Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 18163–18173.
DOI: 10.3934/math.2023922
Received: 23 February 2023
Revised: 14 May 2023
Accepted: 16 May 2023
Published: 26 May 2023

Research article

Blow-up of solutions for nonlinear wave equations on locally finite graphs

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Abstract: Let $G = (V, E)$ be a local finite connected weighted graph, Ω be a finite subset of V satisfying $\Omega^\circ \neq \emptyset$. In this paper, we study the nonexistence of the nonlinear wave equation

$$\partial_t^2 u = \Delta u + f(u)$$

on G. Under the appropriate conditions of initial values and nonlinear term, we prove that the solution for nonlinear wave equation blows up in a finite time. Furthermore, a numerical simulation is given to verify our results.

Keywords: nonlinear wave equation; blow up; locally finite graph
Mathematics Subject Classification: 35L05, 35R02, 58J45

1. Introduction

As is known to us, differential equations have been extensively studied and achieved rich results, including ordinary differential equation, partial differential equation, stochastic differential equation, etc. Among them, wave equations and their related applications have attracted many scholars because of its unique properties and wide applications [1–7]. For a Euclidean space, the nonlinear wave equation reads

$$\partial_t^2 u(t,x) = \Delta u(t,x) + f(u(t,x)), \quad (t,x) \in (0,+\infty) \times \Omega$$

where Ω is bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ and f is a nonlinear function. Kawarada [8] gave the sufficient conditions of nonexistence global solutions with the homogenous Dirichlet boundary and the initial value

$$u(0,x) = a(x), \quad \partial_t u(0,x) = b(x), \quad x \in \Omega,$$

where $a(x)$ and $b(x)$ are sufficiently smooth in Ω with $a(x)|_{\partial\Omega} = b(x)|_{\partial\Omega} = 0$. Matsuya [9] studied a discretization of a nonlinear wave equation with $f(u) = |u|^p$ with $p > 1$ and obtained a blow-up theorem.

Many structures in our real life can be represented by a connected graph whose vertices represent nodes, and whose edges represent their links, such as the brain, organizations, internet, and so on. In recent years, many authors paid attention to study the behavior of solutions of various nonlinear equations on graphs, see for example [10–13] and references therein. In particular, Grigor'yan et al. [14–16] investigated the discrete version of nonlinear elliptic equations, such as Yamabe equation and Kazdan–Warner equation, and obtained the sufficient conditions for the existence of solutions. Lin and Wu [17, 18] studied nonlinear parabolic equations including semilinear heat equation on graphs, and explored the existence and nonexistence results of global solution. However, hyperbolic equations, especially wave equations, have not attracted extensive attention on graphs. Friedman and Tillich [19] studied the linear damped wave equation whose Laplacian is based on the edge. On finite graphs, Ma [20] gave explicit expressions for solutions to the linear wave equation, and Lin and Xie [21, 22] proved the existence and uniqueness of the solution to nonlinear wave equations with and without a damping term. On infinite graphs, Han and Hua [23] obtain a sharp uniqueness class for the solutions of wave equations.

In this paper, we investigate the blow-up phenomenon of nonlinear wave equations on graphs, which is to extend the result of Kawarada [8] to graph.

The rest of the paper is organized as follows. We recall notations from graph theory and state our main results in Section 2. In Section 3, we prove the blow-up theorem. In Section 4, we give an example to explain our results. Meanwhile, we provide a numerical experiment to demonstrate the example.

2. Notations and main results

Let $G = (V, E)$ be a connected graph without loops and multiply edges, where V denotes the set of vertex and $E \subset V \times V$ the set of edge. We write $x \sim y$ if $xy \in E$. In this paper, we allow measures and weights on G. Let $\mu : V \to \mathbb{R}^+$ be a positive measure on V and $\omega : V \times V \to [0, \infty)$ a non-negative weight, which satisfies $\omega_{xy} = \omega_{yx}$ and $\omega_{xy} > 0$ if and only if $x \sim y$. Throughout this paper, we deal with locally finite graph, that is, the number of neighbour of each vertex is finite. Let $\Omega \subseteq V$, the boundary and the interior of Ω are defined by

$$\partial\Omega = \{x \in \Omega : \exists y \in V \setminus \Omega, y \sim x\} \quad \text{and} \quad \Omega^\circ = \Omega \setminus \partial\Omega$$

respectively.

