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Research article

Global regularity to the 3D Cauchy problem of inhomogeneous magnetic Bénard equations with vacuum

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Abstract: This paper deals with the Cauchy problem of 3D inhomogeneous incompressible magnetic Bénard equations. Through some time-weighted a *priori* estimates, we prove the global existence of strong solution provided that the upper boundedness of initial density and initial magnetic field satisfy some smallness condition. Furthermore, we also obtain large time decay rates of the solution.

Keywords: inhomogeneous case; magnetic Bénard equations; global regularity; decay estimates
Mathematics Subject Classification: 35Q35, 76D03, 76W05

1. Introduction

In this paper, we consider the following inhomogeneous incompressible magnetic Bénard equations in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u\cdot\nabla u + \nabla p = \mu\Delta u + b\cdot\nabla b + \rho\theta e_3, \\ \rho\theta_t + \rho u\cdot\nabla\theta - \kappa\Delta\theta = \rho u\cdot e_3, \\ b_t + u\cdot\nabla b = \eta\Delta b + b\cdot\nabla u, \\ \operatorname{div} u = \operatorname{div} b = 0, \end{cases} \tag{1.1}$$

which is equipped with the following initial conditions:

$$(\rho, \rho u, \rho\theta, b)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0\theta_0, b_0)(x) \quad \text{for } x \in \mathbb{R}^3. \tag{1.2}$$

Here, the unknown functions ρ, u, θ, b and p are the density, absolute temperature, magnetic field and pressure of the fluid, respectively. $\mu > 0$ stands for the viscosity constant and $\kappa > 0$ is the heat conductivity coefficient. $\eta > 0$ is the magnetic diffusive coefficient. $e_3 = (0, 0, 1)^T$.

The magnetic Bénard equations (1.1) illuminates the heat convection phenomenon under the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas (see [12] for details). If we ignore the Rayleigh-Bénard convection term $u \cdot e_2$, system (1.1) recovers the inhomogeneous incompressible MHD equations (i.e., $\theta \equiv 0$), which have been discussed in numerous studies on the existence, uniqueness, and regularity of the solutions, please see [1–4, 6, 7, 9, 11, 14–16, 22] and references therein.

When $b \equiv 0$, system (1.1) reduces to inhomogeneous incompressible Bénard system. For the initial density allowing vacuum states, imposing a compatibility condition on the initial data, Cho-Kim [5] showed the local existence of strong solution in bounded domains $\Omega \subset \mathbb{R}^N (N = 2, 3)$. Later on, Zhong [17] removed the compatibility condition, and extended the Cho-Kim's results to the whole space $\mathbb{R}^2$. Meanwhile, Zhong [18, 19] showed the global existence of strong solutions to the 2D Cauchy problem and 2D initial boundary value problem with general large data, respectively. Recently, using time-weighted estimates, Zhong [20] investigated the global existence and exponential decay of strong solutions without a compatibility condition, provided that some initial conditions are suitably small.

Similar to the results achieved for inhomogeneous Bénard system, the authors [8, 21] respectively showed the local and global existence of strong solutions to the Cauchy problem of (1.1) and (1.2) in $\mathbb{R}^2$. However, the global existence of strong solution to the 3D Cauchy problem of (1.1) and (1.2) with vacuum is not addressed. In fact, this is the main aim of this paper.

Before formulating our main results, we first explain the notations and conventions used throughout this paper. For simplicity, we set

$$\int f dx = \int_{\mathbb{R}^3} f dx, \quad \mu = \kappa = \eta = 1.$$

For $1 \le r \le \infty$, $k \ge 1$, the Sobolev spaces are defined in a standard way.

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad D^{k,r} = D^{k,r}(\mathbb{R}^3) = \{v \in L^1_{\text{loc}}(\mathbb{R}^3) | D^k v \in L^r(\mathbb{R}^3)\}, \\ W^{k,p} = W^{k,p}(\mathbb{R}^3), \quad H^k = W^{k,2}, \quad D^k = D^{k,2}. \end{cases}$$

The main result of this paper is formulated in the following theorem.

Theorem 1.1. *For constants $\bar{\rho} > 0$, assume that the initial data $(\rho_0, u_0, \theta_0, b_0)$ satisfies*

$$\begin{cases} 0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in L^1 \cap H^1 \cap W^{1,6}, \quad (\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0) \in L^2, \\ (\nabla u_0, \nabla \theta_0) \in L^2, \quad b_0 \in H^1, \quad \text{div} u_0 = \text{div} b_0 = 0. \end{cases} \tag{1.3}$$

Then, there is a positive constant ϵ_0, which depends only on $\bar{\rho}$ and the initial data, such that if

$$\bar{\rho} + \|b_0\|_{L^3} := \chi_0 \le \epsilon_0, \tag{1.4}$$

then the problems (1.1) *and* (1.2) *has a unique global solution* (ρ, u, θ, b) *on* $\mathbb{R}^3 \times (0, T)$ *satisfying*

$$\begin{cases} 0 \le \rho \in L^\infty(0,T;L^1 \cap L^\infty),\ \nabla\rho \in L^\infty(0,T;L^2 \cap L^6), \\ \rho_t \in L^\infty(0,T;L^2 \cap L^3),\ \rho \in C([0,T];L^q),\ \frac{3}{2} \le q < \infty, \\ \sqrt{\rho}u, \nabla u, t^{\frac{1}{4}}\sqrt{\rho}u_t, t^{\frac{1}{4}}\nabla^2 u, t^{\frac{1}{4}}\nabla p \in L^\infty(0,T;L^2), \\ \sqrt{\rho}\theta, \nabla\theta, t^{\frac{1}{4}}\sqrt{\rho}\theta_t, t^{\frac{1}{4}}\nabla^2\theta \in L^\infty(0,T;L^2), \\ b, \nabla b, t^{\frac{1}{2}}\nabla^2 b, t^{\frac{1}{2}} b_t \in L^\infty(0,T;L^2), \\ \nabla u, \nabla^2 u, t^{\frac{1}{4}}\sqrt{\rho}u_t, t^{\frac{1}{4}}\nabla u_t \in L^2(0,T;L^2), \\ \nabla\theta, \nabla^2\theta, t^{\frac{1}{4}}\sqrt{\rho}\theta_t, t^{\frac{1}{4}}\nabla\theta_t \in L^2(0,T;L^2), \\ \nabla b, \nabla^2 b, t^{\frac{1}{2}} b_t \in L^2(0,T;L^2). \end{cases} \tag{1.5}$$

