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Research article

Three-way decisions with complex q-rung orthopair 2-tuple linguistic decision-theoretic rough sets based on generalized Maclaurin symmetric mean operators

Zeeshan Ali[1], Tahir Mahmood[1,]* and Muhammad Bilal Khan[2,]*

[1] Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan
[2] Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan

* **Correspondence:** Email: tahirbakhat@iiu.edu.pk, bilal42742@gmail.com.

Abstract: In this manuscript, we generalized the notions of three-way decisions (3WD) and decision theoretic rough sets (DTRS) in the framework of Complex q-rung orthopair 2-tuple linguistic variables (CQRO2-TLV) and then deliberated some of its important properties. Moreover, we considered some very useful and prominent aggregation operators in the framework of CQRO2-TLV, while further observing the importance of the generalized Maclurin symmetric mean (GMSM) due to its applications in symmetry analysis, interpolation techniques, analyzing inequalities, measuring central tendency, mathematical analysis and many other real life problems. We initiated complex q-rung orthopair 2-tuple linguistic (CQRO2-TL) information and GMSM to introduce the CQRO2-TL GMSM (CQRO2-TLGMSM) operator and the weighted CQRO2-TL GMSM (WCQRO2-TLGMSM) operator, and then demonstrated their properties such as idempotency, commutativity, monotonicity and boundedness. We also investigated a CQRO2-TL DTRS model. In the end, a comparative study is given to prove the authenticity, supremacy, and effectiveness of our proposed notions.

Keywords: three-way decision; complex q-rung orthopair 2-tuple linguistic variables; rough sets; generalized Maclaurin symmetric mean operators
Mathematics Subject Classification: 03B52, 03E72, 28E10, 68T27, 94D05

1. Introduction

A multi-attribute decision-making (MADM) tool plays an important role in the environment of fuzzy set theory and to evaluate unreliable and awkward information. According to Yukalov and Sornette [1], it is sometimes very much difficult to handle a huge part of data or information because of either the design of the data or due to the uncertainty involved in the data or information. Human beings have faced such issues for the last few decades. To oversee such issues, Zadeh [2] initiated the notion of fuzzy sets (FS), which takes the truth grade from [0, 1]. Moreover, Atanassove [3] modified the technique of FS and initiated the principle of intuitionistic FS (IFS) by including the falsity grade in FS, with the condition that the sum of both the grades must be less than or equal to 1. Sometimes, when a decision maker provides information, the principle of IFS failed, in which the sum of both the grades exceeds 1. For this situation, Yager [4] introduced the principle of Pythagorean FS (PFS), with the condition that instead of sum of both the grades, the sum of squares of both the grades must not exceed 1. Although with the introduction of PFS, decision makers have gained much more flexibility to give the values of both the grades but yet there can arise many situations, where PFS failed as even the sum of squares of both the grades exceeds 1. To handle this ambiguity, Yager [5] improved the rule of PFS and initiated the notion of q-rung orthopair FS (QROFS), with the rule that the sum of the q-powers ($q \geq 1$) of the duplet must not exceeds 1. The QROFS has received massive attraction from many researchers and has utilized in different areas [6–14]. The mathematical portrayal of the IFS and their generalizations are clarified with the assistance of Figure 1.

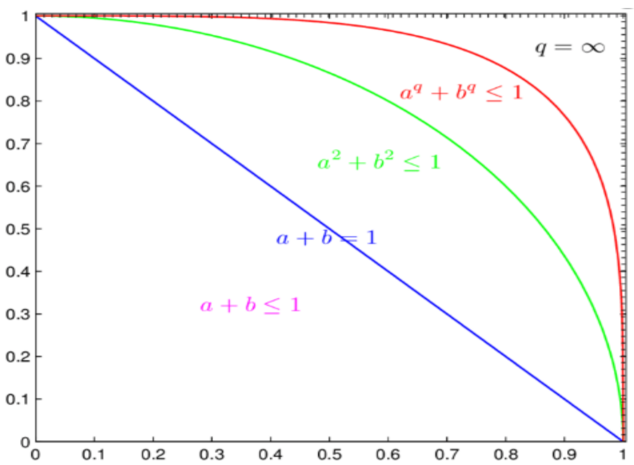

Figure 1. Geometrical representation of IFS, PFS, QROFS.

Ramot et al. [15] investigated the idea of complex FS (CFS) by including the unreal part in the truth grade, called complex truth grade, whose real and unreal parts belong to [0, 1]. Due to certain complications, Alkouri and Salleh [16] modified the principle of CFS by including the falsity grade in the CFS to elaborate the complex IFS (CIFS), with the rule that the sum of the real and unreal parts of the duplet must be limited to [0, 1]. Moreover, Ullah [17] utilized the idea of complex PFS (CPFS) by improving the rule of IFS to initiate a new rule for CPFS, with the condition that the sum of the squares of the real and unreal parts of the duplet must be limited to [0, 1]. Akram and Naz initiated decision making for CPFS [18]. Although CPFS has proved its effectiveness, there are many situations where it fails to perform, for instance, when a decision maker provides such sorts of information whose sum of the squares of the real and unreal parts exceeds from [0, 1]; for this, the complex QROFS (CQROFS) was developed by Liu et al. [19,20]. Furthermore, the CQROFS is portrayed by the interest degree and the non-participation degree, who's total q-powers of the genuine part (likewise for nonexistent part) are not by and large or comparable to 1. The CQROFS is much more generalized and helpful to the researchers of the field as compared to CPFS and CIFS. Mahmood and Ali [21,22] enhanced the effectiveness of the notion of CQROFS by studying the TOPSIS technique, correlation coefficients, and Maclaurin operators in the environment of CQROFS.

In terms of the real-life importance of linguistic terms (LT), Zadeh [23] introduced the theory of LT sets (LTS) for evaluating the awkward and unreliable problems in real-life decisions. Furthermore, Herrera and Martinez [24] pioneered the theory of the 2-tuple LT (2TLT) set, which is more modified than the LTSs. Herrera and Martinez [25,26] utilized the theory of LTS in different fields. Li and Liu [27] initiated the theory of Heronian mean operators for 2TLT.

DTRS is one of the capable methods to assess abnormal and convoluted data. Different researchers have adjusted it to various ways like loss function (LF) [28], trait decrease [29], and further developed models utilizing DTRS [30]. Yao [31] introduced the three-ways decision (3WD), which is changed from DTRS to adapt to practical choice issues. 3WD can isolate all widespread sets into three distinct parts: the positive area (POS(C)), the negative district (NEG(C)), and the limit locale (BND(C)). As a mix of DTRSs and the Bayesian choice method, the procedures of 3WD has successfully overseen numerous request issues. This speculation has been applied in various fields [32,33]. The graphical portrayal of the 3WD dependent on the harsh set is examined with the assistance of Figure 2.

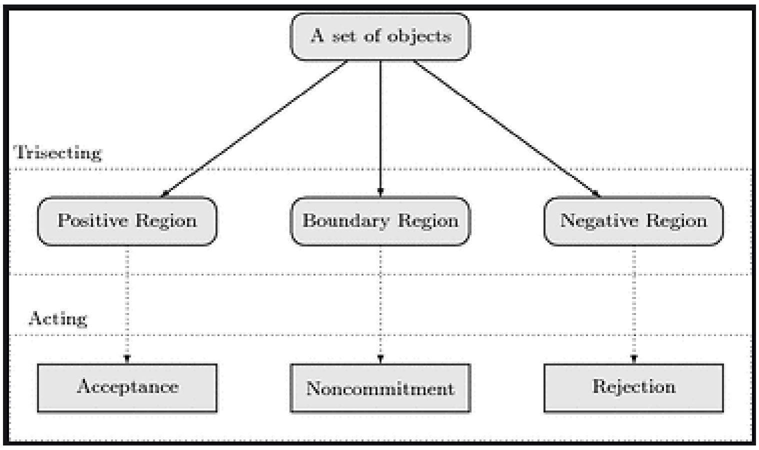

Figure 2. Geometrical interpretation of the 3-WDs.

Liu and Yang [34] introduced the misfortune work in 3WDs dependent on intuitionistic fuzzy linguistic sets. In view of the above examination, we initiated the following: we generalize the massive important techniques of three-way decisions (3WD) [35] and DTRS with Complex q-rung orthopair 2-tuple linguistic variable (CQRO2-TLV), to elaborate certain important properties. Moreover, the generalized Maclurin symmetric mean (GMSM) [36] is a dominant and more flexible method to determine the accuracy and dominancy of real life issues. Therefore, by considering the complex q-rung orthopair 2-tuple linguistic (CQRO2-TL) information and generalized Maclurin symmetric mean (GMSM), we present the CQRO2-TL GMSM (CQRO2-TLGMSM) operator and weighted CQRO2-TL GMSM (WCQRO2-TLGMSM) operator, and demonstrated their effective properties. Finally, a model is applied to exhaustively elucidate the proposed procedure, and the effects of different contingent probabilities on choice results are examined. The investigation of the elaborated approach is likewise discussed regarding certain ways to deal with the capability and capacity of the presented approach. For simplicity, the explored work for this manuscript is explained with the help of Figure 3.

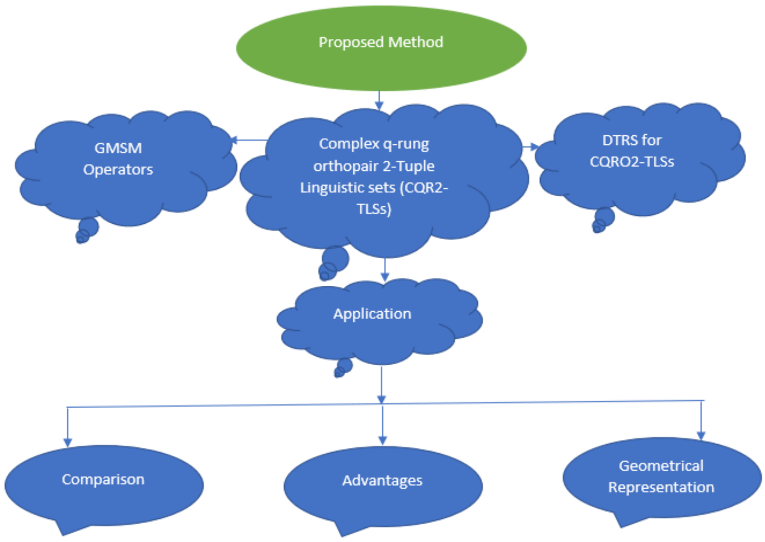

Figure 3. Geometrical representations of the explored approach in this manuscript.

This manuscript is outlined as follows. In Section 2, we briefly audit some helpful ideas of 2-TLF [24], converse 2-TLF [24], CQROFS [19–22,35], and their functional laws. Furthermore, the GMSM operators [36] are discussed. In Section 3, we explored CQRO2-TLV and their algebraic laws. In Section 4, we generalize 3WD and DTRS with CQRO2-TLV to elaborate certain important properties. GMSM is a dominant and more flexible way to determine the accuracy and dominancy of real life issues. Therefore, by considering the CQRO2-TL information and GMSM, in section 5 we present the CQRO2-TLGMSM operator and the WCQRO2-TLGMSM operator and demonstrated their effective properties. In Section 6, a model is applied to exhaustively elucidate the proposed procedure, and the effects of different contingent probabilities on choice results are examined. The conclusion of this manuscript is discussed in Section 7.

2. Preliminaries

For the convenience and improved understanding for the reader, in this section we will recall some basic notions which will be used in the manuscript. The symbols $\tilde{Ư}_{UNI}, \mu_{\mathcal{Q}_{CQ}}(ử)$, and $\eta_{\mathcal{Q}_{CQ}}(ử)$ will denote the universal set, the truth grade, and the falsity grade, respectively, where $\alpha_{SC}, \delta_{SC}, q_{SC} \geq 1$.

