(ii) $\|\mathbb{H}'(\varrho)\|_{L^\infty} \le C$,

(iii) $\|v(\tau)\|^2_{L^2} + \|\varrho(\tau)\|^2_{L^2} + \int_0^\tau \||\mathbf{D}|^{1/2}\varrho(s)\|^2_{L^2} ds \le \mu_0(\tau)$.

Proof. For (i) see [10]. For (ii), we use the fact that $\mathbb{H}$ is a C^1 function, so that

$$\|\mathbb{H}_i'(\varrho)\|_{L^\infty} \le \sup_{|x|\le\|\varrho(\tau)\|_{L^\infty}} |\mathbb{H}_i'(x)|.$$

According to (i) we have $\|\varrho(\tau)\|_{L^\infty} \le \|\varrho_0\|_{L^\infty}$. Thus

$$\|\mathbb{H}_i'(\varrho)\|_{L^\infty} \le \sup_{|x|\le\|\varrho_0\|_{L^\infty}} \|\nabla\mathbb{H}_i(x)\| \le C. \tag{2.1}$$

Regarding (iii), We start first by establishing the L^2-estimate of the density, in fact we take the L^2 inner product of $(1.1)_2$ with ϱ, and integrating with respect to time, we obtain

$$\|\varrho(\tau)\|_{L^2}^2 + \int_0^\tau \||\mathbf{D}|^{1/2}\varrho(s)\|_{L^2}^2 ds \le \mu_0(\tau). \tag{2.2}$$

In the next step, we take the L^2 scalar-product of $(1.1)_1$ with v and keep in mind div$v = 0$, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|v(\tau)\|_{L^2}^2 = \int_{\mathbb{R}^2} \mathbb{H}(\varrho)\cdot v dx \le \|\mathbb{H}(\varrho)\|_{L^2}\|v(\tau)\|_{L^2}.$$

Since $\mathbb{H}(0) = \mathbf{0}$ we use Taylor's formula at order 1, we get

$$\mathbb{H}(\varrho) = \varrho\int_0^1 \mathbb{H}'(\varrho + r\varrho)dr.$$

From (ii), we get

$$\|\mathbb{H}(\varrho)\|_{L^2} \lesssim \|\varrho\|_{L^2}.$$

In view of (2.2) we obtain the desired inequality (iii). □

Proposition 2.2. *Let* (v,ϱ) *be a smooth solution for* (1.1) *such that* div$v_0 = 0$. *Then we have:*

$$\|\nabla\varrho(\tau)\|_{L^2}^2 + \|w(\tau)\|_{L^3}^3 + \int_0^\tau \||\mathbf{D}|^{3/2}\varrho(s)\|_{L^2}^2 ds \le \mu_0(t), \tag{2.3}$$

$$\|w(\tau)\|_{L^\infty} + \|\varrho(\tau)\|_{C^{1+\eta}} \le \mu_0(\tau), \tag{2.4}$$

with $\eta \in (0,\delta)$ *and* $\delta \in (0,\frac{1}{3})$.

Proof. Testing the Eq (1.1) with $\Delta\varrho$ and integrating with respect to space variable, we find

$$\frac{1}{2}\frac{d}{d\tau}\|\nabla\varrho(\tau)\|_{L^2}^2 + \||\mathbf{D}|^{3/2}\varrho(s)\|^2 = |\int_{\mathbb{R}^2}(v\cdot\nabla\varrho)\Delta\varrho dy|.$$

From the incompressibility condition (div$v = 0$), we obtain

$$|\int_{\mathbb{R}^2}(v\cdot\nabla\varrho)\Delta\varrho dy| = |\int_{\mathbb{R}^2} D_k v_j D_j\varrho D_k\varrho dy|.$$

