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Research article

Global regularity and blowup for a class of non-Newtonian polytropic variation-inequality problem from investment-consumption problems

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Abstract: This paper studies variation-inequality problems with fourth order non-Newtonian polytropic operators. First, the test function of the weak solution is constructed by using the difference operator. Then global regularity of the weak solution is obtained by some difference transformation and inequality amplification techniques. The weak solution is transformed into a differential inequality of the energy function. It is proved that the weak solution will blow up in finite time. Then, the upper bound and the blowup rate estimate of the blow up are given by handling some differential inequalities.

Keywords: variation-inequality problems; fourth order non-Newtonian polytropic operator; regularity; blow up
Mathematics Subject Classification: 35K15, 91G15

1. Introduction

The author of this study, given $\Omega \in R_N (N \geq 2)$, a bounded regular domain with Lipschitz boundary and $\Omega_T = [0, T] \times \Omega$, considers a kind of variation-inequality problem

$$\begin{cases} -Lu \geq 0, & (x,t) \in \Omega_T, \\ u - u_0 \geq 0, & (x,t) \in \Omega_T, \\ Lu(u - u_0) = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

with the non-Newtonian polytropic operator

$$Lu = \partial_t u - \Delta^2 u^m + hu^\alpha + f, \; m > 0. \tag{1.2}$$

Here, $u_0 \in H_0^1(\Omega)$, f, h, and α have been used with different conditions in Sections 3 and 4, as specified in Theorem 3.1 and Theorem 4.1.

Variational inequalities, such as problem (1.1), have found widespread application in the field of finance. For example, [1] explores the investment-consumption model, while [2] analyzes dividend optimization and risk control problems through weak solutions of variation-inequality. In [3], a continuous-time, finite horizon, irreversible investment problem is examined, resulting in the emergence of a free boundary that represents the optimal investment boundary.

The behaviours of the free boundary and existence of a weak solution were studied by using the partial differential equation (PDE) approach. Moreover, the regularities of the value function and optimal investment and maintenance policies were considered in [4].

In recent years, there have been much literature on the theoretical research of variation-inequality problems.The authors in [5] studied the following variation-inequality initial-boundary value problems:

$$\begin{cases} \min\{L\phi, \phi - \phi_0\} = 0, & (x,t) \in Q_T, \\ \phi(0,x) = \phi_0(x), & x \in \Omega, \\ \phi(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

with fourth-order p-Laplacian Kirchhoff operators,

$$L\phi = \partial_t\phi - \Delta\left((1 + \lambda\|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)-2}\Delta\phi\right) + \gamma\phi.$$

The existence, stability and uniqueness of solutions are mainly obtained using the Leray Schauder principle. Moreover, Li and Bi in [6] considered the two-dimensional case in [5]. The conditions to ensure the existence of weak solutions are given in [7]. The existence results of weak solutions of variational inequalities can also be found in [8–11]. For the uniqueness of weak solutions of variational inequalities, refer to [9–12]. In addition, the results about the stability of weak solutions on initial values are also worth studying [13]. At present, there are few studies on the regularity of solutions of variation-inequality problems.

In this paper, we study the regularity and blow-up of weak solutions of variational inequalities (1.1). First, we assume that $f \geq 0$ and $h \geq 0$ for any $(x,t) \in \Omega_T$, $u_0 \in H_0^1(\Omega)$, $u^m \in L(0,T;H^2(\Omega))$ and $f \in L(0,T;L^2(\Omega))$. The weak solution equation is transformed into a difference equation by using the difference operator. Under the property of the difference operator, the $L(0,T;H^3(\Omega'))$ estimation inequality is obtained, which is the regularity of the weak solution. Second, we consider the blowup of weak solutions with the restriction that $f < 0$ for any $(x,t) \in \Omega_T$, h is a negative constant and $\alpha > 1$. After defining the energy function $E(t)$, it is proved that the weak solution will blow up in finite time by using Hölder inequality and differential transformation techniques.

