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Research article

Common solutions to some extended system of fuzzy ordered variational inclusions and fixed point problems

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Abstract: The main aim of this work is to use the XOR-operation technique to find the common solutions for a new class of extended system of fuzzy ordered variational inclusions with its corresponding system of fuzzy ordered resolvent equations involving the $\oplus$ operation and fixed point problems, which are slightly different from corresponding problems considered in several recent papers in the literature and are more advantageous. We establish that the system of fuzzy ordered variational inclusions is equivalent to a fixed point problem and a relationship between a system of fuzzy ordered variational inclusions and a system of fuzzy ordered resolvent equations is shown. We prove the existence of a common solution and discuss the convergence of the sequence of iterates generated by the algorithm for a considered problem. The iterative algorithm and results demonstrated in this article have witnessed, a significant improvement for many previously known results of this domain. Some examples are constructed in support of the main results.

Keywords: algorithm; fuzzy inference; iterative methods; nonlinear system; resolvent operator; sequence analysis
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1. Introduction

The variational inclusion problem propelled by Hassouni and Moudafi [16] is a general version of the variational inequality problem introduced by Stampacchia [28] and Fichera [15] in the past decade. As per use of the variational inequalities and inclusions problems, these will help us solve and design various schemes to solve problems that arose in pure and applied sciences (i.e., network equilibrium, traffic network problems, economics, and many more) [10, 11, 13, 24–27, 33].

On the other hand, Zadeh [31] came up with a very interesting and fascinating object called fuzzy sets; as the theory for fuzzy sets evolved, it has extensively been utilized in different disciplines of mathematical research, as well as other areas of pure and applied sciences. The emergence of fuzzy sets were due to a small, notable, and powerful extension as an addition of an interval $[0, 1]$ instead of a set $\{0, 1\}$ to the co-domain of the characteristic function as $\chi : C \subset \mathcal{H} \rightarrow [0, 1]$. After this powerful characterization, this concept will enter into a new zone and the discussion of crisp and fuzzy sets came into existence. It also fulfills the gaps between computer science and mathematics, and even many more subjects too.

Variational inequalities for fuzzy mappings were first introduced and studied by Chang and Zhu [9] in 1989. Following this, many authors have gone through the sandwich concept of variational inequalities and fuzzy mappings for their matter of interest for deep and well mannered details [7, 8, 12, 17, 18, 22, 23].

Another problem, known as the fixed point problem, plays an essential role in the theory of nonlinear analysis, algorithmic development, optimization, and applications across all the discipline of pure and applied sciences, and many more [10, 14, 29, 30, 32]. Therefore, the fixed point problem is the problem of obtaining $p \in \mathcal{H}$ such that $S(p) = p$, where S is a nonlinear mapping on $\mathcal{H}$. In this paper, we use $Fix(S)$ to denote the fixed point set of S, that is, $Fix(S) = \{p \in \mathcal{H} : S(p) = p\}$.

The idea of calculating the number of fixed points in an ordered Banach space was propelled by Amman [1]. Then, people working on variational inclusion and inequalities problems in ordered spaces jumped into the lead and various ways of computing the fixed points/solution of variational inclusion/inequalities problems in the light of ordered Hilbert/Banach spaces. Li and his team has grab the title to first work on ordered resolvent equations and their corresponding ordered variational inequalities/inclusion problems [19–21]. They created a nice line of work regarding the mixture of ordered variational inequalities/inclusion problems involving the concept of operators (e.g., XOR, XNOR, OR and AND).

Motivated by the research of this inclination, Ahmad and his team enrich the work of Li and his team and improvise the structure of resolvent equations corresponding with their variational inequalities/inclusion problems in a broader settings involving XOR, XNOR operator, etc. [2–5].

The whole draft is divided into multiple segments: The first segment is a well equipped collection of basic preliminaries; the second segment is devoted to the formulation of the system of fuzzy ordered variational inclusions with its corresponding system of fuzzy ordered resolvent equations involving $\oplus$ operation and fixed point problems, and discusses the existence of common solution results; a subsegment is also devoted to iterative schemes and a convergence result for the system of fuzzy ordered variational inclusions with its corresponding system of resolvent equations involving $\oplus$ operation and fixed point problems and the last segment is devoted to the conclusion in which the future scope of the problem is discussed and a comprehensive record of references is there.

2. Preliminaries

Throughout the manuscript, we assume that $\mathcal{H}$ is an ordered Banach space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot, \cdot\rangle$. Let $2^{\mathcal{H}}$ (respectively, $CB(\mathcal{H})$) be the family of all non-void (respectively, non-empty closed and bounded) subsets of $\mathcal{H}$.

Let $\mathcal{F}(\mathcal{H})$ be a collection of all fuzzy sets defined over $\mathcal{H}$. A map $F : \mathcal{H} \rightarrow \mathcal{F}(\mathcal{H})$ is said to be fuzzy mapping on $\mathcal{H}$. For each $p \in \mathcal{H}$, $F(p)$ (in the sequel, it will be denoted by F_p) is a fuzzy set on $\mathcal{H}$ and $F_p(q)$ is the membership degree of q in F_p.

A fuzzy mapping $F : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ is said to be closed if for each $p \in \mathcal{H}$, the function $q \to F_p(q)$ is upper semi-continuous, that is, for any given net $\{q_\alpha\} \subset \mathcal{H}$, satisfying $q_\alpha \to q_0 \in \mathcal{H}$, we have

$$\limsup_{\alpha} F_p(q_\alpha) \leq F_p(q_0).$$

For $R \in \mathcal{F}(\mathcal{H})$ and $\lambda \in [0, 1]$, the set $(R)_\lambda = \{p \in \mathcal{H} : R(p) \geq \lambda\}$ is called a λ-cut set of R. Let $F : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ be a closed fuzzy mapping satisfying the following condition:

($*$) If there exists a function $a : \mathcal{H} \to [0, 1]$ such that for each $p \in \mathcal{H}$, the set $(F_p)_{a(p)} = \{q \in \mathcal{H} : F_p(q) \geq a(p)\}$ is a nonempty bounded subset of $\mathcal{H}$.

If F is a closed fuzzy mapping satisfying the condition ($*$), then for each $p \in \mathcal{H}$, $(F_p)_{a(p)} \in CB(\mathcal{H})$. In fact, let $\{q_\alpha\} \subset (F_p)_{a(p)}$ be a net and $q_\alpha \to q_0 \in \mathcal{H}$, then $(F_p)_{a(p)} \geq a(p)$, for each α. Since F is a closed, we have

$$F_q(q_0) \quad \geq \quad \limsup_{\alpha} F_p(q_\alpha) \geq a(p),$$

which implies that $q_0 \in (F_p)_{a(p)}$ and so $(F_p)_{a(p)} \in CB(\mathcal{H})$.

For the presentation of the results, let us demonstrate some known definitions and results.

Definition 2.1. [14, 19] A nonempty subset C of $\mathcal{H}$ is called a normal cone if there exists a constant $\nu > 0$ such that for $0 \leq p \leq q$, we have $\|p\| \leq \nu\|q\|$, for any $p, q \in \mathcal{H}$.

Definition 2.2. [8] Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping. Then,

(i) $\mathcal{G}$ is said to be β-ordered compression mapping, if $\mathcal{G}$ is a comparison mapping and

$$\mathcal{G}(p) \oplus \mathcal{G}(q) \leq \beta(p \oplus q), \text{ for } 0 < \beta < 1.$$

(ii) $\mathcal{G}$ is said to be ϑ-order non-extended mapping, if there exists a constant $\vartheta > 0$ such that

$$\vartheta(p \oplus q) \leq \mathcal{G}(p) \oplus \mathcal{G}(q), \text{ for all } p, q \in \mathcal{H}.$$

Definition 2.3. [21] A mapping $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be (κ, ν)-ordered Lipschitz continuous, if $p \propto q$, $u \propto v$, then $N(p, u) \propto N(q, v)$ and there exist constants $\kappa, \nu > 0$ such that

$$N(p, u) \oplus N(q, v) \leq \kappa(p \oplus q) + \nu(u \oplus v), \text{ for all } p, q, u, v \in \mathcal{H}.$$

Definition 2.4. [19] A compression mapping $h : \mathcal{H} \to \mathcal{H}$ is said to be restricted accretive mapping if there exist two constants $\xi_1, \xi_2 \in (0, 1]$ such that for any $a, z \in \mathcal{H}$,

$$\big(h(p) + I(p)\big) \oplus \big(h(q) + I(q)\big) \leq \xi_1\big(h(p) \oplus h(q)\big) + \xi_2(p \oplus q)$$

holds, where I is the identity mapping on $\mathcal{H}$.

Definition 2.5. [4, 20] A set-valued mapping $A : \mathcal{H} \to CB(\mathcal{H})$ is said to be D-Lipschitz continuous, if for any $p, q \in \mathcal{H}$, $p \propto q$, there exists a constant $\delta_{D_A} > 0$ such that

$$D(A(p), A(q)) \leq \delta_{D_A}(p \oplus q), \text{ for all } p, q, u, v \in \mathcal{H}.$$

Definition 2.6. [4] Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a strong comparison and ϑ-order non-extended mapping. Then, a comparison mapping $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be an ordered (α, λ)-XOR-weak-ANODD set-valued mapping if $\mathcal{B}$ is α-weak-non-ordinary difference mapping and λ-XOR-ordered strongly monotone mapping, and $[\mathcal{G} \oplus \lambda\mathcal{B}](\mathcal{H}) = \mathcal{H}$, for $\lambda, \beta, \alpha > 0$.

Definition 2.7. [4] Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a strong comparison and ϑ-order non-extended mapping. Let $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be an ordered (α, λ)-XOR-weak-ANODD set-valued mapping. The resolvent operator $\mathcal{J}^{\lambda}_{\mathcal{B}} : \mathcal{H} \to \mathcal{H}$ associated with $\mathcal{B}$ is defined by

$$\mathcal{J}^{\lambda}_{\mathcal{B}}(p) = [\mathcal{G} \oplus \lambda\mathcal{B}]^{-1}(p), \forall p \in \mathcal{H}, \tag{2.1}$$

where $\lambda > 0$ is a constant.

Lemma 2.1. [4,20,21] Let $\odot$ be an XNOR operation and $\oplus$ be an XOR operation. Then, the following relations hold:

(i) $p \odot p = p \oplus p = 0$, $p \odot q = q \odot p = -(p \oplus q) = -(q \oplus p)$;
(ii) $(\lambda p) \oplus (\lambda q) = |\lambda|(p \oplus q)$;
(iii) $0 \leq p \oplus q$, if $p \propto q$;
(iv) $(p + q) \odot (u + v) \geq (p \odot u) + (q \odot v)$;
(v) If p, q and w are comparative to each other, then $(p \oplus q) \leq p \oplus w + w \oplus q$;
(vi) $(\alpha p) \oplus (\beta p) = |\alpha - \beta| p = (\alpha \oplus \beta)p$, if $p \propto 0$,
(vii) $\|p \oplus q\| \leq \|p - q\| \leq \nu\|p \oplus q\|$;
$(viii)$ If $p \propto q$, then $\|p \oplus q\| = \|p - q\|$, for all $p, q, u, v, w \in \mathcal{H}$ and $\alpha, \beta, \lambda \in \mathbb{R}$.

