Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 17981–17995.
DOI: 10.3934/math.2023914
Received: 10 February 2023
Revised: 03 May 2023
Accepted: 09 May 2023
Published: 24 May 2023

*Research article*

# On a nonlinear coupled Caputo-type fractional differential system with coupled closed boundary conditions

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**Abstract:** We introduce a novel notion of coupled closed boundary conditions and investigate a nonlinear system of Caputo fractional differential equations equipped with these conditions. The existence result for the given problem is proved via the Leray-Schauder alternative, while the uniqueness of its solutions is accomplished by applying the Banach fixed point theorem. Examples are constructed for the illustration of the main results. Some special cases arising from the present study are discussed.

**Keywords:** Caputo fractional derivative; system; coupled closed boundary conditions; existence; fixed point theorems
**Mathematics Subject Classification:** 34A08, 34B15

## 1. Introduction

An overwhelming interest has been shown in developing the subject of fractional calculus during the past few decades. It has been owing to the application of nonlocal fractional order derivative and integral operators in the mathematical modeling of several problems occurring in scientific and technical disciplines. Examples include fractional diffusion [1–3], immune systems [4], ecology [5], neural networks [6, 7], chaotic synchronization [8, 9], etc. Since the mathematical models associated with physical problems consist of fractional differential equations subject to initial and boundary conditions, therefore, many researchers focused on developing the topic of fractional order initial and boundary value problems. One can find an uptodate account of these problems in the book [10], while a variety of recent results involving different kinds of fractional derivatives can be found in the articles [11–22]. For the basic concepts of fractional calculus, for instance, see the text [23]. Keeping in mind the importance of fractional differential systems appearing in the mathematical models of physical and engineering processes [24–28], many investigators discussed the theoretical aspects of such systems complemented with different boundary conditions, for instance, see the articles [29–37].

Inspired by aforementioned works on fractional differential systems, in this paper, we introduce and investigate a system of nonlinear Caputo fractional differential equations:

$$\begin{cases} {}^{C}D^{q_1}\varphi(t) = \rho_1(t, \varphi(t), \psi(t)), & t \in \mathcal{J} = [0, T], \\ {}^{C}D^{q_2}\psi(t) = \rho_2(t, \varphi(t), \psi(t)), & t \in \mathcal{J} = [0, T], \end{cases} \tag{1.1}$$

complemented with a new class of coupled closed boundary conditions:

$$\begin{cases} \varphi(T) = \alpha_1\psi(0) + \beta_1 T\psi'(0), & T\varphi'(T) = \gamma_1\psi(0) + \delta_1 T\psi'(0), \\ \psi(T) = \alpha_2\varphi(0) + \beta_2 T\varphi'(0), & T\psi'(T) = \gamma_2\varphi(0) + \delta_2 T\varphi'(0), \end{cases} \tag{1.2}$$

where ${}^{C}D^{q_1}$, ${}^{C}D^{q_2}$ denote the Caputo fractional derivatives of order $q_1, q_2$, $1 < q_1, q_2 < 2$, respectively, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$, $T > 0$, and $\rho_1, \rho_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Here, we point out that the closed boundary conditions appear in several practical situations such as Abelian sandpile model [38], honeycomb lattice [39], deblurring problems [40], wavefield decomposition in solid media [41], magneto-electro-elastic cylindrical composite panel [42], etc. In a recent work [43], the authors introduced the concept of nonlocal closed boundary conditions.

The existence and uniqueness results for the problems (1.1) and (1.2) are proved with the aid of standard fixed point theorems. It is worthwhile to mention that the work presented in this paper is new and contributes to the literature on fractional-order systems in a significant manner.

The structure of the remaining paper is as follows. We collect some preliminary definitions and solve a linear version of the problems (1.1) and (1.2) in Section 2. The main results are accomplished in Section 3. Illustrative examples are also given in Section 3. Section 4 contains the concluding remarks and indicates some special cases arising from the present work.

## 2. A preliminary result

Before proceeding for a preliminary result dealing with the linear version of the problems (1.1) and (1.2), we enlist the related definitions from fractional calculus [23].

**Definition 2.1.** *For $\sigma \in L_1[a, b]$, we define the (left) Riemann-Liouville fractional integrals of order $p > 0$ as*

$$I_a^p\sigma(t) = \int_a^t \frac{(t - \bar{t})^{p-1}}{\Gamma(p)}\sigma(\bar{t})d\bar{t}.$$

**Definition 2.2.** *The (left) Caputo fractional derivative for a function $\sigma \in AC^m[a, b]$ of order $p \in (m - 1, m]$, $m \in \mathbb{N}$ is defined by*

$${}^{C}D_a^p\sigma(t) = \int_a^t \frac{(t - \bar{t})^{m-p-1}}{\Gamma(m - p)}\sigma^{(m)}(\bar{t})d\bar{t}.$$

**Lemma 2.1.** *For $F, G \in C(\mathcal{J}, \mathbb{R})$ and $\Delta \neq 0$, the linear system*

$$\begin{cases} {}^C D^{q_1}\varphi(t) = F(t), \quad t \in \mathcal{J}, \\ {}^C D^{q_2}\psi(t) = G(t), \quad t \in \mathcal{J}, \\ \varphi(T) = \alpha_1\psi(0) + \beta_1 T\psi'(0), \quad T\varphi'(T) = \gamma_1\psi(0) + \delta_1 T\psi'(0), \\ \psi(T) = \alpha_2\varphi(0) + \beta_2 T\varphi'(0), \quad T\psi'(T) = \gamma_2\varphi(0) + \delta_2 T\varphi'(0), \end{cases} \tag{2.1}$$