From Hölder inequality, we get

$$|\int_{\mathbb{R}^2} D_k v_j D_j\varrho D_k\varrho dy| \le \|\nabla v\|_{L^3}\|\nabla\varrho\|_{L^3}^2,$$

thus

$$\begin{aligned} |\int_{\mathbb{R}^2} D_k v_j D_j\varrho D_k\varrho dy| &\lesssim \|\nabla v\|_{L^3}\|\nabla\varrho\|_{\dot{B}^{-1}_{\infty,\infty}}^{2\theta}\|\nabla\varrho\|_{\dot{B}^{1/2}_{2,2}}^{2(1-\theta)} \\ &\lesssim \|w\|_{L^3}\|\varrho\|_{\dot{B}^{0}_{\infty,\infty}}^{2\theta}\|\varrho\|_{\dot{B}^{3/2}_{2,2}}^{2(1-\theta)} \\ &\lesssim \|w\|_{L^3}\|\varrho\|_{L^\infty}^{2\theta}\||\mathbf{D}|^{3/2}\varrho\|_{L^2}^{2(1-\theta)}. \end{aligned}$$

From the inequality (ii) in Proposition 2.1 and Young's inequality, we find out

$$|\int_{\mathbb{R}^2} D_k v_j D_j \varrho D_k \varrho dy| \leq C\|w\|_{L^3}^3 \|\varrho_0\|_{L^\infty}^2 + \frac{1}{4}\||\mathbf{D}|^{3/2}\varrho\|_{L^2}^2. \tag{2.5}$$

Hence

$$\frac{1}{2}\frac{d}{d\tau}\|\nabla\varrho(\tau)\|_{L^2}^2 + \||\mathbf{D}|^{3/2}\varrho(\tau)\|_{L^2}^2 \leq C\|w\|_{L^3}^3\|\varrho_0\|_{L^\infty}^2 + \frac{1}{4}\||\mathbf{D}|^{3/2}\varrho\|_{L^2}^2. \tag{2.6}$$

The classical L^3-estimates of the vorticity equation gives:

$$\frac{1}{3}\frac{d}{d\tau}\|w(\tau)\|_{L^3}^3 \leq \|\nabla\mathbb{H}\|_{L^\infty}\|\nabla\varrho\|_{L^3}\|w\|_{L^3}^2.$$

Combining (ii)-Proposition 2.1, Young's inequality and interpolation theorem, we conclude

$$\begin{aligned}\frac{1}{3}\frac{d}{d\tau}\|w(\tau)\|_{L^3}^3 &\lesssim \|\nabla\varrho\|_{\dot{B}^{-1}_{\infty,\infty}}^{2\theta}\|\nabla\varrho\|_{\dot{B}^{1/2}_{2,2}}^{2(1-\theta)}\|w\|_{L^2}^2\\ &\leq C\|w\|_{L^3}^3\|\varrho_0\|_{L^\infty}^{1/2} + \frac{1}{4}\||\mathbf{D}|^{3/2}\varrho\|_{L^2}^2.\end{aligned}$$

Gathering the last inequality with (2.6) and using Gronwall's inequality, we obtain (2.3).

At this stage, we prove that the velocity v belongs to C^δ, with $\delta \in (0, \frac{1}{3})$. In fact, since $C^\delta \approx B^\delta_{\infty,\infty}$, then according to Bernstein inequality, we find

$$\begin{aligned}\|v(\tau)\|_{C^\delta} &\lesssim \|\Delta_{-1}v(\tau)\|_{L^2} + \sup_{j\in\mathbb{N}}\{2^{j\delta}\|\Delta_j v\|_{L^\infty}\}\\ &\lesssim \|v(\tau)\|_{L^2} + \sup_{j\in\mathbb{N}}\{2^{j(\delta+\frac{2}{3})}\|\Delta_j v\|_{L^3}\}.\end{aligned}$$

From the Biot-Savart law, we have $v \triangleq \nabla^\perp\Delta^{-1}w$, then according to the fact that $\delta - \frac{1}{3} < 0$, Propositions 2.1 and 2.2, we infer that

$$\begin{aligned}\|v(\tau)\|_{C^\delta} &\lesssim \|v(\tau)\|_{L^2} + \sup_{j\in\mathbb{N}}\{2^{j(\delta-\frac{1}{3})}\|\Delta_j w\|_{L^3}\}\\ &\lesssim \|v(\tau)\|_{L^2} + \|w(\tau)\|_{L^3} \lesssim \mu_0(\tau).\end{aligned}$$