Moreover, it holds that

$$\sup_{0 \le t < \infty} \|\nabla\rho\|_{L^2 \cap L^6} \le C\|\nabla\rho_0\|_{L^2 \cap L^6}, \tag{1.6}$$

and there exists some positive constant C depending only on $\bar{\rho}$ *and initial data, such that for all* $t \ge 1$,

$$\begin{cases} \|\sqrt{\rho}u\|_{L^2} + \|\sqrt{\rho}\theta\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2} \le Ct^{-\frac{1}{4}}, \\ \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 \le Ct^{-\frac{1}{4}}, \\ \|\nabla b\|_{L^2} + \|b_t\|_{L^2} + \|\nabla^2 b\|_{L^2} \le Ct^{-\frac{1}{2}}. \end{cases} \tag{1.7}$$

Remark 1.1. *Very recently, Zhong [20] establish the unique global strong solutions with vacuum to the Cauchy problem of the inhomogeneous incompressible Bénard equations. However, the corresponding strong solutions admits the exponential decay-in-time property which is quite different from Theorem 1.1 for magnetic Bénard system. This means that the magnetic field acts some significant roles on the large time behaviors of solutions. In particular, this decay rates of solutions are new for the magnetic Bénard system.*

Remark 1.2. *When* $b = 0$, *Theorem 1.1 is different from Zhong's [20] result, since the initial velocity* u_0 *and initial temperature* θ_0 *need not to be small in our result.*

2. Preliminaries

We begin with the local existence and uniqueness of strong solutions whose proof can be performed by strategies as those in [5, 21].

Lemma 2.1. *Assume that* $(\rho_0, u_0, \theta_0, b_0)$ *satisfies* (1.3). *Then there exists a small time* $T_* > 0$ *and a unique strong solution* (ρ, u, θ, b, p) *to the problems* (1.1) *and* (1.2) *in* $\mathbb{R}^3 \times (0, T)$ *satisfying for some constant* $M_* > 1$ *depending on* T_* *and the initial data*

$$\sup_{0 \le t \le T_*} t^{\frac{1}{2}}(\|\nabla u\|_{H^1}^2 + \|\nabla\theta\|_{H^1}^2 + \|b\|_{H^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) \\ + \int_0^{T_*} (\|\nabla u_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2)dt \le M_*. \tag{2.1}$$

Next, the following well-known Gagliardo-Nirenberg inequality (see [13]) will be used later.

Lemma 2.2. *(Gagliardo-Nirenberg inequality) Assume $f \in W^{1,m} \cap L^r$, it holds that*

$$\|f\|_{L^q} \le C\|\nabla f\|_{L^m}^{\vartheta}\|f\|_{L^r}^{1-\vartheta}, \tag{2.2}$$

where $\vartheta = (\frac{1}{r} - \frac{1}{q})/(\frac{1}{r} - \frac{1}{m} + \frac{1}{2})$, and if $m < 2$, then q is between r and $\frac{2m}{2-m}$, if $m = 2$, then $q \in [2, \infty)$, if $m > 2$, then $q \in [2, \infty]$ and constant C depending on q, m and r.

Next, we have some regularity results for the Stokes equations, which have been proven in [9].

Lemma 2.3. *Suppose that $F \in L^r$ with $1 < r < \infty$. Let $(u, p) \in H^1 \times L^2$ be the unique weak solution to the following Stokes problem*

$$\begin{cases} -\mu\Delta u + \nabla p = F, & \text{in } \mathbb{R}^3, \\ \text{div} u = 0, & \text{in } \mathbb{R}^3, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

then $(\nabla^2 u, \nabla p) \in L^r$ and there exists a constant C depending only on μ and r such that

$$\|\nabla^2 u\|_{L^r} + \|\nabla p\|_{L^r} \le C\|F\|_{L^r}. \tag{2.4}$$

3. A priori estimates

In this section, we will establish some necessary a *priori* estimates, which together with the local existence (cf. Lemma 2.1) will complete the proof of Theorem 1.1. To this end, we let (ρ, u, θ, b, p) be a strong solutions of (1.1) and (1.2) in $\mathbb{R}^3 \times [0, T]$. For simplicity, we use the letters C and $C_i (i = 1, 2, \ldots)$ to denote some positive constant which dependent on $\bar{\rho}$ and the initial data, and write $C(\alpha)$ to emphasize that C dependent on α.

We first aim to get the following key a *priori* estimates on (ρ, u, θ, b, p). Set

$$X(t) := \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2, \quad K := \|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + \|\nabla \theta_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

Proposition 3.1. *Assume that*

$$\chi_0 \le \epsilon_0,$$

there exists some small positive constant ϵ_0 depending only on $\bar{\rho}$ and the initial data, such that if (ρ, u, θ, b, p) is a smooth solution of (1.1) and (1.2) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$\bar{\rho} + \|b\|_{L^3} \le 2\chi_0^{\frac{1}{2}}, \quad X(t) \le 6K, \tag{3.1}$$

then the following estimate holds

$$\bar{\rho} + \|b\|_{L^3} \le \frac{3}{2}\chi_0^{\frac{1}{2}}, \quad X(t) \le 5K. \tag{3.2}$$

Moreover, we have

$$\sup_{0\le t\le T} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2) dt \le C. \tag{3.3}$$

Proof of Proposition 3.1. 1) It following from transport Equation $(1.1)_1$, and making use of $(1.1)_5$ (see Lions [10, Theorem 2.1]) that

$$0 \le \rho(x,t) \le \sup_{x\in\mathbb{R}^3} \rho_0(x) = \bar{\rho}. \tag{3.4}$$