*is equivalent to the fractional integral equations:*

$$\begin{aligned} \varphi(t) &= \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon \\ &-\frac{T}{\Delta}\Big\{a_1(t)\int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon + a_2(t)\int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon \\ &+a_3(t)\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}F(\upsilon)\,d\upsilon + a_4(t)\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)\,d\upsilon\Big\}, \end{aligned} \tag{2.2}$$

*and*

$$\begin{aligned} \psi(t) &= \int_0^t \frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon \\ &-\frac{T}{\Delta}\Big\{b_1(t)\int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon + b_2(t)\int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon \\ &+b_3(t)\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}F(\upsilon)\,d\upsilon + b_4(t)\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)\,d\upsilon\Big\}, \end{aligned} \tag{2.3}$$

*where*

$$\begin{aligned} a_1(t) &= [(1-\delta_1\delta_2) + (\delta_2-\beta_2)\gamma_1]T + [\alpha_2\gamma_1 + (\delta_1-\gamma_1)\gamma_2]t, \\ a_2(t) &= [(1-\delta_1\delta_2)\alpha_1 - (1-\beta_1\delta_2)\gamma_1]T + [\gamma_1 + (\alpha_1\delta_1-\beta_1\gamma_1)\gamma_2]t, \\ a_3(t) &= [(\alpha_1\beta_2-1) + (\beta_1-\alpha_1)\delta_2]T + [(1-\alpha_1\alpha_2) + (\alpha_1-\beta_1)\gamma_2]t, \\ a_4(t) &= [(\alpha_1\beta_2-1)\delta_1 + (1-\beta_1\beta_2)\gamma_1 + (\beta_1-\alpha_1)]T + [(\delta_1-\gamma_1) - (\alpha_1\delta_1-\beta_1\gamma_1)\alpha_2]t, \\ b_1(t) &= [(\alpha_2-\gamma_2) - (\alpha_2\delta_2-\beta_2\gamma_2)\delta_1]T + [\gamma_2 + (\alpha_2\delta_2-\beta_2\gamma_2)\gamma_1]t, \\ b_2(t) &= [(1-\delta_1\delta_2) - (\beta_1-\delta_1)\gamma_2]T + [\alpha_1\gamma_2 + (\delta_2-\gamma_2)\gamma_1]t, \\ b_3(t) &= [(\beta_2-\alpha_2) + (\gamma_2-\delta_2) + (\alpha_2\delta_2-\beta_2\gamma_2)\beta_1]T + [(\delta_2-\gamma_2) - (\alpha_2\delta_2-\beta_2\gamma_2)\alpha_1]t, \\ b_4(t) &= [\alpha_2\beta_1-1) + (\beta_2-\alpha_2)\delta_1]T + [(1-\alpha_1\alpha_2) + (\alpha_2-\beta_2)\gamma_1]t, \end{aligned} \tag{2.4}$$

*and*

$$\Delta = T^2[1-\alpha_1\alpha_2-\delta_1\delta_2-\gamma_1\gamma_2+(\alpha_2-\beta_2+\delta_2)\gamma_1+(\alpha_1-\beta_1+\delta_1)\gamma_2+(\alpha_1\delta_1-\beta_1\gamma_1)(\alpha_2\delta_2-\beta_2\gamma_2)]. \tag{2.5}$$

*Proof.* As argued in [23] (see page 199), for some constant $c_0, c_1, c_2$ and $c_3 \in \mathbb{R}$, the solution of the linear system of fractional differential equations in (2.1) can be written as

$$\varphi(t) = \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon - c_0 - c_1 t, \tag{2.6}$$

$$\psi(t) = \int_0^t \frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon - c_2 - c_3 t. \tag{2.7}$$

Using (2.6) and (2.7) in the boundary conditions of the problem (2.1) yields

$$\begin{aligned}
c_0 + Tc_1 - \alpha_1 c_2 - T\beta_1 c_3 &= \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon,\\
Tc_1 - \gamma_1 c_2 - T\delta_1 c_3 &= T\int_0^T \frac{(T-s)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)d\upsilon,\\
-\alpha_2 c_0 - T\beta_2 c_1 + c_2 + Tc_3 &= \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon,\\
-\gamma_2 c_0 - T\delta_2 c_1 + Tc_3 &= T\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)d\upsilon.
\end{aligned} \tag{2.8}$$

Solving the system (2.8) for $c_0, c_1, c_2$ and $c_3$, we obtain

$$\begin{aligned}
c_0 &= \frac{T^2}{\Delta}\Bigg\{[(1-\delta_1\delta_2) + (\delta_2 - \beta_2)\gamma_1] \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon\\
&\quad + [(1-\delta_1\delta_2)\alpha_1 - (1-\beta_1\delta_2)\gamma_1] \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon\\
&\quad + [(\alpha_1\beta_2 - 1) + (\beta_1 - \alpha_1)\delta_2]T \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)d\upsilon\\
&\quad + [(\alpha_1\beta_2 - 1)\delta_1 + (1-\beta_1\beta_2)\gamma_1 + (\beta_1 - \alpha_1)]T \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)d\upsilon\Bigg\},\\
c_1 &= \frac{T}{\Delta}\Bigg\{[\alpha_2\gamma_1 + (\delta_1 - \gamma_1)\gamma_2] \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon\\
&\quad + [\gamma_1 + (\alpha_1\delta_1 - \beta_1\gamma_1)\gamma_2] \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon\\
&\quad + [(1-\alpha_1\alpha_2) + (\alpha_1 - \beta_1)\gamma_2]T \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)d\upsilon\\
&\quad + [(\delta_1 - \gamma_1) - (\alpha_1\delta_1 - \beta_1\gamma_1)\alpha_2]T \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)d\upsilon\Bigg\},\\
c_2 &= \frac{T^2}{\Delta}\Bigg\{[(\alpha_2 - \gamma_2) - (\alpha_2\delta_2 - \beta_2\gamma_2)\delta_1] \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon\\
&\quad + [(1-\delta_1\delta_2) - (\beta_1 - \delta_1)\gamma_2] \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)d\upsilon\\
&\quad + [(\beta_2 - \alpha_2) + (\gamma_2 - \delta_2) + (\alpha_2\delta_2 - \beta_2\gamma_2)\beta_1]T \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)d\upsilon\\
&\quad + [\alpha_2\beta_1 - 1) + (\beta_2 - \alpha_2)\delta_1]T \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)d\upsilon\Bigg\},\\
c_3 &= \frac{T}{\Delta}\Bigg\{[\gamma_2 + (\alpha_2\delta_2 - \beta_2\gamma_2)\gamma_1] \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)d\upsilon
\end{aligned}$$

