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Research article

Medical decision-making techniques based on bipolar soft information

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Abstract: Data uncertainty is a barrier in the decision-making (DM) process. The rough set (RS) theory is an effective approach to study the uncertainty in data, while bipolar soft sets (BSSs) can handle the vagueness and uncertainty as well as the bipolarity of the data in a variety of situations. In this article, we introduce the idea of rough bipolar soft sets (RBSSs) and apply them to find the best decision in two different DM problems in medical science. The first problem is about deciding between the risk factors of a disease. Our algorithm facilitates the doctors to investigate which risk factor is becoming the most prominent reason for the increased rate of disease in an area. The second problem is deciding between the different compositions of a medicine for a particular illness having different effects and side effects. We also propose algorithms for both problems.

Keywords: rough set; soft sets; bipolar soft sets; risk factor of diseases; composition of medicines
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1. Introduction

Data analysis in several domains demands different problems associated with DM. Many algorithms are developed by the researchers, in this respect to get a wise decision. The theories of RSs [36–38] and soft sets (SSs) [33] are constructed to address the uncertainty and vagueness appearing in the data compiled for multiple objectives. These theories synchronized the mathematical designs with incomplete real-world data. In RS theory [36–38], Pawlak investigated the certainty of the information attached to the objects through the lower and upper approximations. Generalization of RS theory by relaxing the condition of an equivalence relation (ER) to any arbitrary relations has been studied by several researchers [5, 8, 21]. This generalization has been discussed using the topological approach as illustrated in [6, 7, 10, 20, 22].

In 1999, the novel concept of SS was offered by Molodtsov [33] as an alternative mathematical strategy to handle inaccuracies and uncertainties. There has been noticed a heightened interest in SS theory. Maji et al. [26] launched algebraic operations of SS theory. Following [26], Ali et al. [3] offered some novel operations on SSs and enhanced the idea of the SS compliment. Al-shami [9] defined a new type of soft subset relation in order to preserve the major properties of the intersection and union operators between sets via their counterparts in soft settings. Fatimah et al. [17] established the idea of N-soft set (N-SS). Al-shami and A. Mhemdi [28] introduced some soft relations via the environment of double-Framed soft sets and explained how it can be applied to treat a practical issues. Topological structures via soft set theory have been explored by many authors like infra soft topology [29], soft somewhat open sets and their applications [30], ordered soft topology [31] and soft Menger spaces [32].

Due to their diverse applications, rapid growth can be seen in the research on RSs and SSs in the last few years. Al-shami et al. [1] provided a new generalization of fuzzy SSs which is known as (a, b)-fuzzy SSs. Dubois and Prade [16] proposed rough fuzzy sets and fuzzy rough sets. Feng et al. [18] initiated a hybridized model of SSs, fuzzy sets, and RSs. Zhan et al. [45] pioneered the idea of Z-soft fuzzy rough set model with applications in DM. For more about the hybridization of SSs, RSs, and their generalization with application, we refer to References [27, 34, 44].

Chen et al. [14] redefined the SS parameterization reduction. Zou and Xiao [43] discussed data analysis techniques of SS under incomplete information. Çağman and Enginoglu [13] pioneered fuzzy parameterized SS theory and its applications in DM problems. The SS theory has a wide range of potential applications in several domains. Çağman and Enginoglu [12] proposed soft matrix theory and its DM applications. Chen et al. [14] provided the parametrization reduction of SSs and their applications. Feng et al. [19] investigated an adjustable approach to fuzzy SS based DM. Ma et al. [25] review some DM methods in the context of fuzzy sets, RSs, and SSs. Zou and Xiao [43] suggested data analysis approaches for SSs under incomplete information. Zhan and Zhu [44] proposed Z-soft rough fuzzy ideals of hemirings with DM application.

Along with uncertainty, the bipolarity of the information is also faced in many real-life problems. Bipolarity speaks about the positive and negative aspects of the data. The positive data reveals what is guaranteed to be possible, while the negative data indicates what is impossible. The idea which lies behind the existence of bipolar information is that a wide variety of human cognition is based on bipolar judgmental thinking. For example, not having high blood pressure does not imply that the person's blood pressure is low. The notions of BSSs, introduced by Shabir and Naz [40], successfully tackled this problem of the bipolarity of the information. Karaaslan and Karataş [24] redefined a variant of BSSs with different approximations allowing a chance to look at the topological structures of BSSs. They also established a DM strategy using BSSs. Al-shami [4] initiated the idea of belong and nonbelong relations between a BSS and an ordinary point. Mahmood [41] redefined a model of BSSs known as T-BSSs and utilized this idea in DM. Dalkılıç and Demirtaş [15] established a decision analysis review on the concept of class for BSS theory. Karaaslan et al. [23] pioneered the notion of bipolar soft groups. Naz and Shabir [34] proposed the conception of fuzzy BSSs and investigated their algebraic structures. In 2017, Shabir and Bakhtawar [39] developed the idea of bipolar soft connected, bipolar soft disconnected, and bipolar soft compact spaces. Then, Öztürk [35] further offered the notions of interior and closure operators, basis, and subspaces in the bipolar soft topological spaces. Recently, Aras et al. [11] have probed the concepts of local compactness and paracompactness via the frame of bipolar soft topological spaces. They also adjusted the definition of bipolar soft points and showed the shortcoming of the previous model. In [27], Malik and Shabir formatted the idea of rough fuzzy BSSs with applications in DM problems.

This article aims to introduce the notion of RBSS and apply this concept to DM problems. We address two different decision problems related to medical science. First, we discuss the problem where the decision is to be taken between the attributes possessed by some objects. We consider the situation in which a particular disease is increasing rapidly in a region. Our study facilitates the doctors to investigate the factor most responsible for increasing the rate of that disease in a region or area. In this way, the most prominent reason for that disease can be identified to control the disease in the area. The second problem is about deciding between some objects according to their attributes (or properties). For this, we discuss the problem of deciding between different compositions of medicine considered by a pharmaceutical company to manufacture. Each composition has some positive effects, as well as some side effects. The company wishes to finalize a single composition to manufacture which can give best results. We provide algorithms for both problems using the concept of RBSS.

The article is organized as follows: In Section 2, we recall some basic concepts. In Section 3, we study the RBSS. In Section 4, we use the RBS approximations of a BSS to design algorithms for two different DM problems in medical science. Section 5 compares the suggested DM methods with some other approaches in the bipolar soft framework. Finally, Section 6 ends with an overview of the current research and a few future perspectives.