2) Multiplying $(1.1)_{2,3,4}$ by u, θ and b, respectively, and integrating by parts over $\mathbb{R}^3$, one obtains by using $\text{div}u = \text{div}b = 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|b\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \\
&\le C\int \rho|u||\theta|dx \le C\|\sqrt{\rho}u\|_{L^2}^2 + C\|\sqrt{\rho}\theta\|_{L^2}^2 \\
&\le C\|\rho\|_{L^{\frac{3}{2}}}(\|u\|_{L^6}^2 + \|\theta\|_{L^6}^2) \\
&\le C\|\rho_0\|_{L^1}^{\frac{2}{3}}\bar{\rho}^{\frac{1}{3}}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) \\
&\le C_1\|\rho_0\|_{L^1}^{\frac{2}{3}}\chi_0^{\frac{1}{6}}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2),
\end{aligned} \tag{3.5}$$

which leads

$$\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|b\|_{L^2}^2) + \mu\|\nabla u\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2 + \eta\|\nabla b\|_{L^2}^2 \le 0 \tag{3.6}$$

provided $\chi_0 \le \epsilon_1 := \min\left\{1, \frac{1}{64C_1^6\|\rho_0\|_{L^1}^4}\right\}$. Integrating (3.23) over $[0,T]$ yields that

$$\sup_{0\le t\le T}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|b\|_{L^2}^2) + \int_0^T(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)dt \le \mathbb{E}_0, \tag{3.7}$$

where

$$\mathbb{E}_0 := \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\sqrt{\rho_0}\theta_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

3) Multiplying $(1.1)_{2,3}$ by u_t and θ_t respectively, and integrating the resulting equality by parts. We infer from Gagliardo-Nirenberg inequality and Young's inequality, (3.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(|\nabla u|^2 + |\nabla\theta|^2)dx + \int(\rho|u_t|^2 + \rho|\theta_t|^2)dx \\
&= -\int \rho u\cdot\nabla u\cdot u_t dx - \int \rho u\cdot\nabla\theta\cdot\theta_t dx - \int \theta\rho e_3\cdot u_t dx \\
&\quad - \int \theta_t\rho u\cdot e_3 dx - \frac{d}{dt}\int b\cdot\nabla u\cdot b dx - \int b_t\cdot\nabla u\cdot b dx \\
&\quad - \int b\cdot\nabla u\cdot b_t dx \\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^3} \\
&+ C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|u\|_{L^6}\|\nabla\theta\|_{L^3} + C\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ C\|b\|_{L^3}\|b_t\|_{L^2}\|\nabla u\|_{L^6} + C\|\sqrt{\rho}\theta\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\\
\le& -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&+ C\|\rho\|_{L^{\frac{3}{2}}}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\rho\|_{L^{\frac{3}{2}}}^{\frac{1}{2}}\|\theta\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}\\
&+ C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}\\
&+ C\|b\|_{L^3}\|b_t\|_{L^2}\|\nabla^2 u\|_{L^2}\\
\le& -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \frac{1}{8}\|b_t\|_{L^2}^2\\
&+ C\bar{\rho}\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2} + C\bar{\rho}\|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\\
&+ C\|b\|_{L^3}^2\|\nabla^2 u\|_{L^2}^2 + C\bar{\rho}^{\frac{1}{3}}\|\nabla u\|_{L^2}^2 + C\bar{\rho}^{\frac{1}{3}}\|\nabla\theta\|_{L^2}^2,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2\\
\le& -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \frac{1}{4}\|b_t\|_{L^2}^2 + C(\bar{\rho}^2 + \|b\|_{L^3}^2)\|\nabla^2 u\|_{L^2}^2 + C\bar{\rho}^2\|\nabla^2\theta\|_{L^2}^2\\
&+ C\|\nabla u\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2. \qquad (3.8)
\end{aligned}
$$

4) Multiplying $(1.1)_4$ by b_t in L^2 and integrating by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla b|^2 dx + \int |b_t|^2 dx = \int (b\cdot\nabla u - u\cdot\nabla b)\cdot b_t dx\\
\le& C\|b_t\|_{L^2}(\|b\|_{L^3}\|\nabla u\|_{L^6} + \|u\|_{L^6}\|\nabla b\|_{L^3})\\
\le& \frac{1}{4}\|b_t\|_{L^2}^2 + C\|b\|_{L^3}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^2 b\|_{L^2}^{\frac{4}{3}}\|b\|_{L^3}^{\frac{2}{3}}\\
\le& \frac{1}{4}\|b_t\|_{L^2}^2 + C\|b\|_{L^3}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6\|b\|_{L^3} + C\|\nabla^2 b\|_{L^2}^2\|b\|_{L^3}^{\frac{1}{2}}\\
\le& \frac{1}{4}\|b_t\|_{L^2}^2 + C\|b\|_{L^3}^2\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|b\|_{L^3}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

5) According to Lemma 2.2 and $F = \rho u_t + \rho u\cdot\nabla u + b\cdot\nabla b + \rho\theta e_3$, we derive

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \le& C\|\rho u_t + \rho u\cdot\nabla u + b\cdot\nabla b + \rho\theta e_3\|_{L^2}\\
\le& C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + C\bar{\rho}\|u\|_{L^6}\|\nabla u\|_{L^3} + C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_{L^2}\\
&+ C\|b\|_{L^3}\|\nabla b\|_{L^6}\\
\le& C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + C\bar{\rho}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}} + C\bar{\rho}^{\frac{2}{3}}\|\theta\|_{L^6}\\
&+ C\|b\|_{L^3}\|\nabla^2 b\|_{L^2}\\
\le& \frac{1}{2}\|\nabla^2 u\|_{L^2} + C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + C\bar{\rho}^2\|\nabla u\|_{L^2}^3\\
&+ C\bar{\rho}^{\frac{2}{3}}\|\nabla\theta\|_{L^2} + C\|b\|_{L^3}\|\nabla^2 b\|_{L^2},
\end{aligned}
$$

which directly leads that

$$\begin{aligned}||\nabla^2 u||_{L^2} + ||\nabla p||_{L^2} &\le C\bar{\rho}^{\frac{1}{2}}||\sqrt{\rho}u_t||_{L^2} + C\bar{\rho}^{\frac{2}{3}}||\nabla\theta||_{L^2} + C\bar{\rho}^2||\nabla u||_{L^2} \\ &\quad + C_1||b||_{L^3}||\nabla^2 b||_{L^2} \\ &\le C||\sqrt{\rho}u_t||_{L^2} + C||\nabla\theta||_{L^2} + C||\nabla u||_{L^2} + C_2\chi_0^{\frac{1}{2}}||\nabla^2 b||_{L^2}. \end{aligned} \tag{3.10}$$