$$+[\alpha_1\gamma_2+(\delta_2-\gamma_2)\gamma_1]\int_0^T\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)d\upsilon$$
$$+[(\delta_2-\gamma_2)-(\alpha_2\delta_2-\beta_2\gamma_2)\alpha_1]T\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}F(\upsilon)d\upsilon$$
$$+[(1-\alpha_1\alpha_2)+(\alpha_2-\beta_2)\gamma_1]T\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)d\upsilon\Big\}.$$

Inserting the above values of $c_i, i = 0,1,2,3$, in (2.6) and (2.7) together with (2.4), we get the solutions (2.2) and (2.3). By direct computation, one can obtain the converse of the lemma.

## 3. Main results

Let $\Theta$ denote the Banach space of all continuous functions from $\mathcal{J}$ to $\mathbb{R}$ equipped with the supremum norm $\|\vartheta\| = \sup_{t\in\mathcal{J}}|\vartheta(t)|$. Then the product space $\Theta\times\Theta$ is also a Banach space endowed with the norm $\|(\vartheta_1,\vartheta_2)\| = \|\vartheta_1\|+\|\vartheta_2\|, (\vartheta_1,\vartheta_2)\in\Theta\times\Theta$.

We define an operator $\mathcal{H}:\Theta\times\Theta\to\Theta\times\Theta$ by

$$\mathcal{H}(\varphi,\psi)(t)=\begin{pmatrix}\mathcal{H}_1(\varphi,\psi)(t)\\ \mathcal{H}_2(\varphi,\psi)(t)\end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}&\mathcal{H}_1(\varphi,\psi)(t)\\ =&\int_0^t\frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon-\frac{T}{\Delta}\Big\{a_1(t)\int_0^T\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\ &+a_2(t)\int_0^T\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon+a_3(t)\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\ &+a_4(t)\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Big\},\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\mathcal{H}_2(\varphi,\psi)(t)\\ =&\int_0^t\frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon-\frac{T}{\Delta}\Big\{b_1(t)\int_0^T\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\ &+b_2(t)\int_0^T\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon+b_3(t)\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\ &+b_4(t)\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Big\}.\end{aligned} \tag{3.3}$$

We need the following hypotheses in the sequel:

$(H_1)$ Assume that $\rho_1,\rho_2\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exist real constants $m_i,n_i\geq 0,(i=1,2)$ and $m_0,n_0>0$ such that

$$|\rho_1(t,\varphi,\psi)|\leq m_0+m_1|\varphi|+m_2|\psi|,\quad |\rho_2(t,\varphi,\psi)|\leq n_0+n_1|\varphi|+n_2|\psi|, \forall\varphi,\psi\in\mathbb{R}.$$

$(H_2)$ There exist constants $\ell_i, \overline{\ell}_i, i = 1, 2$, such that for all $t \in \mathcal{J}, \varphi_i, \psi_i \in \mathbb{R}, i = 1, 2$,

$$|\rho_1(t, \varphi_1, \psi_1) - \rho_1(t, \varphi_2, \psi_2)| \le \ell_1|\varphi_1 - \varphi_2| + \ell_2|\psi_1 - \psi_2|,$$
$$|\rho_2(t, \varphi_1, \psi_1) - \rho_2(t, \varphi_2, \psi_2)| \le \overline{\ell}_1|\varphi_1 - \varphi_2| + \overline{\ell}_2|\psi_1 - \psi_2|.$$

We introduce, for computational convenience, the following notation:

$$\begin{aligned}
\chi_1 &= \max_{t\in[0,T]} \Big\{\frac{t^{q_1}}{\Gamma(q_1+1)} + \frac{1}{|\Delta|}\Big[\frac{T^{q_1+1}|a_1(t)|}{\Gamma(q_1+1)} + \frac{T^{q_1}|a_3(t)|}{\Gamma(q_1)}\Big]\Big\},\\
\chi_2 &= \max_{t\in[0,T]} \frac{1}{|\Delta|}\Big\{\frac{T^{q_2+1}|a_2(t)|}{\Gamma(q_2+1)} + \frac{T^{q_2}|a_4(t)|}{\Gamma(q_2)}\Big\},\\
\chi_3 &= \max_{t\in[0,T]} \Big\{\frac{t^{q_2}}{\Gamma(q_2+1)} + \frac{1}{|\Delta|}\Big[\frac{T^{q_2+1}|b_2(t)|}{\Gamma(q_2+1)} + \frac{T^{q_2}|b_4(t)|}{\Gamma(q_2)}\Big]\Big\},\\
\chi_4 &= \max_{t\in[0,T]} \frac{1}{|\Delta|}\Big\{\frac{T^{q_1+1}|b_1(t)|}{\Gamma(q_1+1)} + \frac{T^{q_1}|b_3(t)|}{\Gamma(q_1)}\Big\},
\end{aligned} \tag{3.4}$$

and

$$\chi_0 = \min\{1 - [m_1(\chi_1 + \chi_4) + n_1(\chi_2 + \chi_3)], 1 - [m_2(\chi_1 + \chi_4) + n_2(\chi_2 + \chi_3)]\}. \tag{3.5}$$

The platform is now set to present our main results. Our first result, dealing with the existence of solutions for the problems (1.1) and (1.2), relies on the Leray-Schauder alternative [44].