Hence by using the differentiability of the drift-diffusion equation (Theorem 1.6), we conclude that there exists $\delta \in (0, \frac{1}{3})$ such that $\varrho \in L^{\infty,loc}([0,\infty); C^{1+\eta}(\mathbb{R}^2))$. In particular $\nabla\varrho \in L^{\infty,loc}([0,\infty); L^\infty(\mathbb{R}^2))$. Therefore, we find

$$\|w(\tau)\|_{L^\infty} \leq \mu_0(\tau).$$

The proof of Proposition 2.2 is now achieved. □

3. Vortex patch issue

In this part, we explore the regular vortex patch issue for the nonlinear Euler-Boussinesq system with general source term.

Proof of Theorem 1.4. Our first step is to control the Lipschitz norm of the velocity, we start by estimating the quantity $\|\partial_{X_{\tau,\lambda}} w\|_{C^{\eta-1}}$, for this aim we will use the commutation beetwen the operator $\partial_{X_{\tau,\lambda}}$ with transport operator, then

$$(\partial_\tau + v \cdot \nabla)\partial_{X_{\tau,\lambda}} w = \partial_{X_{\tau,\lambda}}(\nabla \times (\mathbb{H}(\varrho))).$$

Using Lemma B.10, we obtain

$$\|\partial_{X_{\tau,\lambda}} w\|_{C^{\eta-1}} \leq e^{C\mathcal{V}(\tau)}(\|\partial_{X_{0,\lambda}} w_0\|_{C^{\eta-1}} + C\int_0^\tau e^{-C\mathcal{V}(\tau)}\|\partial_{X_{\tau,\lambda}}(\nabla \times (\mathbb{H}(\varrho)))\|_{C^{\eta-1}} ds, \tag{3.1}$$

where $\mathcal{V}(\tau) = \int_0^\tau \|\nabla v(t)\|_{L^\infty} dt$. Since $\partial_{X_{\tau,\lambda}}(\nabla \times \mathbb{H}(\varrho)) \triangleq \mathrm{div}(\nabla \times (\mathbb{H}(\varrho)) \cdot X_{\tau,\lambda}) - \nabla \times (\mathbb{H}(\varrho))\mathrm{div} X_{\tau,\lambda}$. Then

$$\|\partial_{X_{\tau,\lambda}}(\nabla \times (\mathbb{H}(\varrho)))\|_{C^{\eta-1}} \leq \|\mathrm{div}(\nabla \times (\mathbb{H}(\varrho)) \cdot X_{\tau,\lambda})\|_{C^{\eta-1}} + \|(\nabla \times (\mathbb{H}(\varrho)))\mathrm{div} X_{\tau,\lambda}\|_{C^{\eta-1}}.$$

Since C^η is an algebra, then the embedding $L^\infty \hookrightarrow C^{\eta-1}$ and the paralinearization theorem (Theorem A.5), yellowgive

$$\begin{aligned}
\|\partial_{X_{\tau,\lambda}}(\nabla \times (\mathbb{H}(\varrho)))\|_{C^{\eta-1}} &\lesssim \|\nabla \times (\mathbb{H}(\varrho)) \cdot X_{\tau,\lambda}\|_{C^\eta} + \|(\nabla \times (\mathbb{H}(\varrho)))\mathrm{div} X_{\tau,\lambda}\|_{C^{\eta-1}} \\
&\lesssim \|\varrho\|_{C^{1+\eta}} \widetilde{\|} X_{\tau,\lambda}\|_{C^\eta},
\end{aligned}$$

see the identity (B.1) for the definition of the norm $\widetilde{\|} \cdot \|_{C^\eta}$. Substituting the last estimate into (3.1), we deduce that

$$\|\partial_{X_{\tau,\lambda}} w\|_{C^{\eta-1}} \leq e^{C\mathcal{V}(\tau)}(\|\partial_{X_{0,\lambda}} w_0\|_{C^{\eta-1}} + C\int_0^\tau e^{-C\mathcal{V}(s)}\|\varrho(\tau)\|_{C^{1+\eta}} \widetilde{\|} X_{\tau,\lambda}\|_{C^\eta} ds). \tag{3.2}$$