It follows from $(1.1)_4$, Hölder's and Gagliardo-Nirenberg inequalities, we get

$$\begin{aligned}||\nabla^2 b||_{L^2} &\le C(||b_t||_{L^2} + ||u\cdot\nabla b||_{L^2} + ||b\cdot\nabla u||_{L^2}) \\ &\le C||b_t||_{L^2} + C||u||_{L^6}||\nabla b||_{L^3} + C||b||_{L^3}||\nabla u||_{L^6} \\ &\le C||b_t||_{L^2} + C||\nabla b||_{L^2}^{\frac{1}{2}}||\nabla^2 b||_{L^2}^{\frac{1}{2}}||\nabla u||_{L^2} + C||\nabla u||_{L^2}||\nabla^2 b||_{L^2}^{\frac{2}{3}}||b||_{L^3}^{\frac{1}{3}} \\ &\le \frac{1}{4}||\nabla^2 b||_{L^2} + C||b_t||_{L^2} + C||\nabla b||_{L^2}||\nabla u||_{L^2}^2 \\ &\quad + C||b||_{L^3}||\nabla u||_{L^2}^3, \end{aligned} \tag{3.11}$$

which together with (3.10) implies that

$$||\nabla^2 u||_{L^2} + ||\nabla p||_{L^2} + ||\nabla^2 b||_{L^2} \le C||b_t||_{L^2} + C||\sqrt{\rho}u_t||_{L^2} + C||\nabla u||_{L^2} + C||\nabla\theta||_{L^2}, \tag{3.12}$$

which provided $\chi_0 \le \epsilon_2 = \min\{\epsilon_1, \frac{1}{4C_2^2}\}$.

Similarly, by using the following L^2-estimate of elliptic system, we have

$$\begin{aligned}||\nabla^2\theta||_{L^2} &\le C(||\rho\theta_t||_{L^2} + ||\rho u\cdot\nabla\theta||_{L^2} + ||\rho u\cdot e_3||_{L^2}) \\ &\le C\bar{\rho}^{\frac{1}{2}}||\sqrt{\rho}\theta_t||_{L^2} + C\bar{\rho}||u||_{L^6}||\nabla\theta||_{L^3} + C\bar{\rho}^{\frac{1}{2}}||\rho_0||_{L^{\frac{3}{2}}}^{\frac{1}{2}}||u||_{L^6} \\ &\le C\bar{\rho}^{\frac{1}{2}}||\sqrt{\rho}\theta_t||_{L^2} + C\bar{\rho}^{\frac{1}{2}}||\nabla u||_{L^2}||\nabla\theta||_{L^2}^{\frac{1}{2}}||\nabla^2\theta||_{L^2}^{\frac{1}{2}} + C\bar{\rho}^{\frac{2}{3}}||\nabla u||_{L^2} \\ &\le \frac{1}{2}||\nabla^2\theta||_{L^2} + C\bar{\rho}^{\frac{1}{2}}||\sqrt{\rho}\theta_t||_{L^2} + C\bar{\rho}(||\nabla u||_{L^2} + ||\nabla\theta||_{L^2}),\end{aligned}$$

that is

$$||\nabla^2\theta||_{L^2} \le C\bar{\rho}^{\frac{1}{2}}||\sqrt{\rho}\theta_t||_{L^2} + C\bar{\rho}(||\nabla u||_{L^2} + ||\nabla\theta||_{L^2}). \tag{3.13}$$

6) Combining (3.8), (3.9), (3.12) and (3.13), it is easy to deduce that

$$\begin{aligned}&\frac{d}{dt}(||\nabla u||_{L^2}^2 + ||\nabla\theta||_{L^2}^2 + ||\nabla b||_{L^2}^2) + ||\sqrt{\rho}\theta_t||_{L^2}^2 + ||\sqrt{\rho}u_t||_{L^2}^2 + \frac{1}{2}||b_t||_{L^2}^2 \\ &\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + C(\bar{\rho} + ||b||_{L^3}^2)(||\nabla^2 u||_{L^2}^2 + ||\nabla^2\theta||_{L^2}^2) \\ &\quad + C||b||_{L^3}^{\frac{1}{2}}||\nabla^2 b||_{L^2}^2 + C||\nabla u||_{L^2}^2 + C||\nabla\theta||_{L^2}^2 \\ &\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \chi_0^{\frac{1}{2}}\tilde{C}(||b_t||_{L^2}^2 + ||\sqrt{\rho}u_t||_{L^2}^2 + ||\sqrt{\rho}\theta||_{L^2}^2) \\ &\quad + C||\nabla u||_{L^2}^2 + C||\nabla\theta||_{L^2}^2.\end{aligned}$$

Hence, choosing χ_0 suitably small, we have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2\\&\le-\frac{d}{dt}\int b\cdot\nabla u\cdot bdx+C\|\nabla u\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2.\end{aligned}\tag{3.14}$$

Integrating (3.14) with respect to t, and using (3.1), one obtains

$$\begin{aligned}&\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^T(\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2)dt\\&\le M+C\|b_0\|_{L^3}\|\nabla u_0\|_{L^2}\|b_0\|_{L^6}+C\sup_{0\le t\le T}\|b\|_{L^3}\|\nabla u\|_{L^2}\|b\|_{L^6}\\&\quad+C\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\\&\le 2M+\frac{1}{2}\|\nabla u_0\|_{L^2}^2+C_3\|b_0\|_{L^3}^2\|\nabla b_0\|_{L^2}^2+\frac{1}{2}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\\&\quad+C_3\|b\|_{L^3}^2\|\nabla b\|_{L^2}^2\\&\le\frac{5}{2}M+\frac{1}{2}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2+4C_3\sup_{0\le t\le T}\chi_0\|\nabla b\|_{L^2}^2.\end{aligned}$$

As a consequence, we have

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^T(\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2)dt\le 5M,\tag{3.15}$$

provided $\chi_0\le\epsilon_3:=\min\{\epsilon_2,\frac{1}{8C_3},\frac{1}{16\tilde{C}}\}$.