We denote $C(V)$ by the set of real functions on V. For any function $f \in C(V)$, the graph Laplace operator Δ of f is defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

Given a finite subset $\Omega \subseteq V$, denote by $C(\Omega)$ the set of functions $\Omega \to \mathbb{R}$. The Dirichlet Laplacian Δ_Ω is defined as follows: First extend f to the whole V by setting $f = 0$ outside Ω and then set

$$\Delta_\Omega f = (\Delta f)_\Omega.$$

We recall the smallest non-zero eigenvalue λ_0 of $-\Delta_\Omega$ and the corresponding eigenfunction ϕ_0 in Ω,

$$\begin{cases} -\Delta_\Omega \phi_0 = \lambda_0 \phi_0 & \text{in } \Omega^\circ, \\ \phi_0|_{\partial\Omega} = 0. \end{cases}$$

We assume ϕ_0 is normalized as $\sum_{x\in\Omega} \phi_0(x)\mu(x) = 1$. In particular, $0 < \lambda_0 \leq 1$, see Theorem 4.3 in [24].

In this paper, we deal with the following initial-boundary value problem of nonlinear wave equations on G. Given a non-empty subset $\Omega \subseteq V$,

$$\begin{cases} \partial_t^2 u(t,x) = \Delta_\Omega u(t,x) + f(u(t,x)), & (t,x) \in (0,+\infty)\times\Omega^\circ, \\ u(0,x) = a(x), & x\in\Omega, \\ \partial_t u(0,x) = b(x), & x\in\Omega, \\ u(t,x) = 0, & (t,x)\in[0,+\infty)\times\partial\Omega, \end{cases} \tag{2.1}$$

where f is a nonlinear function, and $a(x)$ and $b(x)$ are sufficiently smooth in Ω with $a(x)|_{\partial\Omega} = b(x)|_{\partial\Omega} = 0$.

Definition 1. A function $u = u(t,x)$ satisfying (2.1) in $[0,+\infty)\times\Omega$ is called a global solution of (2.1) if u is bounded and twice continuously differentiable with respect to t in $[0,+\infty)\times\Omega$. Moreover, we say that the solution of (2.1) blows up in a finite time T, if there exists $x\in\Omega$ such that

$$|u(t,x)| \to +\infty \quad \text{as} \quad t \to T^-.$$

In this paper, we establish the blow-up conditions for solutions of (2.1) on locally finite graphs. For convenience, we use the following notations:

$$\alpha = \sum_{x\in\Omega} a(x)\phi_0(x)\mu(x),$$

and

$$\beta = \sum_{x\in\Omega} b(x)\phi_0(x)\mu(x).$$

Our main results are stated as follows.

Theorem 1. *Let $G = (V,E)$ be a locally finite graph. Suppose that the nonlinear perturbation f and the initial data a, b in* (2.1) *satisfy* (C.1)–(C.5)*. Then the solution of* (2.1) *blows up in a finite time. Specifically,*

(i) when $\beta > 0$, the nonlinear function f and the initial data a, b are required to satisfy (C.1) *and* (C.2)*.*

(ii) when $\beta = 0$, in addition to (C.1) *and* (C.2)*, the nonlinear function f and the initial data a, b need to satisfy* (C.3)*.*

(iii) when $\beta < 0$, the nonlinear function f and the initial data a, b are supposed to satisfy (C.1), (C.2)′, (C.4) *and* (C.5)*.*

Condition (C.1)**.** The non-negative function f is sufficiently smooth and convex in $\mathbb{R}$.

Let

$$F(\sigma) = \beta^2 + 2\int_\alpha^\sigma \{-\lambda_0 s + f(s)\}ds,$$

and

$$F_0(\sigma) = 2\int_\alpha^\sigma \{-\lambda_0 s + f(s)\}ds.$$

Condition $(C.2)$**.** The following inequalities hold:

$$F(\sigma) > 0 \quad \text{for } \sigma > \alpha, \tag{2.2}$$

and

$$\int_{\alpha}^{\infty} F(\sigma)^{-\frac{1}{2}} d\sigma = T < +\infty. \tag{2.3}$$

Condition $(C.2)'$**.** The inequalities (2.2) and (2.3) hold for F_0 in the place of F.
Condition $(C.3)$**.** The following inequality holds:

$$-\lambda_0 \alpha + f(\alpha) > 0. \tag{2.4}$$

Condition $(C.4)$**.** The equation

$$F(\sigma) = 0, \tag{2.5}$$

has at least one root with respect to $\sigma < \alpha$.
Condition $(C.5)$**.** The following inequality

$$-\lambda_0 s + f(s) > 0$$

holds for $s \in [\sigma_m, \alpha]$, where σ_m is the largest root ($< \alpha$) of the Eq (2.5).