2. Preliminaries

The RS theory [36–38] offers a systematic mechanism for tackling uncertainty in data because of indiscernibility in a scenario with incomplete knowledge. Let U $(\neq \varnothing)$ be the initial universe of discourse and ϑ be an ER defined on U. Then, (U, ϑ) is said to be *Pawlak approximation space (PA-space)*. The ER ϑ forms a partition U/ϑ of the universe U into the equivalence classes. These equivalence classes work as the elementary building blocks in the data analysis. In the partition U/ϑ, we denote the equivalence class of the element $u \in U$ by $[u]_\vartheta$ (or by $[u]$, for convenience). For $Q \subseteq U$, the relation ϑ yields the following operators.

$$\underline{Q} = \{u \in U : [u]_\vartheta \subseteq Q\}, \tag{1}$$

$$\overline{Q} = \{u \in U : [u]_\vartheta \cap Q \neq \varnothing\}. \tag{2}$$

These operators assign two subsets, $\underline{Q}$ and $\overline{Q}$ of U, to any $Q \subseteq U$, known as the lower and upper approximations of Q w.r.t ϑ, respectively.

Definition 2.1. *[38] Suppose that (U, ϑ) is a PA-space. A subset Q of U is definable when $\underline{Q} = \overline{Q}$; else, Q is a RS.*

For the non-empty universe U of objects, let 2^U denote the power set of U and let $\mathcal{E}$ denote the non-empty finite set of attributes (characteristics, properties or parameters) the objects of U possess.

Definition 2.2. *[33] A SS over U is a pair $(F, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{E}$ and $F : \mathcal{A} \longrightarrow 2^U$ is a set-valued mapping.*

Thus, the objects of U with the property e are covered by the set $F(e)$. The BSS is built to distinguish between the positive and the negative characteristics of the data, that is, the assured and the prohibited presence of the property e in the objects. A BSS, in contrast to a SS, is constructed with the help of two sets of parameters. One is the attribute set $\mathcal{E}$, while the other set, denoted by $\neg\mathcal{E}$ and named as "not set of $\mathcal{E}$", contains the attributes opposite to those of $\mathcal{E}$.

Definition 2.3. *[40] A BSS over U is a triplet $\eth = (F, G, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{E}$ and $F : \mathcal{A} \longrightarrow 2^U$ and $G : \neg\mathcal{A} \longrightarrow 2^U$ such that $F(e) \cap G(\neg e) = \emptyset$ for all $e \in \mathcal{A}$.*

Here, $F(e)$ denotes the objects in U having a property $e \in A$ and $G(\neg e)$ denotes the objects in U having a property $\neg e$, opposite to e. The condition $F(e) \cap G(\neg e) = \emptyset$ is the consistency constraint. It is worth noting that an object lacking a property e, may not have the property $\neg e$, opposite to e. So, we may have $G(\neg e) \neq U - F(e)$ for some $e \in E$. We write $H(e) = U - (F(e) \cup G(\neg e))$ and call it the grey area of $\eth$ relevant to e. This grey area gives the degree of the hesitancy of the BSS $\eth$ over U, using an allied SS (H, A) over U. The set having all the BSSs over U is denoted by Ω.

Definition 2.4. *[40] A BSS $\eth = (F, G, \mathcal{A}) \in \Omega$ is a bipolar soft subset of a BSS $\eth_1 = (F_1, G_1, \mathcal{B})$, if*

(1) $\mathcal{A} \subseteq \mathcal{B}$,

(2) $F(e) \subseteq F_1(e)$ and $G(\neg e) \supseteq G_1(\neg e)$ for all $e \in \mathcal{A}$.

We denoted it by $\eth \,\widetilde{\subseteq}\, \eth_1$. Two BSSs $\eth$ and $\eth_1$ over the same universe U are equal if each of them is a bipolar soft subset of the other.

Definition 2.5. *[40] The relative whole BSS is $\mathcal{U}_{\mathcal{A}} = (\mathcal{U}, \Phi, \mathcal{A}) \in \Omega$, where $\mathcal{U} : \mathcal{A} \longrightarrow 2^U$ and $\Phi : \neg\mathcal{A} \longrightarrow 2^U$ are defined for all $e \in \mathcal{A}$ as $\mathcal{U}(e) = U$ and $\Phi(\neg e) = \emptyset$. The relative null BSS is $\Phi_{\mathcal{A}} = (\Phi, \mathcal{U}, \mathcal{A}) \in \Omega$, where $\Phi : \mathcal{A} \longrightarrow 2^U$ and $\mathcal{U} : \neg\mathcal{A} \longrightarrow 2^U$ are defined for all $e \in A$ as $\Phi(e) = \emptyset$ and $\mathcal{U}(\neg e) = U$.*

Definition 2.6. *[40] Let $\eth = (F, G, \mathcal{A})$, $\eth_1 = (F_1, G_1, \mathcal{B}) \in \Omega$. Then, their intersections and unions are characterized as:*

(1) The extended union of $\eth$ and $\eth_1$ is a BSS $\eth \cup_\varepsilon \eth_1 = (F \widetilde{\cup} F_1, G \widetilde{\cap} G_1, \mathcal{A} \cup \mathcal{B}) \in \Omega$, where $F \widetilde{\cup} F_1$ and $G \widetilde{\cap} G_1$ are defined as:

$$(F \widetilde{\cup} F_1)(e) = \begin{cases} F(e) & \text{if } e \in \mathcal{A} - \mathcal{B} \\ F_1(e) & \text{if } e \in \mathcal{B} - \mathcal{A} \\ F(e) \cup F_1(e) & \text{if} e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$(G \widetilde{\cap} G_1)(\neg e) = \begin{cases} G(\neg e) & \text{if } e \in \mathcal{A} - \mathcal{B} \\ G_1(\neg e) & \text{if } e \in \mathcal{B} - \mathcal{A} \\ G(\neg e) \cap G_1(\neg e) & \text{if } e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

(2) The extended intersection of $\eth$ and $\eth_1$ is a BSS $\eth \cap_\varepsilon \eth_1 = (F \widetilde{\cap} F_1, G \widetilde{\cup} G_1, \mathcal{A} \cup \mathcal{B}) \in \Omega$, where $F \widetilde{\cap} F_1$ and $G \widetilde{\cup} G_1$ are defined as:

$$(F \widetilde{\cap} F_1)(e) = \begin{cases} F(e) & \text{if } e \in \mathcal{A} - \mathcal{B} \\ F_1(e) & \text{if } e \in \mathcal{B} - \mathcal{A} \\ F(e) \cap F_1(e) & \text{if } e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$(G\widetilde{\cup}G_1)(\neg e)=\begin{cases} G(\neg e) & \text{if } e\in\mathcal{A}-\mathcal{B}\\ G_1(\neg e) & \text{if } e\in\mathcal{B}-\mathcal{A}\\ G(\neg e)\cup G_1(\neg e) & \text{if } e\in\mathcal{A}\cap\mathcal{B}\end{cases}$$

(1) The restricted union of $\eth$ *and* $\eth_1$ *is a BSS* $\eth\cup_r\eth_1=(F\widetilde{\cup}F_1, G\widetilde{\cap}G_1, \mathcal{A}\cap\mathcal{B})\in\Omega$*, where* $(F\widetilde{\cup}F_1)(e)=F(e)\cup F_1(e)$ *and* $(G\widetilde{\cap}G_1)(\neg e)=G(\neg e)\cap G_1(\neg e)$ *for all* $e\in\mathcal{A}\cap\mathcal{B}$*, provided* $\mathcal{A}\cap\mathcal{B}\neq\varnothing$*.*

