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Research article

Supercritical Trudinger-Moser inequalities with logarithmic weights in dimension two

Yony Raúl Santaria Leuyacc*

Universidad Nacional Mayor de San Marcos, Lima, Perú

* **Correspondence:** Email: ysantarial@unmsm.edu.pe.

Abstract: In this work, we are interested in studying the existence of nontrivial weak solutions to the following class of Schrödinger equations

$$\begin{cases} -\operatorname{div}(w(x)\nabla u) = f(x,u), & x \in B_1(0), \\ u = 0, & x \in \partial B_1(0), \end{cases}$$

where $w(x) = (\ln(1/|x|))^{\beta}$ for some $\beta \in [0,1)$, the nonlinearity $f(x,s)$ behaves like $\exp((1+h(|x|))|s|^{2/(1-\beta)})$ and h is a continuous radial function such that $h(r)$ tends to infinity as r tends to 1. The proof involves variational methods and a new version of Trudinger-Moser inequality.

Keywords: Trudinger-Moser inequality; supercritical exponential growth; logarithmic weight variational methods; Schrödinger equation
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1. Introduction

Let consider the following Schrödinger equation

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. In the case, $N \geq 3$ the nonlinearity is of the form $|f(x,u)| \leq c(1+|u|^{q-1})$, with $1 < q \leq 2^* = 2N/(N-2)$ (see Brézis [6], Brézis-Nirenberg [7], Bartsh-Willem [5] and Capozzi-Fortunato-Palmieri [12]). The behaviour of f is related to the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq 2^*$. In the limiting case $N = 2$, one has $2^* = +\infty$, that is, $H_0^1(\Omega) \subset L^q(\Omega)$ for $q \geq 1$, in particular, the nonlinear function f in (1.1) may have any arbitrary polynomial growth. Also, note that $H_0^1(\Omega) \not\subset L^\infty(\Omega)$, for instance $\phi(x) = \ln|\ln|x||$ for $0 < |x| < 1/2$ and $\phi(0) = 0$ in $\Omega = B_{1/2}(0) \subset \mathbb{R}^2$. An important result found independently by Yudovich [37], Pohozaev [25] and Trudinger [35] showed that the maximal growth of the nonlinearity in the bidimensional case is of exponential type. More precisely, it was stated that

$$e^{\alpha u^2} \in L^1(\Omega), \quad \text{for all} \quad u \in H_0^1(\Omega) \text{ and } \alpha > 0. \tag{1.2}$$

Furthermore, Moser [23] stated the existence of a positive constant $C = C(\alpha, \Omega)$ such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \le 1}} \int_\Omega e^{\alpha u^2}\, dx \begin{cases} \le C, & \alpha \le 4\pi, \\ +\infty, & \alpha > 4\pi. \end{cases} \tag{1.3}$$

Estimates (1.2) and (1.3) from now on be referred to as Trudinger-Moser inequalities. The above results motivate to say that the function f has subcritical exponential growth if

$$\lim_{s \to +\infty} \frac{f(x,s)}{e^{\alpha s^2}} = 0, \quad \text{for all } \alpha > 0,$$

and critical exponential growth if there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases} \tag{1.4}$$

Equations of the type (1.1) considering nonlinearities involving subcritical and critical exponential growth were treated by Adimurthi [1], Adimurthi-Yadava [2], de Figueiredo, Miyagaki, and Ruf [15] (see also [13, 14, 16, 19, 20, 27–30]). We shall write $g_1(s) \prec g_2(s)$, if there exist positive constants k and s_0 such that $g_1(s) \le g_2(ks)$ for $s \ge s_0$. Additionally, we shall say that g_1 and g_2 are equivalent and write $g_1(s) \sim g_2(s)$ if $g_1(s) \prec g_2(s)$ and $g_2(s) \prec g_1(s)$. Therefore, f possesses critical exponential growth if only if $f(x,s) = g(s)$ with $g(s) \sim e^{|s|^2}$.

Several extensions of the Trudinger-Moser inequalities were obtained considering weighted Sobolev spaces, weighted Lebesgue measures or Lorentz-Sobolev spaces (see [3, 4, 11, 21, 22, 32–34] among others). In the above mentioned papers, the growth of the nonlinearity is of the type $f(x,s) = Q(x)g(s)$ where $g(s) \sim e^{|s|^p}$ with $p = 2$ on Sobolev spaces and $p > 1$ on Lorentz-Sobolev spaces and for some weight $Q(x)$. Now, we recall some facts about Lorentz-Sobolev spaces. Let $1 < r < +\infty$, $1 \le s < +\infty$ and Ω subset of $\mathbb{R}^N$, the Lorentz space $L^{r,s}(\Omega)$ is the collection of all measurable and finite almost everywhere functions on Ω such that $\|\phi\|_{r,s} < +\infty$, where

$$\|\phi\|_{r,s} = \left(\int_0^{+\infty} [\phi^*(t) t^{1/r}]^s \frac{dt}{t} \right)^{1/s}$$

where $\phi*$ denote the spherically symmetric decreasing rearrangement of ϕ. In addition, if Ω is an open bounded domain in $\mathbb{R}^N$, the Lorentz-Sobolev space $W_0^1 L^{r,s}(\Omega)$ is defined to be the closure of the compactly supported smooth functions on Ω, with respect to the quasinorm

$$\|u\|_{W_0^1 L^{r,s}(\Omega)} := \|\nabla u\|_{r,s}$$

On Lorentz-Sobolev spaces, Brezis and Wainger [8], showed: Let Ω be a bounded domain in $\mathbb{R}^2$ and $s > 1$. Then, $e^{\alpha|u|^{\frac{s}{s-1}}}$ belongs to $L^1(\Omega)$ for all $u \in W_0^1L^{2,s}(\Omega)$ and $\alpha > 0$. Furthermore, Alvino [4] obtained the following refinement of (1.3), there exists a positive constant $C = C(\Omega, s, \alpha)$ such that

$$\sup_{\substack{u\in W_0^1L^{2,s}(\Omega)\\ \|\nabla u\|_{2,s}\le 1}} \int_\Omega e^{\alpha|u|^{\frac{s}{s-1}}}\, dx \begin{cases} \le C, & \alpha \le (4\pi)^{s/(s-1)},\\ = +\infty, & \alpha > (4\pi)^{s/(s-1)}. \end{cases} \tag{1.5}$$