Definition 2.1. [24] For a LS $S_{LT} = \left\{s_{S_{LT-0}}, s_{S_{LT-1}}, s_{S_{LT-2}}, s_{S_{LT-3}}, \dots, s_{S_{LT-g}}\right\}$ with $\beta_{SC} \in [0,1]$, the 2-TL function Δ_{LT} is given by:

$$\Delta_{LT}:[0,1] \rightarrow S_{LT} \times \left[-\frac{1}{2g},\frac{1}{2g}\right\}, \tag{2.1}$$

$$\Delta_{LT}(\beta_{SC}) = \left(s_{S_{LT-j}}, \alpha_{SC}\right) \text{ with } \begin{cases} s_{S_{LT-j}} & j = round(\beta_{SC}, g) \\ \alpha_{SC} = \beta_{SC} - \frac{j}{g} & \alpha_{SC} \in \left[-\frac{1}{2g},\frac{1}{2g}\right\} \end{cases}. \tag{2.2}$$

The 2-TL inverse function Δ_{LT}^{-1} is given by:

$$\Delta_{LT}^{-1}: S_{LT} \times \left[-\frac{1}{2g},\frac{1}{2g}\right\} \rightarrow [0,1], \tag{2.3}$$

$$\Delta_{LT}^{-1}\left(s_{S_{LT-j}}, \alpha_{SC}\right) = \frac{j}{g} + \alpha_{SC} = \beta_{SC}. \tag{2.4}$$

Definition 2.2. [19] A CQROFS is given by

$$\mathcal{Q}_{CQ} = \left\{\left(\mu_{\mathcal{Q}_{CQ}}(\text{ử}), \eta_{\mathcal{Q}_{CQ}}(\text{ử})\right) : \text{ử} \in \tilde{\text{Ư}}_{UNI}\right\} \tag{2.5}$$

where $\mu_{\mathcal{Q}_{CQ}}(\text{ử}) = \mu_{\mathcal{Q}_{CQRP}}(\text{ử}) e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP}}(\text{ử})\right)}$ and $\eta_{\mathcal{Q}_{CQ}}(\text{ử}) = \eta_{\mathcal{Q}_{CQRP}}(\text{ử}) e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP}}(\text{ử})\right)}$, with the conditions: $0 \leq \mu_{\mathcal{Q}_{CQRP}}^{q_{SC}}(\text{ử}) + \eta_{\mathcal{Q}_{CQRP}}^{q_{SC}}(\text{ử}) \leq 1, 0 \leq \mu_{\mathcal{Q}_{CQIP}}^{q_{SC}}(\text{ử}) + \eta_{\mathcal{Q}_{CQIP}}^{q_{SC}}(\text{ử}) \leq 1$.

Moreover, $\zeta_{\mathcal{Q}_{CQ}}(\text{ử}) = \left(1 - \left(\mu_{\mathcal{Q}_{CQRP}}^{q_{SC}}(\text{ử}) + \eta_{\mathcal{Q}_{CQRP}}^{q_{SC}}(\text{ử})\right)^{\frac{1}{q_{SC}}}\right) e^{i2\pi\left(1-\left(\mu_{\mathcal{Q}_{CQIP}}^{q_{SC}}(\text{ử})+\eta_{\mathcal{Q}_{CQIP}}^{q_{SC}}(\text{ử})\right)^{\frac{1}{q_{SC}}}\right)}$ is called the refusal grade. The QROFN is given as $\mathcal{Q}_{CQ} = \left(\mu_{\mathcal{Q}_{CQ}}, \eta_{\mathcal{Q}_{CQ}}\right) = \left(\mu_{\mathcal{Q}_{CQRP}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP}}\right)}, \eta_{\mathcal{Q}_{CQRP}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP}}\right)}\right)$.

Definition 2.3. [20] For any two CQROFNs $\mathcal{Q}_{CQ-1} = \left(\mu_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP-1}}\right)}, \eta_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP-1}}\right)}\right)$ and $\mathcal{Q}_{CQ-2} = \left(\mu_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP-2}}\right)}, \eta_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP-2}}\right)}\right)$, then

(1). $\mathcal{Q}_{CQ-1} \oplus_{CQ} \mathcal{Q}_{CQ-2} = \begin{pmatrix} \begin{pmatrix} \mu_{\mathcal{Q}_{CQRP-1}}^{q_{CQ}} + \mu_{\mathcal{Q}_{CQRP-2}}^{q_{CQ}} - \\ \mu_{\mathcal{Q}_{CQRP-1}}^{q_{CQ}} \mu_{\mathcal{Q}_{CQRP-2}}^{q_{CQ}} \end{pmatrix}^{\frac{1}{q_{CQ}}} e^{i2\pi\begin{pmatrix} \mu_{\mathcal{Q}_{CQIP-1}}^{q_{CQ}} + \mu_{\mathcal{Q}_{CQIP-2}}^{q_{CQ}} - \\ \mu_{\mathcal{Q}_{CQIP-1}}^{q_{CQ}} \mu_{\mathcal{Q}_{CQIP-2}}^{q_{CQ}} \end{pmatrix}^{\frac{1}{q_{CQ}}}}, \\ \eta_{\mathcal{Q}_{CQRP-1}} \eta_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP-1}} \eta_{\mathcal{Q}_{CQIP-2}}\right)} \end{pmatrix}$;

(2). $\mathcal{Q}_{CQ-1} \otimes_{CQ} \mathcal{Q}_{CQ-2} = \left(\begin{array}{c} \mu_{\mathcal{Q}_{CQRP-1}} \mu_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP-1}} \mu_{\mathcal{Q}_{CQIP-2}}\right)}, \\ \left(\begin{array}{c} \eta^{q_{CQ}}_{\mathcal{Q}_{CQRP-1}} + \eta^{q_{CQ}}_{\mathcal{Q}_{CQRP-2}} - \\ \eta^{q_{CQ}}_{\mathcal{Q}_{CQRP-1}} \eta^{q_{CQ}}_{\mathcal{Q}_{CQRP-2}} \end{array} \right)^{\frac{1}{q_{CQ}}} e^{i2\pi \left(\begin{array}{c} \eta^{q_{CQ}}_{\mathcal{Q}_{CQIP-1}} + \eta^{q_{CQ}}_{\mathcal{Q}_{CQIP-2}} - \\ \eta^{q_{CQ}}_{\mathcal{Q}_{CQIP-1}} \eta^{q_{CQ}}_{\mathcal{Q}_{CQIP-2}} \end{array} \right)^{\frac{1}{q_{CQ}}}} \end{array} \right)$;

(3). $\mathcal{Q}^{\delta_{SC}}_{CQ-1} = \left(\begin{array}{c} \mu^{\delta_{SC}}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mu^{\delta_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)}, \\ \left(1 - \left(1 - \eta^{q_{CQ}}_{\mathcal{Q}_{CQRP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(1 - \left(1 - \eta^{q_{CQ}}_{\mathcal{Q}_{CQIP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}}} \end{array} \right)$;

(4). $\delta_{SC}\mathcal{Q}_{CQ-1} = \left(\begin{array}{c} \left(1 - \left(1 - \mu^{q_{CQ}}_{\mathcal{Q}_{CQRP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(1 - \left(1 - \mu^{q_{CQ}}_{\mathcal{Q}_{CQIP-1}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}}}, \\ \eta^{\delta_{SC}}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\eta^{\delta_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)} \end{array} \right)$.

Definition 2.4. [21] For any two CQROFNs

$\mathcal{Q}_{CQ-1} = \left(\mu_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP-1}}\right)}, \eta_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP-1}}\right)}\right)$ and

$\mathcal{Q}_{CQ-2} = \left(\mu_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQIP-2}}\right)}, \eta_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQIP-2}}\right)}\right)$, the score and accuracy functions are given by:

$$Ş\left(\mathcal{Q}_{CQ-1}\right) = \frac{\left(\mu^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} - \eta^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} + \mu^{q_{SC}}_{\mathcal{Q}_{CQIP-1}} - \eta^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)}{2}, \tag{2.6}$$

$$\check{H}\left(\mathcal{Q}_{CQ-1}\right) = \frac{\left(\mu^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} + \eta^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} + \mu^{q_{SC}}_{\mathcal{Q}_{CQIP-1}} + \eta^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)}{2}. \tag{2.7}$$

Based on the two above notions, the compassion between two CQROFNs is given by:

(1).If $Ş\left(\mathcal{Q}_{CQ-1}\right) > Ş\left(\mathcal{Q}_{CQ-2}\right)$, then $\mathcal{Q}_{CQ-1} > \mathcal{Q}_{CQ-2}$;

(2).If $Ş\left(\mathcal{Q}_{CQ-1}\right) = Ş\left(\mathcal{Q}_{CQ-2}\right)$, then:

i) if $\check{H}\left(\mathcal{Q}_{CQ-1}\right) > \check{H}\left(\mathcal{Q}_{CQ-2}\right)$, then $\mathcal{Q}_{CQ-1} > \mathcal{Q}_{CQ-2}$;

ii) if $\check{H}\left(\mathcal{Q}_{CQ-1}\right) = \check{H}\left(\mathcal{Q}_{CQ-2}\right)$, then $\mathcal{Q}_{CQ-1} = \mathcal{Q}_{CQ-2}$.

Definition 2.5. [36] For the family of $\mathcal{Q}_{CQ-j}(j = 1,2,3, \dots, n)$, the MSM operator is given by:

$$MSM^{K_{SC}}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)=\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\mathcal{Q}_{CQ-j_i}}{C_n^{K_{SC}}}\right)^{\frac{1}{K_{SC}}} \tag{2.8}$$

where $K_{SC}=1,2,\dots,n,\left(j_1,j_2,\dots,j_{K_{SC}}\right)$ denotes the K-tuple kind of $(1,2,..,n)$, and the symbol $C_n^{K_{SC}}$ designates the binomial co-efficient (BCO). Furthermore:

(1). $MSM^{K_{SC}}\left(\mathcal{Q}_{CQ},\mathcal{Q}_{CQ},\dots,\mathcal{Q}_{CQ}\right)=\mathcal{Q}_{CQ}$;

(2). $MSM^{K_{SC}}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)\le MSM^{K_{SC}}\left(\mathcal{Q}_{CQ-*1},\mathcal{Q}_{CQ-*2},\dots,\mathcal{Q}_{CQ-*n}\right)$, if $\mathcal{Q}_{CQ-j}\le\mathcal{Q}_{CQ-*j}$ for all j.

(3). $\min_j\mathcal{Q}_{CQ-j}\le MSM^{K_{SC}}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)\le\max_j\mathcal{Q}_{CQ-j}$.

Definition 2.6. [36] For the family of $\mathcal{Q}_{CQ-j}(j=1,2,3,\dots,n)$, the GMSM operator is given by:

$$GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)=\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\mathcal{Q}_{CQ-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}} \tag{2.9}$$

where $K_{SC}=1,2,\dots,n,\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\ge 0,\left(j_1,j_2,\dots,j_{K_{SC}}\right)$ signifies the K-tuple collection of $(1,2,..,n)$. Furthermore:

(1). $GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQ},\mathcal{Q}_{CQ},\dots,\mathcal{Q}_{CQ}\right)=\mathcal{Q}_{CQ}, GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}(0,0,\dots,0)=0$;

(2). $GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)\le GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQ-*1},\mathcal{Q}_{CQ-*2},\dots,\mathcal{Q}_{CQ-*n}\right)$, if $\mathcal{Q}_{CQ-j}\le\mathcal{Q}_{CQ-*j}$ for all j;

(3). $\min_j\mathcal{Q}_{CQ-j}\le GMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQ-1},\mathcal{Q}_{CQ-2},\dots,\mathcal{Q}_{CQ-n}\right)\le\max_j\mathcal{Q}_{CQ-j}$.

2.1. Review of the three-way decisions based on DTRSs

As discussed in [34], DTRS contains two status and three actions, whose information: $\Omega_S=\{\mathcal{F}_B,\sim\mathcal{F}_{NB}\}$ and $A_{AC}=\left\{\chi_{P_{AC}},\chi_{B_{AC}},\chi_{N_{AC}}\right\}$, where Ω_S is utilized for having a place and for to be not having a place and A_{AC} is utilized for +ve, limit and -ve. Table 1 shows the lose capacities and their portrayals.

Table 1. Lose functions and their representations.

Symbols	$\mathcal{F}_B(P_{AC}) = \boldsymbol{Correctly\ solved}$	$\sim \mathcal{F}_{NB}(N_{AC}) = \boldsymbol{wrongly\ solved}$
$\chi_{P_{AC}}$	$\bar{Q}_{CQ-P_{AC}P_{AC}}$	$\bar{Q}_{CQ-P_{AC}N_{AC}}$
$\chi_{B_{AC}}$	$\bar{Q}_{CQ-B_{AC}P_{AC}}$	$\bar{Q}_{CQ-B_{AC}N_{AC}}$
$\chi_{N_{AC}}$	$\bar{Q}_{CQ-N_{AC}P_{AC}}$	$\bar{Q}_{CQ-N_{AC}N_{AC}}$

Note: $\bar{Q}_{CQ-P_{AC}P_{AC}}, \bar{Q}_{CQ-P_{AC}N_{AC}}$ = Expenses of right sort and mistake sort of idea ử in agreed judgment; $\bar{Q}_{CQ-B_{AC}P_{AC}}, \bar{Q}_{CQ-B_{AC}N_{AC}}$ = Expenses of the right sort and mistake sort of the idea ử in abstinace judgment; $\bar{Q}_{CQ-N_{AC}P_{AC}}, \bar{Q}_{CQ-N_{AC}N_{AC}}$ = Expenses of the right sort and mistake sort of the idea ử in non − agreed judgment.