**Theorem 3.1.** *Let $(H_1)$ and the following condition hold:*

$$m_1(\chi_1 + \chi_4) + n_1(\chi_2 + \chi_3)] < 1, \quad m_2(\chi_1 + \chi_4) + n_2(\chi_2 + \chi_3) < 1,$$

*where $\chi_i, i = 1, 2, 3, 4$ are given in (3.4). Then there exists at least one solution for the problems (1.1) and (1.2) on $\mathcal{J}$.*

*Proof.* Let us first establish that the operator $\mathcal{H} : \Theta \times \Theta \to \Theta \times \Theta$ defined by (3.1) is completely continuous. Observe that the operator $\mathcal{H}$ is continuous in view of the continuity of functions $\rho_1$ and $\rho_2$. If $\Upsilon \subset \Theta \times \Theta$ is bounded, then we can find positive constants $L_1$ and $L_2$ such that $|\rho_1(t, \varphi, \psi)| \le L_1$, $|\rho_2(t, \varphi, \psi)| \le L_2$, $\forall(\varphi, \psi) \in \Upsilon$. Then, for any $(\varphi, \psi) \in \Upsilon$, we obtain

$$\begin{aligned}
&\|\mathcal{H}_1(\varphi, \psi)\|\\
\le\;& \max_{t\in\mathcal{J}} \Bigg\{ \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon, \varphi(\upsilon), \psi(\upsilon))|| d\upsilon\\
&+\frac{T}{|\Delta|}\Big[|a_1(t)| \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon, \varphi(\upsilon), \psi(\upsilon))|\, ds + |a_2(t)| \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon, \varphi(\upsilon), \psi(\upsilon))|\, d\upsilon\\
&+|a_3(t)| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}|\rho_1(\upsilon, \varphi(\upsilon), \psi(\upsilon))|\, d\upsilon + |a_4(t)| \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}|\rho_2(\upsilon, \varphi(\upsilon), \psi(\upsilon))|\, d\upsilon\Big]\Bigg\}\\
\le\;& L_1 \max_{t\in\mathcal{J}} \Bigg\{\frac{t^{q_1}}{\Gamma(q_1+1)} + \frac{1}{|\Delta|}\Big[\frac{T^{q_1+1}|a_1(t)|}{\Gamma(q_1+1)} + \frac{T^{q_1}|a_3(t)|}{\Gamma(q_1)}\Big]\Bigg\}\\
&+L_2 \max_{t\in\mathcal{J}} \frac{1}{|\Delta|}\Bigg\{\frac{T^{q_2+1}|a_2(t)|}{\Gamma(q_2+1)} + \frac{T^{q_2}|a_4(t)|}{\Gamma(q_2)}\Bigg\}
\end{aligned}$$

$$\leq \quad L_1\chi_1 + L_2\chi_2. \tag{3.6}$$

In a similar fashion, we can find that

$$\|\mathcal{H}_2(\varphi,\psi)\| \leq L_1\chi_4 + L_2\chi_3. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\|\mathcal{H}(\varphi,\psi)\| = \|\mathcal{H}_1(\varphi,\psi)\| + \|\mathcal{H}_2(\varphi,\psi)\| \leq L_1(\chi_1+\chi_4) + L_2(\chi_2+\chi_3),$$

which shows that $\mathcal{H}(\Upsilon)$ is uniformly bounded.

To show that $\mathcal{H}(\Upsilon)$ is equicontinuous, we take $t_1, t_2 \in \mathcal{J}$ with $t_1 < t_2$. Then, we obtain

$$\begin{aligned}
& |\mathcal{H}_1(\varphi,\psi)(t_2) - \mathcal{H}_1(\varphi,\psi)(t_1)| \\
\leq \ & \Big|\int_0^{t_2} \frac{(t_2-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon - \int_0^{t_1} \frac{(t_1-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon\Big| \\
& + \frac{T|t_2-t_1|}{|\Delta|}\Big\{|[\alpha_2\gamma_1 + (\delta_1-\gamma_1)\gamma_2]| \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon \\
& + |[\gamma_1 + (\alpha_1\delta_1 - \beta_1\gamma_1)\gamma_2]| \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon \\
& + |[(1-\alpha_1\alpha_2) + (\alpha_1-\beta_1)\gamma_2]| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon \\
& + |[(\delta_1-\gamma_1) - (\alpha_1\delta_1-\beta_1\gamma_1)\alpha_2]| \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon\Big\} \\
\leq \ & \frac{L_1}{\Gamma(q_1+1)}\Big[|(t_2-t_1)^{q_1} + t_2^{q_1} - t_1^{q_1}| + |(t_2-t_1)^{q_1}|\Big] \\
& + \frac{L_1|t_2-t_1|}{|\Delta|\Gamma(q_1+1)}\Big\{|[\alpha_2\gamma_1 + (\delta_{-}\gamma_1)\gamma_2]|T^{q_1+1} + |[(1-\alpha_1\alpha_2) + (\alpha_1-\beta_1)\gamma_2]|q_1T^{q_1}\Big\} \\
& + \frac{L_2|t_2-t_1|}{|\Delta|\Gamma(q_2+1)}\Big\{|[\gamma_1 + (\alpha_1\delta_1-\beta_1\gamma_1)\gamma_2]|T^{q_2+1} + |[(\delta_1-\gamma_1) - (\alpha_1\delta_1-\beta_1\gamma_1)\alpha_2]|q_2T^{q_2}\Big\} \\
& \longrightarrow 0 \text{ as } t_2 - t_1 \to 0 \text{ independently of } (\varphi,\psi) \in \Upsilon,
\end{aligned}$$

