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Research article

Global existence and boundedness of chemotaxis-fluid equations to the coupled Solow-Swan model

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Abstract: In this paper, we consider the following Keller-Segel-(Navier)-Stokes system to the coupled Solow-Swan model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \mu_1 n - \mu_2 n^k, & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + \mu_3 c^{\alpha} w^{1-\alpha}, & x \in \Omega,\ t > 0, \\ w_t + u \cdot \nabla w = \Delta w - w + n, & x \in \Omega,\ t > 0, \\ u_t + \kappa (u \cdot \nabla u) = \Delta u - \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with no-flux boundary for n, c, w and no-slip boundary for u, where the parameters $\chi > 0$, $\alpha \in (0, 1)$, $\mu_1 \in \mathbb{R}$, $\mu_2 \geq 0$, $\mu_3 > 0$ and $\kappa \in \{0, 1\}, k \geq N$. Due to the interference of the fractional nonlinear term of the Solow-Swan model, we use the Moser-Trudinger inequality to obtain the global existence of the solution for two-dimensional case without logistic source. For three-dimensional case, we control the required estimation with the help of the negative term of logistic source to obtain the boundedness and asymptotic behavior. In the process of estimating the corresponding term, we find the order of the negative term of the logistic source is related to the spatial dimension, and we give the decay estimate of the corresponding solutions when $\mu_1 < 0$ or $\mu_1 = 0$, $\mu_2 > 0$.

Keywords: Keller-Segel-Solow-Swan; indirect signal production; global existence and boundedness; asymptotic behavior
Mathematics Subject Classification: 35B65, 35Q35, 35Q92, 92C17

1. Introduction

The Keller and Segel model in [22] was introduced in 1970, and the mathematical study of this system has extensively developed the parabolic-parabolic equations in [13, 24, 28, 36, 39] and the parabolic-elliptic equations in [2, 3, 7, 14, 15, 37]. This model is used to describe the chemotaxis-aggregation phenomena in nature.

Cells and microorganisms usually live in fluid, so it is particularly important to consider the interaction of fluids with them. In view of this idea, Tuval et al. considered the experiment of the collective behavior of *Bacillus subtilis* in [49]. Then, a large number of related results of global solvability for chemotaxis-fluid were investigated in recent years. For example, we can see the researches of introducing the Keller-Segel equations in [1,20,34,46,55,78], the Keller-Segel-Navier-Stokes equations in [5, 6, 9, 10, 21, 25, 27, 31, 41–43, 47, 51–54, 56–58, 62–64, 66–71, 73, 76, 77, 79], the rotational flux term in [5,21,31,51,58,59,64,79], the nonlinear diffusion in [8,11,41,48,73], the logistic source in [12,47,54,62,78], the singular sensitivity in [13–15,24,52,65,75] etc. These papers on global existence and boundedness analysis gave a good theoretical and guiding significance for our understanding of biological growth of cells. Due to the global existence of the solution, we do not have to worry about the occurrence of sudden change and other unexpected results, and can achieve the purpose of guiding experiments with theory.

Recently, a macroscopic model called the spatial Solow-Swan was proposed by Juchem Neto et al. in [16–18] for describing economic growth phenomena under capital induction and labor migration. Very recently, Li-Li [26] investigated global boundedness of the following model

$$\begin{cases} n_t = \Delta n - \chi \nabla \cdot (n \nabla c) + \mu_1 n - \mu_2 n^2, & x \in \Omega,\ t > 0, \\ c_t = \Delta c - c + \mu_3 c^{\alpha} n^{1-\alpha}, & x \in \Omega,\ t > 0. \end{cases}$$

Assuming that the dynamic behavior of microscopic particles also meets the above macroscopic model, it is necessary to consider the Keller-Segel-Solow-Swan model. For the above model, there are two difficulties: the first equation contains cross diffusion term $\nabla \cdot (n\nabla c)$, and the second contains the Cobb-Douglas function $\mu_3 c^{\alpha} n^{1-\alpha}$. Therefore, it becomes very interesting to use the corresponding mathematical theory to deal with this problem. Recently, more results in [29,30,32,33,60,72,74] have turned their attention to the indirect signal production model under multi-signal, and the researches on the global solvability of this model have become very important.