Form Table 1, the inequality is given as follows:

$$\bar{Q}_{CQ-P_{AC}P_{AC}} < \bar{Q}_{CQ-B_{AC}P_{AC}} < \bar{Q}_{CQ-N_{AC}P_{AC}}, \tag{2.10}$$

$$\bar{Q}_{CQ-N_{AC}N_{AC}} < \bar{Q}_{CQ-B_{AC}N_{AC}} < \bar{Q}_{CQ-P_{AC}N_{AC}}. \tag{2.11}$$

Given the Bayesian risk choice hypothesis [34], we have:

$$\Pr(\mathcal{F}_B|[\text{ử}]) + \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]) = 1. \tag{2.12}$$

By using the Eq (2.12), the expected losses $Y_{EL}(\chi_{j_{AC}}|[\text{ử}]), j = P, B, N$, and the different actions are expressed below:

$$Y_{EL}(\chi_{P_{AC}}|[\text{ử}]) = \bar{Q}_{CQ-P_{AC}P_{AC}} \Pr(\mathcal{F}_B|[\text{ử}]) + \bar{Q}_{CQ-P_{AC}N_{AC}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]), \tag{2.13}$$

$$Y_{EL}(\chi_{B_{AC}}|[\text{ử}]) = \bar{Q}_{CQ-B_{AC}P_{AC}} \Pr(\mathcal{F}_B|[\text{ử}]) + \bar{Q}_{CQ-B_{AC}N_{AC}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]), \tag{2.14}$$

$$Y_{EL}(\chi_{N_{AC}}|[\text{ử}]) = \bar{Q}_{CQ-N_{AC}P_{AC}} \Pr(\mathcal{F}_B|[\text{ử}]) + \bar{Q}_{CQ-N_{AC}N_{AC}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]). \tag{2.15}$$

From the above information, we have:

$$P_{AC}: \text{When } Y_{EL}(\chi_{P_{AC}}|[\text{ử}]) \le Y_{EL}(\chi_{B_{AC}}|[\text{ử}]) \text{ and } Y_{EL}(\chi_{P_{AC}}|[\text{ử}]) \le Y_{EL}(\chi_{N_{AC}}|[\text{ử}]), \text{ then ử} \in POS(\mathcal{F}_P); \tag{2.16}$$

$$B_{AC}\text{: When } Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right) \leq Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right) \text{ and } Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right) \leq Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right)\text{, then ử} \in BUN(\mathcal{F}_B); \tag{2.17}$$

$$N_{AC}\text{: When } Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right) \leq Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right) \text{ and } Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right) \leq Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right)\text{, then ử} \in NRG(\mathcal{F}_N). \tag{2.18}$$

3. Complex q-rung orthopair 2-tuple linguistic sets

Based on the above analysis, we have explored the novel approach of CQRO2-TLS and its fundamental properties. These properties are also explained with the help of some numerical examples.

Definition 3.1. A CQRO2-TLS is given by

$$\mathcal{Q}_{CQTL} = \left\{\left(\left(s_{S_{LT}(\text{ử})}, \alpha_{SC}\right), \left(\mu_{\mathcal{Q}_{CQTL}}(\text{ử}), \eta_{\mathcal{Q}_{CQTL}}(\text{ử})\right)\right) : \text{ử} \in \tilde{\text{Ư}}_{UNI}\right\} \tag{3.1}$$

where $\mu_{\mathcal{Q}_{CQTL}}(\text{ử}) = \mu_{\mathcal{Q}_{CQTLRP}}(\text{ử})e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\text{ử})\right)}$ and $\eta_{\mathcal{Q}_{CQTL}}(\text{ử}) = \eta_{\mathcal{Q}_{CQTLRP}}(\text{ử})e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\text{ử})\right)}$, with the conditions: $0 \leq \mu^{q_{SC}}_{\mathcal{Q}_{CQTLRP}}(\text{ử}) + \eta^{q_{SC}}_{\mathcal{Q}_{CQTLRP}}(\text{ử}) \leq 1, 0 \leq \mu^{q_{SC}}_{\mathcal{Q}_{CQTLIP}}(\text{ử}) + \eta^{q_{SC}}_{\mathcal{Q}_{CQTLIP}}(\text{ử}) \leq 1$, where q_{SC} expresses the rational numbers and the pair $\left(s_{S_{LT}(\text{ử})}, \alpha_{SC}\right)$ is called 2-TLV with $\alpha_{SC} \in \left[-\frac{1}{2g}, \frac{1}{2g}\right\}$ and $s_{S_{LT}(\text{ử})} \in S_{LT}$.

Moreover,

$$\zeta_{\mathcal{Q}_{CQTL}}(\text{ử}) = \left(1 - \left(\mu^{q_{SC}}_{\mathcal{Q}_{CQTLRP}}(\text{ử}) + \eta^{q_{SC}}_{\mathcal{Q}_{CQTLRP}}(\text{ử})\right)^{\frac{1}{q_{SC}}}\right)e^{i2\pi\left(1-\left(\mu^{q_{SC}}_{\mathcal{Q}_{CQTLIP}}(\text{ử})+\eta^{q_{SC}}_{\mathcal{Q}_{CQTLIP}}(\text{ử})\right)^{\frac{1}{q_{SC}}}\right)}$$ is called the refusal grade. The CQRO2T-LN is shown by:

$$\mathcal{Q}_{CQTL} = \left(\left(s_{S_{LT}}, \alpha_{SC}\right), \left(\mu_{\mathcal{Q}_{CQTL}}, \eta_{\mathcal{Q}_{CQTL}}\right)\right) = \left(\left(s_{S_{LT}}, \alpha_{SC}\right), \left(\mu_{\mathcal{Q}_{CQTLRP}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}\right)}, \eta_{\mathcal{Q}_{CQTLRP}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}\right)}\right)\right).$$

Definition 3.2. For any two CQRO2-TLNs

$$\mathcal{Q}_{CQTL-1} = \left(\left(s_{S_{LT-1}}, \alpha_{SC-1}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-1}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-1}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-1}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-1}}\right)}\right)\right)$$

and $\mathcal{Q}_{CQTL-2} = \left(\left(s_{S_{LT-2}}, \alpha_{SC-2}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-2}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-2}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-2}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-2}}\right)}\right)\right)$, (1). $\mathcal{Q}_{CQTL-1} \oplus_{CQTL} \mathcal{Q}_{CQTL-2} =$

$$\left(\begin{array}{c}\Delta_{LT}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right) + \Delta_{LT}^{-1}\left(s_{S_{LT-2}}, \alpha_{SC-2}\right)\right). \\ \left(\begin{array}{c}\left(\begin{array}{c}\mu_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}} + \mu_{\mathcal{Q}_{CQTLRP-2}}^{q_{CQ}} - \\ \mu_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}} \mu_{\mathcal{Q}_{CQTLRP-2}}^{q_{CQ}}\end{array}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(\begin{array}{c}\mu_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}} + \mu_{\mathcal{Q}_{CQTLIP-2}}^{q_{CQ}} - \\ \mu_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}} \mu_{\mathcal{Q}_{CQTLIP-2}}^{q_{CQ}}\end{array}\right)^{\frac{1}{q_{CQ}}}}, \\ \eta_{\mathcal{Q}_{CQTLRP-1}} \eta_{\mathcal{Q}_{CQTLRP-2}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-1}} \eta_{\mathcal{Q}_{CQTLIP-2}}\right)}\end{array}\right)\end{array}\right);$$

(2). $\mathcal{Q}_{CQTL-1} \otimes_{CQTL} \mathcal{Q}_{CQTL-2} =$

$$\left(\begin{array}{c}\Delta_{LT}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right) \times \Delta_{LT}^{-1}\left(s_{S_{LT-2}}, \alpha_{SC-2}\right)\right). \\ \left(\begin{array}{c}\mu_{\mathcal{Q}_{CQTLRP-1}} \mu_{\mathcal{Q}_{CQTLRP-2}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-1}} \mu_{\mathcal{Q}_{CQTLIP-2}}\right)}, \\ \left(\begin{array}{c}\eta_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}} + \eta_{\mathcal{Q}_{CQTLRP-2}}^{q_{CQ}} - \\ \eta_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}} \eta_{\mathcal{Q}_{CQTLRP-2}}^{q_{CQ}}\end{array}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(\begin{array}{c}\eta_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}} + \eta_{\mathcal{Q}_{CQTLIP-2}}^{q_{CQ}} - \\ \eta_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}} \eta_{\mathcal{Q}_{CQTLIP-2}}^{q_{CQ}}\end{array}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right);$$

(3). $$\mathcal{Q}_{CQTL-1}^{\delta_{SC}} = \left(\begin{array}{c}\Delta_{LT}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right)^{\delta_{SC}}\right), \mu_{\mathcal{Q}_{CQTLRP-1}}^{\delta_{SC}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-1}}^{\delta_{SC}}\right)}, \\ \left(\left(1-\left(1-\eta_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(1-\left(1-\eta_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}}\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}}}\right)\end{array}\right);$$

(4). $$\delta_{SC}\mathcal{Q}_{CQTL-1} = \left(\begin{array}{c}\Delta_{LT}\left(\delta_{SC} \times \Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right)\right), \\ \left(\begin{array}{c}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP-1}}^{q_{CQ}}(\text{ử})\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}} e^{i2\pi\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP-1}}^{q_{CQ}}(\text{ử})\right)^{\delta_{SC}}\right)^{\frac{1}{q_{CQ}}}}, \\ \eta_{\mathcal{Q}_{CQTLRP-1}}^{\delta_{SC}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-1}}^{\delta_{SC}}\right)}\end{array}\right)\end{array}\right).$$

Definition 3.3. For any two QROF2-TLNs

$$\mathcal{Q}_{CQTL-1} = \left(\left(s_{S_{LT-1}}, \alpha_{SC-1}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-1}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-1}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-1}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-1}}\right)}\right)\right)$$

and $\mathcal{Q}_{CQTL-2} = \left(\left(s_{S_{LT-2}}, \alpha_{SC-2}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-2}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-2}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-2}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-2}}\right)}\right)\right)$, the score and accuracy functions are given by:

$$\c{S}\left(\mathcal{Q}_{CQTL-1}\right) = \frac{\Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right) \times \left(1 + \mu_{\mathcal{Q}_{CQTLRP-1}}^{q_{SC}} + \mu_{\mathcal{Q}_{CQTLIP-1}}^{q_{SC}} - \eta_{\mathcal{Q}_{CQTLRP-1}}^{q_{SC}} - \eta_{\mathcal{Q}_{CQTLIP-1}}^{q_{SC}}\right)}{2}, \tag{3.2}$$

$$\check{H}\left(\mathcal{Q}_{CQTL-1}\right) = \Delta_{LT}^{-1}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right) \times \left(\mu_{\mathcal{Q}_{CQTLRP-1}}^{q_{SC}} + \mu_{\mathcal{Q}_{CQTLIP-1}}^{q_{SC}} + \eta_{\mathcal{Q}_{CQTLRP-1}}^{q_{SC}} + \eta_{\mathcal{Q}_{CQTLIP-1}}^{q_{SC}}\right). \tag{3.3}$$

Based on the two above notions, the compassion between two QROF2-TLNs is given by:

(1).If $\text{Ş}(\mathcal{Q}_{CQTL-1}) > \text{Ş}(\mathcal{Q}_{CQTL-2})$, then $\mathcal{Q}_{CQTL-1} > \mathcal{Q}_{CQTL-2}$;

(2).If $\text{Ş}(\mathcal{Q}_{CQTL-1}) = \text{Ş}(\mathcal{Q}_{CQTL-2})$, then:

i) $\check{\mathrm{H}}(\mathcal{Q}_{CQTL-1}) > \check{\mathrm{H}}(\mathcal{Q}_{CQTL-2})$, implies $\mathcal{Q}_{CQTL-1} > \mathcal{Q}_{CQTL-2}$;

ii) $\check{\mathrm{H}}(\mathcal{Q}_{CQTL-1}) = \check{\mathrm{H}}(\mathcal{Q}_{CQTL-2})$, implies $\mathcal{Q}_{CQTL-1} = \mathcal{Q}_{CQTL-2}$.

Table 2. Lose functions and their representations for Complex q-rung orthopair fuzzy 2-tuple linguistic variables.

Symbols	$\boldsymbol{\mathcal{F}_B(P_{AC}) = Correctly\ solved}$	$\sim \boldsymbol{\mathcal{F}_{NB}(N_{AC}) = wrongly\ solved}$
$\boldsymbol{\chi_{P_{AC}}}$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$
$\boldsymbol{\chi_{B_{AC}}}$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$
$\boldsymbol{\chi_{N_{AC}}}$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$	$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}} = \left(\begin{matrix}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}})}, \alpha_{SC}\right), \\ \left(\begin{matrix}\mu_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}) \\ e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}})\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}) \\ e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}})\right)}\end{matrix}\right)\end{matrix}\right)$

Note: $\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}}, \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}} =$ Expenses of the right sort and mistake sort of the idea ử in agreed judgment; $\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}}, \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}} =$ Expenses of the right sort and mistake sort of the idea ử in abstinance judgment; $\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}}, \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}} =$ Expenses of the right sort and mistake sort of the idea ử in non − agreed judgment.

4. A novel DTRS model based on CQRO2-TLVs

The purpose of this section is to explore the novel DTRS model by using the CQROF2-TLV based on three-way decisions. The LF matrix for CQROF2-TLV is presented in Table 2.