2. Statement of the problem and its background

We first give an application of variational inequality in investment and consumption theory. In order to fit optimally the random demand of a good, a social planner needs to control its capacity production at time interval $[0,T]$. Let $\{D_t, t \in [0,T]\}$ be the random demand of a good

$$\mathrm{d}D_t = \mu_1 D_t \mathrm{d}t + \sigma_1 D_t \mathrm{d}w_t,\ D_0 = d,$$

where μ_1 and σ_1 are the expected rate of return and volatility respectively. Further, process $\{C_t, t \in [0,T]\}$ is the production capacity of the firm,

$$\mathrm{d}C_t = \mu_2 C_t \mathrm{d}t + \sigma_2 C_t \mathrm{d}w_t,\ C_0 = c.$$

Here μ_2 and σ_2 are the expected rate of return and volatility of the production process.

A planner is able to create a production plan C_t at any point in time between 0 and T to equilibrate uncertain demand D_t. As such, the planner can use a value function V to determine an optimal policy that minimizes the anticipated total cost within a finite timeframe. According to literature [1–3], the value function V satisfies

$$\begin{cases} \partial_c V \geq -q, & c > 0, d > 0, t \in (0,T), \\ L_1 V + g(c,d) \geq 0, & c > 0, d > 0, t \in (0,T), \\ (\partial_c V + q)(L_1 V + g(c,d)) = 0, & c > 0, d > 0, t \in (0,T), \\ V(c,d,T) = 0, & c > 0, d > 0, \end{cases} \tag{2.1}$$

where $L_1 V$ is a two-dimensional parabolic operator with constant parameters,

$$L_1 V = \partial_t V + \frac{1}{2}\sigma_1^2 c^2 \partial_{cc} V + \frac{1}{2}\sigma_2^2 d^2 \partial_{dd} V + \mu_1 c \partial_c V + \mu_2 d \partial_d V - rV.$$

Here, r represents the risk-free interest rate of the bank. The cost function,

$$g(c,d) = \begin{cases} p_1(c-d), & c \geq d, \\ p_2(d-c), & c < d, \end{cases}$$

is designed to represent the potential expense associated with storing goods, where p_1 and p_2 indicate the per unit costs of having excessive supply and demand, respectively.

If transportation loss and storage costs are taken into account, sigma is dependent on $\partial_c V$, $\partial_d V$, and V itself. This is illustrated by the well-known Leland model, which expresses σ_1 and σ_2 as

$$\sigma_i = \sigma_{0,i}\left(1 - Le\sqrt{\frac{\pi}{2}}\operatorname{sign}(\partial_{SS} V^m)\right), \tag{2.2}$$

where $m > 0$, $i = 1, 2$, $\sigma_{0,1}$ and $\sigma_{0,2}$ represent the original volatility of C_t and D_t, respectively, and Le is the Leland number.

When studying variation-inequality problems, this paper considers cases that are more complex than the example given in Eq 2.2. To do this, we introduce a set of maximal monotone maps that have been defined in previous works [1–3,5,6],

$$G = \{\xi | \xi = 0 \text{ if } u - u_0 > 0;\ \xi \in [-M_0, 0] \text{ if } x = 0\}, \tag{2.3}$$

where M_0 is a positive constant.

Definition 2.1. *A pair (u, ξ) is said to be a generalized solution of variation-inequality (1.1), if (u, ξ) satisfies $u \in L^\infty(0, T, H^1(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$ and $\xi \in G$* for any $(x, t) \in \Omega_T$,
(a) $u(x,t) \geq u_0(x)$, $u(x,0) = u_0(x)$ for any $(x,t) \in \Omega_T$,
(b) for every test-function $\varphi \in C^1(\bar{\Omega}_T)$, there admits the equality

$$\int\int_{\Omega_T} \partial_t u \cdot \varphi + \Delta u^m \Delta\varphi \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} h u^\alpha \varphi \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} f\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t.$$

By a standard energy method, the following existence theorem can be found in [5,6,14,15].