On the other hand, according to Lemma B.10, we find out

$$\|X_{\tau,\lambda}\|_{C^\eta} \leq e^{C\mathcal{V}(\tau)}(\|X_{0,\lambda}\|_{C^\eta} + C\int_0^\tau e^{-C\mathcal{V}(s)}\|\partial_{X_{\tau,\lambda}} v(s)\|_{C^\eta} ds).$$

We make the following emphasis:

$$\|\partial_{X_{\tau,\lambda}} v\|_{C^\eta} \lesssim \|\partial_{X_{\tau,\lambda}} w\|_{C^{\eta-1}} + \|\mathrm{div} X_{\tau,\lambda}\|_{C^\eta}\|w\|_{L^\infty} + \|X_{\tau,\lambda}\|_{C^\eta}\|\nabla v\|_{L^\infty},$$

and for more information, see, e.g. [9, 14]. Therefore

$$\begin{aligned}
e^{-C\mathcal{V}(\tau)}\|X_{\tau,\lambda}\|_{C^\eta} \leq\ & \|X_{0,\lambda}\|_{C^\eta} + C\int_0^\tau e^{-C\mathcal{V}(s)}(\|\partial_{X_{s,\lambda}} w(s)\|_{C^{\eta-1}} + \|\mathrm{div} X_{s,\lambda}\|_{C^\eta}\|w(s)\|_{L^\infty}) ds \\
&+ \int_0^\tau e^{-C\mathcal{V}(s)}\|X_{s,\lambda}\|_{C^\eta}\|\nabla v(s)\|_{L^\infty} ds.
\end{aligned}$$

From Gronwall's inequality, we obtain

$$\|X_{\tau,\lambda}\|_{C^\eta} \leq e^{C\mathcal{V}(\tau)}(\|X_{0,\lambda}\|_{C^\eta} + C\int_0^\tau e^{-C\mathcal{V}(s)}\Big(\|\partial_{X_{s,\lambda}} w(s)\|_{C^{\eta-1}} + \|\mathrm{div} X_{s,\lambda}\|_{C^\eta}\|w(s)\|_{L^\infty}) ds\Big). \tag{3.3}$$

Putting together (B.4) and (3.3), we get

$$\widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} \lesssim e^{C\mathcal{V}(\tau)}(\widetilde{\|X_{0,\lambda}\|}_{C^\eta} + \|\mathrm{div}X_{0,\lambda}\|_{C^\eta}\|w\|_{L^1_t L^\infty} + C\int_0^\tau e^{-C\mathcal{V}(s)}\|\partial_{X_{s,\lambda}}w(s)\|_{C^{\eta-1}}ds).$$

From Proposition 2.2, we find

$$\begin{aligned} e^{-C\mathcal{V}(\tau)}\widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} &\lesssim \widetilde{\|X_{0,\lambda}\|}_{C^\eta} + C\tau\|\mathrm{div}X_{0,\lambda}\|_{C^\eta} + C\int_0^\tau e^{-C\mathcal{V}(s)}\|\partial_{X_{s,\lambda}}w(s)\|_{C^{\eta-1}}ds \\ &\lesssim 1 + \tau + \int_0^\tau e^{-C\mathcal{V}(s)}\|\partial_{X_{s,\lambda}}w(s)\|_{C^{\eta-1}}ds. \end{aligned} \tag{3.4}$$

Inserting (3.2) into (3.4), we obtain

$$e^{-C\mathcal{V}(s)}\widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} \lesssim 1 + \tau + \int_0^\tau e^{-CV(S)}\widetilde{\|X_{s,\lambda}\|}_{C^\eta}\|\varrho(s)\|_{C^{1+\eta}}ds.$$

Gronwall's inequality ensures that

$$\widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} \lesssim e^{C\tau}e^{C\mathcal{V}(\tau)}e^{\|\varrho\|_{L^1_\tau C^{1+\eta}}}. \tag{3.5}$$

However, we also have

$$\|\varrho\|_{L^1_\tau C^{1+\eta}} \lesssim \tau\|\varrho\|_{L^\infty_\tau C^{1+\eta}}.$$