Form Table 2, the inequality is given as:

$$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} < \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}} < \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}}, \tag{4.1}$$

$$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}} < \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}} < \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}}. \tag{4.2}$$

Furthermore, by using the Eq (4.2), we explain the expected losses $Y_{EL}(\chi_{j_{AC}}|[ử]), j = P, B, N$ and the different actions are expressed below:

$$Y_{EL}(\chi_{P_{AC}}|[ử]) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[ử]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[ử]), \tag{4.3}$$

$$Y_{EL}(\chi_{B_{AC}}|[ử]) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[ử]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[ử]), \tag{4.4}$$

$$Y_{EL}(\chi_{N_{AC}}|[ử]) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[ử]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[ử]). \tag{4.5}$$

Assumed that $\Pr(\mathcal{F}_B|[ử]) = \delta_B$ and $\Pr(\sim \mathcal{F}_{NB}|[ử]) = \delta_{NB}$, then the Eq (4.2) is given as: $\delta_B + \delta_{NB} = 1$.

Theorem 4.1. With usual meanings

$$Y_{EL}(\chi_{P_{AC}}|[ử]) = \left(\begin{array}{c} \Delta_{LT}\left(\delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})}, \alpha_{SC}\right) + \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}})}, \alpha_{SC}\right)\right), \\ \left(\begin{array}{c} \left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}})\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}})\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\begin{array}{c}\left(\eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)^{\delta_B} \times \\ \left(\eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}})\right)^{\delta_{NB}}\end{array}\right) e^{i2\pi\left(\begin{array}{c}\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}})\right)^{\delta_B} \times \\ \left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}})\right)^{\delta_{NB}}\end{array}\right)} \end{array}\right) \end{array}\right), \tag{4.6}$$

$$Y_{EL}(\chi_{B_{AC}}|[ử]) = \left(\begin{array}{c} \Delta_{LT}\left(\delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})}, \alpha_{SC}\right) + \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})}, \alpha_{SC}\right)\right), \\ \left(\begin{array}{c} \left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\begin{array}{c}\left(\eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)^{\delta_B} \times \\ \left(\eta_{\mathcal{Q}_{CQTLRP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)^{\delta_{NB}}\end{array}\right) e^{i2\pi\left(\begin{array}{c}\left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}})\right)^{\delta_B} \times \\ \left(\eta_{\mathcal{Q}_{CQTLIP}}(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}})\right)^{\delta_{NB}}\end{array}\right)} \end{array}\right) \end{array}\right), \tag{4.7}$$

$$Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right) = \left(\begin{array}{c} \Delta_{LT}\left(\delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)}, \alpha_{SC}\right) + \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)}, \alpha_{SC}\right)\right), \\ \left(\begin{array}{c} \left(1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\begin{array}{c}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B} \times \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{array}\right) e^{i2\pi\left(\begin{array}{c}\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B} \times \\ \left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{array}\right)} \end{array}\right) \end{array}\right). \quad (4.8)$$

Proof. First, we prove that the Eq (4.6) holds. The proofs of Eq (4.7) and of Eq (4.8) are similar.

$$Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right) = Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[\text{ử}]) \oplus_{CQTL} Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}])$$

$$= Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}}{}^{\delta_B} \oplus_{CQTL} Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}}{}^{\delta_{NB}}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \Delta_{LT}\left(\delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)}, \alpha_{SC}\right)\right), \\ \left(\begin{array}{c} \left(1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}} \end{array}\right) \end{array}\right) \oplus_{CQTL} \\ \left(\begin{array}{c} \Delta_{LT}\left(\delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)}, \alpha_{SC}\right)\right), \\ \left(\begin{array}{c} \left(1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_B}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_B}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_B} e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_B}} \end{array}\right) \end{array}\right) \end{array}\right)$$

$$
= \left(\Delta_{LT}\left(\delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)}, \alpha_{SC}\right) + \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)}, \alpha_{SC}\right)\right), \left(\begin{array}{c} \left(\begin{array}{c} 1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\begin{array}{c} \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \times \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right) e^{i2\pi\left(\begin{array}{c} \left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \times \\ \left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)} \end{array} \right) \right)
$$

Moreover, we diagnose the expected values, such that

$$
Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[ử]\right)\right) = \frac{\left(\left(\begin{array}{c} \delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)}, \alpha_{SC}\right) + \\ \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)}, \alpha_{SC}\right) \end{array}\right) \times \left(\begin{array}{c} \left(\begin{array}{c} 1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)^{\frac{1}{q_{CQ}}} + \\ \left(\begin{array}{c} 1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)^{\frac{1}{q_{CQ}}} - \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} - \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array} \right) \right)}{8}, \tag{4.9}
$$

$$
Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[ử]\right)\right) = \frac{\left(\left(\begin{array}{c} \delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)}, \alpha_{SC}\right) + \\ \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)}, \alpha_{SC}\right) \end{array}\right) \times \left(\begin{array}{c} \left(\begin{array}{c} 1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)^{\frac{1}{q_{CQ}}} + \\ \left(\begin{array}{c} 1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array}\right)^{\frac{1}{q_{CQ}}} - \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}} - \\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array} \right) \right)}{8}, \tag{4.10}
$$

$$Q_{EV}\left(Y_{EL}\left(\chi_{D_{AC}}|\text{[ử]}\right)\right)=\frac{\left(\begin{pmatrix}\delta_B\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}. \tag{4.11}$$

To handle whenever the normal qualities failed to discover the connections between any of two predictable losses, we investigate the ideas of precision work, which is expressed as follows:

$$G_{AF}\left(Y_{EL}\left(\chi_{P_{AC}}|\text{[ử]}\right)\right)$$

$$=\frac{\left(\begin{pmatrix}\delta_B\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}, \tag{4.12}$$

$$G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|\text{[ử]}\right)\right)=\frac{\left(\begin{pmatrix}\delta_B\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{s_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}, \tag{4.13}$$

$$G_{AF}\left(Y_{EL}\left(\chi_{D_{AC}}|[\mathring{u}]\right)\right)$$

$$=\frac{\left(\begin{pmatrix}\delta_B\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{Q_{CQTLRP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\\ \left(1-\mu_{Q_{CQTLIP}}^{q_{CQ}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}. \tag{4.14}$$

Furthermore, we diagnosed the three-way decision rules, such that

P_{AC-1}: When
$$\begin{aligned}
&Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)<Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\vee Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\\
&=Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\wedge G_{AF}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\\
&\leq G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\wedge Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\\
&<Q_{EV}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right)\vee Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\\
&=Q_{EV}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right)\wedge G_{AF}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\\
&\leq G_{AF}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right),\text{ then } \mathring{u}\in POS(\mathcal{F}_P);
\end{aligned}\tag{4.15}$$

B_{AC-1}: When
$$\begin{aligned}
&Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)<Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\vee Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\\
&=Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\wedge G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\\
&\leq G_{AF}\left(Y_{EL}\left(\chi_{P_{AC}}|[\mathring{u}]\right)\right)\wedge Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\\
&<Q_{EV}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right)\vee Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\\
&=Q_{EV}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right)\wedge G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|[\mathring{u}]\right)\right)\\
&\leq G_{AF}\left(Y_{EL}\left(\chi_{N_{AC}}|[\mathring{u}]\right)\right),\text{ then } \mathring{u}\in BUN(\mathcal{F}_P);
\end{aligned}\tag{4.16}$$

N_{AC-1}: When $Q_{EV}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right) < Q_{EV}\left(Y_{EL}(\chi_{P_{AC}}|[ử])\right) \vee Q_{EV}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right)$

$$= Q_{EV}\left(Y_{EL}(\chi_{P_{AC}}|[ử])\right) \wedge G_{AF}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right)$$
$$\leq G_{AF}\left(Y_{EL}(\chi_{P_{AC}}|[ử])\right) \wedge Q_{EV}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right)$$
$$< Q_{EV}\left(Y_{EL}(\chi_{B_{AC}}|[ử])\right) \vee Q_{EV}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right) \tag{4.17}$$
$$= Q_{EV}\left(Y_{EL}(\chi_{B_{AC}}|[ử])\right) \wedge G_{AF}\left(Y_{EL}(\chi_{N_{AC}}|[ử])\right)$$
$$\leq G_{AF}\left(Y_{EL}(\chi_{B_{AC}}|[ử])\right), \text{ then } ử \in NEG(\mathcal{F}_P).$$

Therefore,

$$G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}}\right) < G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-B_{AC}P_{AC}}}\right) < G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-N_{AC}P_{AC}}}\right), \tag{4.18}$$

$$G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-N_{AC}N_{AC}}}\right) < G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-B_{AC}N_{AC}}}\right) < G_{AF}\left(Q_{CQTL-\bar{Q}_{CQ-P_{AC}N_{AC}}}\right). \tag{4.19}$$

5. Certain GMSM operators for CQRO2-TLVs

The interrelationship among the different attributes in real decision theory is ever-present. The MSM and GMSM operators are efficient techniques to perfectly evaluate the interrelation between the characteristics. The purpose of this communication is to explore the GMSM and weighted GMSM based on CQRO2-TLVs.

Definition 5.1. For the family of CQRO2-TLVs $Q_{CQTL-j}(j = 1,2,3,\dots,n)$, the CQRO2-TLGMSM operator is given by:

$$CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q_{CQTL-1}, Q_{CQTL-2},\cdots,Q_{CQTL-n})$$
$$= \left(\frac{\sum_{1\leq j_1\leq\cdots\leq j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}} \tag{5.1}$$

where $K_{SC} = 1,2,\dots,n, \alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}} \geq 0, (j_1, j_2,\dots,j_{K_{SC}})$ designates the K-tuple collection of $(1,2,..,n)$.

Theorem 5.2. For the family of CQRO2-TLVs

$Q_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mu_{Q_{CQTLRP-j}} e^{i2\pi\left(\mu_{Q_{CQTLIP-j}}\right)}, \eta_{Q_{CQTLRP-j}} e^{i2\pi\left(\eta_{Q_{CQTLIP-j}}\right)}\right)\right) (j = 1,2,3,\dots,n)$,

$$CQRO2-TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\dots,\mathcal{Q}_{CQTL-n})$$

$$=\left(\begin{array}{c}\Delta_{LT}\left(\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right),\\ \left(\begin{array}{c}\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\right)},\\ \left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right). \tag{5.2}$$

Proof. Let

$$\prod_{i=1}^{K_{SC}}\mathcal{Q}_{CQTL-j_i}^{\alpha_{SC-i}}=\left(\begin{array}{c}\Delta_{LT}\left(\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}\right),\\ \left(\begin{array}{c}\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLRP-j}}\right)^{\alpha_{SC-i}}e^{i2\pi\left(\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)^{\alpha_{SC-i}}\right)},\\ \left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{q_{CQ}}}e^{i2\pi\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right)$$

$$\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}} = \left(\begin{array}{c} \Delta_{LT}\left(\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}\right), \\ \left(\begin{array}{c}\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLRP-j}}^{q_{SC}}\right)^{\alpha_{SC-i}}\right)\right)^{\frac{1}{q_{SC}}} \\ e^{i2\pi\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLIP-j}}^{q_{SC}}\right)^{\alpha_{SC-i}}\right)\right)^{\frac{1}{q_{SC}}}}, \\ \prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{q_{CQ}}}\right)}\end{array}\right)\end{array}\right)$$

$$\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}} = \left(\begin{array}{c} \Delta_{LT}\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right), \\ \left(\begin{array}{c}\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right)$$

$$\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} \mathcal{Q}_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}$$

$$= \left(\begin{array}{c} \Delta_{LT}\left(\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right), \\ \left(\begin{array}{c} \left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\right)}, \\ \left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}} \end{array}\right) \end{array}\right)$$

Furthermore, by using the investigated operators, we discuss certain properties such as idempotency, commutativity, monotonicity, and boundedness, which are very important for the proposed work.

Theorem 5.3. For the family of CQRO2-TLVs

$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-j}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-j}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-j}}\right)}\right)\right) (j = 1,2,3,\dots,n)$, then

(1). $\mathcal{Q}_{CQTL-j} = \mathcal{Q}_{CQTL} = \left(\left(s_{S_{LT}}, \alpha_{SC}\right), \left(\mu_{\mathcal{Q}_{CQTLRP}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}\right)}, \eta_{\mathcal{Q}_{CQTLRP}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}\right)}\right)\right)$,

implies $CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) = \mathcal{Q}_{CQTL}$.

(2). if $\mathcal{Q}_{CQTL-j}$ is a collection of CQRO2-TLVs and $\mathcal{Q}'_{CQTL-j}$ is any replacement of $\mathcal{Q}_{CQTL-j}$, then

$CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\ldots,\mathcal{Q}_{CQTL-n}\right)=CQRO2-$

$TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}'_{CQTL-1},\mathcal{Q}'_{CQTL-2},\ldots,\mathcal{Q}'_{CQTL-n}\right)$.

(3). if $\mathcal{Q}_{CQTL-j}$ and $\mathcal{Q}'_{CQTL-j}$ are any two collections of CQRO2-TLVs with a situations s.t. $\mathcal{Q}_{CQTL-j}\leq\mathcal{Q}'_{CQTL-j}$,

then $CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\ldots,\mathcal{Q}_{CQTL-n}\right)\leq$ $CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}'_{CQTL-1},\mathcal{Q}'_{CQTL-2},\ldots,\mathcal{Q}'_{CQTL-n}\right)$.