(2) The restricted intersection of $\eth$ *and* $\eth_1$ *is a BSS* $\eth\cap_r\eth_1=(F\widetilde{\cap}F_1, G\widetilde{\cup}G_1, \mathcal{A}\cap\mathcal{B})\in\Omega$*, where* $(F\widetilde{\cap}F_1)(e)=F(e)\cap F_1(e)$ *and* $(G\widetilde{\cup}G_1)(\neg e)=G(\neg e)\cup G_1(\neg e)$ *for all* $e\in\mathcal{A}\cap\mathcal{B}$*, provided* $\mathcal{A}\cap\mathcal{B}\neq\varnothing$*.*

Definition 2.7. *[40] The compliment of a BSS* $\eth=(F,G,\mathcal{A})\in\Omega$*, is a BSS* $\eth^c=(F^c,G^c,\mathcal{A})\in\Omega$*, where* $F^c(e)=G(\neg e)$ *and* $G^c(\neg e)=F(e)$ *for all* $e\in\mathcal{A}$*.*

Example 2.8. *Suppose that* $U=\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5,\hbar_6\}$ *is a universe containing six animals and* $\mathcal{E}=\{e_1=$*tame,* $e_2=$*beautiful,* $e_3=$*big,* $e_4=$*healthy,* $e_5=$*lives on land*$\}$ *is a set of parameters for* U*. Let* $\neg\mathcal{E}=\{\neg e_1=$*wild,* $\neg e_2=$*ugly,* $\neg e_3=$*small,* $\neg e_4=$*weak,* $\neg e_5=$*lives in water*$\}$*. In this example, we construct a BSS and discuss the degree of hesitancy for an attribute. We define the BSS* $\eth=(F,G,\mathcal{A})$ *with* $\mathcal{A}=\{e_1,e_2,e_4\}\subseteq\mathcal{E}$*, which points out the animals having a property* e_i *or* $\neg e_i$ *as follows.*

$$F(e)=\begin{cases}\{\hbar_1,\hbar_3,\hbar_4,\hbar_6\} & \text{if } e=e_1\\ \{\hbar_2,\hbar_3,\hbar_5\} & \text{if } e=e_2\\ \{\hbar_2,\hbar_3,\hbar_4,\hbar_5\} & \text{if } e=e_4\end{cases}\qquad G(\neg e)=\begin{cases}\{\hbar_2,\hbar_5\} & \text{if } \neg e=\neg e_1\\ \varnothing & \text{if } \neg e=\neg e_2\\ \{\hbar_1,\hbar_6\} & \text{if } \neg e=\neg e_4.\end{cases}$$

Here, $H(e_2)=\{\hbar_1,\hbar_4,\hbar_6\}$ *is the degree of hesitancy of the BSS* $\eth$ *w.r.t. the attribute* e_2*, which depicts that although these animals are not beautiful, they are not ugly as well.*

3. Rough bipolar soft sets

In this segment, we establish the idea of RBSSs and investigate their basic properties.

Definition 3.1. *Let* $\eth=(F,G,\mathcal{A})\in\Omega$ *with* (U,ϑ) *as a PA-space. The lower and upper RBS approximations of* $\eth$ *w.r.t* (U,ϑ) *are the BSSs represented by* $\underline{\eth}_\vartheta=(\underline{F}_\vartheta,\underline{G}_\vartheta,\mathcal{A})$ *and* $\overline{\eth}^\vartheta=(\overline{F}^\vartheta,\overline{G}^\vartheta,\mathcal{A})$*, respectively, where* $\underline{F}_\vartheta$, $\overline{F}^\vartheta$ *are defined as:*

$$\underline{F}_\vartheta(e)=\{u\in U:[u]_\vartheta\subseteq F(e)\},\tag{3}$$

$$\overline{F}^\vartheta(e)=\{u\in U:[u]_\vartheta\cap F(e)\neq\varnothing\}\tag{4}$$

for all $e\in\mathcal{A}$*, and* $\underline{G}_\vartheta$, $\overline{G}^\vartheta$ *are defined as:*

$$\underline{G}_\vartheta(\neg e)=\{u\in U:[u]_\vartheta\cap G(\neg e)\neq\varnothing\},\tag{5}$$

$$\overline{G}^\vartheta(\neg e)=\{u\in U:[u]_\vartheta\subseteq G(\neg e)\}\tag{6}$$

for all $\neg e\in\neg\mathcal{A}$*. If* $\underline{\eth}_\vartheta=\overline{\eth}^\vartheta$*, then* $\eth$ *is said to be R-definable; otherwise,* $\eth$ *is a RBS over* U*.*

Furthermore, the knowledge regarding an element $u \in U$, interpreted by these RBS approximations of $\eth$, is as follows:

- If $u \in \underline{F}_{\vartheta}(e)$, then u definitely possess the attribute e.
- If $u \in \overline{G}^{\vartheta}(\neg e)$, then u definitely possess the attribute $\neg e$.
- If $u \in \overline{F}^{\vartheta}(e)$, then u probably has the attribute e.
- If $u \in \underline{G}_{\vartheta}(\neg e)$, then u probably has the attribute $\neg e$.

For simplicity, we write $\underline{\eth}$ and $\overline{\eth}$ for $\underline{\eth}_{\vartheta}$ and $\overline{\eth}^{\vartheta}$, respectively (if there is no confusion in ϑ).

Following are some basic characterizations of the RBSs.

Theorem 3.2. *Let $\eth = (F, G, \mathcal{A}) \in \Omega$ with (U, ϑ) as a PA-space. Then, the following statements hold:*

(1) $\underline{\eth} \,\widetilde{\subseteq}\, \eth \,\widetilde{\subseteq}\, \overline{\eth}$,

(2) $\underline{\Phi_{\mathcal{A}}} = \Phi_{\mathcal{A}} = \overline{\Phi_{\mathcal{A}}}$,

(3) $\underline{\mathcal{U}_{\mathcal{A}}} = \mathcal{U}_{\mathcal{A}} = \overline{\mathcal{U}_{\mathcal{A}}}$,

(4) $\underline{(\underline{\eth})} = \underline{\eth} = \overline{\left(\underline{\eth}\right)}$,

(5) $\underline{(\overline{\eth})} = \overline{\eth} = \overline{(\overline{\eth})}$,

(6) $\overline{\eth^c} = \left(\underline{\eth}\right)^c$,

(7) $\underline{\eth^c} = \left(\overline{\eth}\right)^c$.

Proof. (1)–(5) can be verified by using Definitions 2.4, 2.5 and 3.1.