Lemma 2.2. Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a strong comparison and ϑ-order non-extended mapping. Let $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be an ordered (α, λ)-XOR-weak ANODD set-valued mapping with respect to $\mathcal{J}^{\lambda}_{\mathcal{B}}$, for $\alpha\lambda > 1$. Then, the resolvent operator $\mathcal{J}^{\lambda}_{\mathcal{B}}$ satisfying the following condition:

$$\mathcal{J}^{\lambda}_{\mathcal{B}}(p) \oplus \mathcal{J}^{\lambda}_{\mathcal{B}}(q) \leq \frac{1}{\vartheta(\alpha\lambda \oplus 1)}(p \oplus q), \ \forall p, q \in \mathcal{H}_p,$$

i.e., the resolvent operator $\mathcal{J}^{\lambda}_{\mathcal{B}}$ is $\frac{1}{\vartheta(\alpha\lambda\oplus 1)}$-nonexpansive mapping.

Lemma 2.3. [4] Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a strong comparison and ϑ-order non-extended mapping. Let $\mathcal{B} : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ be an ordered (α, λ)-XOR-weak ANODD set-valued mapping with respect to the first argument. The resolvent operator $\mathcal{J}^{\lambda}_{\mathcal{B}} : \mathcal{H} \to \mathcal{H}$ associated with $\mathcal{B}$ is defined by

$$\mathcal{J}^{\lambda}_{\mathcal{B}(.,z)}(p) = [\mathcal{G} \oplus \lambda\mathcal{B}(., z)]^{-1}(p), \ \text{for } z \in \mathcal{H}. \tag{2.2}$$

Then, for any given $z \in \mathcal{H}$, the resolvent operator $\mathcal{J}^{\lambda}_{\mathcal{B}(.,z)} : \mathcal{H} \to \mathcal{H}$ is well-defined, single valued, continuous, comparison and $\frac{1}{\vartheta(\alpha\lambda\oplus 1)}$-nonexpansive mapping with $\lambda\alpha > 1$, that is

$$\mathcal{J}^{\lambda}_{\mathcal{B}(.,z)}(p) \oplus \mathcal{J}^{\lambda}_{\mathcal{B}(.,z)}(q) \leq \frac{1}{\vartheta(\alpha\lambda \oplus 1)}(p \oplus q), \ \text{for all } p, q \in \mathcal{H}. \tag{2.3}$$

3. Problem and fixed point formulation

For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i$ be an ordered Banach space equipped with the norm $\|.\|_i$ and K_i be a normal cone with normal constant ν_i, and let $h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$

be the ordered single-valued comparison mappings, respectively. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings satisfying the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to [0,1]$ such that for each $p_i \in \mathcal{H}_i$, we have $(S_{i,p_i})_{d_i(p_i)}, (U_{i,p_i})_{c_i(p_i)}$, and $(V_{i,p_i})_{e_i(p_i)}$ in $CB(\mathcal{H}_i)$, respectively. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be the set-valued mapping. We consider the following extended nonlinear system of fuzzy ordered variational inclusions involving the $\oplus$ operation and the solution set is denoted by ENSFOVI$(\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m)$:

For each $i \in \Lambda$ and some $\omega_i > 0$, find $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ such that $S_{i,p_i}(p_i) \geq d_i(p_i)$, $U_{i,p_i}(p_i) \geq c_i(p_i)$ and $V_{i,p_i}(p_i) \geq e_i(p_i)$, i.e., $q_i \in (S_{i,p_i})_{d_i(p_i)}$, $u_i \in (U_{i,p_i})_{c_i(p_i)}$ and $v_i \in (V_{i,p_i})_{e_i(p_i)}$,

$$\begin{cases} 0 \in \mathcal{N}_1(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_1(u_1) + \omega_1 \mathcal{B}_1(h_1(p_1), v_1), \\ 0 \in \mathcal{N}_2(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_2(u_2) + \omega_2 \mathcal{B}_2(h_2(p_2), v_2), \\ 0 \in \mathcal{N}_3(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_3(u_3) + \omega_3 \mathcal{B}_3(h_3(p_3), v_3), \\ \qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad . \\ 0 \in \mathcal{N}_m(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_m(u_m) + \omega_m \mathcal{B}_m(h_m(p_m), v_m). \end{cases} \tag{3.1}$$

Equivalently, for each $i \in \Lambda$,

$$0 \in \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i). \tag{3.2}$$

Some special cases of problem (3.2) are as follows:

(i) For $i = 1$, if $\mathcal{N}_1(q_1, q_2, \cdots, q_m) = \mathcal{N}_1(q_1, q_2)$ and $\omega_1 = 1$, then problem (3.2) reduces to the problem of finding $p_1, q_1, q_2, u_1, z_1 \in \mathcal{H}_1$ such that

$$0 \in \mathcal{N}_1(q_1, q_2) \oplus \mathcal{G}_1(u_1) + \mathcal{B}_1(h_1(p_1), v_1). \tag{3.3}$$

Problem (3.3) was considered and studied by Ahmad et al. [4].

(ii) For $i = 1$, if $S_1, U_1, V_1 = I$ (identity mapping), $\mathcal{B}_1 = -1$, $\mathcal{N}_1$ is single-valued mapping and $\mathcal{N}_1(p_1, p_2, \cdots, p_m) = \mathcal{N}_1(p_1)$, then problem (3.2) reduces to the problem of finding $p_1 \in \mathcal{H}_1$ such that

$$\omega_1 \in \mathcal{N}_1(p_1) \oplus \mathcal{G}_1(p_1). \tag{3.4}$$

Problem (3.4) was considered and studied by Li et al. [21].

By taking suitable choices of the mappings h_i, N_i, $\mathcal{B}_i$, S_i, U_i, V_i and the space $\mathcal{H}_i$, for each $i \in \Lambda$, in above problem (3.1), one can easily obtain the problems considered and studied in [1–4, 19–21] and references therein.

For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, putting $d_i(p_i) = c_i(p_i) = e_i(p_i) = 1$, for all $p_i \in \mathcal{H}_i$, problem (3.1) includes many kinds of variational inequalities and variational inclusion problems [7, 9, 17, 22–24].

In support of our problem (3.2), we provide the following example.

Example 3.1. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i = [0, 11i]$ and $C=\{p_i \in \mathcal{H}_i : 0 \leq p_i \leq 5i\}$ be the normal cone. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be the closed fuzzy mappings and the mappings $d_i, c_i, e_i : \mathcal{H}_i \to [0,1]$ defined by for all $p_i, q_i, u_i, v_i \in \mathcal{H}_i$.

$$S_{i,p_i}(q_i) = \begin{cases} \frac{1}{3i+|q_i-2i|}, & \text{if } p_i \in [0,1], \\ \frac{1}{3i+p_i|q_i-2i|}, & \text{if } p_i \in (1, 11i], \end{cases} \qquad U_{i,p_i}(u_i) = \begin{cases} \frac{1}{2i^2+(u_i-i)^2}, & \text{if } p_i \in [0,1], \\ \frac{1}{2i+p_i(u_i-i)^2}, & \text{if } p_i \in (1, 11i], \end{cases}$$

$$V_{i,p}(v_i)=\begin{cases}\frac{1}{i+p_i|v_i-3i|}, & \text{if } p_i\in[0,1],\\ \frac{1}{2i+|v_i-3i|}, & \text{if } p_i\in(1,11i],\end{cases}\qquad d_i(p_i)=\begin{cases}\frac{1}{5i}, & \text{if } p_i\in[0,1],\\ \frac{1}{3i+2ip_i}, & \text{if } p_i\in(1,11i],\end{cases}$$

$$c_i(p_i)=\begin{cases}\frac{1}{3i^2}, & \text{if } p_i\in[0,1],\\ \frac{1}{i(2+ip_i)}, & \text{if } p_i\in(1,11i],\end{cases}\quad\text{and}\quad e_i(p_i)=\begin{cases}\frac{1}{i+3ip_i}, & \text{if } p_i\in[0,1],\\ \frac{1}{5i}, & \text{if } p_i\in(1,11i].\end{cases}$$

For any $p_i \in [0,1]$, we have

$$\begin{aligned}(S_{i,p_i})_{d_i(p_i)} &= \Big\{q_i : S_{i,p_i}(q_i)\geq \frac{1}{5}\Big\}=\Big\{\{q_i : \frac{1}{3i+|q_i-2i|}\geq\frac{1}{5i}\Big\}=[0,4i],\\ (U_{i,p_i})_{c_i(p_i)} &= \Big\{u_i : U_{i,p_i}(u_i)\geq\frac{1}{3i^2}\Big\}=\Big\{u_i : \frac{1}{2i^2+(u_i-i)^2}\geq\frac{1}{3i^2}\Big\}=[0,2i],\\ (V_{i,p_i})_{e_i(p_i)} &= \Big\{v_i : V_{i,p_i}(v_i)\geq\frac{1}{i+3ip_i}\Big\}=\Big\{v_i : \frac{1}{i+p_i|v_i-3i|}\geq\frac{1}{i+3ip_i}\Big\}=[0,6i],\end{aligned}$$

and for any $p_i \in (1,11i]$, we have

$$\begin{aligned}(S_{i,p_i})_{d_i(p_i)} &= \Big\{q_i : S_{i,p_i}(q_i)\geq\frac{1}{3i+2ip_i}\Big\}=\Big\{\{q_i : \frac{1}{3i+p_i|q_i-2i|}\geq\frac{1}{3i+2ip_i}\Big\}=[0,4i],\\ (U_{i,p_i})_{c_i(p_i)} &= \Big\{u_i : U_{i,p_i}(u_i)\geq\frac{1}{i(2+ip_i)}\Big\}=\Big\{u_i : \frac{1}{2i+p_i(u_i-i)^2}\geq\frac{1}{i(2+ip_i)}\Big\}=[0,2i],\\ (V_{i,p_i})_{e_i(p_i)} &= \Big\{v_i : V_{i,p_i}(v_i)\geq\frac{1}{5i}\Big\}=\Big\{v_i : \frac{1}{2i+|v_i-3i|}\geq\frac{1}{5i}\Big\}=[0,6i].\end{aligned}$$

Now, we define the single-valued mappings $h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m}\mathcal{H}_j \to \mathcal{H}_i$ by

$$h_i(p_i)=\frac{p_i}{5},\quad \mathcal{G}_i(u_i)=\frac{u_i}{7}\quad\text{and}\quad \mathcal{N}_i(q_1,q_2,\cdots,q_m)=\frac{1}{9}\sum_{i=1}^{m}q_i,$$

and the set-valued mapping $\mathcal{B}_i : \mathcal{H}_i\times\mathcal{H}_i \to 2^{\mathcal{H}_i}$ defined by

$$\mathcal{B}_i(h_i(p_i),v_i)=\Big\{h_i(p_i)+\frac{v_i}{5} : p_i\in[0,11i]\ \text{and}\ v_i\in(V_{i,p_i})_{c_i(p_i)}\Big\}.$$

In the above view, it is easy to verify that $0 \in \mathcal{N}_i(q_1,q_2,\cdots,q_m)\oplus\mathcal{G}_i(u_i)+\omega_i\mathcal{B}_i(h_i(p_i),v_i)$, that is, problem (3.2) is satisfied.