Trudinger–Moser type inequalities for radial Sobolev spaces with logarithmic weights were considered by Calanchi and Ruf in [9]. Let B_1 be the unit ball centered at the origin in $\mathbb{R}^2$ and $H^1_{0,\mathrm{rad}}(B_1, w)$ be the subspace of the radially symmetric functions in the closure of $C_0^\infty(B_1)$ with respect to the norm

$$\|u\| = \|u\|_{H^1_{0,\mathrm{rad}}(B_1,w)} := \Big(\int_{B_1} w(x)|\nabla u|^2\, dx\Big)^{\frac{1}{2}},$$

where $w(x) = \big(\ln 1/|x|\big)^\beta$ and $0 \le \beta < 1$.

Proposition 1.1. *(See [9, Calanchi-Ruf]) Suppose that $w(x) = \big(\log 1/|x|\big)^\beta$ and $0 \le \beta < 1$. Then,*

$$\int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\, dx < +\infty, \ \textit{for all}\ \ u \in H^1_{0,rad}(B_1, w)\ \ \textit{and}\ \ \alpha > 0.$$

Furthermore, setting $\alpha^*_\beta = 2\big[2\pi(1-\beta)\big]^{\frac{1}{1-\beta}}$, then there exists a positive constant $C = C(\alpha,\beta)$ such that

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1,w)\\ \|u\|\le 1}} \int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\, dx \begin{cases} \le C, & \alpha \le \alpha^*_\beta,\\ +\infty, & \alpha > \alpha^*_\beta. \end{cases}$$

If the weight is given by $w(x) = \ln(e/|x|)$ the maximal growth is double exponential type, see [9, 10, 31] for more details). Next, we state a Trudinger-Moser inequality proved by Ngô and Nguyen [24], where $H^1_{0,\mathrm{rad}}(B_1)$ denotes the subspace of $H^1_0(B_1)$, which consists of only radially symmetric functions. Let $h : [0, 1) \to \mathbb{R}$ be a continuous radial function such that

(h_1) $h(0) = 0$ and $h(r) > 0$ for $r \in (0, 1)$.
(h_2) There exists some $c > 0$ such that

$$h(r) \le \frac{c}{-\ln r}, \quad \text{near to } 0.$$

(h_3') There exists some $\gamma \in (0, 1)$ such that

$$h(r) \le \frac{2\pi\gamma \ln(1-r)}{\ln r}, \quad \text{near to } 1.$$

Proposition 1.2. *(See [24]) Suppose that h satisfies (h_1), (h_2) and (h_3'). Then,*

$$\int_{B_1} \exp\Big((\alpha + h(|x|))|u|^2\Big)\, dx < +\infty, \quad \textit{for all}\ \ u \in H^1_{0,\mathrm{rad}}(B_1)\ \textit{and}\ \alpha > 0.$$

Furthermore, there exists a positive constant $C = C(\alpha, h)$ such that

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1)\\ \|\nabla u\|_2\le 1}} \int_{B_1} \exp\Big((\alpha + h(|x|))|u|^2\Big)\, dx \begin{cases} \le C, & \alpha \le 4\pi,\\ = +\infty, & \alpha > 4\pi. \end{cases}$$

Before to establish a new version of the Trudinger-Moser inequality which will be used throughout this paper, we give the following condition for the function h.

(h_3) There exists some $\gamma \in (0,1)$ such that

$$h(r) \leq \frac{\gamma \alpha_\beta^* \ln(1-r)}{\ln r}, \quad \text{near to } 1,$$

where α_β^* is given by Proposition 1.1.

Theorem 1.3. *Suppose that h satisfies $(h_1) - (h_3)$ and $w(x) = \ln(1/|x|)^\beta$ for some $\beta \in [0,1)$. If $\alpha \leq \alpha_\beta^*$, there exists a positive constant $C = C(\alpha, \beta, h)$ such that*

$$\sup_{\substack{u \in H^1_{0,\mathrm{rad}}(B_1, w) \\ \|u\| \leq 1}} \int_{B_1} \exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big)\, dx \leq C,$$

and for $\alpha > \alpha_\beta^*$

$$\sup_{\substack{u \in H^1_{0,\mathrm{rad}}(B_1, w) \\ \|u\| \leq 1}} \int_{B_1} \exp\Big((\alpha + h(|x|))|u|^{2/(1-\beta)}\Big)\, dx = +\infty.$$

The proof of the Theorem 1.3 will be presented in next section. In this work, we are interested to find nontrivial weak solutions for the following Schrödinger equation

$$\begin{cases} -\mathrm{div}(w(x)\nabla u) = f(x,u), & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.6}$$

where the nonlinearity is motivated by the Trudinger-Moser inequality given by Proposition 1.3. More precisely, we suppose the following assumptions:

(H_1) $f : \overline{B}_1 \times \mathbb{R} \to \mathbb{R}$ is a continuous and radially symmetric in the first variable function, that is, $f(x,s) = f(y,s)$ for $|x| = |y|$. Moreover $f(x,s) = 0$ for all $x \in B_1$ and $s \leq 0$.

(H_2) There exists a constant $\mu > 2$ such that

$$0 < \mu F(x,s) \leq s f(x,s), \quad \text{for all} \quad x \in B_1 \ \text{ and } \ s > 0,$$

where $F(x,s) = \int_0^s f(x,t)\, dt$.

(H_3) There exists a constant $M > 0$ such that

$$0 < F(x,s) \leq M f(x,s), \quad \text{for all} \quad x \in B_1 \ \text{ and } \ s > 0,$$

(H_4) There holds

$$\limsup_{s \to 0^+} \frac{2F(x,s)}{s^2} < \lambda_1, \quad \text{uniformly in } x \in B_1,$$

where λ_1 is the first eigenvalue associated to $\big(-\mathrm{div}(w(x)\nabla u), H^1_{0,\mathrm{rad}}(B_1, w)\big)$.