(6) We have $\overline{\eth^c} = \left(\overline{F^c}, \overline{G^c}, \mathcal{A}\right)$. As $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in \mathcal{A}$, so using Definitions 2.7 and 3.1, we get

$$\begin{aligned} \overline{F^c}(e) &= \{u \in U : [u] \cap F^c(e) \neq \varnothing\} = \{u \in U : [u] \cap G(\neg e) \neq \varnothing\} \\ &= \underline{G}(\neg e) = (\underline{F})^c(e) \end{aligned}$$

and

$$\begin{aligned} \overline{G^c}(\neg e) &= \{u \in U : [u] \subseteq G^c(\neg e)\} = \{u \in U : [u] \subseteq F(e)\} \\ &= \underline{F}(e) = (\underline{G})^c(\neg e) \end{aligned}$$

for all $e \in \mathcal{A}$. But $\left((\underline{F})^c, (\underline{G})^c, \mathcal{A}\right) = \left(\underline{\eth}\right)^c$. Thus by Definition 2.4, we have $\overline{\eth^c} = \left(\underline{\eth}\right)^c$.

(7) Analogous to the proof of (6). □

Theorem 3.3. *Assume that $\eth = (F, G, \mathcal{A}), \eth_1 = (F_1, G_1, \mathcal{B}) \in \Omega$ with (U, ϑ) as a PA-space. Then, the following properties hold:*

(1) $\eth \,\widetilde{\subseteq}\, \eth_1$ *implies that* $\underline{\eth} \,\widetilde{\subseteq}\, \underline{\eth_1}$ *and* $\overline{\eth} \,\widetilde{\subseteq}\, \overline{\eth_1}$,

(2) $\underline{\eth \cap_{\varepsilon} \eth_1} = \underline{\eth} \cap_{\varepsilon} \underline{\eth_1}$,

(3) $\underline{\eth \cap_{r} \eth_1} = \underline{\eth} \cap_{r} \underline{\eth_1}$,

(4) $\underline{\eth \cup_{\varepsilon} \eth_1} \,\widetilde{\supseteq}\, \underline{\eth} \cup_{\varepsilon} \underline{\eth_1}$,

(5) $\underline{\eth \cup_{r} \eth_1} \,\widetilde{\supseteq}\, \underline{\eth} \cup_{r} \underline{\eth_1}$,

(6) $\overline{\eth \cap_{\varepsilon} \eth_1} \,\widetilde{\subseteq}\, \overline{\eth} \cap_{\varepsilon} \overline{\eth_1}$,

(7) $\overline{\eth \cap_{r} \eth_1} \,\widetilde{\subseteq}\, \overline{\eth} \cap_{r} \overline{\eth_1}$,

(8) $\overline{\eth \cup_{\varepsilon} \eth_1} = \overline{\eth} \cup_{\varepsilon} \overline{\eth_1}$

(9) $\overline{\eth \cup_{r} \eth_1} = \overline{\eth} \cup_{r} \overline{\eth_1}$.

Proof. (1) Assume that $\eth \,\widetilde{\subseteq}\, \eth_1$. Then we have $F(e) \subseteq F_1(e)$ and $G(\neg e) \supseteq G_1(\neg e)$ for all $e \in \mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{B}$. Now for all $e \in \mathcal{A}$, we have

$$\underline{F}(e) = \{u \in U : [u] \subseteq F(e)\} \subseteq \{u \in U : [u] \subseteq F_1(e)\} = \underline{F_1}(e)$$

$$\underline{G}(\neg e) = \{u \in U : [u] \cap G(\neg e) \neq \emptyset\} \supseteq \{u \in U : [u] \cap G_1(\neg e) \neq \emptyset\} = \underline{G_1}(\neg e).$$

Thus according to Definition 2.4, $\underline{\eth} \,\widetilde{\subseteq}\, \underline{\eth 1}$.

Similarly, we can show that $\overline{\eth} \,\widetilde{\subseteq} \overline{\eth_1}$.

(2) In the light of Definition 2.6, we have $\underline{\eth \cap_{\varepsilon} \eth_1} = (\underline{F \widetilde{\cap} F_1}, \underline{G \widetilde{\cup} G_1}, \mathcal{A} \cup \mathcal{B})$. From Definition 3.1, we have

$$\begin{aligned}(\underline{F \widetilde{\cap} F_1})(e) &= \{u \in U : [u] \subseteq F(e) \cap F_1(e)\}\\ &= \{u \in U : [u] \subseteq F(e)\} \cap \{u \in U : [u] \subseteq F_1(e)\} = (\underline{F} \widetilde{\cap} \underline{F_1})(e)\end{aligned}$$

$$\begin{aligned}(\underline{G \widetilde{\cup} G_1})(\neg e) &= \{u \in U : [u] \cap (G(\neg e) \cup G_1(\neg e)) \neq \emptyset\}\\ &= \{u \in U : [u] \cap G(\neg e) \neq \emptyset\} \cup \{u \in U : [u] \cap G_1(\neg e) \neq \emptyset\}\\ &= (\underline{G} \widetilde{\cup} \underline{G_1})(\neg e).\end{aligned}$$

But $(\underline{F} \widetilde{\cap} \underline{F_1}, \underline{G} \widetilde{\cup} \underline{G_1}, \mathcal{A} \cup \mathcal{B}) = \underline{\eth} \cap_{\varepsilon} \underline{\eth_1}$. So, assertion (2) is proved by Definition 2.4.

(3) It can be deduced from (2).

(4) By Definition 2.6, we have $\underline{\eth \cup_{\varepsilon} \eth_1} = (\underline{F \widetilde{\cup} F_1}, \underline{G \widetilde{\cap} G_1}, \mathcal{A} \cup \mathcal{B})$, where

$$\begin{aligned}(\underline{F \widetilde{\cup} F_1})(e) &= \{u \in U : [u] \subseteq F(e) \cup F_1(e)\}\\ &\supseteq \{u \in U : [u] \subseteq F(e)\} \cup \{u \in U : [u] \subseteq F_1(e)\} = (\underline{F} \widetilde{\cup} \underline{F_1})(e)\end{aligned}$$

$$\begin{aligned}(\underline{G \widetilde{\cap} G_1})(\neg e) &= \{u \in U : [u] \cap (G(\neg e) \cap G_1(\neg e)) \neq \emptyset\}\\ &\subseteq \{u \in U : [u] \cap G(\neg e) \neq \emptyset\} \cap \{u \in U : [u] \cap G_1(\neg e) \neq \emptyset\}\\ &= (\underline{G} \widetilde{\cap} \underline{G_1})(\neg e).\end{aligned}$$

But $(\underline{F} \widetilde{\cup} \underline{F_1}, \underline{G} \widetilde{\cap} \underline{G_1}, \mathcal{A} \cup \mathcal{B}) = \underline{\eth} \cup_{\varepsilon} \underline{\eth_1}$. So, assertion (4) is proved by Definition 2.4.

(5) It can be deduced from (4).

