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Research article

On study the fractional Caputo-Fabrizio integro differential equation including the fractional q-integral of the Riemann-Liouville type

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Abstract: The major objective of this scheme is to investigate both the existence and the uniqueness of a solution to an integro-differential equation of the second order that contains the Caputo-Fabrizio fractional derivative and integral, as well as the q-integral of the Riemann-Liouville type. The equation in question is known as the integro-differential equation of the Caputo-Fabrizio fractional derivative and integral. This equation has not been studied before and has great importance in life applications. An investigation is being done into the solution's continued reliance. The Schauder fixed-point theorem is what is used to demonstrate that there is a solution to the equation that is being looked at. In addition, we are able to derive a numerical solution to the problem that has been stated by combining the Simpson's approach with the cubic-b spline method and the finite difference method with the trapezoidal method. We will be making use of the definitions of the fractional derivative and integral provided by Caputo-Fabrizio, as well as the definition of the q-integral of the Riemann-Liouville type. The integral portion of the problem will be handled using trapezoidal and Simpson's methods, while the derivative portion will be solved using cubic-b spline and finite difference methods. After that, the issue will be recast as a series of equations requiring algebraic thinking. By working through this problem together, we are able to find the answer. In conclusion, we present two numerical examples and contrast the outcomes of those examples with the exact solutions to those problems.

Keywords: q-integro-differential equation; fractional Caputo-Fabrizio; existence and uniqueness of solution; numerical solutions
Mathematics Subject Classification: 26Axx, 34Axx, 45Jxx, 45Lxx

1. Introduction

Mathematicians and physicists have become increasingly interested in fractional calculus and quantum calculus (q-calculus), which provide an effective way to describe a wide range of real-world dynamical phenomena encountered in scientific fields and engineering. In addition, they have paid attention to the study of partial differential equations because they are very useful in modelling practical phenomena; for example, for time-fractional stochastic models, see [1, 2], for the time fractional chemotaxis model, see [3], and for the time fractional Rayleigh-Stokes equation, see [4]. Researchers find it difficult to obtain direct solutions to most fractional and q-fractional differential equations. As a result, the researchers are interested in studying the existence and uniqueness of solutions to various fractional integro-differential equations. Researchers have obtained numerous results concerning the existence and uniqueness of solutions to a number of fractional integro-differential equations [5, 6]. Furthermore, many researchers are interested in the existence of solutions to q-fractional integro-differential equations [7, 8]. Simultaneously, numerous numerical solutions to many types of integro-differential equations have been obtained [9–11]. The authors presented analytical and numerical solutions to some ordinary integro-differential equations, as well as fractional q integro-differential equations with nonlocal and initial conditions [12, 13]. We now investigate the nonlocal fractional q integro-differential equations shown below analytically and numerically:

$$\phi''(t) = \mathcal{F}\Big(t, \phi'(t), {}^{CF}I^{\alpha_0}\phi'(t), \phi(t), {}^{CF}D^{\beta_0}\phi(t), I_q^{\gamma_0}\mu(t, \phi'(t))\Big), \quad t \in (0, 1], \tag{1.1}$$

$$(1-q)\rho \sum_{x=0}^{m} q^x \phi(q^x \rho) = \varrho, \quad \phi'(0) = \xi, \quad \rho \in (0, 1], \tag{1.2}$$

where ${}^{CF}I^{\alpha_0}\phi'(t)$, ${}^{CF}D^{\beta_0}\phi(t)$ are the Caputo-Fabrizio fractional integral and derivative of order α_0 and β_0 for the unkwon function respectively, $I_q^{\gamma_0}$ is the Riemann Liouville type's fractional q-integral of order $\gamma_0 \geqslant 0$, ϱ, ξ are constants, and $q, \alpha_0, \beta_0 \in (0, 1)$. We use the definitions of the integral and derivative fractional Caputo-Fabrizio to prove the existence, uniqueness, and continuous dependence of the solution. Then, we solve the proposed equation numerically by using two methods: The first is the merging of the cubic b-spline and Simpson's method, and the second is the merging of the finite difference and trapezoidal methods. Both the cubic b-spline and finite difference methods will be applied to the derivative parts of the equation, and both Simpson's method and the trapezoidal method will be applied to the integral part. These methods will transform the proposed equation into a system of algebraic equations. Therefore, we can obtain the solution to the problem by solving this system together.

This paper is structured as follows: In Section 2, we introduce some key definitions and lemmas that will be needed throughout our paper. We give the main results in Section 3. Section 4 contains an overview of the numerical techniques that will be employed in our paper. In Section 5, we discuss the existence of the solution to some examples, and then we will get the numerical solution to them using the cubic-Simpson's method and the finite-trapezoidal method. Finally, we introduce the conclusion section.

2. Basic concepts

Some key definitions and lemmas related to q-calculus and Fractional calculus will be introduced.

Definition 2.1. *[14] We can define the Caputo–Fabrizio fractional derivative of order* $0 < \mathcal{E} < 1$ *of any function* $\mathfrak{U}(t) \in C[a, b]$ *as follows:*

$$^{CF}D^{\mathcal{E}}\mathfrak{U}(t) = \frac{\psi(\mathcal{E})}{1-\mathcal{E}} \int_0^t e^{-\frac{\mathcal{E}}{1-\mathcal{E}}(t-\varsigma)}\mathfrak{U}'(\varsigma)d\varsigma,$$

where ψ *is a normalization function with the property that* $\psi(0) = \psi(1) = 1$.

Later, the above Caputo-Fabrizio fractional derivative is modified by Losada and Nieto [15] to become

$$^{CF}D^{\mathcal{E}}\mathfrak{U}(t) = \frac{(2-\mathcal{E})\psi(\mathcal{E})}{2(1-\mathcal{E})} \int_0^t e^{-\frac{\mathcal{E}}{1-\mathcal{E}}(t-\varsigma)}\mathfrak{U}'(\varsigma)d\varsigma.$$

They demonstrated that $\psi(\mathcal{E}) = \frac{2}{2-\mathcal{E}}$, *for any* $\mathcal{E} \in (0, 1)$. *Hence, we get*

$$^{CF}D^{\mathcal{E}}\mathfrak{U}(t) = \frac{1}{(1-\mathcal{E})} \int_0^x e^{-\frac{\mathcal{E}}{1-\mathcal{E}}(x-\varsigma)}\mathfrak{U}'(\varsigma)d\varsigma. \tag{2.1}$$

Also, they showed that

$$^{CF}I^{\mathcal{E}}\mathfrak{U}(t) = (1-\mathcal{E})\mathfrak{U}(t) + \mathcal{E}\int_0^t \mathfrak{U}(s)ds, \tag{2.2}$$

where $^{CF}I^{\mathcal{E}}\mathfrak{U}(t)$ *is the fractional integral of order* $\mathcal{E}$ *for the function* $\mathfrak{U}(t)$.