Remark 1. Smoothness of f ensures the existence and uniqueness of solution to the nonlinear wave equation, see [21] in details. If f is non-positive and concave in $\mathbb{R}$, we shall get the same conclusion under the same conditions as above with f, a and b.

3. Proof of Theorem 1

The following lemma is one of the main tools when proving Theorem 1.

Lemma 1. For any $f, g \in C(\Omega)$ and $f = g = 0$ on $\partial\Omega$, we have

$$\sum_{x\in\Omega} \Delta_\Omega f(x) g(x) \mu(x) = \sum_{x\in\Omega} f(x) \Delta_\Omega g(x) \mu(x). \tag{3.1}$$

Proof. For any function $h : V \times V \to \mathbb{R}$,

$$\sum_{x,y\in\Omega} \omega_{xy} h(x,y) = \sum_{x,y\in\Omega} \omega_{xy} h(y,x),$$

which follows from the symmetric weight ω_{xy}. Then we have

$$\begin{aligned}
\sum_{x\in\Omega} \Delta_\Omega f(x) g(x) \mu(x) &= \sum_{x\in\Omega} \sum_{y\sim x} \omega_{xy}(f(y) - f(x)) g(x) \\
&= \sum_{x\in\Omega} \sum_{y\in\Omega} \omega_{xy}(f(y) - f(x)) g(x) + \sum_{x\in\Omega} \sum_{y\in\partial\Omega} \omega_{xy}(f(y) - f(x)) g(x) \\
&= \sum_{x,y\in\Omega} \omega_{xy} f(x) g(y) - \sum_{x,y\in\Omega} \omega_{xy} f(x) g(x) + \sum_{x\in\Omega} \sum_{y\in\partial\Omega} \omega_{xy}(f(y) - f(x)) g(x) \\
&= \sum_{x,y\in\Omega} \omega_{xy}(g(y) - g(x)) f(x) + \sum_{x\in\Omega} \sum_{y\in\partial\Omega} \omega_{xy}(g(y) - g(x)) f(x) \\
&= \sum_{x\in\Omega} f(x) \Delta_\Omega g(x) \mu(x).
\end{aligned}$$

In the forth equality, we utilize $f(y) = g(y) = 0$ for any $y \in \partial\Omega$. $\square$

Proof of Theorem 1. Suppose u is a global solution of (2.1). Let

$$J = J(t) = \sum_{x\in\Omega} u(t,x)\phi_0(x)\mu(x).$$

It is easy to see that

$$J(0) = \sum_{x\in\Omega} u(0,x)\phi_0(x)\mu(x) = \sum_{x\in\Omega} a(x)\phi_0(x)\mu(x) = \alpha.$$

$$\frac{dJ}{dt}(0) = \sum_{x\in\Omega} \frac{\partial_t u(t,x)}{dt}\Big|_{t=0}\phi_0(x)\mu(x) = \sum_{x\in\Omega} b(x)\phi_0(x)\mu(x) = \beta.$$

We *claim* that for all $t > 0$,

$$\frac{d^2J}{dt^2}(t) + \lambda_0 J(t) \geq f(J(t)). \tag{3.2}$$

Indeed, by Lemma 1 and Jensen's inequality, we have for any $t > 0$,

$$\begin{aligned}
\frac{d^2J}{dt^2}(t) &= \sum_{x\in\Omega} \frac{d^2u(t,x)}{dt^2}\phi_0(x)\mu(x)\\
&= \sum_{x\in\Omega} (\Delta_\Omega u(t,x) + f(u(t,x)))\phi_0(x)\mu(x)\\
&= \sum_{x\in\Omega} \Delta_\Omega\phi_0(x)u(t,x)\mu(x) + \sum_{x\in\Omega} f(u(t,x))\phi_0(x)\mu(x)\\
&\geq -\lambda_0 J(t) + f(J(t)),
\end{aligned}$$

which ends the proof of the claim.