(H_5) There exists a constant $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(x,s)}{\exp\Big((\alpha + h(x))|s|^{2/(1-\beta)}\Big)} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases}$$

where h satisfies (h_1), (h_2) and the next condition:

$(\widetilde{h}_3)$ There holds

$$h(r) \le \frac{\gamma \min\{\alpha_\beta^*, \alpha_0\} \ln(1-r)}{\ln r}, \quad \text{near to 1 and for some } \gamma \in (0,1).$$

(H_6) There exist constants $p > 2$ and $C_p > 0$ such that

$$f(x,s) \ge C_p s^{p-1}, \quad \text{for all} \quad s \ge 0,$$

where

$$C_p > \frac{(p-2)^{(p-2)/2} S_p^p}{p^{(p-2)/2}} \Big(\frac{\alpha_0}{\alpha_\beta^*}\Big)^{(1-\beta)(p-2)/2}$$

and

$$S_p := \inf_{0\neq u \in H^1_{0,\mathrm{rad}}(B_1,w)} \frac{\Big(\int_{B_1} w(x)|\nabla u|^2\,dx\Big)^{1/2}}{\Big(\int_{B_1} |u|^p\,dx\Big)^{1/p}}.$$

Throughout what follows, we denote the space $E := H^1_{0,rad}(B_1, w)$ endowed with the inner product

$$\langle u, v\rangle_E = \int_{B_1} w(x)\nabla u \nabla v\,dx, \quad \text{for all} \quad u,\ v \in E,$$

to which corresponds the norm

$$\|u\| = \Big(\int_{B_1} w(x)|\nabla u|^2\,dx\Big)^{1/2},$$

and by E^* the dual space of E with its usual norm. We say that $u \in E$ is a weak solution of (1.6) if

$$\int_{B_1} w(x)\nabla u \nabla \phi\,dx = \int_{B_1} f(x,u)\phi\,dx, \quad \text{for all} \quad \phi \in E. \tag{1.7}$$

Using the function f, we consider the Euler-Lagrange functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_{B_1} w(x)|\nabla u|^2\,dx - \int_{B_1} F(x,u)\,dx.$$

Furthermore, using standard arguments (see [17]), J belongs to $C^1(E,\mathbb{R})$ and

$$J'(u)\phi = \int_{B_1} w(x)\nabla u\nabla\phi\,dx - \int_{B_1} f(x,u)\phi\,dx, \quad \text{for all} \quad u,\ \phi \in E.$$

The following theorem contains our main result.

Theorem 1.4. *Suppose that f satisfies $(H_1) - (H_6)$. Then, the problem (1.6) possesses a nontrivial weak solution.*

First, we observe that (1.6) represents a natural extension of the problem (1.1). From assumption (H_5) the nonlinearity f behaves like $\exp((\alpha + h(|x|))|s|^{2/(1-\beta)})$ as s tends to infinity. Moreover, if $\beta = 0$, we have that $w \equiv 1$ and the equation (1.6) is reduced to problem (1.1), the case for $\beta = 0$ and $h \equiv 0$ were treated in many works considering (see [1, 2, 15] among others). Additionally, condition (h_3) implies that h could be approach to infinity as $|x|$ is close to 1. Also, if β is close to 1 the power of $|s|^p$ where $p = 2/(1-\beta)$ can be sufficiently large. The above properties motivate to say that f possesses supercritical exponential growth and represents an extension of other previously studied works. Finally, note that the class of functions which satisfy conditions $(H_1) - (H_6)$ is not empty, for instance consider the following function $f : B_1 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,s) = \begin{cases} As^{p-1} + p(1+|x|^\eta)s^{p-1}e^{(1+|x|^\eta)s^p}, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

for some positive constants η, $p = 2/(1-\beta)$ and A sufficiently large.

This paper is organized as follows: Section 2 contains some preliminary results and the proof of the supercritical Trudinger-Moser inequality. In section 3, we show that the Euler-Lagrange functional possesses the geometry of the Pass Mountain theorem. In Section 4, It is established some Palais-Smale properties and estimated the minimax level of the functional. Finally, in sections 5, we present the proof of our main theorem.

2. Preliminaries

Let $H^1_{0,rad}(B_1, w)$ denote the subspace of the radially symmetric functions in the closure of $C_0^\infty(B_1)$ with respect to the norm

$$\|u\| = \|u\|_{H^1_{0,\,\mathrm{rad}}(B_1,w)} := \Big(\int_{B_1} w(x)|\nabla u|^2\, dx\Big)^{\frac{1}{2}}.$$

where $w(x) = (\log 1/|x|)^\beta$ for some $0 \leq \beta < 1$. The Sobolev weighted space $H^1_{0,rad}(B, w)$ is a separable Banach space (see [18, Theorem 3.9]). The next lemma presents some embedding results.

Lemma 2.1. *The embedding $H^1_{0,\,\mathrm{rad}}(B_1, w) \hookrightarrow L^p(B_1)$ is continuous and compact for $1 \leq p < \infty$.*

Proof. From the Cauchy-Schwarz inequality, we have

$$\int_{B_1} |\nabla u|\, dx \leq \Big(\int_{B_1} w(x)|\nabla u|^2\, dx\Big)^{1/2} \cdot \Big(\int_{B_1} w(x)^{-1}\, dx\Big)^{1/2}.$$

Using the change of variable $|x| = e^{-s}$, we get

$$\frac{1}{2\pi}\int_{B_1} w(x)^{-1}\, dx = \int_0^{+\infty} e^{-2s}s^{-\beta}\, ds = \int_0^1 e^{-2s}s^{-\beta}\, ds + \int_1^{+\infty} e^{-2s}s^{-\beta}\, ds.$$

Note that

$$\int_0^1 e^{-2s}s^{-\beta}\, ds \leq \int_0^1 s^{-\beta}\, ds = \frac{1}{1-\beta}$$

and

$$\int_1^{+\infty} e^{-2s}s^{-\beta}\,ds \le \int_1^{+\infty} e^{-2s}\,ds = \frac{e^{-2}}{2}.$$