Example 3.2. For $i = 1$, let $\mathcal{H}_1 = \mathbb{R}^n_p$, Ω be a non-empty subset of $\mathbb{R}^n_p$, $\mathcal{B}_1$ is single valued mapping and $V_1 = I$ (identity mapping), and the other functions, that is $\mathcal{G}_1,\ \mathcal{N}_1,\ S_1,\ U_1,\ d_1,\ c_1$ are equal to zero and the fuzzy coalitions of players are identified with the measurable functions e_1 from Ω to $[0,1]$. Define $\mathcal{B} : \mathcal{H}_1\times\mathcal{H}_1 \to \mathcal{H}_1$ by

$$\mathcal{B}(h_1(p_1),p_1)=\int_L P(h_1(u),u)h_1(u)du,$$

we associate each player with its action $P(.,u)$, where $P : \Omega\times\mathcal{H}_1 \to \mathbb{R}^n_p$, Ω is a non-empty subset of $\mathbb{R}^n_p$, and each fuzzy coalition $h_1(u)$ with its action $\int_L P(h_1(u),u)h_1(u)du$. This continuum of players problem can be obtained from xtended system of fuzzy ordered variational inclusions (3.1). For more details see Chapter 13 and Exercise 13.2 of the book "Optima and equilibria" by Aubin [6] and Example 3.1 in [4].

Related to the extended nonlinear system of fuzzy ordered variational inclusions (3.2), we consider the following extended nonlinear system of fuzzy ordered resolvent equations problem:

For each $i \in \Lambda$, find $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ such that $s_i \in \mathcal{H}_i, S_{i,p_i}(p_i) \geq d_i(p_i), U_{i,p_i}(p_i) \geq c_i(p_i)$ and $V_{i,p_i}(p_i) \geq e_i(p_i)$, i.e., $q_i \in (S_{i,p_i})_{d_i(p_i)}, u_i \in (U_{i,p_i})_{c_i(p_i)}$ and $v_i \in (V_{i,p_i})_{e_i(p_i)}$,

$$\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \lambda_i^{-1} \omega_i \mathcal{R}_{\mathcal{B}_i(.,v_i)}(s_i) = \mathcal{G}_i(u_i), \tag{3.5}$$

where $\lambda_i > 0$ is a constant and $\mathcal{R}_{\mathcal{B}_i(.,v_i)}(s_i) = \left[I_i \oplus \mathcal{A}_i \circ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}\right](s_i)$.

The following lemma ensures the equivalence between the extended nonlinear system of fuzzy ordered variational inclusions involving the $\oplus$ operation (3.1) and the extended nonlinear system of fuzzy ordered resolvent equations problem (3.5).

Lemma 3.1. For each $i \in \Lambda$, let $\mathcal{A}_i, h_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be the nonlinear ordered single-valued comparison mappings, respectively. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ and $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be the set-valued mappings. Then, the followings are equivalent for each $i \in \Lambda$,

(*i*) $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a solution of problem (3.1),

(*ii*) for each i, $p_i \in \mathcal{H}_i$ such that $q_i \in (S_{i,p_i})_{d_i(p_i)}$, $u_i \in (U_{i,p_i})_{c_i(p_i)}$ and $v_i \in (V_{i,p_i})_{e_i(p_i)}$ is a fixed point of a mapping $T_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ defined by

$$T_i(p_i) = \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i) + p_i, \tag{3.6}$$

(*iii*) $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a solution of the following equation:

$$h_i(p_i) = \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))], \tag{3.7}$$

(*iv*) $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a solution of the problem (3.5), where

$$\begin{aligned} s_i &= \mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)), \\ h_i(p_i) &= \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}(s_i). \end{aligned} \tag{3.8}$$

Proof. (*i*) $\implies$ (*ii*) For each $i \in \Lambda$, adding p_i to both sides of (3.2), we have

$$\begin{aligned} & 0 \in \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i) \\ \implies \quad & p_i \in \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i) + p_i = T_i(p_i). \end{aligned}$$

Hence, p_i is a fixed point of T_i, for each $i \in \Lambda$.

(*ii*) $\implies$ (*iii*) Let p_i be a fixed point of T_i, then

$$
\begin{aligned}
& p_i \in T_i(p_i) = \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i) + p_i \\
\Longrightarrow \quad & \mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)) \in [\mathcal{A}_i \oplus \lambda_i \mathcal{B}_i(\cdot, v_i)](h_i(p_i)).
\end{aligned}
$$

Hence, $h_i(p_i) = \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))]$, for each $i \in \Lambda$.
$(iii) \implies (iv)$ Taking $s_i = \mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))$, from (3.7), we have $h_i(p_i) = \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}(s_i)$, so,

$$
s_i = \mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)),
$$

which implies that

$$
\begin{aligned}
& s_i \oplus \mathcal{A}_i(\mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}(s_i)) = \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)) \\
\Longrightarrow \quad & [I_i \oplus \mathcal{A}_i \circ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}](s_i) = \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)) \\
\Longrightarrow \quad & \mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \lambda_i^{-1} \omega_i \mathcal{R}_{\mathcal{B}_i(.,v_i)}(s_i) = \mathcal{G}_i(u_i).
\end{aligned}
$$

Consequently, $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a solution of the extended system of fuzzy ordered resolvent equations problem (3.5), for each $i \in \Lambda$.
$(iv) \implies (i)$, from (3.8) we have

$$
\begin{aligned}
h_i(p_i) &= \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}(s_i) \\
&= \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))],
\end{aligned}
$$

so

$$
\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i)) \in [\mathcal{A}_i \oplus \lambda_i \mathcal{B}_i(., v_i)]h_i(p_i),
$$

which implies

$$
0 \in \mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i) + \omega_i \mathcal{B}_i(h_i(p_i), v_i).
$$

Therefore, $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a solution of extended nonlinear system of fuzzy ordered variational inclusions (3.1), for each $i \in \Lambda$. This completes the proof.

4. Main results

In this section, we discuss an existence and convergence result for the extended nonlinear system of fuzzy ordered variational inclusions (3.1) and corresponding its extended nonlinear system of fuzzy ordered resolvent equations problem (3.5).

Theorem 4.1. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i$ be a real Banach space equipped with the norm $\|.\|_i$ and K_i be a normal cone with normal constant ν_i. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings satisfying the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to [0, 1]$ such that for each $p_i \in \mathcal{H}_i$, we have $(S_{i,p_i})_{d_i(p_i)}$, $(U_{i,p_i})_{c_i(p_i)}$ and $(V_{i,p_i})_{e_i(p_i)}$ in $CB(\mathcal{H}_i)$, respectively. Let $\mathcal{A}_i, h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be the nonlinear single-valued mappings. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an ordered (α_i, λ_i)-XOR-weak ANODD set-valued mapping with respect to the first argument. For each $i \in \Lambda$, suppose that the following conditions hold:

(i) h_i is continuous, β_i-oredered compression and (ζ_i, η_i)-ordered restricted-accretive mapping, $\beta_i \in (0, 1)$ and $\zeta_i, \eta_i \in (0, 1]$, respectively;
(ii) $\mathcal{A}_i$ is continuous and τ_i-oredered compression mapping, $\tau_i \in (0, 1)$;
(iii) $\mathcal{G}_i$ is continuous, ϑ_i-order non-extended mapping and μ_i-oredered compression mapping, $\mu_i \in (0, 1)$ and $\vartheta_i > 0$, respectively;
(iv) $\mathcal{N}_i$ is continuous, κ_i-ordered compression mapping in the i^{th}-argument and $\kappa_{i,j}$-ordered compression mapping in the j^{th}-argument for each $j \in \Lambda, i \neq j$, respectively;
(v) S_i, U_i and V_i are ordered Lipschitz type continuous mapping with constants δ_{S_i}, δ_{U_i} and δ_{V_i}, respectively.

If the following conditions

$$(a)\ \ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,x_i)}(p_i) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,y_i)}(p_i) \leq \xi_i(x_i \oplus y_i),\ \text{for all } p_i, x_i, y_i \in \mathcal{H}_i,\ \xi_i > 0, \tag{4.1}$$

$$(b)\ \ \begin{cases} \Theta_i = \omega_i(\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \theta_i(\tau_i\beta_i\omega \oplus \lambda_i\mu_i\delta_{U_i} + \lambda_i\kappa_i\delta_{S_i}) < \omega_i \min\left\{1, \frac{1}{\nu_i}\right\}, \\ \Theta_i + \sum\limits_{\ell\in\Lambda,\ \ell\neq i}^{m} \frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} < 1,\ \theta_i = \frac{1}{\vartheta_i(\alpha_i\lambda_i\oplus 1)} \text{ and } \alpha_i\lambda_i > 1,\ \text{for all } i \in \Lambda \end{cases} \tag{4.2}$$

are satisfied, then there exists $(p_1^*, p_2^*, \cdots, p_m^*) \in \prod_{i=1}^{m} \mathcal{H}_i$ such that $q_i \in (S_{i,p_i})_{d_i(p_i)}, u_i \in (U_{i,p_i})_{c_i(p_i)}$ and $v_i \in (V_{i,p_i})_{e_i(p_i)}$ satisfies the extended nonlinear system of fuzzy ordered resolvent equations problem (3.5) and so $(p_1^*, p_2^*, \cdots, p_m^*)$ is a solution of the extended nonlinear system of fuzzy ordered variational inclusions (3.2), respectively.

Proof. By Lemma 3.1, it is sufficient to prove that there exists $(p_1^*, p_2^*, \cdots, p_m^*)$ satisfying (3.1). For each $i \in \Lambda$, we define $\phi_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ by

$$\phi_i(p_1, p_2, \cdots, p_m) = p_i + h_i(p_i) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))], \tag{4.3}$$

for all $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$. Define $\|.\|_*$ on $\prod_{i=1}^{m} \mathcal{H}_i$ by

$$\|(p_1, p_2, \cdots, p_m)\|_* = \sum_{i=1}^{m} \|p_i\|_i,\ \ \forall\ (p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i. \tag{4.4}$$

It is easy to see that $\left(\prod_{i=1}^{m} \mathcal{H}_i, \|.\|_*\right)$ is a Banach space. Additionally, define $\psi : \prod_{i=1}^{m} \mathcal{H}_i \to \prod_{i=1}^{m} \mathcal{H}_i$ as follows:

$$\psi(p_1, p_2, \cdots, p_m) = (\phi_1(p_1, p_2, \cdots, p_m), \phi_2(p_1, p_2, \cdots, p_m), \cdots, \phi_m(p_1, p_2, \cdots, p_m)), \tag{4.5}$$

for all $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i$. First of all, we prove that ψ is a contraction mapping.