Theorem 2.2. *Assume that* $u_0 \in H_0^1(\Omega)$, $f, h \in L^\infty(0,T;L^2(\Omega))$, $f(x,t) \geq 0$ *and* $h(x,t) \geq 0$ *for any* $(x,t) \in \Omega_T$. *If* $\alpha > 0, m > 0$, *then (1) admits a solution u within the class of Definition 2.1.*

Note that from (1), it follows that $Lu \leq 0$ and $L0 = 0$ for any $(x,t) \in \Omega_T$. Additionally, we have $u_0 \geq 0$ in Ω, and $u = 0$ on $\partial\Omega_T$. Therefore, by the extremum principle [16], we have

$$u \geq 0 \text{ in } \Omega_T.$$

One purpose of this paper is the regularity of weak solutions, so we give some functions and their valuable results. Define the difference operator,

$$\Delta_{\Delta x}^i u(x,t) = \frac{u(x + \Delta x e_i, t) - u(x,t)}{\Delta x},$$

where e_i is the unit vector in the direction x_i. According to literature [14], the difference operator has the following results.

Lemma 2.3. *(1) Let* $\Delta_{\Delta x}^{i*} = -\Delta_{-\Delta x}^i$ *be the conjugate operator of* $\Delta_{\Delta x}^i$, *then we have*

$$\int_{R_n} f(x)\Delta_{\Delta x}^i g(x)\mathrm{d}x = -\int_{R_n} g(x)\Delta_{-\Delta x}^i f(x)\mathrm{d}x,$$

in other words, $\int_{R_n} f(x)\Delta_{\Delta x}^i g(x)\mathrm{d}x = \int_{R_n} g(x)\Delta_{\Delta x}^{i*} f(x)\mathrm{d}x$.
(2) Operator $\Delta_{\Delta x}^i$ *has the following commutative results*

$$D_j\Delta_{\Delta x}^i f(x) = \Delta_{\Delta x}^i D_j f(x),\ j = 1, 2, \cdots, n.$$

(3) If $u \in W^{1,p}(\Omega)$, *for any* $\Omega' \subset\subset \Omega$,

$$\|\Delta_{\Delta x}^i u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega')},\ \|\Delta_{\Delta x}^{i*} u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega')}.$$

(4) Assuming $u \in L^p(\Omega)$ *with* $p \geq 2$, *if h is sufficiently small such that* $\int_\Omega |\Delta_h^i u|^p \mathrm{d}x \leq C$, *where C is independent of h, then we have*

$$\int_\Omega |D_i u|^p \mathrm{d}x \leq C.$$

3. Regularity of solution

This section considers the regularity of weak solutions. Select the sub-region $\Omega' \subset\subset \Omega$, define $d = \mathrm{dist}(\Omega', \Omega)$ and let $\eta \in C_0^\infty(\Omega)$ be the cutoff factor of Ω' in Ω , such that

$$0 \leq \eta \leq 1,\ \eta = 1 \text{ in } \Omega',\ \mathrm{dist}(\mathrm{supp}\eta, \Omega) \geq 2d.$$

Let $\Delta x < d$, define $\varphi = \Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)$, and note that $u \in H_0^1(\Omega)$, then substituting $\varphi = \Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)$ into the weak solution equation gives

$$\begin{aligned}
&\int\int_{\Omega'_T} \partial_t u \cdot \Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u) + \Delta u^m \Delta\Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)\mathrm{d}x\mathrm{d}t + \int\int_{\Omega'_T} hu^\alpha \Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)\mathrm{d}x\mathrm{d}t \\
&+ \int\int_{\Omega'_T} f\Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)\mathrm{d}x\mathrm{d}t \qquad (3.1)\\
&= \int\int_{\Omega'_T} \xi \cdot \Delta_{\Delta x}^{i*}(\eta^2\Delta_{\Delta x}^i u)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now we pay attention to $\int_{\Omega'} \partial_t u \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x$. Using differential transformation techniques,