According to inequality (2.4), we infer that

$$\widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} \lesssim e^{C_0\tau}e^{C\mathcal{V}(\tau)}. \tag{3.6}$$

Consequently, (3.2) becomes

$$\|\partial_{X_{\tau,\lambda}}w\|_{C^{\eta-1}} \lesssim e^{C_0\tau}e^{C\mathcal{V}(\tau)}. \tag{3.7}$$

Gathering together (3.6) and (3.7), we find

$$\|\partial_{X_{\tau,\lambda}}w\|_{C^{\eta-1}} + \widetilde{\|X_{\tau,\lambda}\|}_{C^\eta} \lesssim e^{C_0\tau}e^{C\mathcal{V}(\tau)}. \tag{3.8}$$

According to (B.3), we obtain

$$\|\partial_{X_{\tau,\lambda}}w\|_{C^\eta(X_\tau)} \lesssim e^{C_0\tau}e^{C\mathcal{V}(\tau)}. \tag{3.9}$$

Combining the logarithmic estimate Theorem B.11 with the monotonicity of mapping $y \mapsto y\log(e + \frac{a}{y})$ and (3.9), we find out

$$\begin{aligned} \|\nabla v(\tau)\|_{L^\infty} &\le C(\|w\|_{L^3} + \|w\|_{L^\infty}\log(e + \|w\|_{C^\eta(X_\tau)})) \\ &\lesssim 1 + \|w\|_{L^3} + \|w\|_{L^\infty} + t + \int_0^\tau \|\nabla v(s)\|_{L^\infty}ds. \end{aligned} \tag{3.10}$$

According to Proposition 2.2, then (3.10) takes the following form

$$\|\nabla v(\tau)\|_{L^\infty} \le \mu_0(\tau) + \int_0^\tau \|\nabla v(s)\|_{L^\infty}ds.$$

Using Gronwall's lemma, we obtain

$$\|\nabla v(\tau)\|_{L^\infty} \lesssim \mu_0(\tau).$$

Therefore

$$\|w\|_{C^\eta(X_t)} \lesssim \mu_0(\tau). \tag{3.11}$$

To estimate the flow Φ in $C^\eta(X_\tau)$ we use the identity $\partial_{X_\tau}\Phi(\tau) \triangleq X_{\tau,\lambda}\circ\Phi(\tau)$ and $\partial_{X_\tau}\Phi(\tau) \triangleq X_{\tau,\lambda}\cdot\nabla\Phi(\tau)$, we obtain

$$\|X_{\tau,\lambda} \circ \Phi(\tau)\|_{C^\eta} \leq \|X_{\tau,\lambda}\|_{C^\eta}\|\nabla\Phi(\tau)\|_{L^\infty}.$$

On the other hand we have the classical estimate $\|\nabla\Phi(\tau)\|_{L^\infty} \leq Ce^{C\mathcal{V}(\tau)}$ (see for instance [9]). Therefore, we obtain

$$\|X_{\tau,\lambda} \circ \Phi(\tau)\|_{C^\eta} \lesssim \|X_{\tau,\lambda}\|_{C^\eta}e^{C\mathcal{V}(\tau)}. \tag{3.12}$$

Putting (3.6) into (3.12), we deduce that

$$\|\partial_{X_\tau}\Phi(\tau)\|_{C^\eta} \lesssim \mu_0(\tau).$$

Gathering the last estimate with (3.11), we conclude that

$$\|\partial_{X_\tau}\Phi(\tau)\|_{C^\eta} + \|w\|_{C^\eta(X_t)} \lesssim \mu_0(\tau).$$

Boundary regularity. Since the boundary ∂D_0 is a curve of class $C^{1+\eta}$, there exists a real function h_0 and a neighborhood V_0 such that

(i) $h_0 \in C^{1+\eta}$, $\quad \nabla h_0(x) \neq 0$, on V_0.
(ii) $\partial D_0 = h_0^{-1}(\{0\}) \cap V_0$.