Let $(p_1, p_2, \cdots, p_m), (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ be given. By assumptions (i)–(v) and Lemma 2.1, for each $i \in \Lambda$, we have

$$\begin{aligned}
0 &\leq \phi_i(p_1, p_2, \cdots, p_m) \oplus \phi_i(\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_m)\\
&= \Big[p_i + h_i(p_i) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))]\Big]\\
&\quad \oplus\Big[\hat{p}_i + h_i(\hat{p}_i) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,\hat{v}_i)}[\mathcal{A}_i(h_i(\hat{p}_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m) \odot \mathcal{G}_i(\hat{u}_i))]\Big]\\
&\leq \zeta_i(p_i \oplus \hat{p}_i) + \eta_i(h_i(\hat{p}_i) \oplus h_i(\hat{p}_i)) + \xi_i(v_i \oplus \hat{v}_i)\\
&\quad +\mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))]\\
&\quad \oplus\mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(\hat{p}_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m) \odot \mathcal{G}_i(\hat{u}_i))]\\
&\leq \zeta_i(p_i \oplus \hat{p}_i) + \eta_i(h_i(\hat{p}_i) \oplus h_i(\hat{p}_i)) + \xi_i D_i(V_i(p_i), V_i(\hat{p}_i))\\
&\quad +\theta_i\Big((\mathcal{A}_i(h_i(p_i)) \oplus \mathcal{A}_i(h_i(\hat{p}_i))) \oplus \frac{\lambda_i}{\omega_i}(- (\mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{G}_i(u_i))\\
&\quad \oplus(-\mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m) \oplus \mathcal{G}_i(\hat{u}_i)))\Big)\\
&\leq \zeta_i(p_i \oplus \hat{p}_i) + \eta_i\beta_i(p_i \oplus \hat{p}_i) + \xi_i\delta_{V_i}(p_i \oplus \hat{p}_i) + \theta_i\Big((\tau_i\beta_i(p_i \oplus \hat{p}_i))\\
&\quad \oplus\frac{\lambda_i}{\omega_i}(\mu_i(u_i \oplus \hat{u}_i)) \oplus (\mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m))\Big)\\
&\leq (\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i})(p_i \oplus \hat{p}_i) + \Big(\frac{\theta_i(\tau_i\beta_i\omega_i \oplus \lambda_i\mu_i\delta_{U_i})}{\omega_i}(p_i \oplus \hat{p}_i)\Big)\\
&\quad \oplus\Big(\frac{\lambda_i\theta_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m))\Big).
\end{aligned} \tag{4.6}$$

Since $\mathcal{N}_i$ is a κ_i-ordered comparison mapping in the i^{th} arguments and a κ_{ij}-ordered comparison mapping in the j^{th} arguments $(i \neq j)$, and S_i is ordered δ_{S_i}-Lipschitz continuous mapping.

$$\begin{aligned}
&\mathcal{N}_i(q_1, q_2, \cdots, q_m) \oplus \mathcal{N}_i(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m)\\
\leq\ &\mathcal{N}_i(q_1, q_2, \cdots, q_{i-1}, q_i, q_{i+1}, \cdots, q_m) \oplus \mathcal{N}_i(q_1, q_2, \cdots, q_{i-1}, \hat{q}_i, q_{i+1}, \cdots, q_m)\\
&+ \sum_{j\in\Lambda,\ i\neq j} (\mathcal{N}_i(q_1, q_2, \cdots, q_{j-1}, q_j, q_{j+1}, \cdots, q_m) \oplus \mathcal{N}_i(q_1, q_2, \cdots, q_{j-1}, \hat{q}_j, q_{j+1}, \cdots, q_m))\\
\leq\ &\kappa_i(q_i \oplus \hat{q}_i) + \sum_{j\in\Lambda,\ i\neq j} \kappa_{i,j}(q_j \oplus \hat{q}_j) \leq \kappa_i D_i(S_i(p_i), S_i(\hat{p}_i)) + \sum_{j\in\Lambda,\ i\neq j} \kappa_{i,j} D_j(S_j(p_j), S_j(\hat{p}_j))\\
\leq\ &\kappa_i\delta_{S_i}(p_i \oplus \hat{p}_i) + \sum_{j\in\Lambda,\ i\neq j} \kappa_{i,j}\delta_{S_{i,j}}(p_j \oplus \hat{p}_j).
\end{aligned} \tag{4.7}$$

Using (4.7), (4.6) becomes

$$\begin{aligned}&\phi_i(p_1,p_2,\cdots,p_m)\oplus\phi_i(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\\ \leq\ &(\zeta_i+\eta_i\beta_i+\xi_i\delta_{V_i})(p_i\oplus\hat{p}_i)+\left(\frac{\theta_i((\tau_i\beta_i\omega_i\oplus\lambda_i\mu_i\delta_{U_i})+\lambda_i\kappa_i\delta_{S_i})}{\omega_i}(p_i\oplus\hat{p}_i)\right)\\ &\oplus\left(\frac{\lambda_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}(p_j\oplus\hat{p}_j)\right).\end{aligned}$$

By Definition 2.1 and Lemma 2.2, we have

$$\begin{aligned}&\|\phi_i(p_1,p_2,\cdots,p_m)\oplus\phi_i(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\|_i\\ \leq\ &\Theta_i\|p_i\oplus\hat{p}_i\|_i+\frac{\nu_i\lambda_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\|p_j\oplus\hat{p}_j\|_j,\end{aligned}\tag{4.8}$$

where $\Theta_i=\left(\nu_i(\zeta_i+\eta_i\beta_i+\xi_i\delta_{S_i})+\frac{\nu_i\theta_i(\tau_i\beta_i\omega_i\oplus\lambda_i\mu_i\delta_{U_i}+\lambda_i\kappa_i\delta_{S_i}}{\omega_i}\right)$.

From (4.5) and (4.8), we get

$$\begin{aligned}&\|\psi(p_1,p_2,\cdots,p_m)\oplus\psi(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\|_*\\ =\ &\sum_{i=1}^m\|\phi_i(p_1,p_2,\cdots,p_m)\oplus\phi_i(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\|_i\\ \leq\ &\sum_{i=1}^m\left(\Theta_i\|p_i\oplus\hat{p}_i\|_i+\frac{\nu_i\lambda_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\|p_j\oplus\hat{p}_j\|_j\right)\\ =\ &\left(\Theta_1+\sum_{\ell=2}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,1}\delta_{S_{\ell,1}}\right)\|p_1\oplus\hat{p}_1\|_1+\left(\Theta_2+\sum_{\ell\in\Lambda,\ \ell\neq 2}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,2}\delta_{S_{\ell,2}}\right)\|p_2\oplus\hat{p}_2\|_2\\ &+\left(\Theta_3+\sum_{\ell\in\Lambda,\ \ell\neq 3}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,3}\delta_{S_{\ell,3}}\right)\|p_3-\hat{p}_3\|_3+\cdots+\left(\Theta_m+\sum_{\ell=1}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,m}\delta_{S_{\ell,m}}\right)\|p_m\oplus\hat{p}_m\|_m\\ \leq\ &\max\left\{\Theta_i+\sum_{\ell\in\Lambda,\ \ell\neq i}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}}:i\in\Lambda\right\}\sum_{i=1}^m\|p_i\oplus\hat{p}_i\|_i,\end{aligned}$$

i.e.,

$$\|\psi(p_1,p_2,\cdots,p_m)\oplus\psi(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\|_*\leq\Omega\|(p_1,p_2,\cdots,p_m)\oplus(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_m)\|_*,\tag{4.9}$$

where $\Omega=\max\left\{\Theta_i+\sum_{\ell\in\Lambda,\ \ell\neq i}^m\frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}}:i\in\Lambda\right\}$. The condition (4.2) guarantees that $0\leq\Omega<1$. By the inequality (4.9), we note that ψ is a contraction mapping. Therefore, there exists a unique point $(p_1^*,p_2^*,\cdots,p_m^*)\in\prod_{i=1}^m\mathcal{H}_i$ such that $\psi(p_1^*,p_2^*,\cdots,p_m^*)=(p_1^*,p_2^*,\cdots,p_m^*)$. From (4.3) and (4.5), it follows that $(p_1^*,p_2^*,\cdots,p_m^*)$ such that $q_i^*\in(S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^*\in(U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^*\in(V_{i,p_i^*})_{e_i(p_i^*)}$ satisfies in Eq (3.7), i.e., for each $i\in\Lambda$,

$$h_i(p_i^*)=\mathcal{J}_{\mathcal{B}_i(\cdot,v_i^*)}^{\lambda_i}[\mathcal{A}_i(h_i(p_i^*))\oplus\frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*,q_2^*,\cdots,q_m^*)\odot\mathcal{G}_i(u_i^*))].$$

By Lemma 3.1, we conclude that $(p_1^*, p_2^*, \cdots, p_m^*) \in \prod_{i=1}^{m} \mathcal{H}_i$ is a unique solution of the extended system of fuzzy ordered variational inclusions (3.2) and satisfies the extended system of fuzzy ordered resolvent equations problem (3.5). This completes the proof.

For each $i \in \Lambda$, let $Q_i : \mathcal{H}_i \to \mathcal{H}_i$ be a γ_i-ordered Lipschitz continuous mapping. We define the self-mapping $R : \prod_{i=1}^{m} \mathcal{H}_i \to \prod_{i=1}^{m} \mathcal{H}_i$ by

$$R(p_1, p_2, \cdots, p_m) = (Q_1 p_1, Q_2 p_2, \cdots, Q_m p_m), \ \forall \ (p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i. \tag{4.10}$$

Then, $R = (Q_1, Q_2, \cdots, Q_m) : \prod_{i=1}^{m} \mathcal{H}_i \to \prod_{i=1}^{m} \mathcal{H}_i$ is a $\max\{\gamma_i : i \in \Lambda\}$-ordered Lipschitz continuous mapping with respect to the norm $\|.\|_*$ in $\prod_{i=1}^{m} \mathcal{H}_i$. To see this fact, let $(p_1, p_2, \cdots, p_m), (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_m) \in \prod_{i=1}^{m} \mathcal{H}_i$ be given. Then, we have

$$\begin{aligned}
& \|R(p_1, p_2, \cdots, p_m) \oplus R(\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_m)\|_* \\
= & \sum_{i=1}^{m} \|Q_i p_i \oplus Q_i \hat{p}_i\|_i \leq \sum_{i=1}^{m} \gamma_i \|p_i \oplus \hat{p}_i\|_i \\
\leq & \max\{\gamma_i : i \in \Lambda\} \sum_{i=1}^{m} \|p_i \oplus \hat{p}_i\|_i \\
= & \max\{\gamma_i : i \in \Lambda\} \|(p_1, p_2, \cdots, p_m) \oplus (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_m)\|_*.
\end{aligned}$$

We denote the sets of all fixed points of $Q_i, i \in \Lambda$ and R by $Fix(Q_i)$ and $Fix(R)$, respectively, and the set of all solutions of the extended nonlinear system of fuzzy ordered variational inclusions (3.1) by ENSFOVI($\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m$). In view of (4.10), for any $(p_1, p_2, \cdots, p_m) \in \prod_{i=1}^{m} \mathcal{H}_i, (p_1, p_2, \cdots, p_m) \in Fix(R)$ if and only if $p_i \in Fix(Q_i)$, $i \in \Lambda$, i.e., $Fix(R) = Fix(Q_1, Q_2, \cdots, Q_m) = \prod_{i=1}^{m} Fix(Q_i)$.

If $(p_1^*, p_2^*, \cdots, p_m^*) \in Fix(R) \cap$ ESFOVI($\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m$), then by using Lemma 3.1, one can easily to see that for each $i \in \Lambda$,

$$\begin{cases} p_i^* = Q_i p_i^* = p_i^* - h_i(p_i^*) + \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i^*)}[\mathcal{A}_i(h_i(p_i^*)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*, q_2^*, \cdots, q_m^*) \odot \mathcal{G}_i(u_i^*))] \\ \quad\; = Q_i[p_i^* - h_i(p_i^*) + \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i^*)}[\mathcal{A}_i(h_i(p_i^*)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*, q_2^*, \cdots, q_m^*) \odot \mathcal{G}_i(u_i^*))]]. \end{cases} \tag{4.11}$$

Based on Lemma 3.1, we construct an iterative algorithm for finding the approximate solution of problem (3.1).