(6)–(9) can be verified in the same way as (2)–(5) above. □

The following example illustrates the evaluation of RBS approximations of a BSS.

Example 3.4. *Consider the BSS $\eth = (F, G, \mathcal{A})$ as in Example 2.8. Let ϑ be an ER on U defining classes $\{\hbar_1, \hbar_2, \hbar_3\}$, $\{\hbar_4, \hbar_5\}$ and $\{\hbar_6\}$. Then, the lower RBS approximation of $\eth$ is $\underline{\eth} = (\underline{F}, \underline{G}, \mathcal{A})$, where $\underline{F}$ and $\underline{G}$ are calculated according to Definition 3.1 as:*

$$\underline{F}(e) = \begin{cases} \{\hbar_6\} & \text{if } e = e_1 \\ \varnothing & \text{if } e = e_2 \\ \{\hbar_4, \hbar_5\} & \text{if } e = e_4 \end{cases} \qquad \underline{G}(\neg e) = \begin{cases} \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\} & \text{if } \neg e = \neg e_1 \\ \varnothing & \text{if } \neg e = \neg e_2 \\ \{\hbar_1, \hbar_2, \hbar_3, \hbar_6\} & \text{if } \neg e = \neg e_4 \end{cases}$$

and the upper RBS approximation of $\eth$ is $\overline{\eth} = (\overline{F}, \overline{G}, \mathcal{A})$, calculated as:

$$\overline{F}(e) = \begin{cases} U & \text{if } e = e_1 \\ \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\} & \text{if } e = e_2 \\ \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\} & \text{if } e = e_4 \end{cases} \qquad \overline{G}(\neg e) = \begin{cases} \varnothing & \text{if } \neg e = \neg e_1 \\ \varnothing & \text{if } \neg e = \neg e_2 \\ \{\hbar_6\} & \text{if } \neg e = \neg e_4 \end{cases}.$$

Notice that $\underline{\eth} \neq \overline{\eth}$, so $\eth$ is a RBSS. Also $\underline{F}(e) \subseteq F(e) \subseteq \overline{F}(e)$ and $\underline{G}(e) \supseteq G(e) \supseteq \overline{G}(e)$ for all $e \in \mathcal{A}$. This shows that $\underline{\eth} \mathrel{\widetilde{\subseteq}} \eth \mathrel{\widetilde{\subseteq}} \overline{\eth}$.

The following proposition indicates that the ϑ-definable BSS over U, when ϑ is identity or universal binary relation on U.

Proposition 3.5. *Let (U, ϑ) be a PA-space.*

(1) Each BSS over U is ϑ-definable, whenever, the relation ϑ on U is the identity binary relation.

(2) If the binary relation $\vartheta = U \times U$ and $\eth \in \Omega$ is ϑ-definable, then, $\eth \in \{\mathcal{U}_{\mathcal{A}}, \Phi_{\mathcal{A}} : \mathcal{A} \subseteq \mathcal{E}\}$.

Proof. Straightforward. □

Theorem 3.6. *Let $\eth = (F, G, \mathcal{A}) \in \Omega$ with (U, ϑ) as a PA-space. Then, the following assertions are equivalent:*

(1) $\overline{\eth} \mathrel{\widetilde{\subseteq}} \eth$.

(2) $\eth \mathrel{\widetilde{\subseteq}} \underline{\eth}$.

(3) $\eth$ is ϑ-definable.

Proof. (1)⇒(2) Assume $\overline{\eth} \mathrel{\widetilde{\subseteq}} \eth$. Then, by Theorem 3.3, $\underline{(\overline{\eth})} \mathrel{\widetilde{\subseteq}} \underline{\eth}$. By Theorem 3.2, we get

$$\eth \mathrel{\widetilde{\subseteq}} \overline{\eth} = \underline{(\overline{\eth})} \mathrel{\widetilde{\subseteq}} \underline{\eth}.$$

(2)⇒(3) By Theorem 3.2, $\underline{\eth} \mathrel{\widetilde{\subseteq}} \eth$, also given that $\eth \mathrel{\widetilde{\subseteq}} \underline{\eth}$. So, $\eth = \underline{\eth}$. This gives $\overline{\eth} = \overline{(\underline{\eth})} = \underline{\eth}$. Thus $\eth$ is ϑ-definable.

(3)⇒(1) Straightforward. □

The next result is important because it highlights a remarkable link between lower and upper RBS approximations of a BSS $\eth$ when the ER ϑ in the PA-space is substituted by another ER ϱ on U, which contains ϑ.

Theorem 3.7. *Let $ð = (F, G, \mathcal{A}) \in \Omega$ with (U, ϑ) as a PA-space and let ϱ be an ER on U, such that, $\vartheta \subseteq \varrho$. Then, $\underline{ð}_{\varrho} \widetilde{\subseteq} \underline{ð}_{\vartheta}$ and $\overline{ð}^{\vartheta} \widetilde{\subseteq} \overline{ð}^{\varrho}$.*

Proof. Let $ð \in \Omega$ for some $\mathcal{A} \subseteq \mathcal{E}$. Since $\vartheta \subseteq \varrho$, we have $[u]_{\vartheta} \subseteq [u]_{\varrho}$ for all $u \in U$. Thus, $\underline{F}_{\varrho}(e) \subseteq \underline{F}_{\vartheta}(e)$ and $\underline{G}_{\varrho}(\neg e) \supseteq \underline{G}_{\vartheta}(\neg e)$ for all $e \in \mathcal{A}$. So, we have $\underline{ð}_{\varrho} \widetilde{\subseteq} \underline{ð}_{\vartheta}$. Similarly, one can verify that $\overline{ð}^{\vartheta} \widetilde{\subseteq} \overline{ð}^{\varrho}$. □

4. Applications of RBSSs in DM problems

As we know that, uncertainty is an intrinsic component of medical diagnosis since a symptom is an uncertainty index about whether or not a disease is occurring. In this segment, we use the RBS approximations of the BSSs to solve DM problems related to the medical field. Sometimes, it is required to decide on the best object from a collection of objects. But sometimes, the decision between the attributes of some objects is also needs to be made. We propose algorithms for both situations using RBS approximations of the bipolar soft information about the objects. These algorithms are designed so that the larger data can also be managed. Let the set of attributes be $\mathcal{E} = \{e_i : 1 \leq i \leq p\}$ and the collection of objects be $U = \{u_j : 1 \leq j \leq q\}$. Let the bipolar soft information about the objects and their attributes be expressed by the BSS $ð = (F, G, \mathcal{E})$. We input $ð$ in the form of a matrix M by taking the $(i, j)th$ entry a_{ij} corresponding to the attribute e_i and the object u_j as:

$$a_{ij} = \begin{cases} 1 & \text{if } u_j \in F(e_i) \\ -1 & \text{if } u_j \in G(\neg e_i) \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

Let ϑ be the ER on U, dividing U into q' classes. Surely, $q' \leq q$. We input the relation ϑ as a $q' \times q$ matrix, whose each row corresponds to an equivalence class in such a way that the objects of the class are denoted by 1 and the objects not in that class are given $'0'$. With the help of the relation ϑ, the lower RBS approximation $\underline{ð}$ and the upper RBS approximation $\overline{ð}$ are evaluated. Denote the $(i, j)th$ entry in the matrices B and C of $\underline{ð}$ and $\overline{ð}$ by b_{ij} and c_{ij}, respectively. Then, using the matrix notation and Definition 3.1, we deduce that $b_{ij} = \bigwedge_{[u_j]} a_{ij}$ and $c_{ij} = \bigvee_{[u_j]} a_{ij}$.