Therefore, we can find a positive constant C such that

$$\|\nabla u\|_1 \le C\Big(\int_{B_1} |\nabla u|^2 w(x)\,dx\Big)^{1/2}.$$

Thus, $H^1_{0,\mathrm{rad}}(B_1, w) \hookrightarrow W^{1,1}_0(B_1)$ continuously, which implies the continuous and compact embedding

$$H^1_{0,\mathrm{rad}}(B_1, w) \hookrightarrow L^p(B_1), \quad \text{for all} \quad p \ge 1.$$

□

Lemma 2.2. *(See [9]) Let u be a function in* $H^1_{0,\mathrm{rad}}(B_1, w)$. *Then,*

$$|u(x)| \le \frac{(-\ln|x|)^{\frac{1-\beta}{2}}}{\sqrt{2\pi(1-\beta)}} \cdot \|u\|, \quad \textit{for all} \;\; x \in B_1.$$

2.1. Proof of the Theorem 1.3

Proof. To prove the first statement of the theorem, it is sufficiently to consider $\alpha = \alpha^*_\beta$. From Lemma 2.2 for each $u \in E$ with $\|u\| \le 1$, we have

$$\alpha^*_\beta |u(r)|^{2/(1-\beta)} \le -2\ln r, \quad \text{for all} \quad 0 < r < 1, \tag{2.1}$$

where $r = |x|$, and (2.1) implies that

$$\exp\Big(\alpha^*_\beta |u(r)|^{2/(1-\beta)}\Big) \le \frac{1}{r^2}, \quad \text{for all} \quad 0 < r < 1. \tag{2.2}$$

By (h_3), there exists $r_0 > 0$ such that

$$h(r) \le \frac{\gamma\alpha^*_\beta \ln(1-r)}{2\ln r}, \quad \text{for all} \quad r_0 \le r < 1.$$

Using the above inequality and (2.1), we get

$$h(r)|u(r)|^{\frac{2}{1-\beta}} \le -\ln(1-r)^\gamma, \quad \text{for all} \quad r_0 \le r < 1. \tag{2.3}$$

Note that

$$\int_{B_1\setminus B_{r_0}} \exp\Big((\alpha^*_\beta + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx = 2\pi\int_{r_0}^1 \exp\Big(\alpha^*_\beta|u|^{2/(1-\beta)}\Big)\exp\Big(h(r)|u|^{2/(1-\beta)}\Big) r\,dr.$$

Using (2.2) and (2.3), we obtain

$$\int_{B_1\setminus B_{r_0}} \exp\Big((\alpha^*_\beta + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx \le 2\pi\int_{r_0}^1 \frac{1}{r(1-r)^\gamma}\,dr \le \frac{2\pi}{r_0}\int_{r_0}^1 \frac{1}{(1-r)^\gamma}\,dr.$$

Therefore,

$$\int_{B_1\setminus B_{r_0}} \exp\Big((\alpha_\beta^* + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx \le \frac{2\pi(1-r_0)^{1-\gamma}}{r_0(1-\gamma)}. \tag{2.4}$$

Other other hand, by (h_2) there exist $c > 0$ and $r_1 > 0$ such that

$$h(r) \le \frac{c}{-\ln r}, \quad \text{for all} \quad 0 < r < r_1.$$

Combining last inequality with (2.1), we get

$$h(r)|u(r)|^{2/(1-\beta)} \le \frac{2c}{\alpha_\beta^*}, \quad \text{for all} \quad 0 < r < r_1. \tag{2.5}$$

Also, by (2.1), we have

$$h(r)|u(r)|^{2/(1-\beta)} \le \frac{-2h(r)\ln r}{\alpha_\beta^*}, \quad \text{for all} \quad r_1 \le r \le r_0. \tag{2.6}$$

From (2.5) and (2.6), we can find a constant $M = M(h,\beta)$ such that

$$h(r)|u(r)|^{2/(1-\beta)} \le M, \quad \text{for all} \quad 0 \le r \le r_0.$$

Then,

$$\int_{B_{r_0}} \exp\Big((\alpha_\beta^* + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx \le e^M \int_{B_{r_0}} \exp\Big(\alpha_\beta^*|u|^{2/(1-\beta)}\Big)\,dx.$$

Using Proposition 1.1 there exists $C = C(\beta) > 0$ such that

$$\int_{B_{r_0}} \exp\Big((\alpha_\beta^* + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx \le Ce^M. \tag{2.7}$$

The first two assertions of the theorem follows from (2.4) and (2.7). In order to prove the sharpeness, we consider the following sequence given in [11]

$$\psi_k(x) = \Big(\frac{1}{\alpha_\beta^*}\Big)^{(1-\beta)/2} \begin{cases} k^{\frac{2}{1-\beta}} \ln\Big(\dfrac{1}{|x|^2}\Big)^{1-\beta}, & 0 \le |x| \le e^{-k/2}, \\ k^{\frac{1-\beta}{2}}, & e^{-k/2} \le |x| \le 1. \end{cases}$$

Then, $\|\psi_k\| = 1$ for all $k \in \mathbb{N}$. Moreover, for $\alpha > \alpha_\beta^*$, we have

$$\int_{B_1} \exp\Big((\alpha + h(|x|))|\psi_k|^{2/(1-\beta)}\Big)\,dx \ge \int_{B_1} \exp\Big(\alpha|\psi_k|^{2/(1-\beta)}\Big)\,dx \ge 2\pi \int_{e^{-k/2}}^1 \exp\Big(\frac{\alpha}{\alpha_\beta^*}k\Big) r\,dr$$

Then,

$$\int_{B_1} \exp\Big((\alpha + h(|x|))|\psi_k|^{2/(1-\beta)}\Big)\,dx \ge 2\pi e^{k\left(\frac{\alpha}{\alpha_\beta^*}-1\right)}\Big(e^k - 1\Big) \to +\infty, \quad \text{as } k \to \infty$$

and the proof is complete. □

Corollary 2.3. *Let $\eta > 0$ and $\alpha \le \alpha_\beta^*$, there exists a positive constant C such that*

$$\sup_{\|u\|\le 1} \int_{B_1} \exp\Big((\alpha + |x|^\eta)|u|^{2/(1-\beta)}\Big)\, dx \le C. \tag{2.8}$$

If $\alpha > \alpha_\beta^*$ it holds

$$\sup_{\|u\|\le 1} \int_{B_1} \exp\Big((\alpha + |x|^\eta)|u|^{2/(1-\beta)}\Big)\, dx = +\infty. \tag{2.9}$$

As it was observed in [24], the statements of the Theorem 1.3 and its corollary are no longer true if it is consider the space of non radial functions $H_0^1(B_1, w)$.