Iterative Algorithm 4.1. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{A}_i, h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be the nonlinear ordered single-valued comparison mappings, respectively. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings that satisfy the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to$

$[0,1]$ such that for each $p_i \in \mathcal{H}_i$, $q_i \in (S_{i,p_i})_{d_i(p_i)}, u_i \in (U_{i,p_i})_{c_i(p_i)}$ and $v_i \in (V_{i,p_i})_{e_i(p_i)}$. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be the set-valued mapping. For any given $p_{i,0} \in \mathcal{H}_i$, $q_{i,0} \in (S_{i,p_{i,0}})_{d_i(p_{i,0})}, u_{i,0} \in (U_{i,p_{i,0}})_{c_i(p_{i,0})}$ and $v_{i,0} \in (V_{i,p_{i,0}})_{e_i(p_{i,0})}$, compute the sequences $\{p_{i,n}\}, \{q_{i,n}\}, \{u_{i,n}\}, \{v_{i,n}\}$, and $\{s_{i,n}\}$ by the following iterative schemes with the supposition that $p_{i,n+1} \propto p_{i,n}$, $q_{i,n+1} \propto q_{i,n}$, $u_{i,n+1} \propto u_{i,n}$, $v_{i,n+1} \propto v_{i,n}$, and $s_{i,n+1} \propto s_{i,n}$, for each $i \in \Lambda$ and $n = 0, 1, 2, \cdots,$

$$\begin{cases} s_{i,n+1} = \mathcal{A}_i(h_i(p_{i,n})) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_{1,n}, q_{2,n}, \cdots, q_{m,n}) \odot \mathcal{G}_i(u_{i,n})), \\ p_{i,n+1} = (1-\alpha_n)p_{i,n} + \alpha_n Q_i[p_{i,n} + h_i(p_{i,n}) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n+1})] + r_{i,n}, \\ q_{i,n+1} \in (S_{p_{i,n+1}})_{d_i(p_{i,n+1})},\ q_{i,n+1} \oplus q_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((S_{i,p_{i,n+1}})_{d_i(p_{i,n+1})}, (S_{i,p_{i,n}})_{d_i(p_{i,n})}), \\ u_{i,n+1} \in (U_{p_{i,n+1}})_{c_i(p_{i,n+1})}, u_{i,n+1} \oplus u_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((U_{i,p_{i,n+1}})_{c_i(p_{i,n+1})}, (U_{i,p_{i,n}})_{c_i(p_{i,n})}), \\ v_{i,n+1} \in (V_{p_{i,n+1}})_{e_i(p_{i,n+1})},\ v_{i,n+1} \oplus v_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((V_{i,p_{i,n+1}})_{e_i(p_{i,n+1})}, (V_{i,p_{i,n}})_{e_i(p_{i,n})}), \end{cases} \tag{4.12}$$

where α_n is a sequence in interval $[0,1]$ satisfying $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\{r_{i,n}\}$ are sequences in $\mathcal{H}_i$ introduced to take the possible inexact computation of the resolvent operator point satisfying the following conditions into account: $r_{i,n} \oplus 0 = r_{i,n}$ and $\sum\limits_{n=0}^{\infty} \|(r_{1,n}, r_{2,n}, \cdots, r_{m,n})\| < \infty$.

If for each $i \in \Lambda$, $Q_i = I$, then Algorithm 4.1 reduces to the following algorithm.

Iterative Algorithm 4.2. For each $i \in \Lambda$, let $\mathcal{A}_i,\ h_i,\ \mathcal{G}_i,\ \mathcal{N}_i,\ \mathcal{B}_i,\ S_i,\ U_i,\ V_i,\ d_i,\ c_i,\ e_i$ be the same as in Theorem 4.1 such that all the conditions of Algorithm 4.1 are satisfied. For any given $p_{i,0} \in \mathcal{H}_i$, $q_{i,0} \in (S_{i,p_{i,0}})_{d_i(p_{i,0})}, u_{i,0} \in (U_{i,p_{i,0}})_{c_i(p_{i,0})}$ and $v_{i,0} \in (V_{i,p_{i,0}})_{e_i(p_{i,0})}$, compute the sequences $\{p_{i,n}\}, \{q_{i,n}\}, \{u_{i,n}\}, \{v_{i,n}\}$ and $\{s_{i,n}\}$ by the following iterative schemes with the supposition that $p_{i,n+1} \propto p_{i,n}$, $q_{i,n+1} \propto q_{i,n}$, $u_{i,n+1} \propto u_{i,n}$, $v_{i,n+1} \propto v_{i,n}$ and $s_{i,n+1} \propto s_{i,n}$, for each $i \in \Lambda$ and $n = 0, 1, 2, \cdots,$

$$\begin{cases} s_{i,n+1} = \mathcal{A}_i(h_i(p_{i,n})) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_{1,n}, q_{2,n}, \cdots, q_{m,n}) \odot \mathcal{G}_i(u_{i,n})), \\ p_{i,n+1} = (1-\alpha_n)p_{i,n} + \alpha_n [p_{i,n} + h_i(p_{i,n}) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n+1})] + r_{i,n}, \\ q_{i,n+1} \in (S_{p_{i,n+1}})_{d_i(p_{i,n+1})},\ q_{i,n+1} \oplus q_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((S_{i,p_{i,n+1}})_{d_i(p_{i,n+1})}, (S_{i,p_{i,n}})_{d_i(p_{i,n})}), \\ u_{i,n+1} \in (U_{p_{i,n+1}})_{c_i(p_{i,n+1})}, u_{i,n+1} \oplus u_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((U_{i,p_{i,n+1}})_{c_i(p_{i,n+1})}, (U_{i,p_{i,n}})_{c_i(p_{i,n})}), \\ v_{i,n+1} \in (V_{p_{i,n+1}})_{e_i(p_{i,n+1})},\ v_{i,n+1} \oplus v_{i,n} \leq \left(1 + \frac{1}{n+1}\right) D((V_{i,p_{i,n+1}})_{e_i(p_{i,n+1})}, (V_{i,p_{i,n}})_{e_i(p_{i,n})}), \end{cases} \tag{4.13}$$

where the sequences $\{\alpha_n\}$ and $\{r_{i,n}\}$ are the same as in Algorithm 4.1.

Theorem 4.2. For each $i \in \Lambda$, let $\mathcal{A}_i,\ h_i,\ \mathcal{G}_i,\ \mathcal{N}_i,\ \mathcal{B}_i,\ S_i,\ U_i,\ V_i,\ d_i,\ c_i,\ e_i$ be the same as in Theorem 4.1 such that all the conditions of Theorem 4.1 are satisfied. Let $Q_i : \mathcal{H}_i \to \mathcal{H}_i$ be a γ_i-ordered Lipschitz continuous mapping and $R = (Q_1, Q_2, \cdots, Q_m) : \prod\limits_{i=1}^{m} \mathcal{H}_i \to \prod\limits_{i=1}^{m} \mathcal{H}_i$ be a $\max\{\gamma_i : i \in \Lambda\}$-ordered Lipschitz continuous mapping with respect to the norm $\|.\|_*$ in $\prod\limits_{i=1}^{m} \mathcal{H}_i$. In addition, assume that the following conditions are satisfied:

$$\begin{cases} \Theta_i = \omega_i(\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \theta_i(\tau_i\beta_i\omega \oplus \lambda_i\mu_i\delta_{U_i} + \lambda_i\kappa_i\delta_{S_i}) < \omega_i \min\left\{1, \frac{1}{\nu_i}\right\}, \\ \Theta_i + \sum\limits_{\ell\in\Lambda,\ \ell\neq i}^{m} \frac{\gamma_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} < 1,\ \theta_i = \frac{1}{\vartheta_i(\alpha_i\lambda_i \oplus 1)} \text{ and } \alpha_i\lambda_i > 1 \text{ for all } i \in \Lambda. \end{cases} \tag{4.14}$$

If $\lim\limits_{n\to\infty} \|(r_{1,n} \vee (-r_{1,n}), r_{2,n} \vee (-r_{2,n}), \cdots, r_{m,n} \vee (-r_{m,n}))\|_* = 0$, then there exists $p_i^*, s_i^* \in \mathcal{H}_i$ such that $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_i^*})_{e_i(p_i^*)}$, for each $i \in \Lambda$ satisfying the extended system of fuzzy ordered resolvent equations (3.5) and so $(p_i^*, q_i^*, u_i^*, v_i^*)$ is a common solution of the extended nonlinear system of fuzzy ordered variational inclusions (3.2) and the fixed point of $Fix(Q_1, Q_2, \cdot, Q_m)$, and the iterative sequences $\{p_{i,n}\}, \{q_{i,n}\}, \{u_{i,n}\}$ and $\{v_{i,n}\}$ generated by Algorithm 4.1 converge strongly $p_i^*,\ q_i^*,\ u_i^*$ and v_i^* in $Fix(Q_1, Q_2, \cdots, Q_m) \cap \mathrm{ESFOVI}(\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m)$, for each $i \in \Lambda$, respectively.

Proof. By Algorithm 4.1, Theorem 4.1, Lemmas 2.1 and 2.3, we have

$$\begin{aligned} \|p_{i,n+1} \oplus p_{i,n}\|_i = & \Big\|[(1-\alpha_n)p_{i,n} + \alpha_n Q_i(p_{i,n} + h_i(p_{i,n}) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n+1})) + r_{i,n}] \\ & \oplus[(1-\alpha_n)p_{i,n-1} + \alpha_n Q_i(p_{i,n-1} + h_i(p_{i,n-1}) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n-1})}(s_{i,n}))]\Big\|_i \\ \leq & (1-\alpha_n)\|p_{i,n} \oplus p_{i,n-1}\|_i + \alpha_n\gamma_i\|(p_{i,n} + h_i(p_{i,n})) \oplus (p_{i,n-1} + h_i(p_{i,n-1}))\|_i \\ & +\alpha_n\gamma_i\Big(\Big\|\mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n+1}) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n})\Big\|_i \\ & +\Big\|\mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n})}(s_{i,n}) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_{i,n-1})}(s_{i,n})\Big\|_i\Big) + \alpha_n\|r_{i,n} \oplus 0\|_i \\ \leq & (1-\alpha_n)\|p_{i,n} \oplus p_{i,n-1}\|_i + \alpha_n\gamma_i\|(p_{i,n} + h_i(p_{i,n})) \oplus (p_{i,n-1} + h_i(p_{i,n-1}))\|_i \\ & +\alpha_n\gamma_i\theta_i\|s_{i,n+1} \oplus s_{i,n}\|_i + \alpha_n\gamma_i\xi_i\|v_{i,n} \oplus v_{i,n-1}\|_i + \alpha_n\|r_{i,n} \oplus 0\|_i. \end{aligned} \tag{4.15}$$

Since h_i is a β_i-ordered compression and a (ζ_i, η_i)-restricted-accerative mapping, respectively, and V_i is δ_{V_i}-D-Lipschitz continuous mapping, we have

$$\begin{aligned} (p_{i,n} + h_i(p_{i,n})) \oplus (p_{i,n-1} + h_i(p_{i,n-1})) \leq & \ \zeta_i(p_{i,n} \oplus p_{i,n-1}) + \eta_i(h_i(p_{i,n}) \oplus h_i(p_{i,n-1})) \\ = & \ (\zeta_i + \eta_i\beta_i)(p_{i,n} \oplus p_{i,n-1}), \end{aligned} \tag{4.16}$$

and

$$(v_{i,n} \oplus v_{i,n-1}) \leq \Big(1 + \frac{1}{n+1}\Big)\delta_{V_i}(p_{i,n} \oplus p_{i,n-1}). \tag{4.17}$$