$$\begin{aligned}
&\int\int_{\Omega'_T} \partial_t u \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \\
&= \int\int_{\Omega'_T} \partial_t(\Delta^i_{\Delta x} u)\eta^2 \Delta^i_{\Delta x} u\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int\int_{\Omega'_T} \partial_t((\Delta^i_{\Delta x} u)^2 \eta^2)\mathrm{d}x\mathrm{d}t \\
&= \int_{\Omega'} (\Delta^i_{\Delta x} u(x,T))^2 \eta^2 \mathrm{d}x - \int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x.
\end{aligned} \tag{3.2}$$

Substitute (3.2) into (3.1), so that

$$\begin{aligned}
&\int\int_{\Omega'_T} \Delta\Delta^i_{\Delta x} u^m \Delta(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t + \int\int_{\Omega'_T} h u^\alpha \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t + \int\int_{\Omega'_T} f \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \\
\leq &\int\int_{\Omega'_T} \xi \cdot \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t + \int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x.
\end{aligned} \tag{3.3}$$

Here we use the commutativity of conjugate operator $\Delta^{i*}_{\Delta x}$ in $\int\int_{\Omega'_T} \Delta u^m \Delta\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t$. Further using the differential technique to expand $\Delta\Delta^i_h u^m \Delta(\eta^2 \Delta^i_h u)$, one can get

$$\begin{aligned}
&\int\int_{\Omega'_T} \Delta\Delta^i_{\Delta x} u^m \Delta(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \\
&= 2\int_0^T \int_{\Omega'} \eta\nabla\eta \cdot (\Delta\Delta^i_{\Delta x} u^m)(\Delta^i_{\Delta x} u^m)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.4}$$

Combining formula (3.3) and (3.4), it is easy to verify that

$$\begin{aligned}
&\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \\
= &\int_0^t \int_{\Omega'} \xi \cdot \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t - \int\int_{\Omega'_T} h u^\alpha \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t - \int\int_{\Omega'_T} f \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \\
&+ \int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x - 2\int_0^T \int_{\Omega'} \eta\nabla\eta \cdot (\Delta\Delta^i_{\Delta x} u^m)(\Delta^i_{\Delta x} u^m)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.5}$$

By Hölder and Young inequalities,

$$\int_0^T \int_{\Omega'} f \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int_0^T \int_{\Omega'} f^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \int_{\Omega'} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t, \tag{3.6}$$

$$\begin{aligned}
&2\int_0^T \int_{\Omega'} \eta\nabla\eta \cdot (\Delta\Delta^i_{\Delta x} u^m)(\Delta^i_{\Delta x} u^m)\mathrm{d}x\mathrm{d}t \\
\leq &2\int_0^T \int_{\Omega'} |\nabla\eta|^2 (\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.7}$$

$$\int\int_{\Omega'_T} h u^\alpha \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \leq \frac{1}{2}\int\int_{\Omega'_T} h^2 u^{2\alpha} \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int\int_{\Omega'_T} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Applying Hölder and Young inequalities again and combining with (3.1),

$$\int_0^t \int_{\Omega'} \xi \cdot \Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)\mathrm{d}x\mathrm{d}t \leq \frac{1}{2} M_0^2 T|\Omega| + \frac{1}{2}\int\int_{\Omega'_T} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t. \tag{3.9}$$