(4). if $\mathcal{Q}_{CQTL-j}=\left(\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\mu_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)},\eta_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-j}}\right)}\right)\right)$ is a family of the CQRO2-TLVs, then

$min\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\ldots,\mathcal{Q}_{CQTL-n}\right)\leq CQRO2-$

$TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\ldots,\mathcal{Q}_{CQTL-n}\right)\leq$

$max\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\ldots,\mathcal{Q}_{CQTL-n}\right)$.

Proof.

(1). If $\mathcal{Q}_{CQTL}=\left(\left(s_{S_{LT}},\alpha_{SC}\right),\left(\mu_{\mathcal{Q}_{CQTLRP}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}\right)},\eta_{\mathcal{Q}_{CQTLRP}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}\right)}\right)\right)$, then we have

$$CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(Q_{CQTL-1},Q_{CQTL-2},\ldots,Q_{CQTL-n}\right)$$

$$=\left(\begin{array}{c}\Delta_{LT}\left(\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{s_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right),\\ \left(\begin{array}{c}\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\right)},\\ \left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLRP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLIP-j}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\Delta_{LT}\left(\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Delta_{LT}^{-1}\left(s_{s_{LT}},\alpha_{SC}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right),\\ \left(\begin{array}{c}\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLRP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(\left(\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(\mu_{Q_{CQTLIP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\right)},\\ \left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLRP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(1-\left(1-\prod_{1\le j_1\le\cdots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{Q_{CQTLIP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \Delta_{LT}\left(\left(\left(\frac{\left(\Delta_{LT}^{-1}\left(s_{s_{LT}}, \alpha_{SC}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)\right), \\ \left(\begin{array}{c} \left(\left(1-\left(1-\left(\left(\mu_{Q_{CQTLRP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \\ e^{i2\pi\left(\left(1-\left(1-\left(\left(\mu_{Q_{CQTLIP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}, \\ \left(1-\left(1-\left(1-\left(1-\eta_{Q_{CQTLRP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\left(1-\left(1-\eta_{Q_{CQTLIP}}^{q_{CQ}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}} \end{array} \right) \end{array} \right)$$

$$= \left(\left(s_{s_{LT}}, \alpha_{SC}\right), \left(\mu_{Q_{CQTLRP}} e^{i2\pi\left(\mu_{Q_{CQTLIP}}\right)}, \eta_{Q_{CQTLRP}} e^{i2\pi\left(\eta_{Q_{CQTLIP}}\right)}\right)\right) = Q_{CQTL}.$$

(2). By supposition, we have Q_{CQTL-j}, a collection CQRO2-TLVs and Q'_{CQTL-j} is a slight substitution of Q_{CQTL-j}, then

$$CQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(Q_{CQTL-1}, Q_{CQTL-2}, \ldots, Q_{CQTL-n}\right)$$

$$= \left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}} \prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}$$

$$= \left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}} \prod_{i=1}^{K_{SC}} {Q'}_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}$$

$$= CQRO2 - TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\ldots,\alpha_{SC-K_{SC}}\right)}\left(Q'_{CQTL-1}, Q'_{CQTL-2}, \ldots, Q'_{CQTL-n}\right).$$

(3). By hypothesis, we have if Q_{CQTL-j} and Q'_{CQTL-j} are any two collectionss of CQRO2-TLVs with the settings s.t. $Q_{CQTL-j} \le Q'_{CQTL-j}$, then

$$\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}} \le \prod_{i=1}^{K_{SC}} {Q'}_{CQTL-j_i}^{\alpha_{SC-i}} \Rightarrow \sum_{1\le j_1\le\cdots\le j_{K_{SC}}} \prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}} \le \sum_{1\le j_1\le\cdots\le j_{K_{SC}}} \prod_{i=1}^{K_{SC}} {Q'}_{CQTL-j_i}^{\alpha_{SC-i}}$$

$$\Rightarrow \frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}} \le \frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q'^{\alpha_{SC-i}}_{CQTL-j_i}}{C_n^{K_{SC}}}$$

$$\Rightarrow \left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q_{CQTL-j_i}^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}} \le$$

$$\left(\frac{\sum_{1\le j_1\le\cdots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}} Q'^{\alpha_{SC-i}}_{CQTL-j_i}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\cdots+\alpha_{SC-K_{SC}}}}.$$

Hence

$$CQRO2-TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}) \le CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q'_{CQTL-1},Q'_{CQTL-2},\dots,Q'_{CQTL-n}).$$

(4). By hypothesis, if we have

$$Q_{CQTL-j} = \left(\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\mu_{Q_{CQTLRP-j}}e^{i2\pi\left(\mu_{Q_{CQTLIP-j}}\right)},\eta_{Q_{CQTLRP-j}}e^{i2\pi\left(\eta_{Q_{CQTLIP-j}}\right)}\right)\right),$$

a collection of the CQRO2-TLVs, $Q^{-}_{CQTL-j} = min(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n})$ and $Q^{+}_{CQTL-j} = max(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n})$, then by means of the properties (1) & (3), we have

$$CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}) \ge CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q^{-}_{CQTL-1},Q^{-}_{CQTL-2},\dots,Q^{-}_{CQTL-n}) = Q^{-}_{CQTL}$$

$$CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}) \le CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q^{+}_{CQTL-1},Q^{+}_{CQTL-2},\dots,Q^{+}_{CQTL-n}) = Q^{+}_{CQTL}.$$

Hence

$$min(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}) \le CQRO2 - TLGMSM^{(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}})}(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}) \le max(Q_{CQTL-1},Q_{CQTL-2},\dots,Q_{CQTL-n}).$$

Definition 5.4. For the family of CQRO2-TLVs $Q_{QTL-j}(j = 1,2,3,\dots,n)$, the WGMSM operator is given by:

$$WCQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\dots,\mathcal{Q}_{CQTL-n}\right)$$

$$=\left(\frac{\sum_{1\le j_1\le\dots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Omega_{\mathcal{WV}-j_i}\mathcal{Q}_{CQTL-j_i}\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}} \tag{5.3}$$

where $K_{SC}=1,2,\dots,n,\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\ge 0,\left(j_1,j_2,\dots,j_{K_{SC}}\right)$ represents the K-tuple collection of $(1,2,..,n)$, and $\Omega_{\mathcal{WV}}=\left(\Omega_{\mathcal{WV}-1},\Omega_{\mathcal{WV}-2},\dots,\Omega_{\mathcal{WV}-n}\right)^T$ with the condition $\sum_{j=1}^{n}\Omega_{\mathcal{WV}-j}=1$.

Theorem 5.5. For the family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j}=\left(\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\mu_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)},\eta_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-j}}\right)}\right)\right)(j=1,2,3,\dots,n),$$

$$WCQRO2-TLGMSM^{\left(K_{SC},\alpha_{SC-1},\alpha_{SC-2},\dots,\alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},\dots,\mathcal{Q}_{CQTL-n}\right)$$

$$=\left(\begin{array}{c}\Delta_{LT}\left(\left(\frac{\sum_{1\le j_1\le\dots\le j_{K_{SC}}}\prod_{i=1}^{K_{SC}}\left(\Omega_{\mathcal{WV}-j_i}\Delta_{LT}^{-1}\left(s_{S_{LT-j_i}},\alpha_{SC-j_i}\right)\right)^{\alpha_{SC-i}}}{C_n^{K_{SC}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}}\right),\\ \left(\begin{array}{c}\left(\left(1-\prod_{1\le j_1\le\dots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP-j}}^{q_{CQ}}\right)^{\Omega_{\mathcal{WV}-j_i}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(\left(1-\prod_{1\le j_1\le\dots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP-j}}^{q_{CQ}}\right)^{\Omega_{\mathcal{WV}-j_i}}\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}},\\ \left(1-\left(1-\prod_{1\le j_1\le\dots\le j_{K_{SC}}}\left(1-1-\prod_{i=1}^{K_{SC}}\left(\eta_{\mathcal{Q}_{CQTLRP-j}}^{\Omega_{\mathcal{WV}-j_i}q_{CQ}}(\text{ử})\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}\times\\ e^{i2\pi\left(1-\left(1-\prod_{1\le j_1\le\dots\le j_{K_{SC}}}\left(1-\prod_{i=1}^{K_{SC}}\left(1-\eta_{\mathcal{Q}_{CQTLIP-j}}^{\Omega_{\mathcal{WV}-j_i}q_{CQ}}(\text{ử})\right)^{\alpha_{SC-i}}\right)^{\frac{1}{C_n^{K_{SC}}}}\right)^{\frac{1}{\alpha_{SC-1}+\alpha_{SC-2}+\dots+\alpha_{SC-K_{SC}}}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right). \tag{5.4}$$

Proof. Straightforward.

Theorem 5.6. For the family of CQROF2-TLVs

$$\mathcal{Q}_{CQTL-j}=\left(\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\mu_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)},\eta_{\mathcal{Q}_{CQTLRP-j}}e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-j}}\right)}\right)\right)(j=1,2,3,\dots,n),$$

(1). if $\mathcal{Q}_{CQTL-j} = \mathcal{Q}_{CQTL} = \left(\left(s_{S_{LT}}, \alpha_{SC}\right), \left(\mu_{\mathcal{Q}_{CQTLRP}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}\right)}, \eta_{\mathcal{Q}_{CQTLRP}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}\right)}\right)\right)$, then

$$WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) = \mathcal{Q}_{CQTL}.$$

(2). if $\mathcal{Q}_{CQTL-j}$ is a collection CQRO2-TLVs & $\mathcal{Q}'_{CQTL-j}$ is a replacement for $\mathcal{Q}_{CQTL-j}$, then

$$WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) = WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}'_{CQTL-1}, \mathcal{Q}'_{CQTL-2}, \dots, \mathcal{Q}'_{CQTL-n}\right).$$

(3). if $\mathcal{Q}_{CQTL-j}$ and $\mathcal{Q}'_{CQTL-j}$ are two collections of CQRO2-TLVs with the settings s.t. $\mathcal{Q}_{CQTL-j} \leq \mathcal{Q}'_{CQTL-j}$, then

$$WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) \leq WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}'_{CQTL-1}, \mathcal{Q}'_{CQTL-2}, \dots, \mathcal{Q}'_{CQTL-n}\right).$$

(4). if $\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mu_{\mathcal{Q}_{CQTLRP-j}} e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP-j}}\right)}, \eta_{\mathcal{Q}_{CQTLRP-j}} e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP-j}}\right)}\right)\right)$ is a family of the CQRO2-TLVs, then

$$min\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) \leq WCQRO2-TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \dots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right) \leq max\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, \dots, \mathcal{Q}_{CQTL-n}\right).$$

Proof. Straightforward.

6. Three-way decisions based on the CQRO2-TLVs

We explored the LF based on CQRO2-TLVs and 3WD for CQRO2-TLVs, where the probability vector is represented by: $A_{AC} = \{\chi_{P_{AC}}, \chi_{B_{AC}}, \chi_{N_{AC}}\}$, $\Omega_S = \{\mathcal{F}_B, \sim \mathcal{F}_{NB}\}$ and

$\mathcal{D} = \{\Pr(\mathcal{F}_B|[ử]), \Pr(\sim \mathcal{F}_{NB}|[ử])\}$ with $\Pr(\mathcal{F}_B|[ử]) + \Pr(\sim \mathcal{F}_{NB}|[ử]) = 1$. Then, we have the following:

Step 1: By using the Eqs (6.1)–(6.3), we examine the LFs, which are stated below:

$$Q_{CQTL-\bar{Q}_{CQ-P_{AC}P_{AC}}}=\left(\begin{array}{c}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)},\alpha_{SC}\right),\\ \left(\begin{array}{c}\mu_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)e^{i2\pi\left(\mu_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)},\\ \eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}P_{AC}}\right)\right)}\end{array}\right)\end{array}\right) \text{ and}$$

$$Q_{CQTL-\bar{Q}_{CQ-B_{AC}P_{AC}}}=\left(\begin{array}{c}\left(s_{S_{LT}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)},\alpha_{SC}\right),\\ \left(\begin{array}{c}\mu_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)e^{i2\pi\left(\mu_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)},\\ \eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-B_{AC}P_{AC}}\right)\right)}\end{array}\right)\end{array}\right); \tag{6.1}$$

$$Q_{CQTL-\bar{Q}_{CQ-N_{AC}P_{AC}}}=\left(\begin{array}{c}\left(s_{S_{LT}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)},\alpha_{SC}\right),\\ \left(\begin{array}{c}\mu_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)e^{i2\pi\left(\mu_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)},\\ \eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-N_{AC}P_{AC}}\right)\right)}\end{array}\right)\end{array}\right) \text{ and}$$

$$Q_{CQTL-\bar{Q}_{CQ-P_{AC}N_{AC}}}=\left(\begin{array}{c}\left(s_{S_{LT}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)},\alpha_{SC}\right),\\ \left(\begin{array}{c}\mu_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)e^{i2\pi\left(\mu_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)},\\ \eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-P_{AC}N_{AC}}\right)\right)}\end{array}\right)\end{array}\right); \tag{6.2}$$

$$Q_{CQTL-\bar{Q}_{CQ-B_{AC}N_{AC}}}=\left(\begin{array}{c}\left(s_{S_{LT}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)},\alpha_{SC}\right),\\ \left(\begin{array}{c}\mu_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)e^{i2\pi\left(\mu_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)},\\ \eta_{Q_{CQTLRP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)e^{i2\pi\left(\eta_{Q_{CQTLIP}}\left(\bar{Q}_{CQ-B_{AC}N_{AC}}\right)\right)}\end{array}\right)\end{array}\right) \text{ and} \tag{6.3}$$

$$\mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}} = \left(\begin{array}{c} \left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)}, \alpha_{SC} \right), \\ \left(\begin{array}{c} \mu_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right) e^{i2\pi\left(\mu_{\mathcal{Q}_{CQTLIP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)}, \\ \eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right) e^{i2\pi\left(\eta_{\mathcal{Q}_{CQTLIP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)} \end{array} \right) \end{array} \right).$$