Definition 4.1. *The decision coefficient D has the values d_i for each attribute $e_i \in \mathcal{E}$, given by*

$$d_i = \sum_{j=1}^{q} (b_{ij} + c_{ij}). \tag{8}$$

4.1. Decision for the most responsible risk factor of a disease

In this subsection, we use the RBS approximations of the BSSs to solve a DM problem, where the decision is made between the attributes of some objects. We propose a multipurpose algorithm that helps to decide for an attribute causing the maximum or minimum effect on the objects and use this algorithm to decide for the most responsible risk factor of a disease.

4.2. Algorithm 1

Step 1: Input the BSS $ð = (F, G, \mathcal{E})$ and ages of the patients.

Step 2: Construct an ER ϑ on U.

Step 3: Compute $\underline{ð}$ and $\overline{ð}$.

Step 4: Compute the values d_i for each attribute $e_i \in \mathcal{E}$.

Step 5: Find $d_k = \max_i d_i$.

Step 6: If k has more than one value then choose any of e_k.

Let us use the above algorithm to solve the following problem.

Example 4.2. *We discuss the situation of a city X where many young citizens are suffering from the disease of heart attack, while it is usually considered to be a disease of old age. Dr. Y, a cardiologist, is trying to search out the most prominent reason of occurrence of this disease in early ages so frequently in the city. The common risk factors causing heart attacks are* $\mathcal{E} = \{e_1 =$ *smoking,* $e_2 =$ *heavy drinking of alcohol,* $e_3 =$ *diabetic,* $e_4 =$ *high cholesterol level,* $e_5 =$ *sedentary life style,* $e_6 =$ *high blood pressure*$\}$*, which will serve as the attribute set for U. We take the set* $\neg\mathcal{E}$ *as* $\neg\mathcal{E} = \{\neg e_1 =$ *no smoking,* $\neg e_2 =$ *no drinking of alcohol,* $\neg e_3 =$ *non-diabetic,* $\neg e_4 =$ *normal cholesterol level,* $\neg e_5 =$ *healthy life style,* $\neg e_6 =$ *normal blood pressure*$\}$*. Dr. Y takes a sample of heart patients of age group (30 years - 50 years) admitted in different hospitals of the city. To give an understanding of the procedure, we take a small sample* $U = \{u_1, u_2, ..., u_8\}$ *of eight patients, whose ages are shown in Table 1. Although, this sample is too small for this study, one can take a sufficiently large sample and apply the algorithm.*

Table 1. Ages of patients.

Patients	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
Age (in years)	32	36	38	41	42	42	48	50

We divide the patients into four age groups, whose age (in years) is in interval $[30-35)$, $[35-40)$, $[40-45)$ *or* $[45-50]$. *Define a relation* ϑ *on U such that two patients* u_i *and* u_j *are* ϑ*-related if they belong to the same age group. Then,* ϑ *serves as an ER dividing U into equivalence classes* $\{u_1\}$, $\{u_2, u_3\}$, $\{u_4, u_5, u_6\}$ *and* $\{u_7, u_8\}$. *The history and examination report of patients under consideration taken by Dr. Y, is given by a BSS* $ð_1 = (F, G, \mathcal{E})$ *as shown in Table 2.*

Table 2. History of patients.

$ð_1$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
e_1	1	-1	1	1	1	1	-1	-1
e_2	-1	1	1	-1	-1	1	1	1
e_3	1	1	-1	1	0	-1	-1	1
e_4	0	1	0	1	-1	-1	-1	0
e_5	-1	1	0	-1	-1	1	1	-1
e_6	1	-1	1	1	0	-1	1	-1

Recall that the (i, j)*th entry in the matrices of* $\underline{\eth_1}$ *and* $\overline{\eth_1}$ *is* b_{ij} *and* c_{ij}*, respectively.*

The Algorithm 1 designed to decide for the optimal attribute takes an ER, set of attributes $\mathcal{E}$ *and a BSS* $\eth_1$ *as input. As an output, the decision table is constructed with the columns of* $\mathcal{E}$ *and D, rearranged in the descending order w.r.t* d_i*. Select k, so that,* $d_k = \max_i d_i$*. Then* e_k *is the best optimal attribute. The decision table of* $\eth_1$ *is given in Table 3:*

Table 3. Decision table of $\eth_1$.

$\mathcal{E}$	D
e_2	6
e_1	4
e_3	2
e_6	2
e_4	0
e_5	0

We get $\max_i d_i = d_2 = 6$ *and hence* $k = 2$*. And the second highest decision value is for* $k = 1$*. Thus, Dr. Y comes to the result that the most dominant reason of so frequent heart attacks in younger generation of the city X is firstly heavy drinking of alcohol* (e_2) *and secondly heavy smoking* (e_1)*. This indicates the excessive use of alcoholic drinks and heavy smoking in the city, which is to be controlled on first preference in order to overcome the disease in younger generation. Moreover, the ranking order among the attributes is given as follows:*

$$e_2 \geq e_1 \geq e_3 \geq e_6 \geq e_4 \approx e_5.$$

4.3. Deciding for the best composition of a medicine

While deciding in the favour of an object from a given collection of objects, sometimes it becomes difficult to take the decision which is possibly the best. In that case, detecting the worst object can also aid the decision-makers to sidestep the worst decision. The algorithm proposed in this subsection pinpoints both, the best and the worst decision. This algorithm covers a vast variety of problems where a decision is needed to be taken between some objects having different properties. We discuss here a problem in which a pharmaceutical company requires to decide in the favour of a composition of a particular medicine to be manufactured amongst five different compositions $U = \{m_j : 1 \leq j \leq q\}$ of that medicine having different positive effects, say $\mathcal{E} = \{e_i : 1 \leq i \leq p\}$ and side effects $\neg\mathcal{E} = \{\neg e_i 1 \leq i \leq p\}$. The information about medicines $m_j \in U$ is expressed as a BSS $\eth = (F, G, \mathcal{E})$ over U in matrix form, whose $(i, j)th$ entry a_{ij} is defined as:

$$a_{ij} = \begin{cases} 1 & \text{if } m_j \text{ has effect } e_i \\ -1 & \text{if } m_j \text{ shows side effect } \neg e_i \\ 0 & \text{otherwise} \end{cases}. \qquad (9)$$