Substituting (3.6)–(3.9) to (3.5), it is clear to verify

$$\begin{aligned}
&\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \\
= &M_0^2 T|\Omega| + \frac{1}{2}\int\int_{\Omega'_T} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{2}\int\int_{\Omega'_T} h^2 u^{2\alpha} \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int\int_{\Omega'_T} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{2}\int_0^T \int_{\Omega'} f^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \int_{\Omega'} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t \\
&+ \int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x + 2\int_0^T \int_{\Omega'} |\nabla\eta|^2 (\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Rearranging the above formula, such that

$$\begin{aligned}
&\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \\
\leq\ &2M_0^2 T|\Omega| + \int\int_{\Omega'_T} h^2 u^{2\alpha} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega'} f^2 \mathrm{d}x\mathrm{d}t + \int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x \\
&+ 4\int_0^T \int_{\Omega'} |\nabla\eta|^2 (\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t + 3\int_0^T \int_{\Omega'} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Using the relationship between difference and partial derivative,

$$\int_0^T \int_{\Omega'} |\nabla\eta|^2 (\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \leq C\int_0^T \int_{\Omega'} (\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \leq C\int_0^T \int_{\Omega'} (\nabla u^m)^2 \mathrm{d}x\mathrm{d}t,$$

$$\int_0^T \int_{\Omega'} [\Delta^{i*}_{\Delta x}(\eta^2 \Delta^i_{\Delta x} u)]^2 \mathrm{d}x\mathrm{d}t \leq C\int_0^T \int_{\Omega'} (\Delta u)^2 \mathrm{d}x\mathrm{d}t,$$

$$\int_{\Omega'} (\Delta^i_{\Delta x} u_0)^2 \eta^2 \mathrm{d}x\mathrm{d}t \leq \int_{\Omega'} (\nabla u_0)^2 \mathrm{d}x\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
&\int_0^T \int_{\Omega'} \eta^2 (\Delta\Delta^i_{\Delta x} u^m)^2 \mathrm{d}x\mathrm{d}t \\
\leq\ &C(M_0, T, |\Omega|, h) + C\int\int_{\Omega'_T} u^{2\alpha} \mathrm{d}x\mathrm{d}t + 4\int_0^T \int_{\Omega'} f^2 \mathrm{d}x\mathrm{d}t + C\int_{\Omega'} (\nabla u_0)^2 \mathrm{d}x\mathrm{d}t \\
&+ C\int_0^T \int_{\Omega'} (\nabla u^m)^2 \mathrm{d}x\mathrm{d}t + C\int_0^T \int_{\Omega'} (\Delta u)^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Recall that sub-area Ω' belongs to Ω. It follows from (4) of Lemma 2.3 that

$$\|u\|^2_{L(0,T;H^3(\Omega'))} \le C\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L(0,T;L^2(\Omega))} + \|u\|^{2\alpha}_{L(0,T;L^{2\alpha}(\Omega))} + \|u^m\|^2_{L(0,T;H^2(\Omega))}\right). \tag{3.10}$$

If $\alpha \le 1$, using Hölder inequality gives

$$\|u\|^2_{L(0,T;H^3(\Omega'))} \le C\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L(0,T;L^2(\Omega))} + \|u^m\|^2_{L(0,T;H^2(\Omega))}\right). \tag{3.11}$$

Theorem 3.1. *Assume $f \ge 0$ and $h \ge 0$ for any $(x,t) \in \Omega_T$. If $u_0 \in H^1(\Omega)$, $u^m \in L(0,T;H^2(\Omega))$ and $f \in L(0,T;L^2(\Omega))$, then for any sub-area $\Omega' \subset\subset \Omega$, there holds $u \in L(0,T;H^3(\Omega'))$, and estimate (3.10). Moreover, if $\alpha \le 1$, (3.11) follows.*

Using the finite cover principle and the flattening operator [14], we have the following global regularity result.