Compared with the chemical substance concentration term of the indirect signal model, we found that the system became more difficult to control after adding Cobb-Douglas term. We can explain it by Sturm's comparison theorem in [44] as follows:

$$y'(t) + y = \mu_3 \|c^{\alpha} w^{1-\alpha}\|_{L^1(\Omega)} \le \frac{1}{2} y + (2\mu_3^{\frac{1}{\alpha}})^{\frac{\alpha}{1-\alpha}} \|w\|_{L^1(\Omega)} \quad \text{for all } \alpha \in (0,1),$$

where $y = \|c\|_{L^1(\Omega)}$ and w are the concentrations of another chemical involved in the reaction, which is given in the following model (1.1). Let

$$y'(t) + \frac{1}{2} y = 2^{\frac{\alpha}{1-\alpha}} \|w\|_{L^1(\Omega)} \quad \text{for all } \alpha \in (0,1).$$

If $\alpha = 0$, the above system degenerates into an indirect signal model, and if $\alpha > 0$ increase, then the corresponding solution will be raised. When we assume that the differential equation of the indirect signal model c is

$$\tilde{y}' + \tilde{y} = \|\tilde{w}\|_{L^1(\Omega)}$$

and assume that they have the same initial data and velocity, namely, $y(0) = \tilde{y}(0)$, $\dot{y}(0) = \dot{\tilde{y}}(0)$, as wall as suppose that $y(a) = y(b) = y(0)$, then we have $a \leq b$ and

$$\frac{y(s_1)}{\tilde{y}(s_1)} \geq \frac{y(s_0)}{\tilde{y}(s_0)} \quad \text{and} \quad y(s_1) \geq \tilde{y}(s_1) \quad \text{for all } 0 < s_0 < s_1 < a.$$

This shows that the distance between the two solutions increases gradually during the evolution. Motivated by the above works, we think that the relationship between cells and chemicals also meets the operating mechanism in the Solow-Swan model. In this paper, we let $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be a bounded domain smooth boundary with outer norm vector ν and investigate the following chemotaxis-fluid-Solow-Swan system:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \mu_1 n - \mu_2 n^k, & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + \mu_3 c^\alpha w^{1-\alpha}, & x \in \Omega,\ t > 0, \\ w_t + u \cdot \nabla w = \Delta w - w + n, & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla u) = \Delta u - \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad u = 0, & x \in \partial\Omega,\ t > 0, \\ n(x, 0) = n_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, the unknowns $n = n(t, x), c = c(t, x)$ and $w = w(t, x)$ denote the cell density and the two concentrations of chemical substance, respectively. $u = u(t, x)$ represents the fluid velocity field, and $P = P(t, x)$ denotes the associated pressure. The scalar valued function $\Phi = \Phi(x)$ is given and it accounts the effects of external forces such as gravity or centrifugal forces. The parameters satisfy $\chi > 0, k \geq N, \mu_1 \in \mathbb{R}, \mu_2 \geq 0, \mu_3 > 0, \alpha \in (0, 1),\ \kappa \in \{0, 1\}$. Moser-Trudinger inequality [4, 38, 50] has natural advantage as a priori estimate for dealing with two-dimensional critical cases, and Winkler [68] has promoted it and provided a better version. For the three-dimensional case, we control it with help of the order of logistic source and the estimate of heat semigroup. Based on these results, we describe the work of this paper. For the convenience of this paper, we let

$$m_0 := \int_\Omega n_0 dx > 0.$$

We assume that potential function Φ fulfills

$$\Phi \in W^{2,\infty}(\Omega) \tag{1.2}$$

and that the initial data n_0, c_0, w_0, u_0 satisfies

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \text{ is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative}, \\ w_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative}, \quad \text{and} \\ u_0 \in W^{2,2}(\Omega; \mathbb{R}^2) \cap W^{1,2}_{0,\sigma},\ N = 2 \quad \text{or} \quad u_0 \in W^{2,\frac{22}{4}}(\Omega; \mathbb{R}^3) \cap W^{1,2}_{0,\sigma},\ N = 3, \end{cases} \tag{1.3}$$