Now we define the ERs on U associated with the BSS $\eth$. Due to indiscernibility in U, we can partition U into the following three classes for any $e_i \in \mathcal{E}$.

$$A_i = \{m_j \in U : a_{ij} = 1\}$$

$$\begin{aligned} B_i &= \{m_j \in U : a_{ij} = -1\} \\ C_i &= \{m_j \in U : a_{ij} = 0\}. \end{aligned}$$

These classes (if non-empty) can serve as equivalence classes by considering two objects to be equivalent if they belong to the same class. Then, to each $e_i \in \mathcal{E}$, corresponds an ER, say $\xi(e_i)$ on U. Denote

$$\acute{R} = \bigcap_{e_i \in \mathcal{E}} \xi(e_i). \tag{10}$$

Then, $\acute{R}$ also gives an ER on U. This ER serves as our key tool to calculate the lower RBS approximation $\underline{ð}$ and the upper RBS approximation $\overline{ð}$. Denote the $(i, j)th$ entry in the table of $\underline{ð}$ and $\overline{ð}$ by b_{ij} and c_{ij}, respectively.

Moreover, this ER can also be expressed as

$$m_j \sim m_l \text{ if } a_{im} = a_{il} \ \forall\ i = 1, 2, \cdots, p.$$

Then the lower and upper approximations of the BSS ð are calculated in the similar way.

$$b_{ij} = \min_l \{a_{il}\} \text{ such that } m_j = m_l,$$

$$c_{ij} = \max_l \{a_{il}\} \text{ such that } m_j = m_l.$$

Definition 4.3. *The decision coefficient D has the values d_j for each $m_j \in U$, given by*

$$d_j = \sum_{i=1}^{p} (b_{ij} + c_{ij}). \tag{11}$$

Note that the relation $\acute{R}$ in our problem is the identity relation. So $ð_2$ is $\acute{R}$-definable by Proposition 3.5. The Algorithm 2 is designed to decide for the best, as well as, for the worst object and it takes U and ð as input. As an output, the decision table is constructed with the columns of U and D, rearranged in the descending order w.r.t d_j. Select l and k, so that $d_l = \min_j d_j$ and $d_k = \max_j d_j$. Then m_k is the best decision, while m_l is the worst decision.

4.4. Algorithm 2

Step 1: Input the BSS $ð = (F, G, \mathcal{E})$ in tabular form.

Step 2: Compute the ER $\acute{R}$ according to Eq (10).

Step 3: Compute $\underline{ð}$ and $\overline{ð}$.

Step 4: Evaluate the values d_j for each medicine $m_j \in U$.

Step 5: Find $d_k = \max_j d_j$.

Step 6: If k has more than one value then choose any of m_k.

Example 4.4. *Assume that a pharmaceutical company requires to decide in the favour of a composition of a particular medicine to be manufactured amongst five different compositions* $U = \{m_1, m_2, m_3, m_4, m_5\}$ *of that medicine having different positive effects, say* $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ *and side effects* $\neg\mathcal{E} = \{\neg e_1, \neg e_2, e_3, \neg e_4, \neg e_5, \neg e_6\}$. *The information about medicines* $m_j \in U$ *is expressed as a BSS* $\eth_2 = (F, G, \mathcal{E})$ *over* U *given as follows (see Table 4):*

Table 4. The BSS $\eth_2$.

$\eth_2$	m_1	m_2	m_3	m_4	m_5
e_1	0	−1	1	1	0
e_2	1	1	−1	1	0
e_3	0	−1	1	1	1
e_4	−1	0	1	−1	1
e_5	1	1	1	1	−1
e_6	1	−1	1	1	−1

Now, for each $e_i \in \mathcal{E}$, *corresponds an ERs on* U, *can be calculated using Table 4 as follows:*

- ξ_1 *for* e_1: $\{m_3, m_4\}, \{m_1, m_5\}, \{m_2\}$,
- ξ_2 *for* e_2: $\{m_1, m_2, m_4\}, \{m_5\}, \{m_3\}$,
- ξ_3 *for* e_3: $\{m_3, m_4, m_5\}, \{m_1\}, \{m_2\}$.

Therefore, $\acute{R} = \bigcap_{e_i \in \mathcal{E}} \xi(e_i) =$ *Identity relation.*

Using the ER $\acute{R}$, *we can calculate* $\underline{\eth_2}$ *and* $\overline{\eth_2}$ *same as give in Tables 8 and 9.*

The decision table of $\eth_2$ is given as in Table 5 as follows:

Table 5. Decision table of $\eth_2$.

U	D
m_3	8
m_4	8
m_1	4
m_5	0
m_2	−2

We get $\max_j d_j = d_3 = d_4 = 8$ and $\min_j d_j = d_2 = -2$. Hence $k = 3, 4$ and $l = 2$. Thus, any one of the medicines m_3 and m_4 can be manufactured, while m_2 is the worst decision. Moreover, the ranking among the medicines is given as:

$$m_3 \approx m_4 \geq m_1 \geq m_5 \geq m_2.$$

5. Comparative study

This segment emphasizes the advantages of the proposed DM approaches and conducts a comparison with few other techniques in a bipolar soft environment.

5.1. Advantages

Real-world DM problems typically appear in a complex environment under uncertain and imprecise data, which is difficult to address. The suggested methods are highly suitable for the situation when the data is complex, vague, and uncertain. particularly, when the existing data is relies on the bipolar soft information. A few advantages of the suggested techniques are listed as follows:

(i) The suggested methods incorporates positive and negative aspects of each object in the form of a BSS. These hybrid models are more generalized and suitable to tackle with DM problems consists of bipolar soft information.

(ii) The proposed DM approaches are simple to understand and can be used to DM problems in reality.

(iii) The advantages of suggested model can easily be judged from the Table 6 given below. In Table 6, we compare the characteristics of the established method and the introduced approaches in [12,13,17,24,41]. We conduct qualitative comparison from six features: Membership function (MF), non-membership function (NMF), parametrization, roughness of an information system (IS), ranking of alternatives and ranking of attributes to illustrate its superiority. From Table 6, it can be observed that the proposed method has all listed characteristics, but the mentioned methods do not have all of them.

Table 6. Characteristics comparison of different methods with proposed method.

Methods	Characteristics					
	Handle MF	Handle NMF	Manage parametization	Roughness of an IS	Ranking of alternatives	Ranking of attributes
Çağman and Enginoglu [12]	✓	✗	✓	✗	✓	✗
Çağman and Enginoglu [13]	✓	✗	✓	✗	✓	✗
Fatimah et al. [17]	✓	✗	✓	✗	✓	✗
Karaaslan and Karataş [24]	✓	✓	✓	✗	✓	✗
Mahmood [41]	✓	✓	✓	✗	✓	✗
Proposed Approach	✓	✓	✓	✓	✓	✓

5.2. Comparison with other methods

There are many approaches in the literature useful for addressing various DM problems. Each of these DM approaches has advantages and disadvantages of its own. The capacity of each method depends on the problem under consideration. Here we conduct a comparison of the presented DM techniques with few other DM methods in the fuzzy and bipolar fuzzy, and bipolar soft environments and see the significance of the proposed DM strategies.