7) Mmultiplying $(1.1)_4$ by $3|b|b$ and integrating by parts, we derive

$$\frac{d}{dt}\|b\|_{L^3}^3+3\int|b||\nabla b|^2dx+3\int|b||\nabla|b||^2dx\le\int|b||\nabla b|^2dx+C\|\nabla u\|_{L^2}^2\|b\|_{L^{\frac{9}{2}}}^3.$$

Consequently,

$$\frac{d}{dt}\|b\|_{L^3}^3+2\int|b||\nabla b|^2dx+3\int|b||\nabla|b||^2dx\le C\|\nabla u\|_{L^2}^2\|b\|_{L^{\frac{9}{2}}}^3.\tag{3.16}$$

To deal with the right-hand side of (3.16), we need to use the following variant of the Kato inequality

$$|\nabla|b|^{\frac{3}{2}}|=\frac{3}{2}|b|^{\frac{1}{2}}|\nabla|b||\le\frac{3}{2}|b|^{\frac{1}{2}}|\nabla b|,$$

which combined with Hölder's inequality and Galiardo-Nirenberg inequality leads to

$$\|b\|_{L^{\frac{9}{2}}}^3\le\|b\|_{L^3}^{\frac{3}{2}}\|b\|_{L^9}^{\frac{3}{2}}=\|b\|_{L^3}^{\frac{3}{2}}\||b|^{\frac{3}{2}}\|_{L^6}\le C\|b\|_{L^3}^{\frac{3}{2}}\|\nabla(|b|^{\frac{3}{2}})\|_{L^2}\le C\|b\|_{L^3}^{\frac{3}{2}}\||b|^{\frac{1}{2}}|\nabla b|\|_{L^2}.\tag{3.17}$$

Thus, substituting (3.17) into (3.16), we obtain from Young's inequality that

$$\frac{d}{dt}\|b\|_{L^3}^3+\int|b||\nabla b|^2dx\le C\|\nabla u\|_{L^2}^4\|b\|_{L^3}^3.$$

This together with (3.7), (3.15) and Gronwall's inequality yields

$$\sup_{0\le t\le T}\|b\|_{L^3}\le \exp\left\{C\int_0^T\|\nabla u\|_{L^2}^4dt\right\}^{\frac13}\|b_0\|_{L^3}\le C_4\chi_0\le\frac{\chi_0^{\frac12}}{2},$$

provided $\chi_0\le\epsilon_4=\min\left\{\epsilon_3,\frac{1}{4C_4^2}\right\}$. Thus, choosing $\epsilon_0=\min\{\epsilon_4,\epsilon_7\}$ (ϵ_7 can be chosen in the following lemmas), one obtains

$$\|\rho\|_{L^\infty}+\|b\|_{L^3}\le\|\rho_0\|_{L^\infty}+\|b\|_{L^3}\le\chi_0+\frac12\chi_0^{\frac12}=\frac32\chi_0^{\frac12}. \tag{3.18}$$

Finally, combining (3.15) with (3.7) and (3.12) imply the desired (3.3). We completed the proof of Proposition 3.1. □

Lemma 3.1. *Under the conditions of Proposition 3.1, it holds that*

$$\sup_{0\le t\le T}t\|\nabla b\|_{L^2}^2+\int_0^T t(\|b_t\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)dt\le C, \tag{3.19}$$

$$\sup_{0\le t\le T}t^{\frac12}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\le C, \tag{3.20}$$

$$\begin{aligned}&\sup_{0\le t\le T}t^{\frac12}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)dt\\&\quad+\int_0^T t^{\frac12}(\|\nabla^2 u\|_{L^2}^2+\|\nabla p\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)dt\le C.\end{aligned} \tag{3.21}$$

Proof. 1) Using $(1.1)_4$, Hölder's and Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla b\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2\\&=\int|b_t-\Delta b|^2dx=\int|b\cdot\nabla u-u\cdot\nabla b|^2dx\\&\le C\|b\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|u\|_{L^6}^2\|\nabla b\|_{L^3}^2\\&\le C\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla u\|_{L^2}^2\\&\le\frac12\|\nabla^2 b\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|\nabla b\|_{L^2}^2,\end{aligned}$$

which implies

$$\frac{d}{dt}(t\|\nabla b\|_{L^2}^2)+t\|b_t\|_{L^2}^2+\frac{t}{2}\|\nabla^2 b\|_{L^2}^2\le\|\nabla b\|_{L^2}^2+Ct\|\nabla u\|_{L^2}^4\|\nabla b\|_{L^2}^2. \tag{3.22}$$

This along with Gronwall's inequality, (3.7) and (3.3) yields the desired (3.19).

2) It follows from (3.5) that

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\\&\le\frac14\|\nabla u\|_{L^2}^2+C_5\chi_0^{\frac16}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+C\|b\|_{L^2}\|\nabla b\|_{L^2}^3,\end{aligned}$$

which implies

$$\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\le C\|b\|_{L^2}\|\nabla b\|_{L^2}^3, \tag{3.23}$$

provided $\chi_0\le\epsilon_5=\min\{\epsilon_4,\frac{1}{64C_5^6}\}$. Multiplying it by $t^{\frac12}$, we arrive at

$$\begin{aligned}
&\sup_{0\le t\le T}t^{\frac12}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\\
&\le\sup_{0\le t\le 1}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)\int_0^1 t^{-\frac12}dt+C\int_1^T(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\\
&\quad+C\sup_{0\le t\le T}\|b\|_{L^2}\sup_{0\le t\le T}t^{\frac12}\|\nabla b\|_{L^2}\int_0^T\|\nabla b\|_{L^2}^2dt\\
&\le C.
\end{aligned}\tag{3.24}$$

3) In the view of (3.14), one obtains

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2\\
&\le-\frac{d}{dt}\int b\cdot\nabla u\cdot b dx+C\|\nabla u\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2,
\end{aligned}\tag{3.25}$$

which together with (3.3), (3.7) and (3.19) yields that

$$\begin{aligned}
&\sup_{0\le t\le T}t^{\frac12}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|b_t\|_{L^2}^2)dt\\
&\le\sup_{0\le t\le 1}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\int_0^1 t^{-\frac12}dt+\int_1^T(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)dt\\
&\quad+C\int_0^T t^{-\frac12}\|b\|_{L^3}\|\nabla u\|_{L^2}\|b\|_{L^6}dt+\sup_{0\le t\le T}t^{\frac12}\|b\|_{L^3}\|\nabla u\|_{L^2}\|b\|_{L^6}+C\\
&\le C\sup_{0\le t\le 1}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\int_0^1 t^{-\frac12}dt+\int_1^T(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)dt\\
&\quad+\sup_{0\le t\le T}t^{\frac12}\|b\|_{L^3}\|\nabla u\|_{L^2}\|b\|_{L^6}+C\\
&\le C.
\end{aligned}\tag{3.26}$$