Definition 2.2. *[16, 17] Assume that* $\mathfrak{U}(t)$ *defined on* $[0, 1]$, $q \in (0, 1)$, $\mathcal{E} \geqslant 0$. *Then, we can define the fractional q-integral of the Riemann-Liouville type as*

$$(I_q^{\mathcal{E}}\mathfrak{U})(t) = \begin{cases} \mathfrak{U}(t), & \mathcal{E} = 0, \\ \frac{1}{\Gamma_q(\mathcal{E})}\int_0^t (t-q\varsigma)^{(\mathcal{E}-1)}\mathfrak{U}(\varsigma)d_q\varsigma, & \mathcal{E} > 0, \quad t \in [0, 1], \end{cases} \tag{2.3}$$

where

$$\Gamma_q(\mathcal{E}) = \frac{(1-q)^{(\mathcal{E}-1)}}{(1-q)^{\mathcal{E}-1}}, \quad q \in (0, 1),$$

and satisfy $\Gamma_q(\mathcal{E}+1) = [\mathcal{E}]_q\Gamma_q(\mathcal{E})$, *where* $[\mathcal{E}]_q = \frac{1-q^{\mathcal{E}}}{1-q}$,

$$(\chi-\psi)^{(0)} = 1, \quad (\chi-\psi)^{(l)} = \prod_{j=0}^{l-1}(\chi-q^j\psi), l \in \mathbb{N}, \quad (\chi-\psi)^{(\gamma)} = \chi^\gamma \prod_{j=0}^{\infty} \frac{(\chi-q^j\psi)}{(\chi-q^{j+\gamma}\psi)}, \gamma \in \mathbb{R}.$$

Lemma 2.3. *[16] Using q-integration by parts, we get the following:*

$$(I_q^{\mathcal{E}}1)(t) = \frac{t^{(\mathcal{E})}}{\Gamma_q(\mathcal{E}+1)}, \qquad \mathcal{E} > 0. \tag{2.4}$$

For more details on the properties of q fractional calulus, see [18, 19].

3. Main results

Lemma 3.1. *Assume that* $\nu = (1-q)\rho \sum_{x=0}^{m} q^x$. *The solution of* (1.1) *and* (1.2) *is obtained as follows:*

$$\phi(t) = \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^t z(\varsigma)d\varsigma, \tag{3.1}$$

where,

$$\begin{aligned} z(t) =&\xi + \int_0^t \mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds, \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] \\ &+ \int_0^\varsigma z(s)ds, \frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)} z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big)d\varsigma, \quad t \in (0,1], \end{aligned} \tag{3.2}$$

Proof. Integrating (1.1) first time from $0 \to t$, we obtain

$$\phi'(t) = \phi'(0) + \int_0^t \mathcal{F}\Big(\varsigma, \phi'(\varsigma), {}^{CF}I^{\alpha_0}\phi'(\varsigma), \phi(\varsigma), {}^{CF}D^{\beta_0}\phi(\varsigma), I_q^{\gamma_0}\mu(\varsigma, \phi'(\varsigma))\Big)d\varsigma, \quad t \in (0,1].$$

Using (2.1) and (2.2), we get

$$\begin{aligned} \phi'(t) =&\phi'(0) + \int_0^t \mathcal{F}\Big(\varsigma, \phi'(\varsigma), (1-\alpha_0)\phi'(\varsigma) + \alpha_0 \int_0^\varsigma \phi'(s)ds, \phi(\varsigma), \\ &\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)} \phi'(s)ds, I_q^{\gamma_0}\mu(\varsigma, \phi'(\varsigma))\Big)d\varsigma, \quad t \in (0,1]. \end{aligned} \tag{3.3}$$

Put $\phi'(t) = z(t)$ in (3.3), we get

$$\begin{aligned} z(t) =&\xi + \int_0^t \mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds, \phi(\varsigma), \\ &\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)} z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big)d\varsigma, \quad t \in (0,1], \end{aligned} \tag{3.4}$$

where

$$\phi(t) = \phi(0) + \int_0^t z(\varsigma)d\varsigma, \quad t \in (0,1], \tag{3.5}$$

using (1.2), then

$$(1-q)\rho \sum_{x=0}^{m} q^x \phi(q^x\rho) = \phi(0)(1-q)\rho \sum_{x=0}^{m} q^x + (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma.$$

Therefore,

$$\phi(0) = \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big]. \tag{3.6}$$

Now, we obtain (3.1) and (3.2) from (3.4)–(3.6). The proof is completed. □

Theorem 3.2. *Let the problem* (1.1) *and* (1.2) *satisfy the following conditions:*

1) $\mathcal{F}:[0,1]\times\mathbb{R}^5\to\mathbb{R}$, $\mu:[0,1]\times\mathbb{R}\to\mathbb{R}$ *are measurable and continuous for almost all* $t\in[0,1]$.

2) *There exist functions* $A_1(t), A_2(t)\in L_1[0,1]$ *and a positive constants* $N_1, N_2>0$, *such that for any* $\phi, z, \zeta, u, \upsilon\in\mathbb{R}$, *we have*

$$|\mathcal{F}(t,\phi,z,\zeta,u,\upsilon)|\le A_1(t)+N_1|\phi|+N_1|z|+N_1|\zeta|+N_1|u|+N_1|\upsilon|,$$

$$|\mu(t,z)|\le A_2(t)+N_2|z|.$$

3)

$$\sup_{t\in[0,1]}\int_0^t A_1(\varsigma)d\varsigma\le\mathfrak{a}_1,\quad \sup_{t\in[0,1]}\int_0^t I_q^{\gamma_0}A_2(\varsigma)d\varsigma\le\mathfrak{a}_2.$$

4)

$$4N_1+N_1\frac{\beta_0-(\beta_0-1)\left(e^{\frac{\beta_0}{\beta_0-1}}-1\right)}{\beta_0^2}+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}<1.$$

Then, (3.2) *has at least a solution* $z(t)\in C[0,1]$.

Proof. Define the operator H associated with (3.2) by

$$\begin{aligned}Hz(t)=&\xi+\int_0^t\mathcal{F}\Big(\varsigma, z(\varsigma),(1-\alpha_0)z(\varsigma)+\alpha_0\int_0^\varsigma z(s)ds,\\ &\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^m q^x\int_0^{q^x\rho}z(\varsigma)d\varsigma\Big]+\int_0^\varsigma z(s)ds,\\ &\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma,z(\varsigma))\Big)d\varsigma.\end{aligned}$$

Let $\vartheta_r=\{z(t)\in\mathbb{R}:\|z\|_C\le r\}$, where $r=\dfrac{|\xi|+\mathfrak{a}_1+N_1\nu^{-1}|\varrho|+N_1\mathfrak{a}_2}{1-\left(4N_1+N_1\frac{\beta_0-(\beta_0-1)\left(e^{\frac{\beta_0}{\beta_0-1}}-1\right)}{\beta_0^2}+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\right)}$.