and

$$\begin{aligned}
& |\mathcal{H}_2(\varphi,\psi)(t_2) - \mathcal{H}_2(\varphi,\psi)(t_1)| \\
\leq \ & \Big|\int_0^{t_2} \frac{(t_2-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon - \int_0^{t_1} \frac{(t_1-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))d\upsilon\Big| \\
& + \frac{T|t_2-t_1|}{|\Delta|}\Big\{|[\gamma_2 + (\alpha_2\delta_2-\beta_2\gamma_2)\gamma_1]| \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon \\
& + |[\alpha_1\gamma_2 + (\delta_2-\gamma_2)\gamma_1]| \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon \\
& + |[(\delta_2-\gamma_2) - (\alpha_2\delta_2-\beta_2\gamma_2)\alpha_1]| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\, d\upsilon
\end{aligned}$$

$$
\begin{aligned}
&\quad +|[(1-\alpha_1\alpha_2)+(\alpha_2-\beta_2)\gamma_1]|\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\Big\}\\
&\leq \frac{L_2}{\Gamma(q_2+1)}\Big[|(t_2-t_1)^{q_2}+t_2^{q_2}-t_1^{q_2}|+|(t_2-t_1)^{q_2}|\Big]\\
&\quad +\frac{L_1|t_2-t_1|}{|\Delta|\Gamma(q_1+1)}\Big\{|[\gamma_2+(\alpha_2\delta_2-\beta_2\gamma_2)\gamma_1]|T^{q_1+1}+|[(\delta_2-\gamma_2)-(\alpha_2\delta_2-\beta_2\gamma_2)\alpha_1]|q_1T^{q_1}\Big\}\\
&\quad +\frac{L_2|t_2-t_1|}{|\Delta|\Gamma(q_2+1)}\Big\{|[\alpha_1\gamma_2+(\delta_2-\gamma_2)\gamma_1]|T^{q_2+1}+|[(1-\alpha_1\alpha_2)+(\alpha_2-\beta_2)\gamma_1]|q_2T^{q_2}\Big\}\\
&\quad \longrightarrow 0 \text{ as } t_2-t_1\to 0 \text{ independently of } (\varphi,\psi)\in\Upsilon.
\end{aligned}
$$

Thus, $\mathcal{H}_1(\Upsilon)$ and $\mathcal{H}_2(\Upsilon)$ are equicontinuous and hence $\mathcal{H}(\Upsilon)$ is equicontinuous. Therefore, by Arzelá-Ascoli theorem, $\mathcal{H}(\Upsilon)$ is completely continuous.

In the final step, we consider a set $\Xi = \{(\varphi,\psi)\in\Theta\times\Theta : (\varphi,\psi) = \zeta\mathcal{H}(\varphi,\psi),\ 0<\zeta<1\}$ and show that it is bounded. Let $(\varphi,\psi)\in\Xi$. Then $(\varphi,\psi)=\zeta\mathcal{H}(\varphi,\psi)$ implies that $\varphi(t)=\zeta\mathcal{H}_1(\varphi,\psi)(t)$ and $\psi(t)=\zeta\mathcal{H}_2(\varphi,\psi)(t)$ for $t\in\mathcal{J}$. Then, by the assumption $(H_1)$, we have

$$
\begin{aligned}
|\varphi(t)|\leq\|\mathcal{H}_1(\varphi,\psi)\| &\leq \max_{t\in\mathcal{J}}\Big\{\int_0^t\frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}[m_0+m_1|\varphi|+m_2|\psi|]d\upsilon\\
&\quad +\frac{T}{|\Delta|}\Big[|a_1(t)|\int_0^T\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}[m_0+m_1|\varphi|+m_2|\psi|]\,d\upsilon\\
&\quad +|a_2(t)|\int_0^T\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}[n_0+n_1|\varphi|+n_2|\psi|]d\upsilon\\
&\quad +|a_3(t)|\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}[m_0+m_1|\varphi|+m_2|\psi|]\,d\upsilon\\
&\quad +|a_4(t)|\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}[n_0+n_1|\varphi|+n_2|\psi|]\,d\upsilon\Big]\Big\},
\end{aligned}
$$

which implies that

$$\|\varphi\|\leq[m_0+m_1\|\varphi\|+m_2\|\psi\|]\chi_1+[n_0+n_1\|\varphi\|+n_2\|\psi\|]\chi_2. \tag{3.8}$$

In a similar manner, we can find that

$$\|\psi\|\leq[n_0+n_1\|\varphi\|+n_2\|\psi\|]\chi_3+[m_0+m_1\|\varphi\|+m_2\|\psi\|]\chi_4. \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$\|\varphi\|+\|\psi\|\leq\frac{m_0(\chi_1+\chi_4)+n_0(\chi_2+\chi_3)}{\chi_0},$$

where $\chi_0$ is given in (3.5). In consequence, we have

$$\|(\varphi,\psi)\|\leq\frac{m_0(\chi_1+\chi_4)+n_0(\chi_2+\chi_3)}{\chi_0}.$$

Therefore, the set $\Xi$ is bounded. In consequence, we deduce by the Leray-Schauder alternative [44] that there exists at least one fixed point for the operator $\mathcal{H}$. Hence the problems (1.1) and (1.2) admits a solution on $\mathcal{J}$.