where $W^{1,2}_{0,\sigma} := W^{1,2}_0(\Omega;\mathbb{R}^N) \cap L^2_\sigma(\Omega)$, with $L^2_\sigma := \left\{\varphi \in L^2(\Omega;\mathbb{R}^N) \,\middle|\, \nabla\cdot\varphi = 0 \text{ in } \mathcal{D}(\Omega)\right\}$ denoting the space of all solenoidal vector fields in $L^2(\Omega;\mathbb{R}^N)$.

Under this assumption, our main results on global boundedness and asymptotic behavior of the initial-boundary value problems (1.1) and (1.3) can be formulated as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ $(N = 2,3)$ be a bounded domain with smooth boundary and Φ comply with* (1.2)*, and suppose that n_0, c_0, w_0, and u_0 satisfy* (1.3)*, and if $N = 2$, $\mu_1 \in \mathbb{R}, \mu_2 > 0$ or $\mu_1 = 0, \mu_2 \geq 0$ and if $N = 3, \mu_1 \in \mathbb{R}$, $\mu_2 > 0$, then there exist functions (n, c, w, u, P) satisfying*

$$\begin{cases} n \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ c \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ w \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ u \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ P \in C^{1,0}(\bar\Omega \times [0,\infty)) \end{cases}$$

and fulfill $n > 0, c > 0$ and $w > 0$ in $\bar\Omega \times [0,\infty)$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ $(N = 2,3)$ be a bounded domain with smooth boundary, and let (n, c, w, u, Φ) satisfy the conditions of Theorem 1.1.*
(I) If $\mu_1 < 0$, $\mu_2 \geq 0$, then there exist $C > 0$, suitable small $\delta > 0$, and $t_\star > 1$ satisfying

$$\|n\|_{L^\infty(\Omega)} \leq Ce^{\frac{\mu_1}{N+1}t}$$

and

$$\|c\|_{W^{1,q}(\Omega)} \leq Ce^{\max\{\delta-1,\mu_1\}\cdot\frac{N}{(N+1)q}\cdot t} \quad \text{and} \quad \|w\|_{W^{1,q}(\Omega)} \leq Ce^{\max\{-1,\mu_1\}\cdot\frac{N}{(N+1)q}\cdot t}$$

as well as

$$\|u\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\delta t} \qquad \text{for all } \ t > t_\star.$$

If $\mu_1 = 0$, $\mu_2 < 0$, then there exist $C > 0$, suitable small $\delta > 0$, and $t_\star > 1$ fulfilling

$$\|n\|_{L^\infty(\Omega)} \leq e^{-\frac{1}{N+1}\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|^{k-1}_{L^1(\Omega)}ds}$$

and

$$\|c\|_{W^{1,q}(\Omega)} \leq Ce^{\max\left\{\delta-1,\,-\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|^{k-1}_{L^1(\Omega)}ds\right\}\cdot\frac{N}{(N+1)q}\cdot t} \text{ and } \|w\|_{W^{1,q}(\Omega)} \leq c_2 e^{\max\left\{-1,\,-\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|^{k-1}_{L^1(\Omega)}ds\right\}\cdot\frac{N}{(N+1)q}\cdot t}$$

as well as

$$\|u\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\delta t} \qquad \text{for all } \ t > t_\star.$$

Remark 1.1. *For notational convenience, we do not explain the constants of C_i, $i = 1,2,\cdots,40$ and C_{GN} in the following. Here, C_{GN} is Gagliardo-Nirenberg constant.*