3. The geometry of the Mountain Pass theorem

This section is devoted to establish the geometry of the Mountain Pass theorem for the functional J.

Lemma 3.1. *Suppose that h satisfies $(h_1)-(h_3)$. Then, there exist $0 < r_0 < 1$ and $1 < m < 1/\gamma$ such that*

$$\int_{B_1\setminus B_{r_0}} \exp\Big(mh(|x|)|u|^{2/(1-\beta)}\Big)\, dx \le C,$$

provided $\|u\| \le 1$, where γ is given in (h_3).

Proof. Let u be fixed in E with $\|u\| \le 1$. From Lemma 2.2 and (h_3) there exists $r_0 > 0$ such that

$$\frac{1}{\gamma} h(r)|u(r)|^{2/(1-\beta)} \le -\ln(1-r), \quad \text{for all} \quad r_0 \le r < 1. \tag{3.1}$$

Taking positive constants ξ and m satisfy $1 < m < 1/\xi < 1/\gamma$. Using the Hölder inequality with $s > 1$ such that $ms < 1/\xi$, we obtain

$$\begin{aligned}\int_{B_1\setminus B_{r_0}} \exp\Big(mh(|x|)|u|^{2/(1-\beta)}\Big)\, dx &\le 2\pi\Big(\int_{r_0}^1 \exp\Big(msh(r)|u(r)|^{2/(1-\beta)}\Big)\, dr\Big)^{1/s}\Big(\int_{r_0}^1 r^{s'}\, dr\Big)^{1/s'}\\ &\le 2\pi\Big(\int_{r_0}^1 \exp\Big(\frac{1}{\xi}h(r)|u(r)|^{2/(1-\beta)}\Big)\, dr\Big)^{1/s}.\end{aligned}$$

Combining (3.1) with the fact that the improper integral of $1/(1-r)^{\gamma/\xi}$ is finite on the interval $[r_0, 1]$, there exists $C > 0$ such that

$$\int_{r_0}^1 \exp\Big(\frac{1}{\xi}h(r)|u(r)|^{2/(1-\beta)}\Big)\, dr \le \int_{r_0}^1 \exp\Big(-\frac{\gamma}{\xi}\ln(1-r)\Big)\, dr = \int_{r_0}^1 \frac{1}{(1-r)^{\gamma/\xi}}\, dr \le C,$$

this complete the proof. □

Lemma 3.2. *Suppose that $(H_1), (H_4)$ and (H_5) hold. Then, there exist $\sigma, \rho > 0$ such that*

$$J(u) \ge \sigma, \quad \textit{for all} \quad u \in E \ \textit{ with } \ \|u\| = \rho.$$

Proof. Given $q > 2$ and $0 < \epsilon < \lambda_1/2$. From (H_1) and (H_4), we can find $c > 0$ such that

$$|F(x,s)| \leq \epsilon|s|^2 + c|s|^q \exp\Big((2\alpha_0 + h(|x|))|s|^{2/(1-\beta)}\Big), \quad \text{for all} \quad (x,s) \in B_1 \times \mathbb{R}.$$

Thus,

$$\int_{B_1} F(x,u)\,dx \leq \epsilon\|u\|_2^2 + c\int_{B_1} |u|^q \exp\Big((2\alpha_0 + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx. \tag{3.2}$$

Let $h_0 = \max_{0\leq r\leq r_0} h(r)$ where r_0 is given by Lemma 3.1. By the Cauchy-Schwarz inequality and Theorem 1.3, we have

$$\begin{aligned}\int_{B_{r_0}} |u|^q \exp\Big((2\alpha_0 + h(|x|))|u|^{2/(1-\beta)}\Big)\,dr &\leq \int_{B_1} |u|^q \exp\Big((2\alpha_0 + h_0)|u|^{2/(1-\beta)}\Big)\,dx\\ &\leq \|u\|_{2q}^q \int_{B_1} \exp\Big(2(2\alpha_0 + h_0)\|u\|^{2/(1-\beta)}(\frac{|u|}{\|u\|})^{2/(1-\beta)}\Big)\,dx\\ &\leq c\|u\|_{2q}^q,\end{aligned}$$

provided that $\|u\| \leq \rho_1$ for some $\rho_1 > 0$ such that $2(2\alpha_0 + h_0)\rho_1^{2/(1-\beta)} < \alpha_\beta^*$. From Lemma 3.1 and the Hölder inequality with $m, m_1, m_2 > 1$ such that $1/m_1 + 1/m_2 + 1/m = 1$ where m is given by Lemma 3.1, we obtain

$$\begin{aligned}&\int_{B_1\setminus B_{r_0}} |u|^q \exp\Big((2\alpha_0 + h(|x|))|u|^{2/(1-\beta)}\Big)\,dx\\ &\leq \|u\|_{qm_1}^q\Big(\int_{B_1} \exp\Big(2m_2\alpha_0|u|^{2/(1-\beta)}\Big)\,dx\Big)^{\frac{1}{m_2}}\Big(\int_{B_1\setminus B_{r_0}} \exp\Big(mh(|x|))|u|^{2/(1-\beta)}\Big)\,dx\Big)^{\frac{1}{m}}\\ &\leq c\|u\|_{qm_1}^q\Big(\int_{B_1} \exp\Big(2m_2\alpha_0\|u\|^{2/(1-\beta)}(\frac{|u|}{\|u\|})^{2/(1-\beta)}\Big)\,dx\Big)^{\frac{1}{m_2}}\\ &\leq c\|u\|_{qm_1}^q,\end{aligned}$$

provided that $\|u\| \leq \rho_2$ for some $\rho_2 > 0$ such that $2m_2\alpha_0\rho_2^{2/(1-\beta)} < \alpha_\beta^*$. Replacing the above estimates in (3.2) and using Lemma 2.1, we get $c > 0$ such that

$$\int_{B_1} F(x,u)\,dx \leq \frac{\epsilon}{\lambda_1}\|u\|^2 + c\|u\|^q,$$

provided that $\|u\| \leq \rho_0$ for some $\rho_0 > 0$ such that $\rho_0 < \min\{1, \rho_1, \rho_2\}$. Then,

$$J(u) \geq \frac{1}{2}\|u\|^2 - \int_{B_1} F(x,u)\,dx \geq \Big(\frac{1}{2} - \frac{\epsilon}{\lambda_1}\Big)\|u\|^2 - c\|u\|^q.$$