Theorem 3.2. *Let $f \ge 0$ and $h \ge 0$ for any $(x,t) \in \Omega_T$. If $u_0 \in H^1(\Omega)$, $u^m \in L(0,T;H^2(\Omega))$ and $f \in L(0,T;L^2(\Omega))$, then*

$$\|u\|^2_{L(0,T;H^3(\Omega))} \le C\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L(0,T;L^2(\Omega))} + \|u\|^{2\alpha}_{L(0,T;L^{2\alpha}(\Omega))} + \|u^m\|^2_{L(0,T;H^2(\Omega))}\right).$$

If $\alpha \le 1$,we have

$$\|u\|^2_{L(0,T;H^3(\Omega'))} \le C\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L(0,T;L^2(\Omega))} + \|u^m\|^2_{L(0,T;H^2(\Omega))}\right).$$

4. Blowup of solution

This section discusses the blow-up properties of weak solutions to the variation-inequality problem (1.1), under the constraints that $\alpha \le 1$, $f < 0$, and $h < 0$. As $u > 0$ in Ω_T, we define the function

$$E(t) = \int_\Omega u(x,t)\mathrm{d}x,$$

for this purpose. Choosing the test function $\varphi = \frac{u^m}{u^m+\varepsilon}$ in weak equation, we have

$$\int_\Omega \partial_t u \cdot \frac{u^m}{u^m+\varepsilon} + \varepsilon\frac{|\Delta u^m|^2}{u^m+\varepsilon}\mathrm{d}x + \int_\Omega hu^\alpha \frac{u^m}{u^m+\varepsilon}\mathrm{d}x + \int_\Omega f\frac{u^m}{u^m+\varepsilon}\mathrm{d}x = \int_\Omega \xi \cdot \frac{u^m}{u^m+\varepsilon}\mathrm{d}x. \tag{4.1}$$

It follows from $u \in L^\infty(0,T,H^2(\Omega))$, $\partial_t u \in L^2(\Omega_T)$ and $f \in L(0,T;L^2(\Omega))$ that

$$\int_\Omega \partial_t u \cdot \frac{u^m}{u^m+\varepsilon}\mathrm{d}x \to \int_\Omega \partial_t u\mathrm{d}x \text{ as } \varepsilon \to 0, \tag{4.2}$$

$$\int_\Omega \varepsilon\frac{|\Delta u^m|^2}{u^m+\varepsilon}\mathrm{d}x \to 0 \text{ as } \varepsilon \to 0, \tag{4.3}$$

$$\int_\Omega hu^\alpha \frac{u^m}{u^m+\varepsilon}\mathrm{d}x \to \int_\Omega hu^\alpha\mathrm{d}x \text{ as } \varepsilon \to 0. \tag{4.4}$$

Recall that $u^m \geq 0$ and $\xi \geq 0$ for any $(x,t) \in \Omega_T$. In this section we consider the case that $f \leq 0$ for any $(x,t) \in \Omega_T$ and h is a negative constant, so

$$\int_\Omega \xi \cdot \frac{u^m}{u^m+\varepsilon} \mathrm{d}x \geq 0, \quad \int_\Omega f \frac{u^m}{u^m+\varepsilon} \mathrm{d}x \leq 0. \tag{4.5}$$

Substituting (4.2)–(4.5) to (4.1), one can have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \geq -h \int_\Omega u^\alpha \mathrm{d}x. \tag{4.6}$$

Using Hölder inequality (here, we used the conditions $\alpha > 1$ and $h < 0$),

$$\int_\Omega u \mathrm{d}x \leq \left(\int_\Omega u^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} |\Omega|^{\frac{\alpha-1}{\alpha}} \Leftrightarrow \int_\Omega u^\alpha \mathrm{d}x \geq |\Omega|^{1-\alpha} E(t)^\alpha, \tag{4.7}$$

such that combining (4.6) and (4.7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \geq -h|\Omega|^{1-\alpha} E(t)^\alpha. \tag{4.8}$$

Applying variable separation techniques to above equation, and then integrating from 0 to T gives

$$\frac{1}{1-\alpha} E(t)^{1-\alpha} - \frac{1}{1-\alpha} E(0)^{1-\alpha} \geq -h|\Omega|^{1-\alpha} t. \tag{4.9}$$