(i) If we compare our proposed strategies with the methods offered in [12–14, 17, 25, 26, 43], we can observed that these methods cannot capture bipolarity in DM procedure which is an intrinsic component of human thinking and cognition. Therefore, the suggested techniques in this work have wider practicability and stronger effectiveness.

(ii) Several researchers presented different generalizations of SS theory and BSS theory (see [4, 12, 15, 24, 41]). But the roughness in these models is not investigated. Our work is a conglomeration of RS theory and BSS theory, which is the distinctiveness and novelty of our work.

(iii) In the fuzzy and bipolar fuzzy DM settings, a MF is usually required to fuzzify the data. In the suggested techniques, we have utilized the conception of the RBS approximations to derive the uncertainty from the original set of data without additional adjustment and MFs. In hybrid fuzzy and bipolar fuzzy methods, the MF depends on the choice and thinking of the decision-makers which makes the results more biased.

(iv) When we apply the methods suggested in [4, 24, 40] to our case 4.3, we obtain the following ranking among the objects displayed in Table 7. From Table 7, we observe that the optimal solution via all methods is the same, making our technique is feasible and effective.

Table 7. Comparison with some other methods.

Methods	Ranking
Al-shami [4]	$m_3 \approx m_4 \geq m_1 \geq m_5 \geq m_2$
Karaaslan and Karataş [24]	$m_3 \approx m_4 \geq m_1 \geq m_5 \geq m_2$
Shabir and Naz [40]	$m_3 \approx m_4 \geq m_1 \geq m_5 \geq m_2$
Our proposed method	$m_3 \approx m_4 \geq m_1 \geq m_5 \geq m_2$

6. Conclusions

RS and SS theories are successful tools to handle the uncertainty in the data, while the BSSs are the suitable mathematical approach to handle the uncertainty and the bipolarity of the data. In this work, we have introduced the concept of RBSSs as a hybridization of the RSs with the BSSs and used this concept to two different DM problems. The first problem addresses the case where a decision is needed to be taken between the attributes of some objects. We discussed the situation where the rate of heart attacks in younger generation is increasing rapidly in a city. Algorithm 1 provided the decision between the risk factors of heart attack which is becoming the most prominent cause. So that, the problem could be controlled. Second problem addresses the case where a pharmaceutical company wishes to decide between different compositions of a particular medicine to be manufactured. Each composition has different effects and side effects. Algorithm 2 provided the decision for the best, as well as, the worst composition of that medicine. Both algorithms are multipurpose and can be applied to many other similar problems.

In the future, based on the characterized idea in this article, scholars may also look at the algebraic structures of RBSSs. Another perspective direction is to examine the topological properties of RBSSs to seek a concrete foundation of the research studies and enhancement of working strategies. Moreover, the notions of RBSSs could also be extended to fuzzy and multi-granulation environments, and successful DM strategies can be developed. Furthermore, we will focus on the implementation of the suggested framework in tackling a more extensive scope of selection techniques, like TOPSIS, VIKOR, ELECTRE, AHP, and PROMETHEE.

Appendix

The two matrices $\underline{ð_1}$ and $\overline{ð_1}$ of data given in Example 4.2 are presented by Tables 8 and 9, respectively.

Table 8. Lower approximation $\underline{ð_1}$ of $ð_1 = (F, G, \mathcal{E})$.

$\underline{ð_1}$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	B_i
e_1	1	−1	−1	1	1	1	−1	−1	0
e_2	−1	1	1	−1	−1	−1	1	1	0
e_3	1	−1	−1	−1	−1	−1	−1	−1	−6
e_4	0	0	0	−1	−1	−1	−1	−1	−5
e_5	−1	0	0	−1	−1	−1	−1	−1	−6
e_6	1	−1	−1	−1	−1	−1	−1	−1	−6

Table 9. Upper approximation $\overline{ð_1}$ of $ð_1 = (F, G, \mathcal{E})$.

$\overline{ð_1}$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	C_i
e_1	1	1	1	1	1	1	−1	−1	4
e_2	−1	1	1	1	1	1	1	1	6
e_3	1	1	1	1	1	1	1	1	8
e_4	0	1	1	1	1	1	0	0	5
e_5	−1	1	1	1	1	1	1	1	6
e_6	1	1	1	1	1	1	1	1	8

In Table 10 the values D_j for each $e_j \in \mathcal{E}$, given by

$$D_j = \sum_{j=1}^{p} (B_j + C_j) \tag{111}$$

Table 10. Decision.

$\mathcal{E}$	B_i	C_i	D_i
e_1	0	4	4
e_2	0	6	6
e_3	−6	8	2
e_4	−5	5	0
e_5	−6	6	0
e_6	−6	8	2

The MATLAB code of Algorithms 1 and 2 are given below.

Algorithm 1:
```
R=[1 0 0 0 0 0 0 0;
0 1 1 0 0 0 0 0;
0 0 0 1 1 1 0 0;
0 0 0 0 0 0 1 1];
E=['e1';'e2';'e3';'e4';'e5';'e6'];
M=[1 -1 1 1 1 1 -1 -1;
-1 1 1 -1 -1 1 1 1;
1 1 -1 1 0 -1 -1 1;
0 1 0 1 -1 -1 -1 0;
-1 1 0 -1 -1 1 1 -1;
1 -1 1 1 0 -1 1 -1];
for x=1:size(R,1)
I=find(R(x,:));
for y=1:sum(R(x,:))
Q(:,y)=M(:,I(y));
for y=1:sum(R(x,:))
B(:,I(y))=min(Q,[],2);
C(:,I(y))=max(Q,[],2);
clear Q
Smin=sum(B,2);
Smax=sum(C,2);
D=Smin+Smax;
DecisionTable=sortrows(table(E,D),2,'descend')
```

Algorithm 2: R=[1 0 0 0 0;
```
0 1 0 0 0;
0 0 1 0 0;
0 0 0 1 0;
0 0 0 0 1];
U=['m1';'m2';'m3';'m4';'m5'];
M=[0 -1 1 1 0;
1 1 -1 1 0;
0 -1 1 1 1;
-1 0 1 -1 1;
1 1 1 1 -1;
1 -1 1 1 -1];
for x=1:size(R,1)
I=find(R(x,:));
for y=1:sum(R(x,:))
Q(:,y)=M(:,I(y));
for y=1:sum(R(x,:))
B(:,I(y))=min(Q,[],2);
C(:,I(y))=max(Q,[],2);
clear Q
Smin=sum(B,1)';
Smax=sum(C,1)';
D=Smin+Smax;
DecisionTable=sortrows(table(U,D),2,'descend')
```

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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