Since h_i is a β_i-ordered compression mapping, $\mathcal{G}_i$ is a μ_i-ordered compression mapping, $\mathcal{A}_i$ is a τ_i-ordered compression mapping, U_i is a δ_i-ordered compression mapping, and U_i is a δ_{U_i}-D-Lipschitz continuous mapping, we have

$$\begin{aligned} s_{i,n+1} \oplus s_{i,n} = & \ [\mathcal{A}_i(h_i(p_{i,n})) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_{1,n}, q_{2,n}, \cdots, q_{m,n}) \odot \mathcal{G}_i(u_{i,n})) \\ & \oplus[\mathcal{A}_i(h_i(p_{i,n-1})) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_{1,n-1}, q_{2,n-1}, \cdots, q_{m,n-1}) \odot \mathcal{G}_i(u_{i,n-1}))] \\ \leq & \ \Big(\tau_i\beta_i \oplus \frac{\lambda_i\mu_i\delta_{U_i}}{\omega_i}\Big(1 + \frac{1}{n+1}\Big)\Big)(p_{i,n} \oplus p_{i,n-1}) \\ & \oplus\frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_{1,n}, q_{2,n}, \cdots, q_{m,n}) \oplus \mathcal{N}_i(q_{1,n-1}, q_{2,n-1}, \cdots, q_{m,n-1})). \end{aligned} \tag{4.18}$$

Since $\mathcal{N}_i$ is a κ_i-ordered comparison mapping in the i^{th} arguments and a κ_{ij}-ordered comparison mapping in the j^{th} arguments $(i \neq j)$, and S_i is an ordered δ_{S_i}-Lipschitz continuous mapping.

$$\begin{aligned} & \mathcal{N}_i(q_{1,n}, q_{2,n}, \cdots, q_{m,n}) \oplus \mathcal{N}_i(q_{1,n-1}, q_{2,n-1}, \cdots, q_{m,n-1}) \\ \leq & \ \kappa_i\delta_{S_i}\Big(1 + \frac{1}{n+1}\Big)(p_{i,n} \oplus p_{i,n-1}) + \sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\Big(1 + \frac{1}{n+1}\Big)(p_{j,n} \oplus p_{j,n-1}). \end{aligned} \tag{4.19}$$

Using (4.19), (4.18) becomes

$$\|s_{i,n+1} \oplus s_{i,n}\|_i \leq \Big(\tau_i\beta_i \oplus \tfrac{\lambda_i\mu_i\delta_{U_i}}{\omega_i}\Big(1+\tfrac{1}{n+1}\Big) + \tfrac{\lambda_i\kappa_i\delta_{S_i}}{\omega_i}\Big(1+\tfrac{1}{n+1}\Big)\Big)\|p_{i,n} \oplus p_{i,n-1}\|_i + \tfrac{\lambda_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\Big(1+\tfrac{1}{n+1}\Big)\|p_{j,n} \oplus p_{j,n-1}\|_i. \tag{4.20}$$

From (4.20), (4.15) becomes

$$\begin{aligned}\|p_{i,n+1} \oplus p^*_{i,n}\|_i &\leq (1-\alpha_n)\|p_{i,n} \oplus p_{i,n-1}\|_i + \alpha_n\gamma_i(\zeta_i+\eta_i\beta_i)\|p_{i,n} \oplus p_{i,n-1}\|_i \\ &\quad +\alpha_n\gamma_i\theta_i\Big(\tau_i\beta_i \oplus \frac{\lambda_i\mu_i\delta_{U_i}}{\omega_i}\Big(1+\frac{1}{n+1}\Big) + \frac{\lambda_i\kappa_i\delta_{S_i}}{\omega_i}\Big(1+\frac{1}{n+1}\Big)\Big)\|p_{i,n} \oplus p_{i,n-1}\|_i \\ &\quad +\alpha_n\frac{\lambda_i\gamma_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\Big(1+\frac{1}{n+1}\Big)\|p_{j,n} \oplus p_{j,n-1}\|_i \\ &\quad +\alpha_n\gamma_i\xi_i\delta_{V_i}\Big(1+\frac{1}{n+1}\Big)\|p_{i,n} \oplus p_{i,n-1}\|_i + \alpha_n\|r_{i,n} \oplus 0\|_i \\ &\leq (1-\alpha_n)\|p_{i,n} \oplus p_{i,n-1}\|_i + \Theta_{i,n}\|p_{i,n} \oplus p_{i,n-1}\|_i \\ &\quad +\alpha_n\frac{\lambda_i\gamma_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\Big(1+\frac{1}{n+1}\Big)\|p_{j,n} \oplus p_{j,n-1}\|_i + \alpha_n\|r_{i,n} \oplus 0\|_i,\end{aligned} \tag{4.21}$$

where $\Theta_{i,n} = \Big[\gamma_i(\zeta_i+\eta_i\beta_i) + \gamma_i\theta_i\Big(\tau_i\beta_i \oplus \frac{\lambda_i\mu_i\delta_{U_i}}{\omega_i}\Big(1+\frac{1}{n+1}\Big) + \frac{\lambda_i\kappa_i\delta_{S_i}}{\omega_i}\Big(1+\frac{1}{n+1}\Big)\Big) + \gamma_i\xi_i\delta_{V_i}\Big(1+\frac{1}{n+1}\Big)\Big]$.

Using (4.21), we have

$$\begin{aligned}&\|(p_{1,n+1}, p_{2,n+1}, \cdots, p_{m,n+1}) \oplus (p_{1,n}, p_{2,n}, \cdots, p_{m,n})\|_* = \sum_{i=1}^m \|p_{i,n+1} \oplus p_{i,n}\|_i \\ \leq\ & \sum_{i=1}^m \Big[(1-\alpha_n)\|p_{i,n} \oplus p_{i,n-1}\|_i + \alpha_n\Theta_{i,n}\|p_{i,n} \oplus p_{i,n-1}\|_i \\ &+\alpha_n\frac{\gamma_i\lambda_i\theta_i}{\omega_i}\sum_{j\in\Lambda,\ i\neq j}\kappa_{i,j}\delta_{S_{i,j}}\Big(1+\frac{1}{n+1}\Big)\|p_{j,n} \oplus p_{j,n-1}\|_j + \|r_{i,n} \oplus 0\|_i\Big] \\ \leq\ & (1-\alpha_n)\|(p_{1,n}, p_{2,n}, \cdots, p_{m,n}) \oplus (p_{1,n-1}, p_{2,n-1}, \cdots, p_{m,n-1})\|_* \\ &+\alpha_n \max_{1\leq i\leq m}\Big\{\Theta_{i,n} + \Big(1+\frac{1}{n+1}\Big)\sum_{\ell\in\Lambda,\ \ell\neq i}^m \frac{\gamma_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} : i \in \Lambda\Big\}\sum_{i=1}^m \|p_{i,n} \oplus p_{i,n-1}\|_i \\ &+\|(r_{1,n} \vee (-r_{1,n}), r_{2,n} \vee (-r_{2,n}), \cdots, r_{m,n} \vee (-r_{m,n}))\|_*,\end{aligned}$$

i.e.,

$$\begin{aligned}&\|(p_{1,n+1}, p_{2,n+1}, \cdots, p_{m,n+1}) \oplus (p_{1,n}, p_{2,n}, \cdots, p_{m,n})\|_* \\ \leq\ & [1-\alpha_n(1-\Omega_{i,n})]\|(p_{1,n+1}, p_{2,n+1}, \cdots, p_{m,n+1}) \oplus (p_{1,n}, p_{2,n}, \cdots, p_{m,n})\|_* \\ &+\|(r_{1,n} \vee (-r_{1,n}), r_{2,n} \vee (-r_{2,n}), \cdots, r_{m,n} \vee (-r_{m,n}))\|_*,\end{aligned} \tag{4.22}$$

where $\Omega_{i,n} = \max_{1\leq i\leq m}\Big\{\Theta_{i,n} + \Big(1+\frac{1}{n+1}\Big)\sum_{\ell\in\Lambda,\ \ell\neq i}^m \frac{\gamma_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} : i \in \Lambda\Big\}$.

Letting

$$\Omega = \max_{1\leq i\leq m}\Big\{\Theta_i + \sum_{\ell\in\Lambda,\ \ell\neq i}^m \frac{\nu_\ell\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} : i \in \Lambda\Big\}$$

and

$$\Theta_i = \Big[\gamma_i(\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \gamma_i\theta_i\Big(\tau_i\beta_i \oplus \frac{\lambda_i\mu_i\delta_{U_i}}{\omega_i} + \frac{\lambda_i\kappa_i\delta_{S_i}}{\omega_i}\Big)\Big].$$

By condition (4.2), we have $0 \leq \Omega < 1$, thus $\{(p_{1,n}, p_{2,n}, \cdots, p_{m,n})\}$ is a Cauchy sequence in $\prod_{i=1}^{m}\mathcal{H}_i$ and as $\prod_{i=1}^{m}\mathcal{H}_i$ is complete, there exists $(p_1^*, p_2^*, \cdots, p_m^*) \in \prod_{i=1}^{m}\mathcal{H}_i$ such that $(p_{1,n}, p_{2,n}, \cdots, p_{m,n}) \to (p_1^*, p_2^*, \cdots, p_m^*)$ as $n \to \infty$. Additionally, for each $i \in \Lambda$, $p_{i,n} \to p_i^*$ as $n \to \infty$. From (4.12) of Algorithm 4.1 and D-Lipschitz continuity of S_i, U_i and V_i, we have

$$q_{i,n+1} \oplus q_{i,n} \leq \Big(1 + \frac{1}{n+1}\Big)\delta_{D_{S_i}}(p_{i,n+1} \oplus p_{i,n}), \tag{4.23}$$

$$u_{i,n+1} \oplus u_{i,n} \leq \Big(1 + \frac{1}{n+1}\Big)\delta_{D_{U_i}}(p_{i,n+1} \oplus p_{i,n}), \tag{4.24}$$

$$v_{i,n+1} \oplus v_{i,n} \leq \Big(1 + \frac{1}{n+1}\Big)\delta_{D_{V_i}}(p_{i,n+1} \oplus p_{i,n}). \tag{4.25}$$

It is clear from (4.23)–(4.25) that $\{q_{i,n}\}$, $\{u_{i,n}\}$ and $\{v_{i,n}\}$ are also Cauchy sequences in $\mathcal{H}_i$, so there exist q_i^*, u_i^* and v_i^* in $\mathcal{H}_i$ such that $q_{i,n} \to q_i^*$, $u_{i,n} \to u_i^*$ and $v_{i,n} \to v_i^*$ as $n \to \infty$, for each $i \in \Lambda$. Additionally, for each $i \in \Lambda$, by using the continuity of the operators h_i, S_i, U_i, V_i, $\mathcal{J}^{\lambda}_{B(.,v_i^*)}$ and Algorithm 4.1, we have

$$\begin{aligned} p_i^* &= Q_i[p_i^* + h_i(p_i^*) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i^*)}[\mathcal{A}_i(h_i(p_i^*)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*, q_2^*, \cdots, q_m^*) \odot \mathcal{G}_i(u_i^*))]] \\ &= p_i^* + h_i(p_i^*) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i^*)}[\mathcal{A}_i(h_i(p_i^*)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*, q_2^*, \cdots, q_m^*) \odot \mathcal{G}_i(u_i^*))], \end{aligned}$$

which implies that

$$h_i(p_i^*) = \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i^*)}[\mathcal{A}_i(h_i(p_i^*)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1^*, q_2^*, \cdots, q_m^*) \odot \mathcal{G}_i(u_i^*))].$$

By Lemma 3.1, we conclude that $(p_1^*, p_2^*, \cdots, p_m^*)$ is a solution of problem (3.2). It remains to show that $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_I^*})_{e_i(p_i^*)}$. Using Lemma 2.1, in fact,

$$\begin{aligned} \mathbf{d}_i(q_i^*, (S_{i,p_i^*})_{d_i(p_i^*)}) &\leq \|q_i^* \oplus q_{i,n}\|_i + \mathbf{d}_i(q_{i,n}, (S_{i,p_i^*})_{d_i(p_i^*)}) \\ &\leq \|q_i^* \oplus q_{i,n}\|_i + D_i((S_{i,p_{i,n}})_{d_i(p_{i,n})}, (S_{i,p_i^*})_{d_i(p_i^*)}) \\ &\leq \|q_{i,n} \oplus q_i^*\|_i + \delta_{D_{S_i}}\|p_{i,n} \oplus p_i^*\|_i \to 0, \ as\ n \to \infty. \end{aligned}$$

Hence $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$. Similarly, we can show that $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_i^*})_{e_i(p_i^*)}$, for each $i \in \Lambda$. This completes the proof.