Let ς be a smooth function satisfying

$$\text{supp}\varsigma \subset V_0, \quad \varsigma(x) = 1, \quad \forall x \in V_1$$

with V_0 is a small nighbrohood of V_1. Now we construct an admissible family as follows, we set

$$X_{0,0}(x) = \begin{pmatrix} -\partial_2 h_0(x) \\ \partial_1 h_0(x) \end{pmatrix}, \quad X_{0,1}(x) = (1 - \varsigma(x))\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Set $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$, we have $X_{0,\lambda}$ belongs to C^η as well as its divergence, we can check that $X_{0,\lambda}$ is admissible in the sense of Definition B.7, see for instance [19]. Since $w_0 = \mathbf{1}_{D_0}$, we obtain

$$\partial_{\nabla^\perp h_0} w_0 \quad = \quad 0.$$

On the other hand $1 - \varsigma \equiv 0$ on V_1 then $\partial_{X_{0,1}} w_0 = 0$ in view of Theorem 1.1, the system (1.1) has a unique local solution

$$(v, \varrho) \in L^\infty_{loc}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^2)) \times L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)).$$

Next, we move to prove the regularity of the transported initial domain D_τ. We parametrize the boundary ∂D_0 as follows:

Let $y_0 \in \partial D_0, \gamma^0 \in C^\eta(\mathbb{R}_+, \mathbb{R}^2)$ solves the following equation:

$$\begin{cases} \partial_\sigma \gamma^0(s) = X_{0,0}(\gamma^0(s)) \\ \gamma^0(0) = x_0 \end{cases}$$

for every $t \geq 0$,

$$\gamma(\tau, s) \triangleq \Phi(\tau, \gamma^0(s)).$$

Differentiating with respect to the variable s, we get

$$\partial_s \gamma(\tau, s) = (\partial_{X_{0,0}} \Phi)(\tau, \gamma^0(s)).$$

Thus using that $\partial_{X_{0,0}} \Phi \in L^\infty([0, T_0], C^\eta(\mathbb{R}^2))$, we infer that $\gamma(\tau) \in L^\infty([0, T_0], C^{1+\eta}(\mathbb{R}^2))$. This completes the regularity persistence of the boundary ∂D_τ. □

4. Conclusions

In this work, we have shown that the two-dimensional Boussinesq system with critical dissipation and general source term admits a unique global solution of the Yudovich type. Furthermore, the regular vortex patch issue has been successfully addressed.

Appendix

Appendix A

In this appendix, we give some useful tools and notions. We begin with the definition of the dyadic blocks.

Definition A.1. *There exist two radial positive functions* $(\phi, \psi) \in \mathfrak{D}(\mathbb{R}^2) \times \mathfrak{D}(\mathbb{R}^2 - \{0\})$ *such that*

$$\forall x \in \mathbb{R}^2, \qquad \phi(x) + \sum_{j \geq 0} \psi(2^j x) = 1.$$

For every $\varpi \in S'(\mathbb{R}^d)$, the homogenous dyadic blocks are defined as follows: Let $j \in \mathbb{Z}$,

$$\dot{\Delta}_j \varpi \triangleq \psi(2^{-j}\partial)\varpi,$$

and the nonhomogenous dyadic blocks are defined as follows:

$$\Delta_j \varpi \triangleq \begin{cases} 0 & \text{if } j \leq -2 \\ \phi(\partial)\varpi & \text{if } j = -1, \\ \psi(2^{-j}\partial)\varpi & \text{if } j \in \mathbb{N}. \end{cases}$$

The following result is the famous Bernstein's inequality, for the proof see [3].

Proposition A.2. *Let* $1 \leq p \leq q \leq \infty$*. Suppose* $\varpi \in L^p$ *then for every* $\gamma \in \mathbb{N}^d$*, there exists two constants* C_1, C_2 *such that*

$$\text{supp}\widehat{\varpi} \subset \{|\xi| \leq A_0 2^q\} \Rightarrow \|\partial^\gamma \varpi\|_{L^q} \leq C_1 2^{q(|\gamma| + d(\frac{1}{p} - \frac{1}{q})} \|\varpi\|_{L^p},$$

$$\text{supp}\widehat{\varpi} \subset \{A_1 2^q \leq |\xi| \leq A_2 2^q\} \Rightarrow \|\varpi\|_{L^p} \leq C_2 2^{-qk} \sup_{|\gamma|=k} \|\partial^k \varpi\|_{L^p},$$

where A_0, A_1 *and* A_2 *are non-negative constant.*

Now we recall the homogeneous and nonhomogeneous Besov spaces in terms of the Littlewood-Paley operators.