Now we accomplish a uniqueness result for the problems (1.1) and (1.2) by means of a fixed point theorem due to Banach.

**Theorem 3.2.** *Suppose that* $(H_2)$ *is satisfied. Then, the problems (1.1) and (1.2) has a unique solution on* $\mathcal{J}$*, provided that*

$$(\ell_1 + \ell_2)(\chi_1 + \chi_4) + (\overline{\ell}_1 + \overline{\ell}_2)(\chi_2 + \chi_3) < 1, \tag{3.10}$$

*where* $\chi_i, i = 1, 2, 3, 4,$ *are given in (3.4).*

*Proof.* By $(H_2)$, we have

$$\begin{aligned} |\rho_1(\upsilon, \varphi(\upsilon), \psi(\upsilon))| &= |\rho_1(\upsilon, \varphi(\upsilon), \psi(\upsilon)) - \rho_1(t, 0, 0) + \rho_1(t, 0, 0)| \leq \ell_1\|\varphi\| + \ell_2\|\psi\| + N_1, \\ |\rho_2(\upsilon, \varphi(\upsilon), \psi(\upsilon))| &= |\rho_2(\upsilon, \varphi(\upsilon), \psi(\upsilon)) - \rho_2(t, 0, 0) + \rho_2(t, 0, 0)| \leq \overline{\ell}_1\|\varphi\| + \overline{\ell}_2\|\psi\| + N_2, \end{aligned} \tag{3.11}$$

where $\sup_{t\in\mathcal{J}} \rho_1(t, 0, 0) = N_1 < \infty$, $\sup_{t\in\mathcal{J}} \rho_2(t, 0, 0) = N_2 < \infty$.

Now, we establish that $\mathcal{H}(\mathcal{U}_r) \subset \mathcal{U}_r$, where $\mathcal{U}_r = \{(\varphi, \psi) \in \Theta \times \Theta : \|(\varphi, \psi)\| \leq r\}$ is a closed ball with

$$r \geq \frac{N_1(\chi_1 + \chi_4) + N_2(\chi_2 + \chi_3)}{1 - [(\ell_1 + \ell_2)(\chi_1 + \chi_4) + (\overline{\ell}_1 + \overline{\ell}_2)(\chi_2 + \chi_3)]}. \tag{3.12}$$

For $(\varphi, \psi) \in \Theta \times \Theta$, it follows by (3.11) that

$$\begin{aligned} \|\mathcal{H}_1(\varphi, \psi)\| \leq\ & \max_{t\in\mathcal{J}} \Bigg\{ \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} [\ell_1|\varphi| + \ell_2|\psi| + N_1] d\upsilon \\ & + \frac{T}{|\Delta|} \Bigg\{ |a_1(t)| \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} [\ell_1|\varphi| + \ell_2|\psi| + N_1]\, d\upsilon \\ & + |a_2(t)| \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} [\overline{\ell}_1|\varphi| + \overline{\ell}_2|\psi| + N_2]\, d\upsilon \\ & + |a_3(t)| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} [\ell_1|\varphi| + \ell_2|\psi| + N_1]\, d\upsilon \\ & + |a_4(t)| \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} [\overline{\ell}_1|\varphi| + \overline{\ell}_2|\psi| + N_2]\, d\upsilon \Bigg\}\Bigg\} \\ \leq\ & [(\ell_1 + \ell_2)r + N_1]\chi_1 + [(\overline{\ell}_1 + \overline{\ell}_2)r + N_2]\chi_2. \end{aligned}$$

Likewise, we can find that

$$\|\mathcal{H}_2(\varphi, \psi)\| \leq [(\overline{\ell}_1 + \overline{\ell}_2)r + N_2]\chi_3 + [(\ell_1 + \ell_2)r + N_1]\chi_4.$$

Therefore, we get

$$\begin{aligned} \|\mathcal{H}(\varphi, \psi)\| &= \|\mathcal{H}_1(\varphi, \psi)\| + \|\mathcal{H}_2(\varphi, \psi)\| \\ &\leq [(\ell_1 + \ell_2)(\chi_1 + \chi_4) + (\overline{\ell}_1 + \overline{\ell}_2)(\chi_2 + \chi_3)]r + (\chi_1 + \chi_4)N_1 + (\chi_2 + \chi_3)N_2 \leq r, \end{aligned}$$

which shows that $\mathcal{H}(\varphi, \psi) \in \mathcal{U}_r$. Hence, $\mathcal{H}(\mathcal{U}_r) \subset \mathcal{U}_r$.