2. Local existence of $N = 2$ and $N = 3$

First of all, we give the local existence result. This proof is based on the Banach's fixed point theorem in a bounded closed set in $L^\infty((0,T);C^0(\bar{\Omega})\times(W^{1,q}(\Omega))^2\times D(A^\gamma))$ for all $\gamma\in(\frac{1}{2},1)$ and suitably small T, where A is the realization of the stokes operator in the solenoidal subspace. Additionally, here we omit the details of the proof, which can be found in [1, 20, 63]. For the positive solutions, we can obtain them using the principle of comparison. Because $\underline{n}\equiv 0$ is a sub-solution of the first equation in (1.1) and $n(x,0)\geq 0$, we have $n(x,t)\geq 0$. Furthermore, we can obtain $n(x,t)>0$ due to $n_0(x)\not\equiv 0$. Therefore, we can get $w(x,t)>0$ and $c(x,t)>0$, respectively.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^N\,(N=2,3)$ be a bounded domain with smooth boundary and Φ comply with (1.2), and suppose that n_0, c_0, w_0, and u_0 satisfy (1.3), then there exist functions*

$$\begin{cases} n\in C^0(\bar{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{\max})),\\ c\in C^0(\bar{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{\max})),\\ w\in C^0(\bar{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{\max})),\\ u\in C^0(\bar{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{\max})),\\ P\in C^{1,0}(\bar{\Omega}\times[0,T_{\max})) \end{cases}$$

and fulfill $n>0, c>0$ and $w>0$ in $\bar{\Omega}\times[0,T_{\max})$. Moreover, if $T_{\max}<\infty$, then for all $q>N$, $\gamma\in(\frac{1}{2},1)$ we have

$$\lim_{t\to T_{\max}}\sup\Big(\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,q}(\Omega)}+\|w(\cdot,t)\|_{W^{1,q}(\Omega)}+\|A^\gamma u(\cdot,t)\|_{L^2(\Omega)}\Big)=\infty.$$

3. Global existence of $N = 2,\ \mu_1 = \mu_2 = 0$

For the treatment of the global existence for two-dimensional Keller-Segel-Navier-Stokes-Solow-Swan system, we adopt the following Moser-Trudinger inequalities.

Lemma 3.1. *([68]) Suppose that $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary. Then for all $\epsilon>0$ there exists $M=M(\epsilon,\Omega)>0$ such that if $0\not\equiv\phi\in C^0(\bar{\Omega})$ is nonnegative and $\psi\in W^{1,2}(\Omega)$, then for each $a>0$,*

$$\int_\Omega\phi|\psi|dx\leq\frac{1}{a}\int_\Omega\phi\ln\frac{\phi}{\bar{\phi}}dx+\frac{(1+\epsilon)a}{8\pi}\cdot\left\{\int_\Omega\phi dx\right\}\cdot\int_\Omega|\nabla\psi|^2dx+Ma\cdot\left\{\int_\Omega\phi dx\right\}\cdot\left\{\int_\Omega|\psi|dx\right\}^2+\frac{M}{a}\int_\Omega\phi dx, \tag{3.1}$$

where $\bar{\phi}:=\frac{1}{|\Omega|}\int_\Omega\phi dx$.

Lemma 3.2. *([68]) Suppose that $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, and let $0\not\equiv\phi\in C^0(\bar{\Omega})$ is nonnegative. Then for any choice of $\epsilon>0$,*

$$\int_\Omega\phi\ln(\phi+1)dx\leq\frac{1+\epsilon}{2\pi}\cdot\left\{\int_\Omega\phi dx\right\}\cdot\int_\Omega\frac{|\nabla\phi|^2}{(\phi+1)^2}dx+4M\cdot\left\{\int_\Omega\phi dx\right\}^3+\{M-\ln\bar{\phi}\}\cdot\int_\Omega\phi dx,$$

where $M=M(\epsilon,\Omega)>0$ is as in Lemma 3.1.

Next, we give the required a prior estimates.