Definition A.3. *The homogeneous Besov space* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *is the set of all tempered distributions* $v \in \mathcal{S}'(\mathbb{R}^d)$ *with*

$$\|\varpi\|_{\dot{B}^s_{p,r}} \triangleq (2^{js}\|\dot{\Delta}_j \varpi\|_{L^p})_{\ell^r(\mathbb{Z})} < \infty,$$

for $(s, p) \in \mathbb{R} \times [1, +\infty]$ *and* $1 \leq r < \infty$*. Besides, if* $r = \infty$

$$\|\varpi\|_{\dot{B}^s_{p,\infty}} \triangleq \sup_{j\in\mathbb{Z}} (2^{js}\|\dot{\Delta}_j \varpi\|_{L^p}) < \infty.$$

Definition A.4. *The nonhomogeneous Besov space* $B^s_{p,r}(\mathbb{R}^d)$ *is the set of all tempered distributions* $\varpi \in \mathcal{S}'(\mathbb{R}^d)$ *with*

$$\|\varpi\|_{B^s_{p,r}} \triangleq (2^{js}\|\Delta_j \varpi\|_{L^p})_{\ell^r(\mathbb{Z})} < \infty,$$

for $(s, p) \in \mathbb{R} \times [1, +\infty]$ *and* $1 \leq r < \infty$*. Besides, if* $r = \infty$

$$\|\varpi\|_{B^s_{p,\infty}} \triangleq \sup_{j\in\mathbb{N}\cup\{-1\}} (2^{js}\|\Delta_j \varpi\|_{L^p}) < \infty.$$

The next Theorem is the paralinearization theorem, see [2].

Theorem A.5. *Let* $\mathbb{H} \in C^{[s]+2}$*, with* $\mathbb{H}(0) = 0$ *and* $s \in [0, \infty]$*. Assume that* $f \in B^s_{p,r} \cap L^\infty$*, with* $(p, r) \in [1, +\infty]^2$*, then* $\mathbb{H} \circ f \in B^s_{p,r}$ *and satisfying*

$$\|\mathbb{H} \circ f\|_{B^s_{p,r}} \leq C(s) \sup_{|y|\leq C\|f\|_{L^\infty}} \|\mathbb{H}^{[s]+2}(x)\|_{L^\infty}\|f\|_{B^s_{p,r}}.$$

We end this part with an interpolation theorem, for more details see [3].

Theorem A.6. *Let* $1 \leq a < b < \infty$*, and* α *positive number. A constant exists, such that*

$$\|\varpi\|_{L^b} \leq C\|\varpi\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|\varpi\|^{\theta}_{\dot{B}^{-\beta}_{a,a}}, \quad \beta = \alpha(\frac{b}{a} - 1); \theta = \frac{a}{b}.$$

Appendix B

In this part, we provide some definitions and notations concerning the admissible family of vector fields and the anisotropic Hölder space. These quantities constitute the main ingredients concerning the vortex patch issue.

Definition B.7. *Let* $\eta \in (0,1)$*. Let* $X = (X_\lambda)_{(\lambda\in\Lambda)}$ *be a family of vector fields. We assert that this family is admissible of class* C^η *if and only if:*

- *Regularity:* X_λ, *div* $X_\lambda \in C^\eta$.
- *Non-degeneracy:*

$$J(X) \triangleq \inf_{x\in\mathbb{R}^d} \sup_{\lambda\in\Lambda} |X_\lambda(x)| > 0.$$

We set

$$\widetilde{\|X_\lambda\|}_\eta \triangleq \|X_\lambda\|_{C^\eta} + \|\mathrm{div}\ X_\lambda\|_{C^{\eta-1}}, \tag{B.1}$$

and

$$M_\eta(X) \triangleq \sup_{\lambda\in\Lambda} \frac{\widetilde{\|X_\lambda\|}_\eta}{J(X)}.$$