Therefore, we can find $\rho > 0$ and $\sigma > 0$ with $0 < \rho < \rho_1$ sufficiently small such that $J(u) \geq \sigma > 0$, for all $u \in E$ satisfying $\|u\| = \rho$. □

Lemma 3.3. *Suppose that $(H_1) - (H_2)$ hold. Then, there exists $e \in E$ such that*

$$J(e) < 0 \quad \textit{and} \quad \|e\| > \rho,$$

where $\rho > 0$ is given by Lemma 3.2.

Proof. Let $s_0 = \mu M > 0$, from (H_2), we have

$$0 < \frac{1}{M} \leq \frac{f(x,s)}{F(x,s)} \quad \text{for all } \ x \in B_1 \ \text{ and } \ s \geq s_0.$$

For $x \in \overline{B}_1$ fixed, the above inequality implies that the function

$$h(s) = F(x,s)\exp\Big(-\frac{s}{M}\Big),$$

is increasing in $[s_0, +\infty)$. Consequently,

$$F(x,s)\exp\Big(-\frac{s}{M}\Big) \geq F(x,s_0)\exp\Big(-\frac{s_0}{M}\Big) > 0 \qquad \text{for all } \ s \geq s_0.$$

Taking $C = \inf_{x\in\overline{B}_1} F(x,s_0)\exp\Big(-\frac{s_0}{M}\Big)$, we obtain

$$F(x,s) \geq C\exp\Big(\frac{s}{M}\Big), \quad \text{for all } \ x \in B_1 \ \text{ and } \ s \geq s_0$$

Let $e_0 \geq 0$ and $e_0 \neq 0$ fixed. Then, there exists $\delta > 0$ such that $|\{x \in B_1 : e_0(x) \geq \delta\}| \geq \delta$. Thus, for $t \geq s_0/\delta$, we have

$$J(te_0) \leq \frac{t^2}{2}\|e_0\|^2 - \int_{\{x\in B_1:e_0\geq\delta\}} F(x,te_0)\,dx \leq \frac{t^2}{2}\|e_0\|^2 - C\delta\exp\Big(\frac{t\delta}{M}\Big),$$

which implies that $J(te_0) \to -\infty$, as $t \to +\infty$. Therefore, we can take $e = t_0e_0$ with $t_0 > 0$ sufficiently large such that $J(e) < 0$ and $\|e\| > \rho$. □

4. Palais-Smale sequence

By Lemmas 3.2 and 3.3 in Mountain Pass theorem (see [26, 36]), there exists a Palais-Smale sequence at level $d \geq \sigma$, where σ is given by Lemma 3.2, that is, there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \to d \quad \text{and} \quad \|J'(u_n)\|_{E^*} \to 0, \tag{4.1}$$

where $d > 0$ can be characterized as

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)), \tag{4.2}$$

and

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 4.1. *Let $(u_n) \subset E$ be a Palais-Smale sequence for the functional J satisfying (4.1). Then, $\|u_n\| \leq c$, for every $n \in \mathbb{N}$ and for some positive constant c.*

Proof. From (H_2), we have

$$J(u_n) - \frac{1}{\mu}J'(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \frac{1}{\mu}\int_{B_1}(\mu F(x,u_n) - f(x,u_n)u_n)\,dx \geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2.$$

Using (4.1), for n sufficiently large, we have

$$J(u_n) \leq d + 1 \quad \text{and} \quad \|J'(u_n)\|_{E^*} \leq \mu,$$

Therefore, for n sufficiently large, we obtain

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 \leq d + 1 + \|u_n\|,$$

which implies that the sequence (u_n) is bounded. □

Lemma 4.2. *(See [15, Lemma 2.1]) Let Ω be a bounded subset in $\mathbb{R}^N$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function and (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u(x))$ and $f(x, u_n(x))$ are also $L^1(\Omega)$ functions. If*

$$\int_\Omega |f(x, u_n)u_n| \, dx \leq C,$$

then, $f(x, u_n)$ converges in $L^1(\Omega)$ to $f(x, u)$.

Lemma 4.3. *Let (u_n) be a Palais-Smale sequence for the functional J satisfying (4.1) and suppose that $u_n \rightharpoonup u$ in E. Then, there exists a subsequence, still denoted by (u_n), such that*

$$f(x, u_n) \to f(x, u) \quad in \quad L^1(B_1) \tag{4.3}$$

and

$$F(x, u_n) \to F(x, u) \quad in \quad L^1(B_1). \tag{4.4}$$

Proof. According to Lemma 2.1, we can assume that $u_n \to u$ in $L^1(B_1)$. By assumption on f and Theorem 1.3, we have that $f(x, u_n) \in L^1(B_1)$. Using Lemma 4.1, the sequence $(\|u_n\|)$ is bounded and the fact that $\|J'(u_n)\|_{E^*} \to 0$, we obtain

$$|J'(u_n)u_n| \leq \|J'(u_n)\|_{E^*}\|u_n\| \to 0.$$

Thus,

$$J'(u_n)u_n = \frac{\|u_n\|^2}{2} - \int_{B_1} f(x, u_n)u_n \, dx \to 0.$$

Therefore, the sequence $f(x, u_n)u_n$ is bounded in $L^1(B_1)$. Applying Lemma 4.2, we conclude that $f(x, u_n) \to f(x, u)$ in $L^1(B_1)$. On the other hand, by the convergence (4.3), there exists $p \in L^1(B_1)$ such that

$$f(x, u_n) \leq p(x), \quad \text{almost everywhere in } B_1 \text{ and } n \text{ sufficiently large.}$$

From (H_3), we have

$$F(x, u_n) \leq Mp(x), \quad \text{almost everywhere in } B_1 \text{ and } n \text{ sufficiently large.}$$

By Lebesgue's dominated convergence theorem, the convergence (4.4) follows. □

Lemma 4.4. *Let $(u_n) \subset E$ be a Palais-Smale sequence for the functional J satisfying (4.1). Then,*

$$d < \frac{1}{2}\Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta},$$

where d is the minimax level given by (4.2).