Thus, for $z(t)\in\vartheta_r$, we get

$$\begin{aligned}\|Hz(t)\|_C\le&\Big|\xi+\int_0^t\mathcal{F}\Big(\varsigma, z(\varsigma),(1-\alpha_0)z(\varsigma)+\alpha_0\int_0^\varsigma z(s)ds,\\ &\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^m q^x\int_0^{q^x\rho}z(\varsigma)d\varsigma\Big]+\int_0^\varsigma z(s)ds,\\ &\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma,z(\varsigma))\Big)d\varsigma\Big|\\ \le&|\xi|+\int_0^t\Big|\mathcal{F}\Big(\varsigma, z(\varsigma),(1-\alpha_0)z(\varsigma)+\alpha_0\int_0^\varsigma z(s)ds,\\ &\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^m q^x\int_0^{q^x\rho}z(\varsigma)d\varsigma\Big]+\int_0^\varsigma z(s)ds,\\ &\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma,z(\varsigma))\Big)\Big|d\varsigma\end{aligned}$$

$$
\begin{aligned}
&\leq |\xi| + \int_0^t \Big[A_1(\varsigma) + N_1|z(\varsigma)| + N_1|(1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds| \\
&\quad + N_1 \Big| \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^\varsigma z(s)ds \Big| \\
&\quad + \frac{N_1}{1-\beta_0} \int_0^\varsigma |e^{-\frac{\beta_0}{1-\beta_0}(\varsigma - s)} z(s)|ds + N_1 I_q^{\gamma_0}|\mu(\varsigma, z(\varsigma))|\Big] d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\xi| + \mathfrak{a}_1 + \int_0^t \Big[N_1|z(\varsigma)| + N_1(1-\alpha_0)|z(\varsigma)| + N_1\alpha_0 \int_0^\varsigma |z(s)|ds \\
&\quad + N_1\nu^{-1}\Big[|\varrho| + (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} |z(\varsigma)|d\varsigma\Big] + N_1 \int_0^\varsigma |z(s)|ds \\
&\quad + \frac{N_1}{1-\beta_0} \int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma - s)} |z(s)|ds + N_1 I_q^{\gamma_0}(A_2(\varsigma) + N_2|z(\varsigma)|)\Big] d\varsigma \\
&\leq |\xi| + \mathfrak{a}_1 + \int_0^t \Big[N_1\|z\| + N_1(1-\alpha_0)\|z\| + N_1\alpha_0\|z\| + N_1\nu^{-1}|\varrho| + N_1\|z\| + N_1\|z\| \\
&\quad + \frac{N_1\|z\|(1 - e^{\frac{\beta_0\varsigma}{\beta_0 - 1}})}{\beta_0} + N_1\mathfrak{a}_2 + N_1N_2\|z\| \frac{\varsigma^{(\gamma_0)}}{\Gamma_q(\gamma_0+1)}\Big] d\varsigma \\
&\leq |\xi| + \mathfrak{a}_1 + 4N_1 r + N_1\nu^{-1}|\varrho| + N_1 r \frac{\beta_0 - (\beta_0 - 1)\Big(e^{\frac{\beta_0}{\beta_0-1}} - 1\Big)}{\beta_0^2} \\
&\quad + N_1\mathfrak{a}_2 + \frac{N_1N_2 r}{(\gamma_0+1)\Gamma_q(\gamma_0+1)} = r.
\end{aligned}
$$

This proves that $H : \vartheta_r \to \vartheta_r$ and $\{Hz(t)\}$ is uniformly bounded in ϑ_r.

Now, Assume that $0 < t_1, t_2 \leq 1$ and $|t_2 - t_1| < \delta$; therefore,

$$
\begin{aligned}
|Hz(t_2) - Hz(t_1)| &= \Big|\xi + \int_0^{t_2} \mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds, \\
&\quad \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^\varsigma z(s)ds, \\
&\quad \frac{1}{1-\beta_0} \int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma - s)} z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big) d\varsigma - \\
&\quad \xi - \int_0^{t_1} \mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds, \\
&\quad \nu^{-1}\Big[\varrho - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^\varsigma z(s)ds, \\
&\quad \frac{1}{1-\beta_0} \int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma - s)} z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big) d\varsigma\Big| \\
&\leq \int_{t_1}^{t_2} \Big|\mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0 \int_0^\varsigma z(s)ds,
\end{aligned}
$$

$$\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^{m}q^{x}\int_{0}^{q^{x}\rho}z(\varsigma)d\varsigma\Big]+\int_{0}^{\varsigma}z(s)ds,$$
$$\frac{1}{1-\beta_0}\int_{0}^{\varsigma}e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds,I_q^{\gamma_0}\mu(\varsigma,z(\varsigma))\Big)\Big|d\varsigma$$
$$\leq \int_{t_1}^{t_2}A_1(\varsigma)d\varsigma+4N_1r(t_2-t_1)+N_1\nu^{-1}|\varrho|(t_2-t_1)+\frac{N_1r(1-e^{\frac{\beta_0\varsigma}{\beta_0-1}})(t_2-t_1)}{\beta_0}$$
$$+N_1\int_{t_1}^{t_2}I_q^{\gamma_0}A_2(\varsigma)d\varsigma+N_1N_2r\int_{t_1}^{t_2}\frac{\varsigma^{(\gamma_0)}}{\Gamma_q(\gamma_0+1)}d\varsigma.$$

As a result, $\{Hz(t)\}$ is equi-continuous in ϑ_r.

Assume that $z_k(t)\in\vartheta_r$, $z_k(t)\to z(t)(k\to\infty)$. Therefore, the continuity of the two functions $\mathcal{F}$ and μ, implies that $\mathcal{F}(t,\phi_k,z_k,\zeta_k,u_k,\upsilon_k)\to\mathcal{F}(t,\phi,z,\zeta,u,\upsilon)$ and $\mu(t,z_k)\to\mu(t,z)$ as $k\to\infty$. Also,

$$\lim_{k\to\infty}Hz_k(t)=\lim_{k\to\infty}\Big[\xi+\int_0^t\mathcal{F}\Big(\varsigma,z_k(\varsigma),(1-\alpha_0)z_k(\varsigma)+\alpha_0\int_0^{\varsigma}z_k(s)ds,$$
$$\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^{m}q^{x}\int_{0}^{q^{x}\rho}z_k(\varsigma)d\varsigma\Big]+\int_{0}^{\varsigma}z_k(s)ds,$$
$$\frac{1}{1-\beta_0}\int_{0}^{\varsigma}e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z_k(s)ds,I_q^{\gamma_0}\mu(\varsigma,z_k(\varsigma))\Big)d\varsigma\Big].$$

Using assumptions 1 and Lebesgue dominated convergence theorem [20], then

$$\lim_{k\to\infty}Hz_k(t)=\xi+\int_0^t\lim_{k\to\infty}\mathcal{F}\Big(\varsigma,z_k(\varsigma),(1-\alpha_0)z_k(\varsigma)+\alpha_0\int_0^{\varsigma}z_k(s)ds,$$
$$\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^{m}q^{x}\int_{0}^{q^{x}\rho}z_k(\varsigma)d\varsigma\Big]+\int_{0}^{\varsigma}z_k(s)ds,$$
$$\frac{1}{1-\beta_0}\int_{0}^{\varsigma}e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z_k(s)ds,I_q^{\gamma_0}\mu(\varsigma,z_k(\varsigma))\Big)d\varsigma=Hz(t).$$

Then, $Hz_k(t)\to Hz(t)$ as $k\to\infty$. As a result, the operator H is continuous in ϑ_r. Therefore, Schauder's fixed point Theorem implies that there exists at least a solution $z(t)\in C[0,1]$ of (3.2). As a result, Lemma 3.1 implies that (1.1) and (1.2) possess a solution $\phi(t)\in C[0,1]$.