Step 2: By using the Eq (6.4), we aggregate the decision matrix which is construct by decision experts.

$$WCQRO2 - TLGMSM^{\left(K_{SC}, \alpha_{SC-1}, \alpha_{SC-2}, \ldots, \alpha_{SC-K_{SC}}\right)}\left(\mathcal{Q}_{CQTL-ij}^{1}, \mathcal{Q}_{CQTL-ij}^{2}, \ldots, \mathcal{Q}_{CQTL-ij}^{n}\right) = \mathcal{Q}_{CQTL-ij}. \tag{6.4}$$

Step 3: By using the Eqs (6.5)–(6.7), we examine the expected losses $Y_{EL}\left(\chi_{j_{AC}}|[\text{ử}]\right), j = P, B, N$ and the different actions are expressed below:

$$Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[\text{ử}]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]) \tag{6.5}$$

$$Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[\text{ử}]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]) \tag{6.6}$$

$$Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right) = \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}} \Pr(\mathcal{F}_B|[\text{ử}]) \oplus_{CQTL} \mathcal{Q}_{CQTL-\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}} \Pr(\sim \mathcal{F}_{NB}|[\text{ử}]) \tag{6.7}$$

Step 4: By using the Eqs (6.8)–(6.10), we examine the expected values, which are stated below:

$$\mathcal{Q}_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right)\right) = \frac{\left(\left(\begin{array}{c} \delta_B \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)}, \alpha_{SC}\right) + \\ \delta_{NB} \times \Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)}, \alpha_{SC}\right) \end{array} \right) \times \left(\begin{array}{c} \left(\begin{array}{c} 1 - \left(1 - \mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1 - \mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array} \right)^{\frac{1}{q_{CQ}}} + \\ \left(\begin{array}{c} 1 - \left(1 - \mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \\ \left(1 - \mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array} \right)^{\frac{1}{q_{CQ}}} - \\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} - \\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_B} \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}} \end{array} \right) \right)}{8}, \tag{6.8}$$

$$Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right)\right)=\frac{\left(\begin{pmatrix}\delta_{B}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\\ \left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\\ \left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}, \tag{6.9}$$

$$Q_{EV}\left(Y_{EL}\left(\chi_{D_{AC}}|[\text{ử}]\right)\right)=\frac{\left(\begin{pmatrix}\delta_{B}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\begin{pmatrix}\begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_{B}}\\ \left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}+\\ \begin{pmatrix}1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_{B}}\\ \left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}^{\frac{1}{q_{CQ}}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}-\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\right)}{8}. \tag{6.10}$$

Moreover, we investigate the accuracy function, which is stated below:

$$G_{AF}\left(Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right)\right)=\frac{\begin{pmatrix}\begin{pmatrix}\delta_{B}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\\ \begin{pmatrix}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-P_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\end{pmatrix}}{8}, \tag{6.11}$$

$$G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right)\right)=\frac{\begin{pmatrix}\begin{pmatrix}\delta_{B}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{pmatrix}\times\\ \begin{pmatrix}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}P_{AC}}\right)\right)^{\delta_{B}}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-B_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{pmatrix}\end{pmatrix}}{8}, \tag{6.12}$$

$$G_{AF}\left(Y_{EL}\left(\chi_{D_{AC}}|[\text{ử}]\right)\right)=\frac{\left(\begin{array}{c}\left(\begin{array}{c}\delta_B\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)},\alpha_{SC}\right)+\\ \delta_{NB}\times\Delta_{LT}^{-1}\left(s_{S_{LT}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)},\alpha_{SC}\right)\end{array}\right)\times\\ \left(\begin{array}{c}\left(1-\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLRP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(1-\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(1-\mu_{\mathcal{Q}_{CQTLIP}}^{q_{CQ}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\right)^{\frac{1}{q_{CQ}}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}+\\ \left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}P_{AC}}\right)\right)^{\delta_B}\left(\eta_{\mathcal{Q}_{CQTLRP}}\left(\bar{\mathcal{Q}}_{CQ-N_{AC}N_{AC}}\right)\right)^{\delta_{NB}}\end{array}\right)\end{array}\right)}{8}. \tag{6.13}$$

Step 5: By using the Eqs (6.14)–(6.16), we examine the three-way decision rules, which are listed below:

$$P_{AC-1}\text{: When } Q_{EV}\left(Y_{EL}\left(\chi_{P_{AC}}|[\text{ử}]\right)\right)\leq G_{AF}\left(Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right)\right)\text{, then } \text{ử}\in POS(\mathcal{F}_P); \tag{6.14}$$

$$B_{AC-1}\text{: When } Q_{EV}\left(Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right)\right)\leq G_{AF}\left(Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right)\right)\text{, then } \text{ử}\in BUN(\mathcal{F}_P); \tag{6.15}$$

$$N_{AC-1}\text{: When } Q_{EV}\left(Y_{EL}\left(\chi_{N_{AC}}|[\text{ử}]\right)\right)\leq G_{AF}\left(Y_{EL}\left(\chi_{B_{AC}}|[\text{ử}]\right)\right)\text{, then } \text{ử}\in NEG(\mathcal{F}_P); \tag{6.16}$$

Step 6: This ends the proof.

Example 6.1. Three decision experts $\{\text{ử}_1,\text{ử}_2,\text{ử}_3\}$ and $D_{DE-k}(k=1,2,3)$ with weight vectors $\Omega_{\mathcal{WV}}=(0.4,0.35,0.25)^{\boldsymbol{T}}$ are given. Then, $\Pr\left(\mathcal{F}_B|[\text{ử}_j]\right)=0.3, j=1,2,3,4.$

Step 1: By using the Eqs (6.1)–(6.3), we examine the LFs of the Tables 3, 4, and 5, which are stated below:

Table 3. Information matrix $\mathcal{D}_1$.

$\text{ử}_{\mathbf{1}}$			$\text{ử}_{\mathbf{2}}$	
	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$
$\boldsymbol{\chi_{P_{AC}}}$	$\left(\begin{array}{c}(s_1,0.01),\\ \left(\begin{array}{c}0.6e^{i2\pi(0.6)},\\ 0.4e^{i2\pi(0.4)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_4,0.02),\\ \left(\begin{array}{c}0.5e^{i2\pi(0.5)},\\ 0.4e^{i2\pi(0.4)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_0,0.03),\\ \left(\begin{array}{c}0.7e^{i2\pi(0.7)},\\ 0.1e^{i2\pi(0.1)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_4,0.04),\\ \left(\begin{array}{c}0.6e^{i2\pi(0.6)},\\ 0.2e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$
$\boldsymbol{\chi_{B_{AC}}}$	$\left(\begin{array}{c}(s_3,0.011),\\ \left(\begin{array}{c}0.7e^{i2\pi(0.7)},\\ 0.2e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_2,0.021),\\ \left(\begin{array}{c}0.6e^{i2\pi(0.6)},\\ 0.4e^{i2\pi(0.4)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_5,0.031),\\ \left(\begin{array}{c}0.6e^{i2\pi(0.6)},\\ 0.3e^{i2\pi(0.3)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_2,0.041),\\ \left(\begin{array}{c}0.8e^{i2\pi(0.8)},\\ 0.2e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$
$\boldsymbol{\chi_{N_{AC}}}$	$\left(\begin{array}{c}(s_4,0.012),\\ \left(\begin{array}{c}0.8e^{i2\pi(0.8)}\\ ,0.1e^{i2\pi(0.1)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_1,0.022),\\ \left(\begin{array}{c}0.7e^{i2\pi(0.7)},\\ 0.2e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_4,0.032),\\ \left(\begin{array}{c}0.7e^{i2\pi(0.7)},\\ 0.2e^{i2\pi(0.2)}\end{array}\right)\end{array}\right)$	$\left(\begin{array}{c}(s_1,0.042),\\ \left(\begin{array}{c}0.7e^{i2\pi(0.7)},\\ 0.3e^{i2\pi(0.3)}\end{array}\right)\end{array}\right)$

Continued on next page

	ữ$_3$		ữ$_4$	
$\chi_{P_{AC}}$	$\left((s_0, 0.013), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_4, 0.023), \left(0.5e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_1, 0.033), \left(0.6e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_4, 0.04), \left(0.6e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_2, 0.014), \left(0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_2, 0.024), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2, 0.034), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2, 0.043), \left(0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_3, 0.015), \left(0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_1, 0.025), \left(0.8e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_3, 0.035), \left(0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_1, 0.044), \left(0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.2)}\right)\right)$

Table 4. Information matrix by $\mathcal{D}_2$.

ữ$_1$			ữ$_2$	
	$\mathcal{F}_B$	$\sim \mathcal{F}_{NB}$	$\mathcal{F}_B$	$\sim \mathcal{F}_{NB}$
$\chi_{P_{AC}}$	$\left((s_0, 0.02), \left(0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_4, 0.03), \left(0.6e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_0, 0.04), \left(0.5e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_5, 0.05), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_1, 0.021), \left(0.81e^{i2\pi(0.81)}, 0.11e^{i2\pi(0.11)}\right)\right)$	$\left((s_3, 0.031), \left(0.61e^{i2\pi(0.61)}, 0.21e^{i2\pi(0.21)}\right)\right)$	$\left((s_1, 0.041), \left(0.1e^{i2\pi(0.1)}, 0.41e^{i2\pi(0.41)}\right)\right)$	$\left((s_4, 0.01), \left(0.71e^{i2\pi(0.71)}, 0.21e^{i2\pi(0.21)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_2, 0.022), \left(0.82e^{i2\pi(0.82)}, 0.12e^{i2\pi(0.12)}\right)\right)$	$\left((s_2, 0.032), \left(0.62e^{i2\pi(0.62)}, 0.22e^{i2\pi(0.22)}\right)\right)$	$\left((s_2, 0.042), \left(0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_1, 0.042), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$
	ữ$_3$		ữ$_4$	
$\chi_{P_{AC}}$	$\left((s_0, 0.03), \left(0.7e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_4, 0.04), \left(0.6e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_4, 0.03), \left(0.6e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_0, 0.04), \left(0.5e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.4)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_5, 0.031), \left(0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2, 0.041), \left(0.8e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_3, 0.031), \left(0.61e^{i2\pi(0.61)}, 0.21e^{i2\pi(0.21)}\right)\right)$	$\left((s_1, 0.041), \left(0.1e^{i2\pi(0.1)}, 0.41e^{i2\pi(0.41)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_4, 0.032), \left(0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_1, 0.042), \left(0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2, 0.032), \left(0.62e^{i2\pi(0.62)}, 0.22e^{i2\pi(0.22)}\right)\right)$	$\left((s_2, 0.042), \left(0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.2)}\right)\right)$

Table 5. Information matrix by $\mathcal{D}_3$.

	$ử_1$		$ử_2$	
	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$
$\chi_{P_{AC}}$	$\left((s_0,0.013),\left(0.7e^{i2\pi(0.7)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_4,0.023),\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_1,0.033),\left(0.6e^{i2\pi(0.6)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_4,0.04),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_2,0.014),\left(0.7e^{i2\pi(0.7)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_2,0.024),\left(0.7e^{i2\pi(0.7)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2,0.034),\left(0.7e^{i2\pi(0.7)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_2,0.043),\left(0.8e^{i2\pi(0.8)},0.1e^{i2\pi(0.1)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_0,0.03),\left(0.7e^{i2\pi(0.7)},0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_4,0.04),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_4,0.03),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_0,0.04),\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$
	$ử_3$		$ử_4$	
$\chi_{P_{AC}}$	$\left((s_0,0.013),\left(0.7e^{i2\pi(0.7)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_4,0.023),\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_1,0.033),\left(0.6e^{i2\pi(0.6)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_4,0.04),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_0,0.02),\left(0.8e^{i2\pi(0.8)},0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_4,0.03),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_0,0.04),\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left((s_5,0.05),\left(0.7e^{i2\pi(0.7)},0.3e^{i2\pi(0.3)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_1,0.021),\left(0.81e^{i2\pi(0.81)},0.11e^{i2\pi(0.11)}\right)\right)$	$\left((s_3,0.031),\left(0.61e^{i2\pi(0.61)},0.21e^{i2\pi(0.21)}\right)\right)$	$\left((s_1,0.041),\left(0.1e^{i2\pi(0.1)},0.41e^{i2\pi(0.41)}\right)\right)$	$\left((s_4,0.01),\left(0.71e^{i2\pi(0.71)},0.21e^{i2\pi(0.21)}\right)\right)$

Table 6. By using the Eq (6.4), we aggregate the information of Tables 3, 4, and 5.