Proof. Let $u_p \in E$ be a nonnegative function with $\|u_p\|_p = 1$ such that

$$S_p = \inf_{0\neq u\in H^1_{0,rad}(B_1,w)} \frac{\Big(\int_{B_1} w(x)|\nabla u|^2\,dx\Big)^{1/2}}{\Big(\int_{B_1} |u|^p\,dx\Big)^{1/p}} = \|u_p\|.$$

From (H_6), we get

$$J(tu_p) = \frac{t^2}{2}\|u_p\|^2 - \int_{B_1} F(x,tu_p)\,dx \leq \frac{t^2}{2}\|u_p\|^2 - \frac{C_p t^p}{p}\int_{B_1} |u_p|^p\,dx.$$

Therefore, by the estimate of C_p, we have

$$\sup_{t\geq 0} J(tu_p) \leq \max_{t\geq 0}\Big\{\frac{t^2 S_p^2}{2} - \frac{C_p t^p}{p}\Big\} = \frac{(p-2)S_p^{2p/(p-2)}}{2pC_p^{2/(p-2)}} < \frac{1}{2}\Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta}, \tag{4.5}$$

where we have used that the function $\lambda(t) = t^2S_p^2/2 - C_pt^p/p$, possesses a critical point at $t_0 = (S_p/C_p)^{1/(p-2)}$. Now, taking $e_0 = u_p$ in Lemma 3.3, that is, we consider $e = t_0u_p$ with $t_0 > 0$ given by Lemma 3.3. Setting $\gamma_0(t) = tt_0u_p$, in particular, we have $\gamma_0 \in \Gamma = \{\gamma \in C([0,1],E)\colon \gamma(0) = 0, \gamma(1) = e\}$. Using (4.2) and (4.5), we obtain

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(\gamma_0(t)) \leq \max_{t\in[0,1]} J(tt_0u_p) \leq \max_{t\geq 0} J(tu_p) < \frac{1}{2}\Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta}.$$

□

5. Proof of the main theorem

Let $(u_n) \subset E$ be a Palais-Smale sequence of the functional J satisfying (4.1). Then,

$$J'(u_n)\phi = \int_{B_1} w(x)\nabla u_n\nabla\phi\,dx - \int_{B_1} f(x,u_n)\phi\,dx = o_n(1), \tag{5.1}$$

for all $\phi \in C^\infty_{0,\mathrm{rad}}(B_1)$. By Lemma 4.1, the sequence (u_n) is bounded in E. Thus, we may assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E, using this together with Lemma 4.3 in (5.1), we obtain passing to limit

$$\int_{B_1} w(x)\nabla u\nabla\phi\,dx - \int_{B_1} f(x,u)\phi\,dx = 0, \quad \text{for all} \quad \phi \in C^\infty_{0,\mathrm{rad}}(B_1).$$

Using the fact that $C_{0,\text{rad}}^{\infty}(B_1)$ is dense in E, yields

$$\int_{B_1} w(x)\nabla u\nabla\phi\, dx = \int_{B_1} f(x,u)\phi\, dx, \quad \text{for all} \quad \phi \in E.$$

Therefore, $u \in E$ is a critical point of J. To conclude the proof, it only remains to prove that u is nontrivial. Suppose, by contradiction, that $u \equiv 0$. From Lemma 2.1, we can assume that

$$u_n \to 0 \quad \text{in} \quad L^p(B_1), \quad \text{for all} \quad p \geq 1. \tag{5.2}$$

Using the fact that $J(u_n) \to d$, we have

$$J(u_n) = \frac{\|u_n\|^2}{2} - \int_{B_1} F(x,u_n)\, dx = d + o_n(1). \tag{5.3}$$

Since, we suppose that $u_n \rightharpoonup 0$, by Lemma 4.3, we obtain

$$\int_{B_1} F(x,u_n)\, dx \to \int_{B_1} F(x,0)\, dx = 0.$$

Replacing in (5.3) one has

$$\frac{\|u_n\|^2}{2} = d + o_n(1). \tag{5.4}$$

By Lemma 4.4, we get

$$\|u_n\|^2 = 2d + o_n(1) < \Big(\frac{\alpha_\beta^*}{\alpha_0}\Big)^{1-\beta} + o_n(1).$$

Thus, we can assume that there exists $\delta > 0$ sufficiently small such that

$$\|u_n\|^{2/(1-\beta)} \leq \frac{\alpha_\beta^*}{\alpha_0} - \delta, \quad \text{for all } n \geq 1 \tag{5.5}$$

Taking $m > 1$ sufficiently close to 1 and $\epsilon > 0$ sufficiently small such that

$$m(\alpha_0 + 2\epsilon)\Big(\frac{\alpha_\beta^*}{\alpha_0} - \delta\Big) < \alpha_\beta^*. \tag{5.6}$$

From assumptions on f there exists a positive constant C such that

$$|f(x,s)| \leq C\exp\Big((\alpha_0 + \epsilon + h(|x|))|s|^{2/(1-\beta)}\Big), \quad \text{for all} \quad (x,s) \in B_1 \times \mathbb{R}.$$

By the Hölder inequality and the above inequality, we have

$$\int_{B_1} f(x,u_n)u_n\, dx \leq C\|u_n\|_{m'}\Big(\int_{B_1} \exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big)\, dx\Big)^{1/m}. \tag{5.7}$$