□

Theorem 3.3. *Assume that $\mathcal{F}$ and μ are measurable and continuous for all $t\in[0,1]$ and satisfy the following conditions:*

(i)

$$|\mathcal{F}(t,\phi,z,\zeta,u,\upsilon)-\mathcal{F}(t,\phi_1,z_1,\zeta_1,u_1,\upsilon_1)|\leq N_1|\phi-\phi_1|+N_1|z-z_1|+N_1|\zeta-\zeta_1|+N_1|u-u_1|+N_1|\upsilon-\upsilon_1|,$$

(ii)

$$|\mu(t,z)-\mu(t,z_1)|\leq N_2|z-z_1|.$$

Therefore (3.2)*, has a unique solution.*

Proof. Assume that (3.2) has two solutions $z(t), z^*(t)$. Therefore, we have

$$\begin{aligned}
|z(t)-z^*(t)| &\leq \int_0^t \Big|\mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma)+\alpha_0\int_0^\varsigma z(s)ds,\\
&\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^{m} q^x\int_0^{q^x\rho} z(\varsigma)d\varsigma\Big]+\int_0^\varsigma z(s)ds,\\
&\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big)d\varsigma-\\
&\mathcal{F}\Big(\varsigma, z^*(\varsigma), (1-\alpha_0)z^*(\varsigma)+\alpha_0\int_0^\varsigma z^*(s)ds,\\
&\nu^{-1}\Big[\varrho-(1-q)\rho\sum_{x=0}^{m} q^x\int_0^{q^x\rho} z^*(\varsigma)d\varsigma\Big]+\int_0^\varsigma z^*(s)ds,\\
&\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z^*(s)ds, I_q^{\gamma_0}\mu(\varsigma, z^*(\varsigma))\Big)\Big|d\varsigma\\
&\leq \int_0^t \Big[N_1|z(\varsigma)-z^*(\varsigma)|+N_1\Big|(1-\alpha_0)(z(\varsigma)-z^*(\varsigma))\\
&+\alpha_0\int_0^\varsigma (z(s)-z^*(s))ds\Big|+N_1\Big|\frac{1}{(1-q)\rho\sum_{x=0}^{m}q^x}(1-q)\rho\sum_{x=0}^{m}q^x\\
&\int_0^{q^x\rho}(z(\varsigma)-z^*(\varsigma))d\varsigma+\int_0^\varsigma (z(s)-z^*(s))ds\Big|\\
&+N_1\frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}|z(s)-z^*(s)|ds\\
&+N_1 I_q^{\gamma_0}|\mu(\varsigma, z(\varsigma))-\mu(\varsigma, z^*(\varsigma))|\Big]d\varsigma\\
&\leq N_1\int_0^t\Big[4|z(\varsigma)-z^*(\varsigma)|+\frac{(1-e^{\frac{\beta_0\varsigma}{\beta_0-1}})}{\beta_0}|z(\varsigma)-z^*(\varsigma)|\\
&+\frac{N_2\varsigma^{(\gamma_0)}}{\Gamma_q(\gamma_0+1)}|z(\varsigma)-z^*(\varsigma)|\Big]d\varsigma\\
&\leq 4N_1\|z-z^*\|+N_1\frac{\beta_0-(\beta_0-1)\Big(e^{\frac{\beta_0}{\beta_0-1}}-1\Big)}{\beta_0^2}\|z-z^*\|\\
&+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\|z-z^*\|\\
&\leq \Big(4N_1+N_1\frac{\beta_0-(\beta_0-1)\Big(e^{\frac{\beta_0}{\beta_0-1}}-1\Big)}{\beta_0^2}+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\Big)\|z-z^*\|.
\end{aligned}$$

Hence,

$$\Big[1-(4N_1+N_1\frac{\beta_0-(\beta_0-1)\Big(e^{\frac{\beta_0}{\beta_0-1}}-1\Big)}{\beta_0^2}+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)})\Big]\|z-z^*\|\leq 0.$$

Since $4N_1 + N_1 \frac{\beta_0-(\beta_0-1)\left(e^{\frac{\beta_0}{\beta_0-1}}-1\right)}{\beta_0^2} + \frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)} < 1$, this implies that $z(t) = z^*(t)$. Therefore, the solution of (3.2) is unique. Thus, Lemma 3.1 implies that the proplem (1.1) and (1.2) possess a unique solution $\phi(t) \in C[0, 1]$. □

3.1. Continuous dependence on ϱ

Definition 3.4. *The solution $\phi(t) \in C[0,1]$ of (1.1) and (1.2) depends continuously on ϱ, if*

$$\forall\epsilon > 0, \quad \exists \quad \delta_0(\epsilon) \quad s.t \quad |\varrho - \varrho^*| < \delta_0 \Rightarrow \|\phi - \phi^*\| < \epsilon,$$

where $\phi^*(t)$ is the solution of

$$\phi^{*\prime\prime}(t) = \mathcal{F}\Big(t, \phi^{*\prime}(t), {}^{CF}I^{\alpha_0}\phi^{*\prime}(t), \phi^*(t), {}^{CF}D^{\beta_0}\phi^*(t), I_q^{\gamma_0}\mu(t, \phi^{*\prime}(t))\Big), \quad t \in (0, 1], \tag{3.7}$$

$$(1-q)\rho\sum_{x=0}^{m} q^x\phi^*(q^x\rho) = \varrho^*, \quad \phi^{*\prime}(0) = \xi. \tag{3.8}$$

Theorem 3.5. *Assume that conditions 1–4 of the Theorem 3.3 are satisfied. Therefore, the solution of (1.1) and (1.2) is continuously dependent on ϱ.*

Proof. Assume that $z(t)$, $z^*(t)$ are two solutions of (1.1) and (1.2) and (3.7) and (3.8) respectively. Then,