Lemma 3.3. *Assume that* (1.3) *holds. Then we have*

$$\int_\Omega n(x,t)dx = m_0 \tag{3.2}$$

and

$$\int_\Omega c(x,t)dx \le \int_\Omega c_0(x)dx + C_0\left(m_0 + \left\{\int_\Omega w_0(x)dx\right\}\cdot e^{-t}\right)$$

as well as

$$\int_\Omega w(x,t)dx \le m_0 + \left\{\int_\Omega w_0(x)dx\right\}\cdot e^{-t}. \tag{3.3}$$

Proof. Since $\mu_1 = \mu_2 = 0$, we integrate the first equation of (1.1) to get (3.2) and integrate the third equation of (1.1) and use the ODE argument to obtain (3.3). Then, using the similar method for the second equation of (1.1), we can complete the proof of the Lemma 3.3. □

Lemma 3.4. *Suppose that* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there exists* $C(T) > 0$ *such that*

$$\int_\Omega \left(c^2(x,t) + w^2(x,t)\right)dx \le C(T) \tag{3.4}$$

and

$$\int_0^T \int_\Omega \left(|\nabla c(x,t)|^2 + |\nabla w(x,t)|^2 + \frac{|\nabla n(x,t)|^2}{(n+1)^2}\right)dxdt \le C(T) \tag{3.5}$$

as well as

$$\int_0^T \int_\Omega n(x,t)\ln\frac{n(x,t)}{\bar{n}_0}dxdt \le C(T). \tag{3.6}$$

Proof. We first integrate by parts in the first equation from (1.1) and use $\nabla\cdot u = 0$ and the Young's inequality to deduce that

$$\begin{aligned}
-\frac{d}{dt}\int_\Omega \ln(n+1)dx &= -\int_\Omega \frac{n_t}{n+1}dx \\
&= -\int_\Omega \frac{1}{n+1}\Big[\Delta n - \chi\nabla\cdot(n\nabla c) - u\cdot\nabla n\Big]dx \\
&= -\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx + \chi\int_\Omega \frac{n\nabla n\cdot\nabla c}{(n+1)^2}dx \\
&\le -\frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx + \frac{\chi^2}{2}\int_\Omega \frac{n^2}{(n+1)^2}|\nabla c|^2dx \\
&\le -\frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx + \frac{\chi^2}{2}\int_\Omega |\nabla c|^2dx.
\end{aligned} \tag{3.7}$$

Multiplying the second equation of (1.1) by c, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega c^2dx &= \int_\Omega c(\Delta c - c + \mu_3 c^\alpha w^{1-\alpha} - u\cdot\nabla c)\\ &= -\int_\Omega |\nabla c|^2dx - \int_\Omega c^2dx + \mu_3\int_\Omega c^{1+\alpha}w^{1-\alpha}dx\\ &\le -\int_\Omega |\nabla c|^2dx - \int_\Omega c^2dx + \mu_3\|c^{1+\alpha}\|_{L^{\frac{2}{1+\alpha}}(\Omega)}\|w^{1-\alpha}\|_{L^{\frac{2}{1-\alpha}}(\Omega)}\\ &= -\int_\Omega |\nabla c|^2dx - \int_\Omega c^2dx + \mu_3\|c\|^{1+\alpha}_{L^2(\Omega)}\|w\|^{1-\alpha}_{L^2(\Omega)}\\ &\le -\int_\Omega |\nabla c|^2dx - \frac{1}{2}\int_\Omega c^2dx + C_1\|w\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.8}$$

Multiplying (3.8) by χ^2 and then substituting it into (3.7), we have

$$\frac{d}{dt}\Big(-\int_\Omega \ln(n+1)dx + \frac{\chi^2}{2}\int_\Omega c^2dx\Big) + \frac{\chi^2}{2}\Big(\int_\Omega c^2dx + \int_\Omega |\nabla c|^2dx\Big) + \frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx \le \chi^2 C_1\|w\|^2_{L^2(\Omega)}. \tag{3.9}$$

For the right hand side of (3.9), using the Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\Big(-\int_\Omega \ln(n+1)dx + \frac{\chi^2}{2}\int_\Omega c^2dx\Big) + \frac{\chi^2}{2}\Big(\int_\Omega c^2dx + \int_\Omega \nabla c|^2dx\Big) + \frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx\\ &\le 2\chi^2 C_1 C_{GN}\left(\|w\|_{L^1(\Omega)}\|\nabla w\|_{L^2(\Omega)} + \|w\|^2_{L^1(\Omega)}\right) \le \epsilon_1\|\nabla w\|^2_{L^2(\Omega)} + C_2,\end{aligned} \tag{3.10}$$

where $\epsilon_1 > 0$ is small enough and to be determined.
Multiplying the third equation of (1.1) by w, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx &= \int_\Omega w(\Delta w - w + n - u\cdot\nabla w)\\ &= -\int_\Omega |\nabla w|^2dx - \int_\Omega w^2dx + \int_\Omega nwdx.\end{aligned} \tag{3.11}$$