	$ử_1$		$ử_2$	
	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$	$\mathcal{F}_B$	$\sim\mathcal{F}_{NB}$
$\chi_{P_{AC}}$	$\left((s_0,0.012),\left(0.299e^{i2\pi(0.299)},0.678e^{i2\pi(0.678)}\right)\right)$	$\left((s_{1.41},0.021),\left(0.275e^{i2\pi(0.275)},0.675e^{i2\pi(0.675)}\right)\right)$	$\left((s_0,0.031),\left(0.251e^{i2\pi(0.251)},0.622e^{i2\pi(0.622)}\right)\right)$	$\left((s_{1.39},0.042),\left(0.274e^{i2\pi(0.274)},0.66e^{i2\pi(0.66)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_{0.72},0.0112),\left(0.293e^{i2\pi(0.293)},0.632e^{i2\pi(0.632)}\right)\right)$	$\left((s_{0.76},0.022),\left(0.274e^{i2\pi(0.274)},0.606e^{i2\pi(0.606)}\right)\right)$	$\left((s_{0.85},0.033),\left(0.274e^{i2\pi(0.274)},0.656e^{i2\pi(0.656)}\right)\right)$	$\left((s_{0.74},0.043),\left(0.269e^{i2\pi(0.269)},0.659e^{i2\pi(0.659)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_{1.04},0.013),\left(0.34e^{i2\pi(0.34)},0.599e^{i2\pi(0.599)}\right)\right)$	$\left((s_{0.18},0.024),\left(0.318e^{i2\pi(0.318)},0.563e^{i2\pi(0.563)}\right)\right)$	$\left((s_{1.41},0.032),\left(0.297e^{i2\pi(0.297)},0.657e^{i2\pi(0.657)}\right)\right)$	$\left((s_{0.14},0.042),\left(0.318e^{i2\pi(0.318)},0.619e^{i2\pi(0.619)}\right)\right)$
	$ử_3$		$ử_4$	
$\chi_{P_{AC}}$	$\left((s_0,0.0142),\left(0.279e^{i2\pi(0.279)},0.56e^{i2\pi(0.56)}\right)\right)$	$\left((s_{1.443},0.0221),\left(0.25e^{i2\pi(0.25)},0.75e^{i2\pi(0.75)}\right)\right)$	$\left((s_0,0.0321),\left(0.21e^{i2\pi(0.21)},0.62e^{i2\pi(0.62)}\right)\right)$	$\left((s_{1.59},0.043),\left(0.254e^{i2\pi(0.254)},0.70e^{i2\pi(0.70)}\right)\right)$
$\chi_{B_{AC}}$	$\left((s_{0.69},0.013),\left(0.288e^{i2\pi(0.288)},0.612e^{i2\pi(0.612)}\right)\right)$	$\left((s_{0.81},0.023),\left(0.287e^{i2\pi(0.287)},0.599e^{i2\pi(0.599)}\right)\right)$	$\left((s_{0.81},0.032),\left(0.257e^{i2\pi(0.257)},0.670e^{i2\pi(0.670)}\right)\right)$	$\left((s_{0.57},0.0434),\left(0.273e^{i2\pi(0.273)},0.69e^{i2\pi(0.69)}\right)\right)$
$\chi_{N_{AC}}$	$\left((s_{1.31},0.023),\left(0.349e^{i2\pi(0.349)},0.615e^{i2\pi(0.615)}\right)\right)$	$\left((s_{0.78},0.029),\left(0.389e^{i2\pi(0.389)},0.569e^{i2\pi(0.569)}\right)\right)$	$\left((s_{1.84},0.038),\left(0.302e^{i2\pi(0.302)},0.686e^{i2\pi(0.686)}\right)\right)$	$\left((s_{0.35},0.0423),\left(0.319e^{i2\pi(0.319)},0.617e^{i2\pi(0.617)}\right)\right)$

Step 2: Consider Eq (6.4), for $K_{SC} = 3$, and the values of $\alpha_{SC-1} = \alpha_{SC-2} = \alpha_{SC-3} = 1$, we have Table 6.

Step 3: By using the Eqs (6.5)–(6.7), we examine the expected losses $Y_{EL}(\chi_{j_{AC}}|[ử]), j = P, B, N$, for $\delta_{B-j} = 0.4, j = 1,2,3$ and $q_{SC} = 1$, and the separate actions are expressed and discussed in Table 7.

Table 7. By using again the Eqs (6.5)–(6.7), we aggregate the information of Table 6.

Symbols	$\boldsymbol{Y_{EL}(\chi_{P_{AC}}\|[ử])}$	$\boldsymbol{Y_{EL}(\chi_{B_{AC}}\|[ử])}$	$\boldsymbol{Y_{EL}(\chi_{N_{AC}}\|[ử])}$
$\boldsymbol{ử_1}$	$\begin{pmatrix}(s_0, 0.1866),\\ \begin{pmatrix}0.2504e^{i2\pi(0.2504)},\\ 0.6762e^{i2\pi(0.6762)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.1656),\\ \begin{pmatrix}0.2817e^{i2\pi(0.2817)},\\ 0.6163e^{i2\pi(0.6163)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.1244),\\ \begin{pmatrix}0.3269e^{i2\pi(0.3269)},\\ 0.5771e^{i2\pi(0.5771)}\end{pmatrix}\end{pmatrix}$
$\boldsymbol{ử_2}$	$\begin{pmatrix}(s_1, 0.0044),\\ \begin{pmatrix}0.2649e^{i2\pi(0.2649)},\\ 0.6445e^{i2\pi(0.6445)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.1346),\\ \begin{pmatrix}0.271e^{i2\pi(0.271)},\\ 0.6578e^{i2\pi(0.6578)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.1676),\\ \begin{pmatrix}0.3097e^{i2\pi(0.3097)},\\ 0.6339e^{i2\pi(0.6339)}\end{pmatrix}\end{pmatrix}$
$\boldsymbol{ử_3}$	$\begin{pmatrix}(s_0, 0.1921),\\ \begin{pmatrix}0.2617e^{i2\pi(0.2617)},\\ 0.6673e^{i2\pi(0.6673)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.1714),\\ \begin{pmatrix}0.2874e^{i2\pi(0.2874)},\\ 0.6042e^{i2\pi(0.6042)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_1, 0.025),\\ \begin{pmatrix}0.3733e^{i2\pi(0.3733)},\\ 0.587e^{i2\pi(0.587)}\end{pmatrix}\end{pmatrix}$
$\boldsymbol{ử_4}$	$\begin{pmatrix}(s_1, 0.0166),\\ \begin{pmatrix}0.2367e^{i2\pi(0.2367)},\\ 0.6668e^{i2\pi(0.6668)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.172),\\ \begin{pmatrix}0.2666e^{i2\pi(0.2666)},\\ 0.6819e^{i2\pi(0.6819)}\end{pmatrix}\end{pmatrix}$	$\begin{pmatrix}(s_0, 0.192),\\ \begin{pmatrix}0.3123e^{i2\pi(0.3123)},\\ 0.6437e^{i2\pi(0.6437)}\end{pmatrix}\end{pmatrix}$

Step 4: By using the Eqs (6.8)–(6.10), we examine the expected values, which are stated in Table 8.

Table 8. Expected values of the information, which are shown in Table 7.

Symbols	$\boldsymbol{Q_{EV}\left(Y_{EL}(\chi_{P_{AC}}\|[ử])\right)}$	$\boldsymbol{Q_{EV}\left(Y_{EL}(\chi_{B_{AC}}\|[ử])\right)}$	$\boldsymbol{Q_{EV}\left(Y_{EL}(\chi_{N_{AC}}\|[ử])\right)}$
$\boldsymbol{ử_1}$	0.0137	0.017	0.0119
$\boldsymbol{ử_2}$	0.0185	0.0096	0.0095
$\boldsymbol{ử_3}$	0.0136	0.0186	0.0089
$\boldsymbol{ử_4}$	0.0211	0.0088	0.0085

When the predictable values are unsuccessful in catching the relationships amongst any two predictable losses, we discover the concepts of accuracy function, which is quantified in Table 9.

Table 9. Accuracy values of the information, which are shown in Table 6.

Symbols	$\boldsymbol{G_{AF}\left(Y_{EL}(\chi_{P_{AC}}\|[ử])\right)}$	$\boldsymbol{G_{AF}\left(Y_{EL}(\chi_{B_{AC}}\|[ử])\right)}$	$\boldsymbol{G_{AF}\left(Y_{EL}(\chi_{N_{AC}}\|[ử])\right)}$
$\boldsymbol{ử_1}$	0.266	0.2222	0.1555
$\boldsymbol{ử_2}$	0.282	0.1867	0.2219
$\boldsymbol{ử_3}$	0.27	0.2257	0.2731
$\boldsymbol{ử_4}$	0.3127	0.2435	0.2656

Step 5: By using the Eqs (6.11)–(6.13), we examine the three-way decision rules, as discussed in Table 10.

Table 10. Three-ways decision based on Eqs (6.11)–(6.13).

Enterprises	Decision rule
$ử_1$	P_{AC-1}
$ử_2$	P_{AC-1}
$ử_3$	P_{AC-1}
$ử_4$	P_{AC-1}

Step 6: This ends the proof.

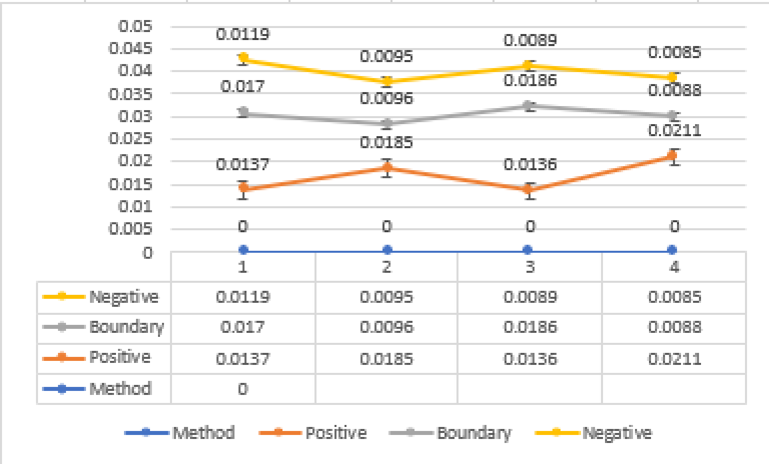

Figure 4. Graphical representation for the information of Table 8.

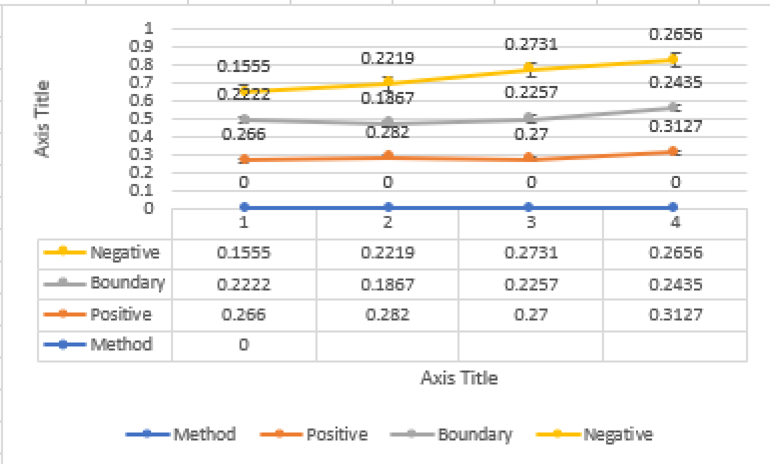

Figure 5. Graphical representation for the information of Table 9.

The obtained result states that all these alternatives belong to positive opinions, which are P_{AC-1}. Furthermore, the comparison of the explored work with some existing work is discussed in Table 11.

Table 11. Comparative analysis for explored work with existing work

Enterprises	$\mathring{ư}_{1}$	$\mathring{ư}_{2}$	$\mathring{ư}_{3}$	$\mathring{ư}_{4}$
Proposed work for q=1	P_{AC-1}	P_{AC-1}	P_{AC-1}	P_{AC-1}
Proposed work for q=2	P_{AC-1}	P_{AC-1}	P_{AC-1}	P_{AC-1}
Proposed work for q=3	P_{AC-1}	P_{AC-1}	P_{AC-1}	P_{AC-1}

From the above analysis, we conclude that all approaches have provided similar consequences and are exposed in Table 11; additionally, all alternatives belong to the +ve regions. The graphical interpretation for the information's score and the expected positive, boundary, and negative values of Table 8 and Table 9 are shown in Figure 4 and Figure 5, respectively.