Case (i): $\beta > 0$. We first show that

$$\frac{dJ}{dt}(t) > 0, \quad t \in [0,\infty). \tag{3.3}$$

Suppose not, the equation

$$\frac{dJ}{dt}(t) = 0 \tag{3.4}$$

has at least one root for $t > 0$. Let t_1 be the smallest positive root of (3.4). Due to $\frac{dJ}{dt}(0) = \beta > 0$, it follows that $\frac{dJ}{dt}(t) > 0$ in $[0, t_1)$ by the continuity of $\frac{dJ}{dt}(t)$ in $[0, +\infty)$. For any $t \in [0, t_1)$, multiplying with $\frac{dJ}{ds}$ in the both sides of (3.2) and integrating from 0 to t with respect to s, we obtain

$$\int_0^t \frac{d^2J}{ds^2}\cdot\frac{dJ}{ds}ds \geq -\int_0^t \lambda_0 J\cdot\frac{dJ}{ds}ds + \int_0^t f(J)\cdot\frac{dJ}{ds}ds.$$

Rewrite the above inequality to get

$$\int_0^t \frac{dJ}{ds}d\left(\frac{dJ}{ds}\right) \geq -\int_\alpha^{J(t)} \lambda_0 J dJ + \int_\alpha^{J(t)} f(J)dJ,$$

which implies

$$\frac{dJ}{dt}(t) \geq \sqrt{F(J(t))}, \quad t \in [0, t_1).$$

By the continuity of $\frac{dJ}{dt}(t)$ and $\sqrt{F(J(t))}$ in $[0, +\infty)$, we thus get

$$\frac{dJ}{dt}(t_1) \geq \sqrt{F(J(t_1))}.$$

And, it is easy to see that $J(t_1) > J(0) = \alpha$ by $\frac{dJ}{dt}(t) > 0$ in $[0, t_1)$, which implies that $F(J(t_1)) > 0$ from (2.2) in Condition (C.2). Combining with the above inequality, we get

$$\frac{dJ}{dt}(t_1) > 0,$$

which contradicts that t_1 is a root of the Eq (3.4). Then (3.3) is proved, which also implies that for all $t \in [0, +\infty)$

$$\frac{dJ}{dt}(t) \geq \sqrt{F(J(t))} > 0,$$

by following the above computations. It can be rewritten to

$$\frac{dt}{dJ} \leq F(J(t))^{-\frac{1}{2}}.$$

Integrating the above inequality from α to $J(t)$, we conclude that for all $t \geq 0$,

$$\int_{\alpha}^{J(t)} F(\sigma)^{-\frac{1}{2}} d\sigma \geq t. \tag{3.5}$$

However, (2.3) in Condition $(C.2)$ implies the integral in (3.5) converges to a finite value T as $J \to +\infty$. Then $J = J(t)$ goes to infinite in some $t \leq T$. This contracts that u is a global solution of (2.1).

Case (ii): $\beta = 0$**.** The inequality (2.4) in Condition $(C.3)$ implies $\frac{dJ}{dt}(t) > 0$ for $t \in (0, \delta)$ with sufficiently small δ. Indeed, according to (3.2) and (2.4), we have $\frac{d^2J}{dt^2}(0) > 0$. It follows that $\frac{dJ}{dt}(t) > 0, t \in (0, \delta)$ due to $\frac{dJ}{dt}(0) = \beta = 0$. Thus, it is similar to the above arguments, we conclude that $J = J(t)$ becomes infinite in a finite time. Then u blows up in a finite time.

Case (iii): $\beta < 0$**.** According to the continuity of $\frac{dJ}{dt}(t)$, we can find some $t_0 > 0$ such that $\frac{dJ}{dt}(t) < 0$ and $\alpha > J(t) > \sigma_m$ in $[0, t_0)$. By following the computations similar to the Case (i), we thus get

$$-\sqrt{F(J(t))} \leq \frac{dJ}{dt}(t) < 0 \quad \text{in } [0, t_0). \tag{3.6}$$

According to whether σ_m can be reached when J is decreasing, it can be separated into three cases:

(a) $\frac{dJ}{dt}(t) < 0$ in $[0, +\infty)$, $J(t) \to \sigma_2, t \to +\infty$ with $\sigma_m \leq \sigma_2 < \alpha$.

(b) There is some $t_1 (t_0 \leq t_1 < +\infty)$ such that $\frac{dJ}{dt}(t) < 0$ in $[0, t_1)$, $\frac{dJ}{dt}(t_1) = 0$ and $J(t_1) = \sigma_1$ with $\sigma_m < \sigma_1 < \alpha$.