Next, it will be established that the operator $\mathcal{H}$ is a contraction. For that, let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \Theta \times \Theta$. Then, for any $t \in \mathcal{J}$, it follows by means of the assumption $(H_2)$ that

$$\|\mathcal{H}_1(\varphi_2, \psi_2) - \mathcal{H}_1(\varphi_1, \psi_1)\|$$

$$
\begin{aligned}
\leq\ & \max_{t\in\mathcal{J}}\Bigg\{\int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))|d\upsilon \\
& +\frac{T}{|\Delta|}\Big[|a_1(t)|\int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))|\,d\upsilon \\
& +|a_2(t)|\int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))|\,d\upsilon \\
& +|a_3(t)|\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(s,\varphi_1(\upsilon),\psi_1(\upsilon))|\,d\upsilon \\
& +|a_4(t)|\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(s,\varphi_1(\upsilon),\psi_1(\upsilon))|d\upsilon\Big]\Bigg\} \\
\leq\ & (\ell_1\|\varphi_2-\varphi_1\|+\ell_2\|\psi_2-\psi_1\|)\max_{t\in\mathcal{J}}\Bigg\{\int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}d\upsilon \\
& +\frac{T}{|\Delta|}\Big[|a_1(t)|\int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,d\upsilon+|a_3(t)|\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\,d\upsilon\Big]\Bigg\} \\
& +\frac{T}{|\Delta|}(\overline{\ell}_1\|\varphi_2-\varphi_1\|+\overline{\ell}_2\|\psi_2-\psi_1\|)\max_{t\in\mathcal{J}}\Bigg\{|a_2(t)|\int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,d\upsilon \\
& +|a_4(t)|\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\,d\upsilon\Bigg\} \\
\leq\ & \Big[(\ell_1+\ell_2)\chi_1+(\overline{\ell}_1+\overline{\ell}_2)\chi_2\Big](\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|).
\end{aligned}
$$

Similarly, we can get

$$
\|\mathcal{H}_2(\varphi_2,\psi_2)-\mathcal{H}_2(\varphi_1,\psi_1)\|\leq\Big[(\ell_1+\ell_2)\chi_4+(\overline{\ell}_1+\overline{\ell}_2)\chi_3\Big](\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|).
$$

From the foregoing two inequalities, it follows that

$$
\|\mathcal{H}(\varphi_2,\psi_2)-\mathcal{H}(\varphi_1,\psi_1)\|\leq\Big[(\ell_1+\ell_2)(\chi_1+\chi_4)+(\overline{\ell}_1+\overline{\ell}_2)(\chi_2+\chi_3)\Big](\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|).
$$

By the condition (3.10), we deduce from the preceding inequality that the operator $\mathcal{H}$ is a contraction. So, by Banach's fixed point theorem, there exists a unique fixed point for the operator $\mathcal{H}$. In consequence, the problems (1.1) and (1.2) has a unique solution on $\mathcal{J}$.

### *3.1. Examples*

Consider a fully coupled fractional boundary value problem:

$$
\begin{cases}
{}^{C}D^{1.07}\varphi(t)=\rho_1(t,\varphi(t),\psi(t)), \qquad t\in[0,2], \\
{}^{C}D^{1.4}\psi(t)=\rho_2(t,\varphi(t),\psi(t)), \qquad t\in[0,2] \\
\varphi(2)=\frac{3}{2}\psi(0)+\frac{10}{9}\psi'(0), \quad 2\varphi'(2)=\frac{-3}{8}\psi(0)+\frac{8}{9}\psi'(0), \\
\psi(2)=\frac{2}{5}\varphi(0)+\frac{-14}{3}\varphi'(0), \quad 2\psi'(2)=\frac{-5}{9}\varphi(0)+2\varphi'(0),
\end{cases}
\tag{3.13}
$$

where $T = 2, q_1 = 1.07, q_2 = 1.4, \alpha_1 = 3/2, \alpha_2 = 2/5, \beta_1 = 5/9, \beta_2 = -7/3, \gamma_1 = -3/8, \gamma_2 = -5/9, \delta_1 = 4/9, \delta_2 = 1$. With the given data, it is found that $\chi_1 \approx 4.749052648, \chi_2 \approx 2.425952097, \chi_3 \approx 3.909949767$ and $\chi_4 \approx 2.482951479$ ($\chi_i, i = 1, 2, 3, 4$, are given in (3.4)).

**(a)** We illustrate Theorem 3.1 by choosing

$$\begin{aligned}
\rho_1(t, \varphi(t), \psi(t)) &= \frac{1}{3(t^2+7)} \frac{\varphi(t)|\varphi(t)|}{(1+|\varphi(t)|)} + \frac{1}{27} \sin\psi(t) + \frac{1}{8\sqrt{t^2+9}}, \\
\rho_2(t, \varphi(t), \psi(t)) &= \frac{1}{30} \frac{\varphi(t)|\psi(t)|}{(1+|\psi(t)|)} + \frac{\psi(t)\cos\varphi(t)}{(9+t^2)^2} + \frac{1}{2(t+4)^2}.
\end{aligned} \tag{3.14}$$

From (3.14), it is easy to find that $m_0 = 1/24, m_1 = 1/21, m_2 = 1/27, n_0 = 1/32, n_1 = 1/30, n_2 = 1/81$. Moreover, $m_1(\chi_1+\chi_4)+n_1(\chi_2+\chi_3) \approx 0.5555778778 < 1$ and $m_2(\chi_1+\chi_4)+n_2(\chi_2+\chi_3) \approx 0.3460730155 < 1$. Thus, the hypotheses of Theorem 3.1 are satisfied and consequently there exists at least one solution for the problem (3.13) with $\rho_1(t, \varphi(t), \psi(t))$ and $\rho_2(t, \varphi(t), \psi(t))$ given by (3.14) on $[0, 2]$.