In the explored work, if we choose the values of $q = 1,2$ and the imaginary part is zero, then the explored work is reduced for intuitionistic 2-tuple and Pythagorean 2-tuple linguistic sets. Similarly, if we choose the values of $q = 1$, 2, then the explored will be reduced for complex intuitionistic 2-tuple and complex Pythagorean 2-tuple linguistic sets. The presented approach is more powerful and more proficient than the existing ones, as given in [34–37].

7. Conclusions

We modified the notions of 3WD and DTRS in the environment of CQRO2-TLV and elaborated certain important properties. Moreover, GMSM is a dominant and more flexible method to determine the accuracy and dominancy of real life issues. Therefore, by considering the CQRO2-TL information and GMSM, we presented CQRO2-TLGMSM operator and the WCQRO2-TLGMSM operator, and demonstrated their effective properties. We also elaborated a q-rung orthopair 2-tuple linguistic DTRS model and discussed its applications. Furthermore, we discussed some important and well known properties of the defined aggregation operators like idempotency, commutativity, monotonicity, and boundedness. We discussed some examples to explain our proposed methods and techniques. To prove the authenticity, workability, effectiveness, and supremacy of our proposed methods, techniques, and notions, we initiated a comparative analysis and proved that our initiated notions are much better as compared to certain existing notions.

In the future, we will discuss the proposed notions for CQRO2-TLVSs in the framework of complex q-rung orthopair fuzzy sets [38], picture hesitant fuzzy sets [39,40], neutrosophic sets [41,42], and some more useful frameworks and notions, as given in [43–49].

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Conflict of interest

The authors declare no conflicts of interest about the publication of the research article.

References

1. V. I. Yukalov, D. Sornette, Physics of risk and uncertainty in quantum decision making, *Eur. Phys. J. B*, **71** (2009), 533–548. https://doi.org/10.1140/epjb/e2009-00245-9

2. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
3. K. Atanassov, Intuitionistic fuzzy sets, In: *Intuitionistic Fuzzy Se*ts, Heidelberg: Physica, 1986. https://doi.org/10.1007/978-3-7908-1870-3_1
4. R. R. Yager, Pythagorean fuzzy subsets, *2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)*, 2013, 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375
5. R. R. Yager, Generalized orthopair fuzzy sets, *IEEE T. Fuzzy Syst.*, **25** (2016), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005
6. P. Liu, P. Wang, Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 259–280. https://doi.org/10.1002/int.21927
7. H. Garg, S. M. Chen, Multiattribute group decision making based on neutrality aggregation operators of q-rung orthopair fuzzy sets, *Inform. Sci.*, **517** (2020), 427–447. https://doi.org/10.1016/j.ins.2019.11.035
8. X. Peng, J. Dai, H. Garg, Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function, *Int. J. Intell. Syst.*, **33** (2018), 2255–2282. https://doi.org/10.1002/int.22028
9. Y. Xing, R. Zhang, Z. Zhou, J. Wang, Some q-rung orthopair fuzzy point weighted aggregation operators for multi-attribute decision making, *Soft Comput.*, **23** (2019), 11627–11649. https://doi.org/10.1007/s00500-018-03712-7
10. P. Wang, J. Wang, G. Wei, C. Wei, Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications, *Mathematics*, **7** (2019), 340. https://doi.org/10.3390/math7040340
11. W. S. Du, Minkowski-type distance measures for generalized orthopair fuzzy sets, *Int. J. Intell. Syst.*, **33** (2018), 802–817. https://doi.org/10.1002/int.21968
12. D. Liu, X. Chen, D. Peng, Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets, *Int. J. Intell. Syst.*, **34** (2019), 1572–1587. https://doi.org/10.1002/int.22108
13. X. Peng, L. Liu, Information measures for q-rung orthopair fuzzy sets, *Int. J. Intell. Syst.*, **34** (2019), 1795–1834. https://doi.org/10.1002/int.22115
14. P. Liu, P. Wang, Multiple-attribute decision-making based on Archimedean Bonferroni Operators of q-rung orthopair fuzzy numbers, *IEEE T. Fuzzy Syst.*, **27** (2018), 834–848. https://doi.org/10.1109/TFUZZ.2018.2826452
15. D. Ramot, R. Milo, M. Friedman, A. Kandel, Complex fuzzy sets, *IEEE T. Fuzzy Syst.*, **10** (2002), 171–186. https://doi.org/10.1109/91.995119
16. A. M. D. J. S. Alkouri, A. R. Salleh, Complex intuitionistic fuzzy sets, *AIP Conf. Proc.*, **1482** (2012), 464–470. https://doi.org/10.1063/1.4757515
17. K. Ullah, T. Mahmood, Z. Ali, N. Jan, On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition, *Complex Intell. Syst.*, **6** (2019), 15–27. https://doi.org/10.1007/s40747-019-0103-6
18. M. Akram, S. Naz, A novel decision-making approach under complex Pythagorean fuzzy environment, *Math. Comput. Appl.*, **24** (2019), 73–103. https://doi.org/10.3390/mca24030073

19. P. Liu, Z. Ali, T. Mahmood, A method to multi-attribute group decision-making problem with complex q-rung orthopair linguistic information based on heronian mean operators, *Int. J. Comput. Intell. Syst.*, **12** (2019), 1465–1496. https://doi.org/10.2991/ijcis.d.191030.002
20. P. Liu, T. Mahmood, Z. Ali, Complex Q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making, *Information*, **11** (2019), 5–27. https://doi.org/10.3390/info11010005
21. T. Mahmood, Z. Ali, Entropy measure and TOPSIS method based on correlation coefficient using complex q-rung orthopair fuzzy information and its application to multi-attribute decision making, *Soft Comput.*, **25** (2021), 1249–1275. https://doi.org/10.1007/s00500-020-05218-7
22. Z. Ali, T. Mahmood, Maclaurin symmetric mean operators and their applications in the environment of complex q-rung orthopair fuzzy sets, *Comput. Appl. Math.*, **39** (2020), 161–187. https://doi.org/10.1007/s40314-020-01145-3
23. L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, *Inform. Sci.*, **9** (1975), 43–80. https://doi.org/10.1016/0020-0255(75)90017-1
24. F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, *IEEE T. Fuzzy Syst.*, **8** (2000), 746–752. https://doi.org/10.1109/91.890332
25. F. Herrera, L. Martinez, An approach for combining linguistic and numerical information based on the 2-tuple fuzzy linguistic representation model in decision-making, *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, **8** (2000), 539–562. https://doi.org/10.1142/S0218488500000381
26. L. Martı, F. Herrera, An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges, *Inform. Sci.*, **207** (2012), 1–18. https://doi.org/10.1016/j.ins.2012.04.025
27. Y. Li, P. Liu, Some Heronian mean operators with 2-tuple linguistic information and their application to multiple attribute group decision making, *Technol. Eco. Develop. Econ.*, **21** (2007), 797–814.
28. D. Liang, D. Liu, W. Pedrycz, P. Hu, Triangular fuzzy decision-theoretic rough sets, *Int. J. Approx. Reason.*, **54** (2013), 1087–1106. https://doi.org/10.1016/j.ijar.2013.03.014
29. X. Jia, W. Liao, Z. Tang, L. Shang, Minimum cost attribute reduction in decision-theoretic rough set models, *Inform. Sci.*, **219** (2013), 151–167. https://doi.org/10.1016/j.ins.2012.07.010
30. D. Liu, T. Li, D. Ruan, Probabilistic model criteria with decision-theoretic rough sets, *Inform. Sci.*, **181** (2011), 3709–3722. https://doi.org/10.1016/j.ins.2011.04.039
31. Y. Yao, Y. Zhao, Attribute reduction in decision-theoretic rough set models, *Inform. Sci.*, **178** (2008), 3356–3373. https://doi.org/10.1016/j.ins.2008.05.010
32. D. Liu, Y. Yao, T. Li, Three-way investment decisions with decision-theoretic rough sets, *Int. J. Comput. Intell. Syst.*, **4** (2011), 66–74.
33. D. Liu, T. Li, D. Liang, Three-way government decision analysis with decision-theoretic rough sets, *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, **20** (2012), 119–132. https://doi.org/10.1142/S0218488512400090
34. P. Liu, H. Yang, Three-way decisions with intuitionistic uncertain linguistic decision-theoretic rough sets based on generalized Maclaurin symmetric mean operators, *Int. J. Fuzzy Syst.*, **22** (2020), 653–667. https://doi.org/10.1007/s40815-019-00718-7
35. Y. Ju, A. Wang, J. Ma, H. Gao, E. D. Santibanez Gonzalez, Some q-rung orthopair fuzzy 2-tuple linguistic Muirhead mean aggregation operators and their applications to multiple-attribute group decision making, *Int. J. Intell. Syst.*, **35** (2020), 184–213. https://doi.org/10.1002/int.22205

36. P. Liu, H. Gao, Multicriteria decision making based on generalized Maclaurin symmetric means with multi-hesitant fuzzy linguistic information, *Symmetry*, **10** (2018), 81–107. https://doi.org/10.3390/sym10040081
37. D. Liang, D. Liu, Deriving three-way decisions from intuitionistic fuzzy decision-theoretic rough sets, *Inform. Sci.*, **300** (2015), 28–48. https://doi.org/10.1016/j.ins.2014.12.036
38. D. E. Tamir, N. D. Rishe, A. Kandel, Complex fuzzy sets and complex fuzzy logic an overview of theory and applications, In: *Fifty years of fuzzy logic and its applications*, Cham: Springer, 2015. https://doi.org/10.1007/978-3-319-19683-1_31
39. W. Ullah, M. Ibrar, A. Khan, M. Khan, Multiple attribute decision making problem using GRA method with incomplete weight information based on picture hesitant fuzzy setting, *Int. J. Intell. Syst.*, **36** (2021), 866–889. https://doi.org/10.1002/int.22324
40. L. Yang, X. H. Wu, J. Qian, A Novel Multicriteria Group Decision-Making Approach with Hesitant Picture Fuzzy Linguistic Information, *Math. Probl. Eng.*, **2020** (2020), 6394028. https://doi.org/10.1155/2020/6394028
41. M. Ali, F. Smarandache, Complex neutrosophic set, *Neural Comput. Appl.*, **28** (2017), 1817–1834. https://doi.org/10.1007/s00521-015-2154-y
42. S. Broumi, A. Bakali, M. Talea, F. Smarandache, P. K. Singh, V. Uluçay, et al., Bipolar complex neutrosophic sets and its application in decision making problem, In: *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, Cham: Springer, 2019. https://doi.org/10.1007/978-3-030-00045-5_26
43. M. Q. Wu, T. Y. Chen, J. P. Fan, Divergence measure of T-spherical fuzzy sets and its applications in pattern recognition, *IEEE Access*, **8** (2019), 10208–10221. https://doi.org/10.1109/ACCESS.2019.2963260
44. F. Zhao, Z. Zeng, H. Liu, R. Lan, J. Fan, Semi-supervised Approach to Surrogate-assisted Multiobjective Kernel Intuitionistic Fuzzy Clustering Algorithm for Color Image Segmentation, *IEEE T. Fuzzy Syst.*, **28** (2020), 1023–1034. https://doi.org/10.1109/TFUZZ.2020.2973121
45. T. Mahmood, A Novel Approach toward Bipolar Soft Sets and Their Applications, *J. Math.*, **2020** (2020), 4690808. https://doi.org/10.1155/2020/4690808
46. D. Wu, X. Tan, Multitasking Genetic Algorithm (MTGA) for Fuzzy System Optimization, *IEEE T. Fuzzy Syst.*, **28** (2020), 1050–1061. https://doi.org/10.1109/TFUZZ.2020.2968863
47. A. Caliskan, Z. A. Cil, H. Badem, D. Karaboga, Regression Based Neuro-Fuzzy Network Trained by ABC Algorithm for High-Density Impulse Noise Elimination, *IEEE T. Fuzzy Syst.*, **28** (2020), 1084–1095. https://doi.org/10.1109/TFUZZ.2020.2973123
48. S. Harifi, M. Khalilian, J. Mohammadzadeh, S. Ebrahimnejad, Optimizing a Neuro-Fuzzy System based on nature inspired Emperor Penguins Colony optimization algorithm, *IEEE T. Fuzzy Syst.,* **28** (2020), 1110–1124. https://doi.org/10.1109/TFUZZ.2020.2984201
49. P. Dziwiński, L. Bartczuk, A new hybrid particle swarm optimization and genetic algorithm method controlled by fuzzy logic, *IEEE T. Fuzzy Syst.*, **28** (2019), 1140–1154. https://doi.org/10.1109/TFUZZ.2019.2957263