(c) $\frac{dJ}{dt}(t) < 0$ in $[0, t_1)$, $J(t_1) = \sigma_m$.

It is easy to see that (a) does not occur. Indeed, since $\sigma_2 \leq J(t) \leq \alpha$ and Condition $(C.5)$, we get

$$\frac{d^2J}{dt^2}(t) \geq -\lambda_0 J(t) + f(J(t)) > c > 0,$$

which contradicts $\frac{dJ}{dt}(t) < 0$ in $[0, +\infty)$. In (c), by virtue of (3.6) and the definition of σ_m, we conclude that

$$\frac{dJ}{dt}(t_1) = 0.$$

Thus for (b) and (c), we shall deal with the following initial-boundary value problem for $J = J(t)$:

$$\begin{cases} \frac{d^2J}{dt^2}(t) \geq -\lambda_0 J(t) + f(J(t)), \quad t > t_1; \\ J(t_1) = \sigma'; \\ \frac{dJ}{dt}(t_1) = 0, \end{cases} \tag{3.7}$$

where $\sigma_m \leq \sigma' < \alpha$. Similar as the case $\beta = 0$, we can solve the Eq (3.7) by virtue of Condition $(C.5)$. Indeed, by following the computations of the cases $\beta \geq 0$, the solution $J = J(t)(t > t_1)$ is given implicitly by

$$t - t_1 \leq \int_{\sigma'}^{J(t)} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma \tag{3.8}$$

for any $t \geq t_1$. Conditions $(C.5)$ and $(C.2)'$ imply the integral in (3.8) converges to a finite value as $J(t) \to +\infty$. In fact,

$$\begin{aligned} \int_{\sigma'}^{+\infty} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma = & \int_{\sigma'}^{\alpha} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma \\ & + \int_{\alpha}^{+\infty} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma. \end{aligned} \tag{3.9}$$

By Conditions $(C.5)$ and $(C.2)'$, we obtain

$$\begin{aligned} \int_{\alpha}^{+\infty} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma &= \int_{\alpha}^{+\infty} [2\int_{\sigma'}^{\alpha} \{-\lambda_0 s + f(s)\}ds + F_0(\sigma)]^{-\frac{1}{2}} d\sigma \\ &< \int_{\alpha}^{+\infty} F_0(\sigma)^{-\frac{1}{2}} d\sigma \\ &< +\infty, \end{aligned} \tag{3.10}$$

and

$$\int_{\sigma'}^{\alpha} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma < +\infty. \tag{3.11}$$

From (3.9)–(3.11), we thus get

$$\int_{\sigma'}^{+\infty} [2\int_{\sigma'}^{\sigma} \{-\lambda_0 s + f(s)\}ds]^{-\frac{1}{2}} d\sigma < +\infty.$$

From (3.8), $J = J(t)$ goes to infinity in a finite time. This contradicts that u is a global solution of (2.1). □

4. A numerical simulation

In this section, we take the case of $f(u) = e^u$ in (2.1). By Theorem 1, we assert that the solution of (2.1) blows up with the initial conditions

$$a(x) = 0, \tag{4.1}$$

$$b(x) = \frac{\sqrt{2}\phi_0(x)}{||\phi_0||_{\ell^2}^2}. \tag{4.2}$$

Indeed, it is obvious that Condition $(C.1)$ is satisfied. The initial conditions (4.1) and (4.2) imply that $\alpha = 0$ and $\beta = \sqrt{2}$. Since $0 < \lambda_0 \leq 1$, we have $-\lambda_0 s + e^s > 0$ for $\sigma \in [0, +\infty)$. It follows that

$$F(\sigma) = 2 + 2\int_0^\sigma \{-\lambda_0 s + e^s\} ds > 0.$$

For all $\sigma \in [L, +\infty)$ with a sufficiently large constant $L(> 0)$, we have

$$2e^\sigma - \lambda_0 \sigma^2 > e^\sigma,$$

which implies $F(\sigma) > e^\sigma$. We thus get

$$\int_L^{+\infty} F(\sigma)^{-\frac{1}{2}} d\sigma < \int_L^{+\infty} e^{-\frac{\sigma}{2}} d\sigma = 2e^{-\frac{L}{2}} < +\infty.$$

Since $\int_0^L F(\sigma)^{-\frac{1}{2}} d\sigma < +\infty$, we conclude

$$\int_0^{+\infty} F(\sigma)^{-\frac{1}{2}} d\sigma = \int_0^L F(\sigma)^{-\frac{1}{2}} d\sigma + \int_L^{+\infty} F(\sigma)^{-\frac{1}{2}} d\sigma < +\infty.$$

Thus, we see that Condition $(C.2)$ is satisfied.