$$\begin{aligned}
|z(t) - z^*(t)| = \; & \Bigg|\int_0^t \Bigg[\mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0\int_0^\varsigma z(s)ds, \\
& \nu^{-1}\Big[\varrho - (1-q)\rho\sum_{x=0}^{m} q^x\int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^\varsigma z(s)ds, \\
& \frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big) \\
& -\mathcal{F}\Big(\varsigma, z^*(\varsigma), (1-\alpha_0)z^*(\varsigma) + \alpha_0\int_0^\varsigma z^*(s)ds, \\
& \nu^{-1}\Big[\varrho^* - (1-q)\rho\sum_{x=0}^{m} q^x\int_0^{q^x\rho} z^*(\varsigma)d\varsigma\Big] + \int_0^\varsigma z^*(s)ds, \\
& \frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z^*(s)ds, I_q^{\gamma_0}\mu(\varsigma, z^*(\varsigma))\Big)\Bigg]d\varsigma\Bigg| \\
\leq \; & \int_0^t \Big|\mathcal{F}\Big(\varsigma, z(\varsigma), (1-\alpha_0)z(\varsigma) + \alpha_0\int_0^\varsigma z(s)ds, \\
& \nu^{-1}\Big[\varrho - (1-q)\rho\sum_{x=0}^{m} q^x\int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^\varsigma z(s)ds, \\
& \frac{1}{1-\beta_0}\int_0^\varsigma e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}z(s)ds, I_q^{\gamma_0}\mu(\varsigma, z(\varsigma))\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\mathcal{F}\Big(\varsigma, z^*(\varsigma), (1-\alpha_0)z^*(\varsigma) + \alpha_0 \int_0^{\varsigma} z^*(s)ds, \\
&\nu^{-1}\Big[\varrho^* - (1-q)\rho \sum_{x=0}^{m} q^x \int_0^{q^x\rho} z^*(\varsigma)d\varsigma\Big] + \int_0^{\varsigma} z^*(s)ds, \\
&\frac{1}{1-\beta_0}\int_0^{\varsigma} e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)} z^*(s)ds, I_q^{\gamma_0}\mu(\varsigma, z^*(\varsigma))\Big)\Big|d\varsigma \\
\leq\ & \int_0^t \Big[4N_1|z(\varsigma) - z^*(\varsigma)| + N_1\nu^{-1}|\varrho - \varrho^*| \\
&+N_1\frac{1}{1-\beta_0}\int_0^{\varsigma} e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}|z(s) - z^*(s)|ds \\
&+N_1 I_q^{\gamma_0}|\mu(\varsigma, z(\varsigma)) - \mu(\varsigma, z^*(\varsigma))|\Big]d\varsigma \\
\leq\ & \int_0^t \Big[4N_1\|z - z^*\| + N_1\nu^{-1}|\varrho - \varrho^*| \\
&+\frac{N_1}{1-\beta_0}\int_0^{\varsigma} e^{-\frac{\beta_0}{1-\beta_0}(\varsigma-s)}|z(s) - z^*(s)|ds + N_1N_2\frac{\varsigma^{(\gamma_0)}}{\Gamma_q(\gamma_0+1)}\|z - z^*\|\Big]d\varsigma \\
\leq\ & 4N_1\|z - z^*\| + N_1\nu^{-1}|\varrho - \varrho^*| + N_1\frac{\beta_0 - (\beta_0 - 1)\Big(e^{\frac{\beta_0}{\beta_0-1}} - 1\Big)}{\beta_0^2}\|z - z^*\| \\
&+\frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\|z - z^*\| \\
\leq\ & N_1\nu^{-1}\delta_0 + \Big(4N_1 + N_1\frac{\beta_0 - (\beta_0 - 1)\Big(e^{\frac{\beta_0}{\beta_0-1}} - 1\Big)}{\beta_0^2} + \frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\Big)\|z - z^*\|.
\end{aligned}
$$

Hence,

$$
\|z - z^*\| \leq \frac{N_1\nu^{-1}\delta_0}{1 - \Big(4N_1 + N_1\frac{\beta_0-(\beta_0-1)\big(e^{\frac{\beta_0}{\beta_0-1}}-1\big)}{\beta_0^2} + \frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\Big)}.
$$

Therefore,

$$
\begin{aligned}
|\phi(t) - \phi^*(t)| &= \Big|\nu^{-1}\Big[\varrho - (1-q)\rho\sum_{x=0}^{m} q^x \int_0^{q^x\rho} z(\varsigma)d\varsigma\Big] + \int_0^t z(\varsigma)d\varsigma \\
&\quad -\nu^{-1}\Big[\varrho^* - (1-q)\rho\sum_{x=0}^{m} q^x \int_0^{q^x\rho} z^*(\varsigma)d\varsigma\Big] + \int_0^t z^*(\varsigma)d\varsigma\Big| \\
&\leq \nu^{-1}|\varrho - \varrho^*| + 2\|z - z^*\|.
\end{aligned}
$$

Hence,

$$
\|\phi - \phi^*\| \leq \nu^{-1}\delta_0 + \frac{2N_1\nu^{-1}\delta_0}{1 - \Big(4N_1 + N_1\frac{\beta_0-(\beta_0-1)\big(e^{\frac{\beta_0}{\beta_0-1}}-1\big)}{{\beta_0}^2} + \frac{N_1N_2}{(\gamma_0+1)\Gamma_q(\gamma_0+1)}\Big)} = \epsilon.
$$

As a result, the solution of (1.1) and (1.2) is continually dependent on ϱ. □

4. Methodology of numerical technique

The problem (1.1) and (1.2) can be expressed as follows:

$$\phi''(t) - N_1\kappa_1(\phi'(t)) - N_1{}^{CF}I^{\alpha_0}\phi'(t) - N_1\kappa_2(\phi(t)) - N_1{}^{CF}D^{\beta_0}\phi(t) - N_1 I_q^{\gamma_0}\mu(t,\phi'(t)) = A_1(t), \tag{4.1}$$

$$(1-q)\rho\sum_{x=0}^{m} q^x\phi(q^x\rho) = \varrho, \quad \phi'(0) = \xi.$$

Assume that $\mu(t,\phi'(t)) = A_2(t) + N_2\kappa_3(\phi'(t))$, where $\kappa_1(\phi'(t)), \kappa_2(\phi(t)), \kappa_3(\phi'(s))$ are nonlinear terms for the unknown function. Then, by using (2.1)–(2.3), Equation (4.1) become:

$$\begin{aligned}&\phi''(t) - N_1\kappa_1(\phi'(t)) - N_1(1-\alpha_0)\phi'(t) - N_1\alpha_0\int_0^t \phi'(s)ds - N_1\kappa_2(\phi(t))\\ &\quad - \frac{N_1}{1-\beta_0}\int_0^t e^{\frac{-\beta_0(t-s)}{1-\beta_0}}\phi'(s)ds - \frac{N_1}{\Gamma_q(\gamma_0)}\int_0^t (t-qs)^{(\gamma_0-1)}(A_2(s) + N_2\kappa_3(\phi'(s)))d_qs = A_1(t),\end{aligned} \tag{4.2}$$

Now, the interval of integration $[0,t]$ of Eq (4.2) is subdivided into l equally spaced intervals of width $h = (t_i - 0)/i, i \geqslant 1$ [21]. Taking $\phi''(t_i) = \phi_i''$, $\phi'(t_i) = \phi_i'$, $\kappa_1(\phi'(t_i)) = \kappa_1(\phi_i')$, $\kappa_2(\phi(t_i)) = \kappa_2(\phi_i)$, $\kappa_3(\phi'(s_j)) = \kappa_3(\phi_j')$, $A_1(t_i) = A_{1,i}, A_2(s_j) = A_{2,j}$, let $k_{i,j} = \frac{1}{\Gamma_q(\gamma_0)}(t_i - qs_j)^{(\gamma_0-1)}$, $K_{i,j} = \frac{1}{1-\beta_0}e^{\frac{-\beta_0(t_i-s_j)}{1-\beta_0}}$. Therefore, (4.2) can be expressed as follows:

$$\begin{aligned}&\phi_i'' - N_1\kappa_1(\phi_i') - N_1(1-\alpha_0)\phi_i' - N_1\alpha_0\int_0^{t_i}\phi_j'ds - N_1\kappa_2(\phi_i) - N_1\int_0^{t_i}K_{i,j}\phi_j'ds\\ &\quad - N_1N_2\int_0^{t_i}k_{i,j}\kappa_3(\phi_j')d_qs = B_i,\end{aligned} \tag{4.3}$$

where $B_i = A_{1,i} + N_1\int_0^{t_i}k_{i,j}A_{2,j}d_qs$.