Taking $Q_i = I$ (identity mapping), for each $i \in \Lambda$ in Algorithm 4.1, we can also prove the existence and convergence result for the extended nonlinear system of fuzzy ordered variational inclusions involving the $\oplus$ operation (3.1) and the extended nonlinear system of fuzzy ordered resolvent equations problem (3.5).

Corollary 4.1. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i$ be a real Banach space equipped with the norm $\|.\|_i$ and K_i be a normal cone with normal constant ν_i. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings that satisfies the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to [0,1]$ such that for each $p_i \in \mathcal{H}_i$, we have $(S_{i,p_i})_{d_i(p_i)}, (U_{i,p_i})_{c_i(p_i)}$ and $(V_{i,p_i})_{e_i(p_i)}$ in $CB(\mathcal{H}_i)$, respectively. Let $\mathcal{A}_i, h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be nonlinear single-valued mappings. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an ordered (α_i, λ_i)-XOR-weak-ANODD set-valued mapping with respect to the first argument. Suppose that the following conditions hold:

(i) h_i is continuous, β_i-oredered compression and (ζ_i, η_i)-ordered restricted-accretive mapping, $\beta_i \in (0,1)$ and $\zeta_i, \eta_i \in (0,1]$, respectively;
(ii) $\mathcal{A}_i$ is continuous and τ_i-oredered compression mapping, $\tau_i \in (0,1)$;
(iii) $\mathcal{G}_i$ is continuous, ϑ_i-order non-extended mapping and μ_i-oredered compression mapping, $\mu_i \in (0,1)$ and $\vartheta_i > 0$, respectively;
(iv) $\mathcal{N}_i$ is continuous, κ_i-ordered compression mapping in the i^{th}-argument and $\kappa_{i,j}$-ordered compression mapping in the j^{th}-argument for each $j \in \Lambda, i \neq j$, respectively;
(v) S_i, U_i and V_i are ordered Lipschitz type continuous mapping with constants δ_{S_i}, δ_{U_i} and δ_{V_i}, respectively.

In addition, the following conditions hold:

$$(a)\ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,x_i)}(p_i) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,y_i)}(p_i) \leq \xi_i(x_i \oplus y_i),\ \text{for all } p_i, x_i, y_i \in \mathcal{H}_i,\ \xi_i > 0, \tag{4.26}$$

$$(b)\ \begin{cases} \Theta_i = \omega_i(\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \theta_i(\tau_i\beta_i\omega_i \oplus \lambda_i\mu_i\delta_{U_i} + \lambda_i\kappa_i\delta_{S_i}) < \omega_i, \\ \Theta_i + \sum\limits_{\ell\in\Lambda,\ \ell\neq i}^{m} \frac{\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} < 1,\ \theta_i = \frac{1}{\vartheta_i(\alpha_i\lambda_i \oplus 1)} \text{ and } \alpha_i\lambda_i > 1,\ \text{for all } i \in \Lambda. \end{cases} \tag{4.27}$$

If $\lim\limits_{n\to\infty} \|(r_{1,n} \vee (-r_{1,n}), r_{2,n} \vee (-r_{2,n}), \cdots, r_{m,n} \vee (-r_{m,n}))\|_* = 0$, then there exists $p_i^*, s_i^* \in \mathcal{H}_i$ such that $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_i^*})_{e_i(p_i^*)}$, for each $i \in \Lambda$ that satisfies the extended system of fuzzy ordered resolvent equations (3.5) and so $(p_i^*, q_i^*, u_i^*, v_i^*)$ is a solution of the extended system of fuzzy ordered variational inclusions (3.2), and the iterative sequences $\{p_{i,n}\}, \{q_{i,n}\}, \{u_{i,n}\}$, and $\{v_{i,n}\}$ generated by Algorithm 4.2 converge strongly p_i^*, q_i^*, u_i^* and v_i^* in ESFOVI$(\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m)$, for each $i \in \Lambda$, respectively.

Taking $\mathcal{G}_i = I$ (identity mapping), for each $i \in \Lambda$ in Algorithm 4.1, we can also prove the existence and convergence results for the extended nonlinear system of fuzzy ordered variational inclusions involving the $\oplus$ operation (3.1) and the extended nonlinear system of fuzzy ordered resolvent equations problem (3.5).

Corollary 4.2. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i$ be a real Banach space equipped with the norm $\|.\|_i$ and K_i be a normal cone with normal constant ν_i. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings satisfying the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to [0,1]$ such that for each $p_i \in \mathcal{H}_i$, we have $(S_{i,p_i})_{d_i(p_i)}, (U_{i,p_i})_{c_i(p_i)}$ and $(V_{i,p_i})_{e_i(p_i)}$ in $CB(\mathcal{H}_i)$, respectively. Let $\mathcal{A}_i, h_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be the nonlinear single-valued mappings. Let $Q_i : \mathcal{H}_i \to \mathcal{H}_i$ be a γ_i-ordered Lipschitz continuous mapping and $R = (Q_1, Q_2, \cdots, Q_m) : \prod_{i=1}^{m} \mathcal{H}_i \to \prod_{i=1}^{m} \mathcal{H}_i$ be a $\max\{\gamma_i : i \in \Lambda\}$-ordered Lipschitz continuous mapping with respect to the norm $\|.\|_*$ in $\prod_{i=1}^{m} \mathcal{H}_i$. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be a ordered (α_i, λ_i)-XOR-weak-ANODD set-valued mapping with respect to the first argument. Suppose that the following conditions hold:

(i) h_i is continuous, β_i-oredered compression and (ζ_i, η_i)-ordered restricted-accretive mapping, $\beta_i \in (0, 1)$ and $\zeta_i, \eta_i \in (0, 1]$, respectively;

(ii) $\mathcal{A}_i$ is continuous and τ_i-oredered compression mapping, $\tau_i \in (0, 1)$;

(iii) $\mathcal{N}_i$ is continuous, κ_i-ordered compression mapping in the i^{th}-argument and $\kappa_{i,j}$-ordered compression mapping in the j^{th}-argument for each $j \in \Lambda, i \neq j$, respectively;

(iv) S_i, U_i and V_i are ordered Lipschitz type continuous mapping with constants δ_{S_i}, δ_{U_i} and δ_{V_i}, respectively.

In addition, the following conditions hold:

$$(a)\ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,x_i)}(p_i) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,y_i)}(p_i) \leq \xi_i(x_i \oplus y_i),\ \text{for all } p_i, x_i, y_i \in \mathcal{H}_i,\ \xi_i > 0, \tag{4.28}$$

$$(b)\ \begin{cases} \Theta_i = \omega_i(\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \theta_i(\tau_i\beta_i\omega_i \oplus \lambda_i\delta_{U_i} + \lambda_i\kappa_i\delta_{S_i}) < \omega_i \min\left\{1, \frac{1}{\nu_i}\right\}, \\ \Theta_i + \sum\limits_{\ell\in\Lambda,\ \ell\neq i}^{m} \frac{\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} < 1,\ \theta_i = \frac{1}{\vartheta_i(\alpha_i\lambda_i\oplus 1)} \text{ and } \alpha_i\lambda_i > 1,\ \text{for all } i \in \Lambda. \end{cases} \tag{4.29}$$

If $\lim\limits_{n\to\infty} \|(r_{1,n} \vee (-r_{1,n}), r_{2,n} \vee (-r_{2,n}), \cdots, r_{m,n} \vee (-r_{m,n}))\|_* = 0$, then there exists $p_i^*, s_i^* \in \mathcal{H}_i$ such that $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_i^*})_{e_i(p_i^*)}$, for each $i \in \Lambda$ satisfying the extended nonlinear system of fuzzy ordered resolvent equation (3.5) and so $(p_i^*, q_i^*, u_i^*, v_i^*)$ is a common solution of the extended nonlinear system of fuzzy ordered variational inclusions (3.2) and the fixed point of $Fix(Q_1, Q_2, \cdot, Q_m)$, and the iterative sequences $\{p_{i,n}\}$, $\{q_{i,n}\}$, $\{u_{i,n}\}$ and $\{v_{i,n}\}$ generated by Algorithm 4.1 converge strongly p_i^*, q_i^*, u_i^* and v_i^* in $Fix(Q_1, Q_2, \cdot, Q_m) \cap$ ENSFOVI$(\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2, \cdots, m)$, for each $i \in \Lambda$, respectively.

Taking $\alpha_n = 1$, for all $n \in \mathbb{N}$ in Algorithm 4.1, we can also prove the existence and convergence result for the extended nonlinear system of fuzzy ordered variational inclusions involving the $\oplus$ operation (3.1) and the extended nonlinear system of fuzzy ordered resolvent equations problem (3.5).

Corollary 4.3. For each $i \in \Lambda = \{1, 2, 3, \cdots, m\}$, let $\mathcal{H}_i$ be a real Banach space equipped with the norm $\|.\|_i$ and K_i be a normal cone with normal constant ν_i. Let $S_i, U_i, V_i : \mathcal{H}_i \to \mathcal{F}_i(\mathcal{H}_i)$ be closed fuzzy mappings satisfying the following condition $(*)$, with functions $d_i, c_i, e_i : \mathcal{H}_i \to [0, 1]$ such that for each $p_i \in \mathcal{H}_i$, we have $(S_{i,p_i})_{d_i(p_i)}$, $(U_{i,p_i})_{c_i(p_i)}$ and $(V_{i,p_i})_{e_i(p_i)}$ in $CB(\mathcal{H}_i)$, respectively. Let $\mathcal{A}_i, h_i, \mathcal{G}_i : \mathcal{H}_i \to \mathcal{H}_i$ and $\mathcal{N}_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ be the nonlinear single-valued mappings. Let $\mathcal{B}_i : \mathcal{H}_i \times \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be a ordered (α_i, λ_i)-XOR-weak-ANODD set-valued mapping with respect to the first argument. Suppose that the following conditions hold:

(i) h_i is continuous, β_i-oredered compression and (ζ_i, η_i)-ordered restricted-accretive mapping, $\beta_i \in (0, 1)$ and $\zeta_i, \eta_i \in (0, 1]$, respectively;

(ii) $\mathcal{A}_i$ is continuous and τ_i-oredered compression mapping, $\tau_i \in (0, 1)$;

(iii) $\mathcal{G}_i$ is continuous, ϑ_i-order non-extended mapping and μ_i-oredered compression mapping, $\mu_i \in (0, 1)$ and $\vartheta_i > 0$, respectively;

(iv) $\mathcal{N}_i$ is continuous, κ_i-ordered compression mapping in the i^{th}-argument and $\kappa_{i,j}$-ordered compression mapping in the j^{th}-argument for each $j \in \Lambda, i \neq j$, respectively;

(v) S_i, U_i and V_i are ordered Lipschitz type continuous mapping with constants δ_{S_i}, δ_{U_i} and δ_{V_i}, respectively.