In order to control the last term at the right end of (3.11), using Lemma 3.1, we obtain

$$\int_\Omega nwdx \le \frac{1}{a}\int_\Omega n\ln\frac{n}{\bar{n}_0}dx + \frac{(1+\epsilon)m_0 a}{8\pi}\int_\Omega |\nabla w|^2dx + Mm_0 a\left\{\int_\Omega wdx\right\}^2 + \frac{Mm_0}{a} \quad \text{for all } t>0. \tag{3.12}$$

For the first term at the right end of (3.12), using Lemma 3.2, we can get

$$\int_\Omega n\ln\frac{n}{\bar{n}_0}dx \le \frac{(1+\epsilon)m_0}{2\pi}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx + 4Mm_0^3 + m_0\cdot(M - \ln\frac{m_0}{|\Omega|}). \tag{3.13}$$

Multiplying (3.13) by $\frac{1}{a}$, that is

$$\frac{1}{a}\int_\Omega n\ln\frac{n}{\bar{n}_0}dx \le \frac{(1+\epsilon)m_0}{2\pi a}\int_\Omega \frac{|\nabla n|^2}{(n+1)^2}dx + \frac{4Mm_0^3}{a} + \frac{m_0}{a}\cdot(M - \ln\frac{m_0}{|\Omega|}). \tag{3.14}$$

We now substituting (3.12) and (3.14) into (3.11) to deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx+\int_\Omega w^2dx+\left(1-\frac{(1+\epsilon)m_0a}{8\pi}\right)\int_\Omega|\nabla w|^2dx\\&\le\frac{(1+\epsilon)m_0}{2\pi a}\int_\Omega\frac{|\nabla n|^2}{(n+1)^2}dx+Mm_0a\left\{\int_\Omega wdx\right\}^2+\frac{2Mm_0}{a}+\frac{m_0}{a}\cdot(4Mm_0^2-\ln\frac{m_0}{|\Omega|}).\end{aligned}\tag{3.15}$$

Let $\lambda_0:=\frac{4(1+\epsilon)m_0}{\pi a}>0$. Multiplying (3.10) by λ_0 and adding it to (3.15), we can see that

$$\begin{aligned}&\frac{d}{dt}\left\{-\lambda_0\int_\Omega\ln(n+1)dx+\frac{\lambda_0\chi^2}{2}\int_\Omega c^2dx+\int_\Omega w^2dx\right\}+\frac{\lambda_0\chi^2}{2}\left(\int_\Omega c^2dx+\int_\Omega|\nabla c|^2dx\right)\\&\quad+\left(2-\frac{(1+\epsilon)m_0a}{4\pi}-\epsilon_1\lambda_0\right)\int_\Omega|\nabla w|^2dx+\frac{(1+\epsilon)m_0}{\pi a}\int_\Omega\frac{|\nabla n|^2}{(n+1)^2}dx+2\int_\Omega w^2dx\\&\le 2Mam_0\left(m_0+\left\{\int_\Omega w_0dx\right\}\cdot e^{-t}\right)^2+\frac{4Mm_0}{a}+\frac{2m_0}{a}\left(4Mm_0^2-\ln\frac{m_0}{|\Omega|}\right)+C_2\lambda_0.\end{aligned}$$