4.1. A summary of the finite-trapezoidal method

1) We use the first and second order central finite difference method to approximate the derivative part of (4.3) as follows:

$$\phi_i'' \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}, \qquad \phi_i' \approx \frac{\phi_{i+1} - \phi_{i-1}}{2h}.$$

2) We use the trapezoidal rule to approximate the integral part of (4.3) as follows:

$$\int_0^{t_i} K_{i,j}\phi_j'ds \approx \frac{h}{2}\Big[K_{i,0}\phi_0' + 2\sum_{j=1}^{i-1}K_{i,j}\phi_j' + K_{i,i}\phi_i'\Big],$$

$$\int_0^{t_i} k_{i,j}\kappa_3(\phi_j')d_qs \approx \frac{h}{2}\Big[k_{i,0}\kappa_3(\phi_0') + 2\sum_{j=1}^{i-1}k_{i,j}\kappa_3(\phi_j') + k_{i,i}\kappa_3(\phi_i')\Big], \qquad i = 0,1,2,3,\dots l.$$

3) Therefore, (4.3) becomes:

$$
\begin{aligned}
&\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2}-N_1\kappa_1(\frac{\phi_{i+1}-\phi_{i-1}}{2h})-N_1(1-\alpha_0)(\frac{\phi_{i+1}-\phi_{i-1}}{2h})\\
&-N_1\alpha_0\frac{h}{2}\Big[\frac{\phi_1-\phi_{-1}}{2h}+2\sum_{j=1}^{i-1}\frac{\phi_{j+1}-\phi_{j-1}}{2h}+\frac{\phi_{i+1}-\phi_{i-1}}{2h}\Big]\\
&-N_1\kappa_2(\phi_i)-N_1\frac{h}{2}\Big[K_{i,0}(\frac{\phi_1-\phi_{-1}}{2h})+2\sum_{j=1}^{i-1}K_{i,j}(\frac{\phi_{j+1}-\phi_{j-1}}{2h}) \\
&+K_{i,i}(\frac{\phi_{i+1}-\phi_{i-1}}{2h})\Big]-N_1N_2\frac{h}{2}\Big[k_{i,0}\kappa_3(\frac{\phi_1-\phi_{-1}}{2h})+2\sum_{j=1}^{i-1}k_{i,j}\kappa_3(\frac{\phi_{j+1}-\phi_{j-1}}{2h})\\
&+k_{i,i}\kappa_3(\frac{\phi_{i+1}-\phi_{i-1}}{2h})\Big]=B_i, \qquad i=0,1,\dots l.
\end{aligned}
\tag{4.4}
$$

4.2. A summary of the cubic-Simpson's method

We can obtain the numerical solution of (4.3) by using a merge of cubic b-spline with the Simpson's method as follows:

1) The unkown function $\phi(t)$ and its derivatives can be approximated by using cubic b-spline as follows [22]:

$$
\phi_i\approx\Omega_{i-1}+4\Omega_i+\Omega_{i+1},\quad \phi_i'\approx\frac{3}{h}(\Omega_{i+1}-\Omega_{i-1}),\quad \phi_i''\approx\frac{6}{h^2}(\Omega_{i-1}-2\Omega_i+\Omega_{i+1}),
$$

where Ω_i are constants to be determined.

2) We use the Simpson's method [23] to approximate the integral part of (4.3).

3) As a result, we can write (4.3) as follows:

$$
\begin{aligned}
&\frac{6}{h^2}(\Omega_{i-1}-2\Omega_i+\Omega_{i+1})-N_1\kappa_1(\frac{3\Omega_{i+1}-3\Omega_{i-1}}{h})-N_1(1-\alpha_0)\frac{3\Omega_{i+1}-3\Omega_{i-1}}{h}\\
&-N_1\alpha_0\frac{h}{3}\Big[\frac{3\Omega_1-3\Omega_{-1}}{h}+2\sum_{j=1}^{\frac{i}{2}-1}\frac{3\Omega_{2j+1}-3\Omega_{2j-1}}{h}+4\sum_{j=1}^{\frac{i}{2}}\frac{3\Omega_{2j}-3\Omega_{2j-2}}{h}+\frac{3\Omega_{i+1}-3\Omega_{i-1}}{h}\Big]\\
&-N_1\kappa_2(\Omega_{i-1}+4\Omega_i+\Omega_{i+1})-N_1\frac{h}{3}\Big[K_{i,0}(\frac{3\Omega_1-3\Omega_{-1}}{h})+2\sum_{j=1}^{\frac{i}{2}-1}K_{i,2j}(\frac{3\Omega_{2j+1}-3\Omega_{2j-1}}{h})\\
&+4\sum_{j=1}^{\frac{i}{2}}K_{i,2j-1}(\frac{3\Omega_{2j}-3\Omega_{2j-2}}{h})+K_{i,i}(\frac{3\Omega_{i+1}-3\Omega_{i-1}}{h})\Big]-N_1N_2\frac{h}{3}\Big[k_{i,0}\kappa_3(\frac{3\Omega_1-3\Omega_{-1}}{h})\\
&+2\sum_{j=1}^{\frac{i}{2}-1}k_{i,2j}\kappa_3(\frac{3\Omega_{2j+1}-3\Omega_{2j-1}}{h})+4\sum_{j=1}^{\frac{i}{2}}k_{i,2j-1}\kappa_3(\frac{3\Omega_{2j}-3\Omega_{2j-2}}{h})+k_{i,i}\kappa_3(\frac{3\Omega_{i+1}-3\Omega_{i-1}}{h})\Big]\\
&=B_i,\quad i=0,1,\dots l.
\end{aligned}
$$

5. Test problems

Now, two numerical examples will be introduced by using the following two methods:

1) Finite-trapezoidal method.

2) cubic-Simpson method.

Problem 1. In (4.2), taking $A_1(t) = t\left(-0.0468103t^{4/3} - 0.0333333t - 0.377778\right) - 0.0833333\cos\left(t^2\right) - 0.185185e^{-1.5t} - 0.166667\sin(t)\cos(t) + 2.18519, A_2(t) = t, N_1 = \frac{1}{12}, N_2 = \frac{1}{10}, \alpha_0 = 0.4, \beta_0 = 0.6, \gamma_0 = \frac{4}{3}, q = 0.5, \kappa_1(\phi'(t)) = \sin(\phi'(t)), \kappa_2(\phi(t)) = \cos(\phi(t)), \kappa_3(\phi'(t)) = \phi'(t)$. Then,

$$\left|\mathcal{F}\left(t, \phi'(t), {}^{CF}I^{\alpha_0}\phi'(t), \phi(t), {}^{CF}D^{\beta_0}\phi(t), I_q^{\gamma_0}\mu(t, \phi'(t))\right)\right| \leq \left|t\left(-0.0468103t^{4/3} - 0.0333333t - 0.377778\right)\right.$$
$$\left. - 0.0833333\cos\left(t^2\right) - 0.185185e^{-1.5t} - 0.166667\sin(t)\cos(t) + 2.18519\right| + \frac{1}{12}|\sin(\phi'(t))|$$
$$+ \frac{1}{12}|{}^{CF}I^{\alpha_0}\phi'(t)| + \frac{1}{12}|\cos(\phi(t))| + \frac{1}{12}|{}^{CF}D^{\beta_0}\phi(t)| + \frac{1}{12}|I_q^{\gamma_0}\mu(t, \phi'(t))|,$$
$$|I_q^{\gamma_0}\mu(t, \phi'(t))| \leq t + \frac{1}{10}|\phi'(t)|, 4N_1 + N_1\frac{\beta_0 - (\beta_0 - 1)\left(e^{\frac{\beta_0}{\beta_0 - 1}} - 1\right)}{\beta_0^2} + \frac{N_1 N_2}{(\gamma_0 + 1)\Gamma_q(\gamma_0 + 1)} = 0.403493 < 1.$$