A numerical simulation is given to illustrate the effectiveness of Theorem 1. We consider the lattice graph $\mathbb{Z}^2$. Let Ω be the subset of $\mathbb{Z}^2$ such that $\Omega^\circ = \{x_1, x_2\}$ with $x_1 \sim x_2$. Take the weights $\omega_{xy} \equiv 1$ for any $x, y \in V$ with $x \sim y$ and the measure $\mu(x) = \deg(x) = \frac{1}{4}$ for all $x \in \mathbb{Z}^2$. It is easy to compute that $\phi_0(x_1) = \phi_0(x_2) = 2$. It follows that

$$b(x_i) = \frac{\sqrt{2}\phi_0(x_i)}{||\phi_0||_{\ell^2}^2} = \sqrt{2},$$

for $i = 1, 2$. Moreover, we take $a(x) = 0$ for all $x \in \Omega$, which implies $\alpha = 0$ and $\beta = \sqrt{2}$. Let $f(u) = e^u$, the Eq (2.1) can be rewritten as

$$\begin{cases} \partial_t^2 u(t, x_1) = \frac{1}{4}u(t, x_2) - u(t, x_1) + e^{u(t,x_1)}, & t \in (0, +\infty), \\ \partial_t^2 u(t, x_2) = \frac{1}{4}u(t, x_1) - u(t, x_2) + e^{u(t,x_2)}, & t \in (0, +\infty), \\ u(0, x_1) = u(0, x_2) = 0, \\ \partial_t u(0, x_1) = \partial_t u(0, x_2) = \sqrt{2}, \\ u(t, x) = 0, & (t, x) \in [0, \infty) \times \partial\Omega. \end{cases} \tag{4.3}$$

There is no global solution to the Eq (4.3) from the previous discussion. By using the finite difference method, we give the numerical simulation of the solution to the Eq (4.3). The numerical simulation result is shown in Figure 1.

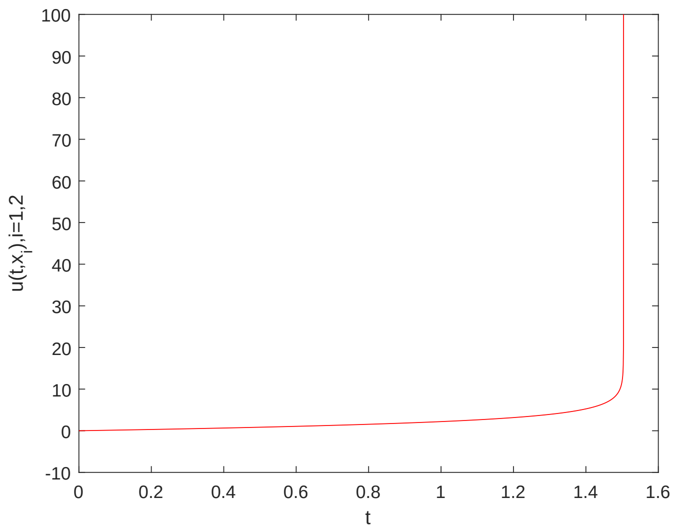

Figure 1. Blow-up phenomenon of the Eq (4.3)

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Figure 1 shows that there exists $x \in V$, a function $u(t, x)$ satisfying (4.3) and becoming infinite in a finite time t, which means the solution of (4.3) blows up by Definition 1.

5. Conclusions

In this paper, we have investigated nonexistence of global solutions for nonlinear wave equations on locally finite graphs. We extended the results of Kawarada onto the graph and enriched the results of Kawarada. In addition, we have given a numerical simulation to verify our results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express sincere thanks to the anonymous referees. The authors also gratefully acknowledges the many helpful suggestions and discussions of Prof. Yong Lin and Dr. Shuang Liu during the preparation of this paper.

This work was supported by the National Science Foundation of China (No. 11671401).

Conflict of interest

This work does not have any conflict of interest.

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