Thus, we directly obtain (3.21) from (3.12), (3.13) and (3.26). The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Under the conditions of Proposition 3.1, it holds that*

$$\sup_{0\le t\le T}t^{\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\le C, \tag{3.27}$$

$$\sup_{0\le t\le T}t(\|b_t\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)+\int_0^T t\|\nabla b_t\|_{L^2}^2dt\le C, \tag{3.28}$$

$$\sup_{0\le t\le T}t^{\frac12}(\|\nabla^2 u\|_{L^2}^2+\|\nabla p\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)\le C. \tag{3.29}$$

Proof. 1) Differentiating $(1.1)_{2,3}$ with respect to time variable t give

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \Delta u_t + \nabla p_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + (\rho\theta e_3)_t + (b \cdot \nabla b)_t, \tag{3.30}$$

$$\rho\theta_{tt} + \rho u \cdot \nabla\theta_t - \Delta\theta_t = -\rho_t(\theta_t + u \cdot \nabla\theta) - \rho u_t \cdot \nabla\theta + (\rho u)_t \cdot e_3. \tag{3.31}$$

Multiplying (3.30), (3.31) by u_t, θ_t respectively, and integrating it by parts, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2\\
&= -2\int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t)dx\\
&\quad + \int (b \cdot \nabla b)_t \cdot u_t dx - 2\int \rho u \cdot \nabla\theta_t\theta_t dx - \int \rho u_t \cdot \nabla\theta\theta_t dx\\
&\quad - \int \rho u \cdot \nabla(u \cdot \nabla\theta\theta_t)dx + \int (\rho\theta e_3)_t \cdot u_t dx + \int (\rho u)_t\theta_t \cdot e_3 dx\\
&=: \sum_{i=1}^{9} I_i.
\end{aligned} \tag{3.32}$$

By using Hölder's, Gagliardo-Nirenberg inequalities, and (3.4), one gets

$$\begin{aligned}
I_1 &\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}\\
&\le \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,\\
I_2 &\le C\|\sqrt{\rho}u_t\|_{L^4}^2\|\nabla u\|_{L^2}\\
&\le C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\\
&\le \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,\\
I_3 &\le C\bar{\rho}\|u\|_{L^6}\|u_t\|_{L^6}(\|\nabla u\|_{L^3}^2 + \|u\|_{L^6}\|\nabla^2 u\|_{L^2})\\
&\quad + C\bar{\rho}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\bar{\rho}^2\|\nabla^2 u\|_{L^2}^2\|\nabla u\|_{L^2}^4,\\
I_4 &\le C\|b_t\|_{L^6}\|\nabla u_t\|_{L^2}\|b\|_{L^3}\\
&\le \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + C\|b\|_{L^3}^2\|\nabla b_t\|_{L^2}^2,\\
I_5 &\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^3}\|u\|_{L^6}\|\nabla\theta_t\|_{L^2}\\
&\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla\theta_t\|_{L^2}\\
&\le C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\\
&\le \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2 + C\bar{\rho}^3\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned}$$

$$
\begin{aligned}
I_6 &\le C\bar{\rho}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3}\|\theta_t\|_{L^6}\\
&\le C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|u_t\|_{L^6}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}\\
&\le \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2+\frac{1}{10}\|\nabla u_t\|_{L^2}^2+C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^4,\\
I_7 &\le C\int\rho|u||\theta_t|(|\nabla u||\nabla\theta|+|u||\nabla^2\theta|)dx+C\int\rho|u|^2|\nabla\theta||\nabla\theta_t|dx\\
&\le C\bar{\rho}\|u\|_{L^6}^2\|\nabla\theta_t\|_{L^2}\|\nabla\theta\|_{L^6}+C\bar{\rho}\|u\|_{L^6}\|\nabla u\|_{L^2}\|\theta_t\|_{L^6}\|\nabla\theta\|_{L^6}\\
&\quad+C\bar{\rho}\|\theta_t\|_{L^6}\|\nabla^2\theta\|_{L^2}\|u\|_{L^6}^2\\
&\le C\bar{\rho}\|\nabla u\|_{L^2}^2\|\nabla\theta_t\|_{L^2}\|\nabla^2\theta\|_{L^2}\\
&\le \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2+C\bar{\rho}^2\|\nabla u\|_{L^2}^4\|\nabla^2\theta\|_{L^2}^2,\\
I_8+I_9 &\le C\rho|u||u_t||\nabla\theta|dx+C\int\rho|\theta||u||\nabla u_t|dx+C\int\rho|u||\theta_t||\nabla u|dx\\
&\quad+C\int\rho|u|^2|\nabla\theta_t|dx+C\int\rho|\theta_t||u_t|dx\\
&\le C\bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla\theta\|_{L^2}+C\bar{\rho}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^3}\|\theta\|_{L^6}\\
&\quad+C\bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla u\|_{L^2}+C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla\theta_t\|_{L^2}\\
&\quad+C\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\\
&\le \frac{1}{10}\|\nabla u_t\|_{L^2}^2+\frac{1}{8}\|\nabla\theta_t\|_{L^2}^2+C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla\theta\|_{L^2}^4\\
&\quad+C\bar{\rho}^3\|\sqrt{\rho}\theta_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+C\|\nabla\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\\
&\quad+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2.
\end{aligned}
$$

Putting all above estimates into (3.32), we show

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\bar{\rho}^2(\|\nabla^2u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)\\
&\le C\bar{\rho}^{\frac{1}{3}}(\|u_t\|_{L^6}^2+C\|\theta_t\|_{L^6}^2)+C(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)\\
&\quad+C\|b_t\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&\le C_6\chi_0^{\frac{1}{6}}(\|\nabla u_t\|_{L^2}^2+C\|\nabla\theta_t\|_{L^2}^2)+C(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)\\
&\quad+C\|b_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\frac{1}{2}\|\nabla u_t\|_{L^2}^2+\frac{1}{2}\|\nabla\theta_t\|_{L^2}^2\\
&\le C\|b_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+C(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2),
\end{aligned}
\tag{3.33}
$$

provided $\chi_0\le\epsilon_6:=\min\{\epsilon_5,\left(\frac{1}{2C_6}\right)^6\}$. Hence, multiplying (3.33) by $t^{\frac{1}{2}}$, and integrating by parts, we infer from Lemma 3.1 and (3.3) that

$$
\begin{aligned}
&\sup_{0\le t\le T} t^{\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\\
&\le C\int_0^T t^{-\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)dt+C\int_0^T t^{\frac12}\|b_t\|_{L^2}^2\|\nabla u\|_{L^2}^4dt\\
&\quad +C\int_0^T t^{\frac12}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)dt\\
&\le C\sup_{0\le t\le t_0} t^{\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)\int_0^{t_0}t^{-1}dt+C\bar{\rho}^{\frac13}\int_{t_0}^T(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\\
&\quad +C\sup_{0\le t\le T} t^{\frac12}\|\nabla u\|_{L^2}^2\int_0^T\|b_t\|_{L^2}^2dt\\
&\le C(M_*)t_0+C_7\chi_0^{\frac16}\int_0^T t^{\frac12}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt+C,
\end{aligned}
\tag{3.34}
$$

that is

$$
\sup_{0\le t\le T} t^{\frac12}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)+\int_0^T t^{\frac12}(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2)dt\le C,
\tag{3.35}
$$

provided $\chi_0\le\epsilon_7:=\min\{\epsilon_6,\left(\frac{1}{2C_7}\right)^6\}$.