Since h is continuous, there exists $r_1 > 0$ such that

$$h(|x|) < \epsilon, \quad \text{for all} \quad |x| \leq r_1.$$

Thus,

$$\int_{B_{r_1}} \exp\Big(m(\alpha_0+\epsilon+h(|x|))|u_n|^{2/(1-\beta)}\Big)\,dx \le \int_{B_{r_1}} \exp\Big(m(\alpha_0+2\epsilon)\|u_n\|^{2/(1-\beta)}(\frac{|u_n|}{\|u_n\|})^{2/(1-\beta)}\Big)\,dx.$$

Using (5.5), (5.6) and Theorem 1.3, we obtain $C_1 > 0$ such that

$$\int_{B_{r_1}} \exp\Big(m(\alpha_0+\epsilon+h(|x|))|u_n|^{2/(1-\beta)}\Big)\,dx \le \int_{B_{r_1}} \exp\Big(\alpha_\beta^*(\frac{|u_n|}{\|u_n\|})^{2/(1-\beta)}\Big)\,dx \le C_1. \tag{5.8}$$

By the boundedness of the sequence $(\|u_n\|)$ given in (5.5) and Lemma 2.2, we have

$$|u_n(x)| \le \frac{(-\ln r)^{\frac{1-\beta}{2}}}{\sqrt{2\pi(1-\beta)}}\Big(\frac{\alpha_\beta^*}{\alpha_0}-\delta\Big)^{(1-\beta)/2}. \quad \text{for all } n\ge 1 \quad \text{and} \quad |x| = r,$$

which implies

$$(\alpha_0+\epsilon)|u_n(x)|^{2/(1-\beta)} \le \Big(\frac{\alpha_0+\epsilon}{\alpha_\beta^*}\Big)\Big(\frac{\alpha_\beta^*}{\alpha_0}-\delta\Big)(-2\ln r), \quad \text{for all} \quad n\ge 1 \quad \text{and} \quad |x| = r.$$

By (5.6), we obtain

$$m(\alpha_0+\epsilon)|u_n(x)|^{2/(1-\beta)} \le -2\ln r, \quad \text{for all} \quad 0<r<1 \quad \text{and} \quad n\ge 1, \tag{5.9}$$

and

$$\exp\Big(m(\alpha_0+\epsilon)|u_n(x)|^{2/(1-\beta)}\Big) \le \frac{1}{r^2}, \quad \text{for all} \quad 0<r<1 \quad \text{and} \quad n\ge 1. \tag{5.10}$$

By $(\widetilde{h}_3)$, we can find $r_2 > r_1$ such that

$$h(r) \le \frac{\gamma\alpha_0\ln(1-r)}{2\ln r}, \quad \text{for all} \quad r_2\le r<1.$$

Combining above inequality with (5.9), we have

$$mh(r)|u_n(x)|^{\frac{2}{1-\beta}} \le -\gamma\Big(\frac{\alpha_0}{\alpha_0+\epsilon}\Big)\ln(1-r) \le -\ln(1-r)^{\gamma}, \quad \text{for all} \quad r_2\le r<1 \quad \text{and} \quad n\ge 1. \tag{5.11}$$

Using (5.10) and (5.11), for $n \ge 1$, we obtain

$$\begin{aligned}
&\int_{B_1\setminus B_{r_2}} \exp\Big(m(\alpha_0+\epsilon+h(|x|))|u_n|^{2/(1-\beta)}\Big)\,dx \\
&\qquad = 2\pi\int_{r_0}^{1} \exp\Big(m(\alpha_0+\epsilon)|u_n|^{2/(1-\beta)}\Big)\exp\Big(mh(r)|u_n|^{2/(1-\beta)}\Big) r\,dr \\
&\qquad \le 2\pi\int_{r_2}^{1} \frac{1}{r(1-r)^{\gamma}}\,dr \le \frac{2\pi(1-r_2)^{1-\gamma}}{r_2(1-\gamma)} = C_2.
\end{aligned} \tag{5.12}$$

On the other hand, using the boundedness of $(\|u_n\|)$ and Lemma 2.2, we have

$$|u_n(x)| \le M_0, \quad \text{for all} \quad r_1\le |x|\le r_2 \quad \text{and} \quad n\ge 1.$$

Moreover, by the continuity of h, we can find $C_3 > 0$ such that

$$\int_{B_{r_2}\setminus B_{r_1}} \exp\Big(m(\alpha_0 + \epsilon + h(|x|))|u_n|^{2/(1-\beta)}\Big)\, dx \le C_3. \tag{5.13}$$

Replacing (5.8), (5.12) and (5.13) in (5.7), we obtain

$$\int_{B_1} f(x, u_n)u_n\, dx \le C\|u_n\|_{m'}.$$

By (5.2), we get

$$\int_{B_R} f(x, u_n)u_n\, dx \to 0. \tag{5.14}$$

Using the fact that $(\|u_n\|)$ is bounded and $\|J'(u_n)\|_{E^*} \to 0$, we obtain

$$|J'(u_n)u_n| \le \|J'(u_n)\|_{E^*}\|u_n\| \to 0. \tag{5.15}$$

Since,

$$J'(u_n)u_n = \|u_n\|^2 - \int_{B_1} f(x, u_n)u_n\, dx.$$

By (5.14) and (5.15), we have

$$\|u_n\|^2 = J'(u_n)u_n + \int_{B_1} f(x, u_n)u_n\, dx \to 0.$$

From (5.4), we have $\|u_n\|^2 \to 2d$. Hence, $d = 0$ which is a contradiction, according to the fact that $d \ge \sigma > 0$. Thus, u is a nontrivial critical point of J. Therefore, u is a nontrivial weak solution of the problem (1.6).

6. Conclusions

In this work, we apply variational methods to find a nontrivial solution for a class of Schrödinger equations where the nonlinearities possess maximal growth in the sense of Trudinger-Moser. It is established a new version of Trudinger-Moser inequality with logarithmic weight which allows to treat supercritical nonlinearities that generalizes previous results in the literature. According to our definition of logarithm weight, we restricted the domain to the unit ball. It is of interest to further our results to extend our Trudinger-Moser inequality on the whole space $\mathbb{R}^2$.

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Conflict of interest

The author declares no conflicts of interest.

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