The action of the family X_λ on bounded real-valued functions u in the weak meaning as follows:

$$\partial_{X_\lambda} u \triangleq \mathrm{div}(uX_\lambda) - u\mathrm{div}X_\lambda.$$

Now, for all $\tau \in [0,T]$ the transported $X_\tau = (X_{\tau,\lambda})$ of an initial family $X_0 = (X_{0,\lambda})$ by the flow Φ, is defined by

$$X_{\tau,\lambda}(x) \triangleq (\partial_{X_{0,\lambda}}\Phi(t))(\Phi^{-1}(\tau,x)). \tag{B.2}$$

The next definition deals with the concept of anisotropic Hölder space, denoted by $C^{\eta+k}(X)$. We refer the reader to [9] for more details.

Definition B.8. *Let $\eta \in (0,1)$ and $k \in \mathbb{N}$. We assume that a family of vector fields $X = (X_\lambda)_\lambda$ be as in Definition B.7. Then, the space $C^{\eta+k}(X_\lambda)$ is defined as the space of functions $v \in W^{k,\infty}$ such that*

$$\sum_{|\alpha|\le k} \|\partial^\alpha v\|_{L^\infty} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} v\|_{C^{\eta+k-1}} < \infty,$$

and we set

$$\|v\|_{C^{\eta+k}(X)} \triangleq M_\eta(X) \sum_{|\alpha|\le k} \|\partial^\alpha v\|_{L^\infty} + \sup_{\lambda\in\Lambda} \frac{\|\partial_{X_\lambda} v\|_{C^{\eta+k-1}}}{J(X)}.$$

Following that, we list a few family properties. For more information, we reference [9].

Corollary B.9. *There exists a constant C such that for any smooth solution (v,ϱ) of (1.1) on $[0,T]$ and any time dependent family of vector field X_τ transported by the flow of v, we have for all $\tau \in [0,T]$,*

$$J(X_\tau) \ge J(X_0)e^{-C\mathcal{V}(\tau)} \tag{B.3}$$

$$\|\mathrm{div}X_{\tau,\lambda}\|_{C^\eta} \le \|\mathrm{div}X_{0,\lambda}\|_{C^\eta} e^{C\mathcal{V}(\tau)}. \tag{B.4}$$

The next lemma is a useful result used in our proof, for the proof see [9].

Lemma B.10. *Let v be a smooth divergence-free vector field, and $s \in]-1,1[$. Consider (ϖ, f) a couple of functions belonging to $L^{\infty,loc}(\mathbb{R},C^s) \times L^{1,loc}(\mathbb{R},C^s)$ and such that*

$$\begin{cases} \partial_\tau\varpi + v\cdot\nabla\varpi = f. \\ \varpi_{|\tau=0} = \varpi_0. \end{cases}$$

Then, we have

$$\|\varpi(\tau)\|_{C^s} \lesssim \|\varpi_0\|_{C^s} e^{C\mathcal{V}(\tau)} + \int_0^\tau \|f(r)\|_{C^s} e^{C\mathcal{V}(r)} dr. \tag{B.5}$$

Where the constant C depends only on s and

$$\mathcal{V}(\tau) \triangleq \int_0^\tau \|\nabla v(r)\|_{L^\infty} dr.$$

Next, we state the main tool used in the proof of Theorem B.11, which is the logarithmic estimate introduced by Chemin [9].

Theorem B.11. *Let $a \in (1,\infty), \eta \in (0,1)$ and X be a family of vector fields as in Definition B.7. Consider $w \in C^{\eta}(X) \cap L^{a}$. Assume v be a divergence-free vector field with vorticity w. Then there exists C such that*

$$\|\nabla v(\tau)\|_{L^\infty} \leq C(a,\eta)\Big(\|w(\tau)\|_{L^a} + \|w(t)\|_{L^\infty}\log\Big(e + \frac{\|w(\tau)\|_{C^{\eta}(X)}}{\|w(t)\|_{L^\infty}}\Big)\Big).$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No conflicts of interest are related to this work.

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