2) Differentiating $(1.1)_4$ with respect to t, and multiplying the resulting equality with b_t and then integrating by parts over $\mathbb{R}^3$, we arrive at

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int|b_t|^2dx+\int|\nabla b_t|^2dx\\
&\le C(\||u_t||b|\|_{L^2}+\||u||b_t|\|_{L^2})\|\nabla b_t\|_{L^2}\\
&\le C\|u_t\|_{L^6}\|b\|_{L^3}\|\nabla b_t\|_{L^2}+C\|u\|_{L^6}\|b_t\|_{L^3}\|\nabla b_t\|_{L^2}\\
&\le C\|\nabla b_t\|_{L^2}\|\nabla u_t\|_{L^2}\|b\|_{L^3}+C\|\nabla u\|_{L^2}\|b_t\|_{L^2}^{\frac12}\|\nabla b_t\|_{L^2}^{\frac32}\\
&\le \frac12\|\nabla b_t\|_{L^2}^2+C\|b\|_{L^3}^2\|\nabla u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|b_t\|_{L^2}^2,
\end{aligned}
$$

which leads

$$
\frac{d}{dt}\int|b_t|^2dx+\int|\nabla b_t|^2dx\le C\|b\|_{L^3}^2\|\nabla u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|b_t\|_{L^2}^2.
\tag{3.36}
$$

Multiplying (3.36) by t, and using Gronwall's inequality, we infer from (3.35) that

$$
\begin{aligned}
\sup_{0\le t\le T} t\|b_t\|_{L^2}^2+\int_0^T t\|\nabla b_t\|_{L^2}^2dt&\le C\int_0^T t\|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^2dt+\int_0^T\|b_t\|_{L^2}^2dt\\
&\le \sup_{0\le t\le T}(t\|\nabla b\|_{L^2}^2)^{\frac12}\int_0^T t^{\frac12}\|\nabla u_t\|_{L^2}^2dt+C\\
&\le C.
\end{aligned}
\tag{3.37}
$$

Then, the desired (3.27) follows from (3.12) and (3.13). We completed the proof of lemma. □

Lemma 3.3. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T}(\|\nabla\rho\|_{L^2\cap L^6}+\|\rho_t\|_{L^2\cap L^3})+\int_0^T\|\nabla u\|_{L^\infty}dt\le C(T). \tag{3.38}$$

Proof. 1) It follows from the Lemma 2.3, Hölder's and Gagliardo-Nirenberg inequalities that for $r\in(3,\min\{q,6\})$,

$$\begin{aligned}
\|\nabla^2 u\|_{L^r}+\|\nabla p\|_{L^r}\le& C\|\rho u_t\|_{L^r}+C\|\rho u\cdot\nabla u\|_{L^r}+C\|\rho\theta\|_{L^r}+C\|b\cdot\nabla b\|_{L^r}\\
\le& C(\bar\rho)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+C\bar\rho\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6r}{6-r}}}\\
&+C(\bar\rho)\|\sqrt{\rho}\theta\|_{L^2}^{\frac{6-r}{2r}}\|\sqrt{\rho}\theta\|_{L^6}^{\frac{3r-6}{2r}}+C\|b\|_{L^\infty}\|\nabla b\|_{L^r}\\
\le& C(\bar\rho)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+C(\bar\rho)\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}\\
&+C(\bar\rho)(\|\rho\|_{L^{\frac32}}^{\frac12}\|\theta\|_{L^6})^{\frac{6-r}{2r}}(\bar\rho\|\theta\|_{L^6})^{\frac{3r-6}{2r}}+\frac12\|\nabla^2 u\|_{L^r}\\
&+C\|\nabla b\|_{L^2}^{\frac3r}\|\nabla^2 b\|_{L^2}^{\frac{2r-3}{r}}\\
\le& C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+C\|\nabla\theta\|_{L^2}\\
&+\frac12\|\nabla^2 u\|_{L^r}+C\|\nabla b\|_{L^2}^{\frac3r}\|\nabla^2 b\|_{L^2}^{\frac{2r-3}{r}}
\end{aligned}$$

which yields

$$\begin{aligned}
\|\nabla^2 u\|_{L^r}+\|\nabla p\|_{L^r}\le& C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+C\|\nabla\theta\|_{L^2}\\
&+C\|\nabla b\|_{L^2}^{\frac3r}\|\nabla^2 b\|_{L^2}^{\frac{2r-3}{r}}.
\end{aligned}\tag{3.39}$$

Then, one derives from the Gagliardo-Nirenberg inequality and (3.12) that

$$\begin{aligned}
\|\nabla u\|_{L^\infty}\le& C\|\nabla^2 u\|_{L^r}^{\frac{3r}{5r-6}}\|\nabla u\|_{L^2}^{\frac{2r-6}{5r-6}}\le C\|\nabla u\|_{L^2}+C\|\nabla^2 u\|_{L^r}\\
\le& C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla u\|_{L^2}^{\frac{6r-6}{r}}+C\|\nabla\theta\|_{L^2}\\
&+C\|\nabla u\|_{L^2}+C\|\nabla b\|_{L^2}^{\frac3r}\|\nabla^2 b\|_{L^2}^{\frac{2r-3}{r}},
\end{aligned}$$

which together with Lemma 3.2, (3.7) and (3.3) implies

$$\begin{aligned}
\int_0^T\|\nabla u\|_{L^\infty}dt\le& C\sup_{0\le t\le T}\Big(t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2}^2\Big)^{\frac{6-r}{4r}}\Big(\int_0^T t^{\frac12}\|\nabla u_t\|_{L^2}^2dt\Big)^{\frac{3r-6}{4r}}\Big(\int_0^T t^{-\frac{r}{r+6}}dt\Big)^{\frac{r+6}{4}}\\
&+C\Big(\int_0^T\|\nabla\theta\|_{L^2}^2dt\Big)^{\frac12}+C\Big(\int_0^T\|\nabla u\|_{L^2}^2dt\Big)^{\frac12}+C\int_0^T\|\nabla^2 b\|_{L^2}^2dt\\
&+C\Big(\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\Big)^{\frac{2r-3}{r}}\int_0^T\|\nabla u\|_{L^2}^2dt+C\\
\le& C.
\end{aligned}\tag{3.40}$$