Therefore, the conditions of the Theorem 3.2 are clearly satisfied. As a result, this problem has a unique solution. Now, to solve this problem, we take $l = 20, \rho = 0.2, m = 2, \varrho = 0.0045625, \xi = 0$. Then, we apply two methods: the first is finite-trapezoidal method and the second is the cubic-Simpson's method. The exact solution of this problem is $\phi(t) = t^2$.

Table 1 and Figure 1 above demonstrate the comparison between the exact solutions and the numerical solutions of the problem using two numerical methods. We can see from the results that both numerical methods are effective. Furthermore, the continuous dependence on ϱ using the cubic-Simpson's method will be studied. Taking $|\varrho - \varrho^*| = 10^{-4} \Rightarrow |\phi(0.2) - \phi^*(0.2)| = 5.71426 \times 10^{-4}$. Therfore, $\phi(t)$ is continuous dependence on ϱ.

Table 1. The exact and numerical solutions to Problem 1.

t_i	Exact solutions	Finite-trap.	Abs. error (Finite-trap)	cubic-Sim.	Abs. error (cubic-Sim.)
0.1	0.01	0.009999	1.4886×10^{-6}	0.010004	3.66028×10^{-6}
0.2	0.04	0.040001	1.2012×10^{-6}	0.039997	2.84075×10^{-6}
0.3	0.09	0.090010	1.0445×10^{-5}	0.089977	2.34169×10^{-5}
0.4	0.16	0.160035	3.5320×10^{-5}	0.159939	6.12610×10^{-4}
0.5	0.25	0.250091	9.1196×10^{-5}	0.249887	1.13342×10^{-4}
0.6	0.36	0.360201	2.0080×10^{-4}	0.359831	1.69313×10^{-4}
0.7	0.49	0.490395	3.9505×10^{-4}	0.489789	2.10604×10^{-3}
0.8	0.64	0.640714	7.1376×10^{-4}	0.639790	2.09675×10^{-3}
0.9	0.81	0.811206	1.2062×10^{-3}	0.809871	1.29396×10^{-3}

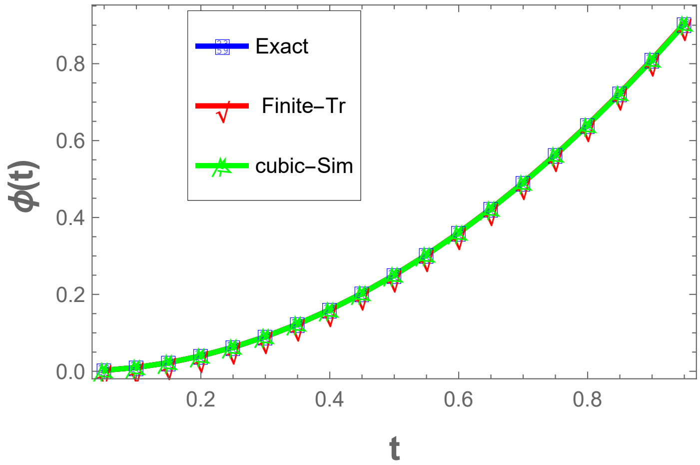

Figure 1. Comparison between the numerical and exact solutions of test Problem 1.

Problem 2. In (4.2), taking $A_1(t) = -0.0541126t^2 + \left(0.0415584t^2 - 1.19919\right)\cos(t) + 0.0840336e^{-0.25t} + (0.0121212t + 0.121008)\sin(t) + 0.0437229, A_2(t) = \sin(t), N_1 = \frac{1}{14}, N_2 = \frac{1}{11}, \alpha_0 = 0.6, \beta_0 = 0.2, \gamma_0 = 3, q = 0.2, \kappa_1(\phi'(t)) = \phi'(t), \kappa_2(\phi(t)) = \phi(t), \kappa_3(\phi'(t)) = \phi'(t)$. Then,

$$\left|\mathcal{F}\left(t, \phi'(t), {}^{CF}I^{\alpha_0}\phi'(t), \phi(t), {}^{CF}D^{\beta_0}\phi(t), I_q^{\gamma_0}\mu(t, \phi'(t))\right)\right| \leq \left| -0.0541126t^2 + \left(0.0415584t^2 - 1.19919\right)\cos(t) + 0.0840336e^{-0.25t} + (0.0121212t + 0.121008)\sin(t) + 0.0437229\right| + \frac{1}{14}|\phi'(t)| + \frac{1}{14}|{}^{CF}I^{\alpha_0}\phi'(t)| + \frac{1}{14}|\phi(t)| + \frac{1}{14}|{}^{CF}D^{\beta_0}\phi(t)| + \frac{1}{14}|I_q^{\gamma_0}\mu(t, \phi'(t))|,$$

$$|I_q^{\gamma_0}\mu(t, \phi'(t))| \leq \sin(t) + \frac{1}{11}|\phi'(t)|,$$

$$4N_1 + N_1\frac{\beta_0 - (\beta_0 - 1)\left(e^{\frac{\beta_0}{\beta_0 - 1}} - 1\right)}{\beta_0^2} + \frac{N_1 N_2}{(\gamma_0 + 1)\Gamma_q(\gamma_0 + 1)} = 0.327949 < 1.$$

Therefore, the conditions of the Theorem (3.2) are clearly satisfied. As a result, this problem has a unique solution. Now, to solve this problem, we take $l = 20, \rho = 0.5, m = 1, \varrho = 0.430633, \xi = 0$. Then, we apply two methods: the first is finite-trapezoidal method and the second is the cubic-Simpson's method. The exact solution of this problem is $\phi(t) = \cos(t)$.

Table 2 and Figure 2 above demonstrate the comparison between the exact solutions and the numerical solutions of the problem using two numerical methods. The results tabulated in the above table demonstrate that the finite difference-trapezoidal method is better than cubic-Simpson method.