**(b)** For demonstrating the application of Theorem 3.2, we take

$$\begin{aligned}
\rho_1(t, \varphi(t), \psi(t)) &= \frac{e^{-t^2}}{(t^2+40)} \tan^{-1}\varphi(t) + \frac{1}{(35+t^3)} \frac{|\psi(t)|}{(1+|\psi(t)|)} + \frac{\cos t}{8\sqrt{t^2+1}}, \\
\rho_2(t, \varphi(t), \psi(t)) &= \frac{1}{(t^4+36)} \frac{|\varphi(t)|}{(1+|\varphi(t)|)} + \frac{1}{\sqrt{t^2+625}} \cos\psi(t) + \frac{e^t}{2(\sin^2 t+3)}.
\end{aligned} \tag{3.15}$$

It follows from (3.15) that $\ell_1 = \frac{1}{40}, \ell_2 = \frac{1}{35}, \overline{\ell}_1 = \frac{1}{36}, \overline{\ell}_2 = \frac{1}{25}$ and

$$[(\ell_1 + \ell_2)(\chi_1 + \chi_4) + (\overline{\ell}_1 + \overline{\ell}_2)(\chi_2 + \chi_3)] \approx 0.8168621412 < 1.$$

Clearly the hypothesis of Theorem 3.2 is satisfied and hence its conclusion implies that there exists a unique solution for the problem (3.13) with $\rho_1(t, \varphi(t), \psi(t))$ and $\rho_2(t, \varphi(t), \psi(t))$ given in (3.15) on $[0, 2]$.

**(c)** Here we consider a problem for the values of of $q_1$ and $q_2$ close to 2 ($q_1 = 1.95, q_2 = 1.98$):

$$\begin{cases}
{}^{C}D^{1.95}\varphi(t) = \rho_1(t, \varphi(t), \psi(t)), \quad t \in [0, 2], \\
{}^{C}D^{1.98}\psi(t) = \rho_2(t, \varphi(t), \psi(t)), \quad t \in [0, 2], \\
\varphi(2) = \frac{3}{2}\psi(0) + \frac{10}{9}\psi'(0), \quad 2\varphi'(2) = \frac{-3}{8}\psi(0) + \frac{8}{9}\psi'(0), \\
\psi(2) = \frac{2}{5}\varphi(0) + \frac{-14}{3}\varphi'(0), \quad 2\psi'(2) = \frac{-5}{9}\varphi(0) + 2\varphi'(0),
\end{cases} \tag{3.16}$$

where

$$\begin{aligned}
\rho_1(t, \varphi(t), \psi(t)) &= \frac{1}{(t^4+50)} \sin\varphi(t) + \frac{1}{\sqrt{t^2+1600}} \cos\psi(t) + \frac{1}{8\sqrt{t+8}}, \\
\rho_2(t, \varphi(t), \psi(t)) &= \frac{1}{(t^4+49)} \tan^{-1}\varphi(t) + \frac{1}{(55+t^2)} \frac{|\psi(t)|}{(1+|\psi(t)|)} + \frac{e^{-t}}{2(t+3)}.
\end{aligned}$$

Using the given values, we find that that $\chi_1 \approx 6.263094290$, $\chi_2 \approx 2.985105239$, $\chi_3 \approx 4.172733302$ and $\chi_4 \approx 3.794487362$ ($\chi_i, i = 1, 2, 3, 4$, are given in (3.4)). Moreover, $\ell_1 = \frac{1}{50}, \ell_2 = \frac{1}{40}, \overline{\ell}_1 = \frac{1}{49}, \overline{\ell}_2 = \frac{1}{55}$ and

$$[(\ell_1 + \ell_2)(\chi_1 + \chi_4) + (\overline{\ell}_1 + \overline{\ell}_2)(\chi_2 + \chi_3)] \approx 0.7288120308 < 1.$$

As the assumptions of Theorem 3.2 hold true, therefore, it follows by its conclusion that the problem (3.16) has a unique solution on $[0, 2]$.

## 4. Conclusions

Applying the standard fixed point theorems, we studied a new class of fully coupled boundary value problems of nonlinear Caputo type fractional differential equations supplemented with closed boundary conditions. It is interesting to notice that our work gives rise to several new results by specializing the parameters involved in the conditions (1.2) appropriately. For example, by selecting $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$ and $\alpha_1 = \alpha_2 = 1$, we enlist some new results arising from the present ones for a nonlinear coupled system (1.1) subject to

(1) semi-periodic coupled boundary conditions ($\psi(T) = \varphi(0),\ \varphi(T) = \psi(0),\ \varphi'(T) = \delta_1\psi'(0), \psi'(T) = \delta_2\varphi'(0)$) when $\alpha_1 = \alpha_2 = 1$;
(2) combination of coupled periodic and anti-periodic boundary conditions ($\varphi(0) = \psi(T),\ \varphi'(0) = \psi'(T),\ \psi(0) = -\varphi(T),\ \psi'(0) = -\varphi'(T)$) when $\alpha_1 = -1 = \delta_1$ and $\alpha_2 = 1 = \delta_2$;
(3) combination of coupled anti-periodic and periodic boundary conditions ($\varphi(0) = -\psi(T),\ \varphi'(0) = -\psi'(T),\ \psi(0) = \varphi(T),\ \psi'(0) = \varphi'(T)$) when $\alpha_1 = 1 = \delta_1$ and $\alpha_2 = -1 = \delta_2$.

Moreover, letting $\alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$ and $\beta_1 = \beta_2 = 1/T, \gamma_1 = \gamma_2 = T$, our results become the ones associated with mixed boundary conditions ($\varphi(T) = \psi'(0),\ \varphi'(0) = \psi(T),\ \varphi'(T) = \psi(0),\ \varphi(0) = \psi'(T)$).

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgements

The Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU), Jeddah, Saudi Arabia has funded this project, under grant No. (KEP-PhD: 41-130-1443). The authors thank the reviewers for their constructive remarks on their work.

## Conflict of interest

All authors declare no conflicts of interest in this paper.

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