In addition, the following conditions hold:

$$(a)\ \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,x_i)}(p_i) \oplus \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,y_i)}(p_i) \leq \xi_i(x_i \oplus y_i),\ \text{for all } p_i, x_i, y_i \in \mathcal{H}_i,\ \xi_i > 0, \tag{4.30}$$

$$(b)\ \begin{cases}\Theta_i = (\zeta_i + \eta_i\beta_i + \xi_i\delta_{V_i}) + \theta_i(\tau_i\beta_i \oplus \lambda_i\mu_i\delta_{U_i} + \lambda_i\kappa_i\delta_{S_i}) < 1,\\ \Theta_i + \sum\limits_{\ell\in\Lambda,\ \ell\neq i}^{m} \frac{\lambda_\ell\theta_\ell}{\omega_\ell}\kappa_{\ell,i}\delta_{S_{\ell,i}} < 1,\ \theta_i = \frac{1}{\vartheta_i(\alpha_i\lambda_i\oplus 1)} \text{ and } \alpha_i\lambda_i > 1, \text{ for all } i\in\Lambda.\end{cases} \tag{4.31}$$

If $\lim\limits_{n\to\infty} \|(r_{1,n}\vee(-r_{1,n}), r_{2,n}\vee(-r_{2,n}),\cdots, r_{m,n}\vee(-r_{m,n}))\|_* = 0$, then there exists $p_i^*, s_i^* \in \mathcal{H}_i$ such that $q_i^* \in (S_{i,p_i^*})_{d_i(p_i^*)}$, $u_i^* \in (U_{i,p_i^*})_{c_i(p_i^*)}$ and $v_i^* \in (V_{i,p_i^*})_{e_i(p_i^*)}$, for each $i\in\Lambda$ satisfying the extended nonlinear system of fuzzy ordered resolvent equation (3.5) and so $(p_i^*, q_i^*, u_i^*, v_i^*)$ is a common solution of the extended nonlinear system of fuzzy ordered variational inclusions (3.2) and the fixed point of $Fix(Q_1, Q_2, \cdot, Q_m)$ and the iterative sequences $\{p_{i,n}\}$, $\{q_{i,n}\}$, $\{u_{i,n}\}$ and $\{v_{i,n}\}$ generated by Algorithm 4.1 converge strongly p_i^*, q_i^*, u_i^* and v_i^* in $Fix(Q_1, Q_2,\cdots, Q_m)\cap$ ENSFOVI($\mathcal{N}_i, \mathcal{G}_i, \mathcal{B}_i, h_i, i = 1, 2,\cdots, m$), for each $i\in\Lambda$, respectively.

The following numerical example gives the guarantee that all the proposed conditions of Theorems 4.1 and 4.2 are satisfied.

Example 4.1. For each $i\in\Lambda = \{1, 2, 3,\cdots, m\}$, and let $\mathcal{H}_i = \mathbb{R}$, with the usual inner product and norm and $K_i = \{p_i \in \mathcal{H}_i : 0 \le p_i \le 1\}$ be a normal cone with normal constant $\delta_i = \frac{1}{i}$. Let S_i, U_i, V_i and d_i, c_i, e_i be defined the same as in Example 3.1. Let $h_i, \mathcal{A}_i, \mathcal{G}_i, Q_i : \mathcal{H}_i \to \mathcal{H}_i$, and $\mathcal{N}_i : \prod\limits_{j=1}^{m}\mathcal{H}_j \to \mathcal{H}_i$ be the mappings defined by for all $p_i \in \mathcal{H}_i$ and $j\in\Lambda$,

$$h_i(p_i) = \frac{p_i}{13i},\ \mathcal{A}_i(p_i) = \frac{p_i}{3i},\ \mathcal{G}_i(p_i) = \frac{p_i}{7i},\ Q_i(p_i) = \frac{p_i}{2^i} \text{ and } T_i(p_1, p_2,\cdots, p_j,\cdots, p_m) = \frac{x_j}{30ij}.$$

It is easy to verify that h_i is a $\frac{1}{10i}$-ordered compression and an $(\frac{1}{11i}, 1)$-ordered restricted-accretive mapping, $\mathcal{G}_i$ is $\frac{1}{9i}$-ordered compression and $\frac{1}{5i}$-ordered non-extended mapping, and $\mathcal{A}_i$ is $\frac{1}{2i}$-ordered compression mapping. Further,

$$\begin{aligned}&\mathcal{N}_i(p_1, p_2,\cdots, p_{j-1}, p_j, p_{j+1},\cdots, p_m)\oplus\mathcal{N}_i(p_1, p_2,\cdots, p_{i-1}, \hat{p}_i, p_{i+1},\cdots, p_m)\\ =\ & \frac{p_i}{30i^2}\oplus\frac{\hat{p}_i}{30i^2} \le \frac{1}{30i}(p_i\oplus\hat{p}_i).\end{aligned}$$

Hence, $\mathcal{N}_i$ is a $\frac{1}{30i}$-ordered compression mapping in the i^{th} argument.

$$\begin{aligned}&\mathcal{N}_i(p_1, p_2,\cdots, p_m)\oplus\mathcal{N}_i(\hat{p}_1, \hat{p}_2,\cdots, \hat{p}_m)\\ \le\ & \mathcal{N}_i(p_1, p_2,\cdots, p_{i-1}, p_i, p_{i+1},\cdots, p_m)\oplus\mathcal{N}_i(p_1, p_2,\cdots, p_{i-1}, \hat{p}_i, p_{i+1},\cdots, p_m)\\ &+\sum_{j\in\Lambda,\ i\neq j}(\mathcal{N}_i(p_1, p_2,\cdots, p_{j-1}, p_j, p_{j+1},\cdots, p_m)\oplus\mathcal{N}_i(p_1, p_2,\cdots, p_{j-1}, \hat{p}_j, p_{j+1},\cdots, p_m))\\ =\ & \frac{1}{30i^2}(p_i\oplus\hat{p}_i) + \sum_{j\in\Lambda,\ i\neq j}\frac{1}{30ij}(p_j\oplus\hat{p}_j) \le \frac{1}{30i}(p_i\oplus\hat{p}_i) + \sum_{j\in\Lambda,\ i\neq j}\frac{1}{30ij}(p_j\oplus\hat{p}_j).\end{aligned}$$

Suppose that the mappings $\mathcal{B}_i : \mathcal{H}_i\times\mathcal{H}_i \to 2^{\mathcal{H}_i}$ are defined by

$$\mathcal{B}_i(h_i(p_i), p_i) = \{13i^3 h_i(p_i) + 4i^2 p_i\} = \{5i^2 p_i\},\ \forall\ p_i \in \mathcal{H}_i.$$

It is easy to verify that $\mathcal{B}_i$ is a $2i^2$-ordered rectangular compression mapping and a $\frac{1}{i}$-weak-ordered different comparison mapping. Additionally, it is clear that for $\lambda_i = \frac{1}{i}$, $[\mathcal{G}_i\oplus\lambda_i\mathcal{B}_i](\mathcal{H}_i) = \mathcal{H}_i$, for each $i\in\Lambda$. Hence, $\mathcal{B}_i$ is an ordered $(2i^2, \frac{1}{i})$-XOR-weak ANODD set-valued mapping.

The resolvent operator defined by (2.1) associated with $\mathcal{B}_i$ is given by

$$\mathcal{R}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}(p_i) = \frac{7i}{1 \oplus 35i^2} p_i, \ \forall p_i \in \mathcal{H}_i, \tag{4.32}$$

It is easy to examine that the resolvent operator defined above is a comparison, a single-valued mapping, and $\mathcal{R}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}$ is $\frac{55i^2}{11i-1}$-ordered Lipschitz continuous.

For each $i \in \Lambda$, in particular $\omega_i = 2i$ and we define $\phi_i : \prod_{j=1}^{m} \mathcal{H}_j \to \mathcal{H}_i$ by

$$\begin{aligned}
\phi_i(p_1, p_2, \cdots, p_m) &= p_i + h_i(p_i) - \mathcal{J}^{\lambda_i}_{\mathcal{B}_i(.,v_i)}[\mathcal{A}_i(h_i(p_i)) \oplus \frac{\lambda_i}{\omega_i}(\mathcal{N}_i(q_1, q_2, \cdots, q_m) \odot \mathcal{G}_i(u_i))] \\
&= \Big(\frac{13i+1}{13i} - \frac{7i}{35i^2-1}\Big(\frac{(60i^2-7i)}{420i^5} - \frac{1}{39i^2}\Big)\Big)p_i.
\end{aligned}$$

It also confirms that assumptions (4.2) and (4.14) are fulfilled, where $\beta_i = \frac{1}{10i}$, $\zeta_i = \frac{1}{11i}$, $\eta_i = 1$, $\tau_i = \frac{1}{2i}$, $\mu_i = \frac{1}{9i}$, $\vartheta_i = \frac{1}{5i}$, $\xi_i = 1$, $\kappa_i = \frac{1}{30i}$, $\kappa_{ij} = \frac{1}{30ij}$, $\alpha_i = 2i^2$, $\lambda_i = \frac{1}{i}$, $\omega_i = 2i$, $\delta_{S_i} = \frac{1}{4i}$, $\delta_{U_i} = \frac{1}{2i}$, $\delta_{V_i} = \frac{1}{6i}$ and $\theta_i = \frac{55i^2}{11i-1}$. Therefore, all the conditions of Theorems 4.1 and 4.2 are satisfied. Therefore, $(0, 0, \cdots, 0)$ is a fixed point of the mapping $\psi(., ., \cdots, .) = (\phi_1(.), \phi_2(.), \cdots, \phi_p(.))$ defined by (4.5) as well as the fixed point of $R = (Q_1, Q_2, \cdot, Q_m)$. By Lemma 3.1, $(0, 0, \cdots, 0)$ is a common solution of the extended nonlinear system of fuzzy ordered variational inclusions (3.2) and the fixed point of $R = (Q_1, Q_2, \cdot, Q_m)$.

5. Conclusions

In the draft, we had discussed an extended system of fuzzy ordered variational inclusions and its corresponding extended system of fuzzy ordered resolvent equations with very suitable binary structures in an ordered Banach space. We had looked upon the existence of the solution of an extended system of fuzzy ordered variational inclusions and its corresponding extended system of fuzzy ordered resolvent equations. On the basis of fixed point formulation, we formulated iterative schemes for the said system of problems corresponding the resolvent equations involving special binary operations and the fixed point problem. Furthermore, we discussed the existence of common solution and discuss the convergence of the sequence of iterates generated by the algorithm for a considered problems. At the end, we discussed some consequences of our main results. Notice that the benefits of such systems on future research may work upon the forward-backward splitting method based on the inertial technique for solving ordered inclusion problems and also develop some better versions of the algorithms for solving the ordered inclusion problems in real ordered product Banach spaces with XOR and XNOR operations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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