Rearranging (4.9), one can get

$$E(t) \geq [E(0)^{1-\alpha} - (1-\alpha)h|\Omega|^{1-\alpha} t]^{\frac{1}{1-\alpha}}.$$

Note that $\alpha < 1$ and $h < 0$. As t approaches $\frac{1}{(\alpha-1)} h^{-1} |\Omega|^{\alpha-1} E(0)^{1-\alpha}$, $E(t)$ tends to infinity. This indicates that the weak solution of the equation will experience a finite-time blow up at T^*, and T^* satisfies

$$T^* \leq \frac{1}{(\alpha-1)} h^{-1} |\Omega|^{\alpha-1} E(0)^{1-\alpha}. \tag{4.10}$$

Further, we analyze the rate of Blowup. Integrating the value of (4.8) from t to T^* gives

$$\int_t^{T^*} \frac{1}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} E(t)^{1-\alpha} \geq -h|\Omega|^{1-\alpha}(T^* - t), \tag{4.11}$$

which (note that $E(T^*)^{1-\alpha} = 0$) implies that

$$\frac{1}{\alpha-1} E(t)^{1-\alpha} \geq |h| \cdot |\Omega|^{1-\alpha}(T^* - t). \tag{4.12}$$

Rearranging (4.12), it is easy to see that

$$E(t)^{1-\alpha} \geq (\alpha-1)|h| \cdot |\Omega|^{1-\alpha}(T^* - t). \tag{4.13}$$

Theorem 4.1. *Assume that $f < 0$ for any $(x,t) \in \Omega_T$ and h is a negative constant. If $\alpha > 1$, then the weak solution (u, ξ) of variation-inequality problem (1) at time T^* in which T^* is bounded by (4.13). Moreover, the rate of blowup is given by*

$$E(t) \leq C(T^* - t)^{\frac{1}{1-\alpha}},$$

where $C = (\alpha-1)^{\frac{1}{1-\alpha}} |h|^{\frac{1}{1-\alpha}} |\Omega|$.

5. Conclusions

This article investigates the global regularity and blow-up of weak solutions for the following variational inequality (1.1) with the non-Newtonian polytropic operator

$$Lu = \partial_t u - \Delta^2 u^m + hu^\alpha + f,\ m > 0.$$

Firstly, this article analyzes the $H^3(\Omega)$ regularity of weak solutions for variational inequality (1.1). We assume that $f \geq 0$ and $h \geq 0$ for any $(x,t) \in \Omega_T$, $u_0 \in H_0^1(\Omega)$, $u^m \in L(0,T;H^2(\Omega))$ and $f \in L(0,T;L^2(\Omega))$. Since using $\partial_{xx}u$ as test function does not comply with the definition of weak solution, this article introduces spatial difference operator and constructs test functions with it to approximate the second-order spatial gradient of u. Additionally, with the aid of spatial cutoff factor, Hölder's inequality and Young's inequality, two $H^3(\Omega)$ regularity estimates for weak solutions of variational inequality (1.1) are obtained. The specific results can be seen in Theorem 3.1 and Theorem 3.2.

Secondly, we analyze the blow-up properties of weak solutions for variational inequality (1.1) within a finite time under the assumption that $f < 0$ for any $(x,t) \in \Omega_T$, h is a negative constant and $\alpha \leq 1$. Considering that u is non-negative, we define an energy function

$$E(t) = \int_\Omega u(x,t)\mathrm{d}x,$$

and obtain the differential inequality of the energy function, as shown in (4.8). By using differential transform techniques, we obtain the lower bound of the blow-up point and the blow-up rate. The results are presented in Theorem 4.1.

Currently, there are still some limitations in this article: (1) Equations (4.6) and (4.10) can only hold when h is a non-negative parameter; (2) Equations (4.10)–(4.13) can only hold when $\alpha \leq 1$. In future research, we will attempt to overcome these limitations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that he has no conflict of interest.

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