2) Differentiating the continuity equation $(1.1)_1$ with respect to x_i gives rise to

$$(\rho_{x_i})_t+\nabla\rho_{x_i}\cdot u+\nabla\rho\cdot u_{x_i}=0. \tag{3.41}$$

Multiplying (3.41) by $s|\rho_{x_i}|^{s-2}\rho_{x_i}$ ($s = \{2, 6\}$) and integrating the resulting equation over $\mathbb{R}^3$ indicate that

$$\frac{d}{dt}\|\nabla\rho\|_{L^2\cap L^6} \leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^2\cap L^6}. \tag{3.42}$$

It follows from the Gronwall's inequality and (3.40) that

$$\|\nabla\rho\|_{L^2\cap L^6} \leq C\|\nabla\rho_0\|_{L^2\cap L^6}. \tag{3.43}$$

Noticing the following facts

$$\begin{aligned}\|\rho_t\|_{L^2\cap L^3} &\leq C\|u\|_{L^6}(\|\nabla\rho\|_{L^3} + \|\nabla\rho\|_{L^6})\\ &\leq C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^2\cap L^6} \leq C\|\nabla\rho_0\|_{L^2\cap L^6}.\end{aligned} \tag{3.44}$$

This ends the proof of Lemma 3.3. □

Lemma 3.4. *Under the assumption of Theorem 1.1, it holds that for*

$$\int_0^T t^{\frac{3}{2}}\|\nabla^3 b\|_{L^2}^2 dt \leq C. \tag{3.45}$$

Proof. Taking ∇ operator to $(1.1)_4$, we get

$$-\nabla\Delta b = \nabla(b\cdot\nabla u - u\cdot\nabla b - b_t). \tag{3.46}$$

Using the L^2-estimates of elliptic system, we derive

$$\begin{aligned}\|\nabla^3 b\|_{L^2}^2 &\leq C(\|\nabla b_t\|_{L^2}^2 + \|\nabla(u\cdot\nabla b)\|_{L^2}^2 + \|\nabla(b\cdot\nabla u)\|_{L^2}^2)\\ &\leq C\|\nabla b_t\|_{L^2}^2 + C\||\nabla u||\nabla b|\|_{L^2}^2 + C\||u||\nabla^2 b|\|_{L^2}^2 + C\||b||\nabla^2 u|\|_{L^2}^2\\ &\leq C\|\nabla b_t\|_{L^2}^2 + C\|\nabla u\|_{L^3}^2\|\nabla b\|_{L^6}^2 + C\|u\|_{L^6}^2\|\nabla^2 b\|_{L^2}^2\\ &\quad + C\|b\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}^2\\ &\leq C\|\nabla b_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2 b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2\\ &\quad + C\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^2 u\|_{L^2}^2,\end{aligned} \tag{3.47}$$

which yields to

$$\begin{aligned}\int_0^T t^{\frac{3}{2}}\|\nabla^3 b\|_{L^2}^2 dt &\leq C\sup_{0\leq t\leq T}(t^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}t^{\frac{1}{4}}\|\nabla u\|_{L^2})\int_0^T t\|\nabla^2 b\|_{L^2}^2 dt\\ &\quad + C\sup_{0\leq t\leq T}(t^{\frac{1}{2}}\|\nabla b\|_{L^2}t^{\frac{1}{2}}\|\nabla^2 b\|_{L^2})\int_0^T t^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^2 dt\\ &\quad + C\sup_{0\leq t\leq T} t\|\nabla^2 b\|_{L^2}^2\int_0^T t^{\frac{1}{2}}\|\nabla u\|_{L^2}^2 dt\\ &\leq C.\end{aligned} \tag{3.48}$$

We complete the proof of this lemma. □

4. The proof of Theorem 1.1

By Lemma 2.1, there exists a T_* such that the problems (1.1) and (1.2) has a unique local strong solution (ρ, u, θ, b) on $\mathbb{R}^3 \times (0, T_*]$. In what follows, we shall extend the local solution to all the time.

Set

$$T^* = \sup\{T|\, (\rho, u, \theta, b) \text{ is a strong solution of (1.1) } \textit{and} \text{ (1.2) on } \mathbb{R}^3 \times (0, T]\}. \tag{4.1}$$

First, for any $0 < \tau < T_* < T < T^*$ with T finite, it follows from Proposition 3.1, and Lemmas 3.1–3.4 that for any $p \geq 2$,

$$\nabla u, \nabla\theta, \nabla b \in C([\tau, T]; L^2), \tag{4.2}$$

where we used the following standard Sobolev embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C(\tau, T; L^2).$$

Moreover, one deduces from (3.4) and (3.38) that

$$\rho \in C(0, T; L^{\frac{3}{2}} \cap W^{1,q}). \tag{4.3}$$

Now, we claim that

$$T^* = \infty. \tag{4.4}$$

Otherwise, if $T^* < \infty$, in the view of Lemmas 3.1–3.4, we have

$$(\rho, u, \theta, b)(T^*, x) = \lim_{t \to T^*} (\rho, u, \theta, b)(t, x) \tag{4.5}$$

satisfies (1.3) at $t = T^*$. Thus, we can take $(\rho, u, \theta, b)(T^*, x)$ as the initial data, and Lemma 2.1 implies that one can extend the local solutions beyond T^*. This contradicts the assumption of T^* in (4.4). The proof of Theorem 1.1 is completed.

5. Conclusions

This paper deals with the Cauchy problem of 3D inhomogeneous incompressible magnetic Bénard equations. Through some time-weighted a *priori* estimates, we prove the global existence of strong solution provided that the upper boundedness of initial density and initial magnetic field satisfy some smallness condition. Furthermore, we also obtain large time decay rates of the solution.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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