Table 2. The exact and numerical solutions to test Problem 2.

t_i	Exact solutions	Finite-trap.	Abs. error (Finite-trap)	cubic-Sim.	Abs. error (cubic-Sim.)
0.1	0.995004	0.995024	2.01940×10^{-5}	0.985487	9.51696×10^{-3}
0.2	0.980067	0.980084	1.71111×10^{-5}	0.970545	9.52204×10^{-3}
0.3	0.955336	0.955348	1.20079×10^{-5}	0.945809	9.52678×10^{-3}
0.4	0.921061	0.921066	4.93372×10^{-6}	0.911533	9.52798×10^{-3}
0.5	0.877583	0.877579	4.03881×10^{-6}	0.868060	9.52251×10^{-3}
0.6	0.825336	0.825321	1.48091×10^{-5}	0.815828	9.50730×10^{-3}
0.7	0.764842	0.764815	2.72459×10^{-5}	0.755363	9.47939×10^{-3}
0.8	0.696707	0.696666	4.11851×10^{-5}	0.687271	9.43589×10^{-3}
0.9	0.621609	0.621554	5.64288×10^{-5}	0.612236	9.37407×10^{-3}

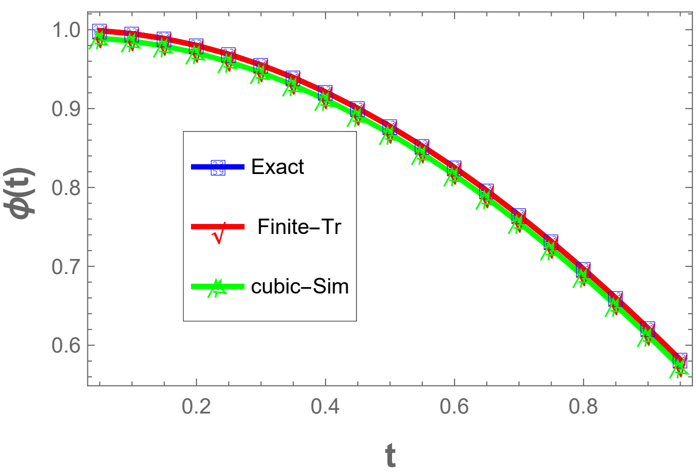

Figure 2. Comparison between the numerical and exact solutions of test Problem 2.

6. Conclusions

We have demonstrated the existence and uniqueness of a solution for a nonlocal fractional q-integro differential equation. We investigated whether or not the answer has a continuous reliance on ϱ. This section provides a synopsis of the finite difference, trapezoidal, and cubic Simpson's methods. The numerical solution is applied to two cases, and the results of those solutions are compared with the exact solution. The findings indicated that the approach is not only successful but also straightforward to put into practice.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. P. N. Duc, E. Nane, O. Nikan, N. A. Tuan, Approximation of the initial value for damped nonlinear hyperbolic equations with random Gaussian white noise on the measurements, *AIMS Mathematics*, **7** (2022), 12620–12634. https://doi.org/10.3934/math.2022698
2. T. N. Thach, N. H. Tuan, Stochastic pseudo-parabolic equations with fractional derivative and fractional Brownian motion, *Stoch. Anal. Appl.*, **40** (2022), 328–351. https://doi.org/10.1080/07362994.2021.1906274
3. A. T. Nguyen , N. H. Tuan, C.Yang, On Cauchy problem for fractional parabolic-elliptic Keller-Segel model, *Adv. Nonlinear Anal.*, **12** (2023), 97–116. https://doi.org/10.1515/anona-2022-0256
4. N. H. Tuan, N. D. Phuong, T. N. Thach, New well-posedness results for stochastic delay Rayleigh-Stokes equations, *AIMS Mathematics*, **28** (2023), 347–358. https://doi.org/10.3934/dcdsb.2022079
5. F. M. Gaafar. The Existence of solution for a nonlocal problem of an implicit fractional-order differential equation, *J. Egy. Math. Soc.*, **26** (2018), 184–195. https://doi.org/10.21608/JOEMS.2018.9473
6. S. Abbas, M. Benchohra, J. J. Nieto, Caputo-Fabrizio fractional differential equations with instantaneous impulses, *AIMS Mathematics*, **6** (2021), 2932–2946. https://doi.org/10.3934/math.2021177
7. S. K. Ntouyas, M. E. Samei, Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus, *Adv. Differ. Equ.*, **2019** (2019), 475. https://doi.org/10.1186/s13662-019-2414-8
8. M. E. Samei, L. Karimi, M. K. A. Kaabar, To investigate a class of multi-singular pointwise defined fractional q–integro-differential equation with applications, *AIMS Mathematics*, **7** (2022), 7781–7816. https://doi.org/10.3934/math.2022437
9. E. Cuesta, C. Palencia, A fractional trapezoidal rule for integro-differential equations of fractional order in Banach spaces, *Appl. Numer. Math.*, **45** (2003), 139–159. https://doi.org/10.1016/S0168-9274(02)00186-1
10. S. S. Ahmed, Solving a system of fractional-order Volterra integro-differential equations based on the explicit finite difference approximation via the trapezoid method with error analysis, *Symmetry*, **14** (2022), 575. https://doi.org/10.3390/sym14030575
11. F. Mirzaee, S. Alipour, Cubic B-spline approximation for linear stochastic integro-differential equation of fractional order, *J. Comput. Appl. Math.*, **366** (2020), 112440. https://doi.org/10.1016/j.cam.2019.112440

12. A. A. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, On the analytical and numerical study for nonlinear Fredholm integro-differential equations, *Appl. Math. Inform. Sci.*, **14** (2020), 921–929. https://doi.org/10.18576/amis/140520
13. A. A. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, K. K. Ali, On study nonlocal integro differetial equation involving the Caputo-Fabrizio Fractional derivative and q-integral of the Riemann Liouville Type, *Appl. Math. Inform Sci.*, **16** (2022), 983–993. https://doi.org/10.18576/amis/160615
14. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. https://doi.org/10.12785/pfda/010201
15. J. Losada, J. J. Nieto, Properties of a new fractional derivative without Singular Kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 87–92. https://doi.org/10.12785/pfda/010202
16. B. Ahmad, J. J. Nieto, A. Alsaedi, H. Al-Hutami, Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions, *J. Franklin I.*, **351** (2014), 2890–2909. https://doi.org/10.1016/j.jfranklin.2014.01.020
17. S. Suantai, S. K. Ntouyas, S. Asawasamrit, J. Tariboon, A coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions, *Adv. Differ. Equ.*, **2015** (2015), 124. https://doi.org/10.1186/s13662-015-0462-2
18. V. Kac, P. Cheung, *Quantum calculus*, New York: Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7
19. M. H. Annaby, Z. S. Mansour, *q-fractional calculus and equations*, Springer, 2012. https://doi.org/10.1007/978-3-642-30898-7
20. A. N. Kolomogorov, S. V. Fomin, *Inroductory real analysis*, Dover Publications, Inc, 1975.
21. R. Saadati, B. Raftari, H. Adibi, S. M. Vaezpour, S. Shakeri, A comparison between the variational iteration method and trapezoidal rule for solving linear integro-differential equations, *Word Appl. Sci. J.*, **4** (2008), 321–325.
22. R. C. Mittal, R. K. Jain, Numerical solutions of nonlinear Burgers' equation with modified cubic B-splines collocation method, *Appl. Math. Comput.*, **218** (2012), 7839–7855. https://doi.org/10.1016/j.amc.2012.01.059
23. K. E. Atkinson, *An introduction to numerical analysis*, Cambridge University Press, 1988. https://doi.org/10.1017/CBO9780511801181

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