Therefore, we only need to select the appropriate positive numbers ϵ,ϵ_1 and a such that $2-\frac{(1+\epsilon)m_0a}{4\pi}-\epsilon_1\lambda_0>0$. If ϵ is fixed, we can take $a=\frac{2\pi}{(1+\epsilon)m_0}$ and $\epsilon_1=\frac{\pi a}{4(1+\epsilon)m_0}$, which can meet the conditions we need. Then we use the inequality $\int_\Omega\ln(n+1)dx\le\int_\Omega ndx=m_0$ to get (3.4) and (3.5). Finally, we use (3.5), (3.13) and the fact that $n\ln n\ge -e^{-1}$ to arrive at (3.6). □

Lemma 3.5. *Assume* (1.3) *is satisfied. Then, for all* $T\in(0,T_{\max})$ *there exists* $C(T)>0$ *such that*

$$\int_\Omega|u(x,t)|^2dx\le C(T)\tag{3.16}$$

and

$$\int_0^T\int_\Omega|\nabla u(x,t)|^2dxdt\le C(T).\tag{3.17}$$

Proof. We test the fourth equation of (1.1) by u and use the Hölder's inequality and Moser-Trudinger inequality to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2dx+\int_\Omega|\nabla u|^2dx=\int_\Omega n\nabla\Phi\cdot u\le\|\nabla\Phi\|_{L^\infty(\Omega)}\left\{\sum_{i=1}^2\int_\Omega|n||u_i|\right\}\\&\le\frac{\|\nabla\Phi\|_{L^\infty(\Omega)}}{a_1}\int_\Omega n\ln\frac{n}{\bar n}+\frac{(1+\epsilon_2)m_0a_1\|\nabla\Phi\|_{L^\infty(\Omega)}}{8\pi}\int_\Omega|\nabla u|^2dx+\|\nabla\Phi\|_{L^\infty(\Omega)}\left(Mm_0a_1\left\{\int_\Omega|u|dx\right\}^2+\frac{Mm_0}{a_1}\right),\end{aligned}\tag{3.18}$$

where

$$a_1:=\frac{1}{\left(2Mm_0\kappa_1|\Omega|+\frac{(1+\epsilon_2)m_0}{4\pi}\right)\|\nabla\Phi\|_{L^\infty(\Omega)}}>0,$$

and $\kappa_1>0$ is to be determined, it will be given by the following Poincaré's inequality.
On the other hand, using Poincaré's inequality and Hölder's inequality we have

$$\left(\int_\Omega|u|dx\right)^2\le|\Omega|\int_\Omega u^2dx\le\kappa_1|\Omega|\int_\Omega|\nabla u|^2dx.\tag{3.19}$$

Therefore, (3.18) together with (3.19) shows that

$$\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \leq \frac{2\|\nabla\Phi\|_{L^\infty(\Omega)}}{a_1}\left(\int_\Omega n\ln\frac{n}{\bar{n}} + Mm_0\right).$$

So, using Gronwall's inequality and (3.6), we have the descried results. □

Lemma 3.6. *If* (1.3) *holds, then for all* $T \in (0, T_{\max})$ *there exists* $C(T) > 0$ *such that*

$$\int_\Omega |\nabla c(x,t)|^2 dx \leq C(T).$$

Moreover, we have

$$\int_0^T \int_\Omega \left(|\Delta c(x,t)|^2 + |\nabla c(x,t)|^4\right) dxdt \leq C(T). \tag{3.20}$$

Proof. We multiply the Eq $(1.1)_2$ with $-\Delta c$ and use the integration by parts and Hölder's inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx &= \int_\Omega (u\cdot\nabla c)\Delta c dx - \mu_3\int_\Omega c^\alpha w^{1-\alpha}\Delta c dx\\ &\leq \frac{1}{4}\int_\Omega |\Delta c|^2 dx + 2\|u\|^2_{L^4(\Omega)}\|\nabla c\|^2_{L^4(\Omega)} + 2\mu_3^2\|c\|^{2\alpha}_{L^2(\Omega)}\|w\|^{2(1-\alpha)}_{L^2(\Omega)}\\ &\leq \frac{1}{4}\int_\Omega |\Delta c|^2 dx + 2\|u\|^2_{L^4(\Omega)}\|\nabla c\|^2_{L^4(\Omega)} + \|c\|^2_{L^2(\Omega)} + C_3\|w\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.21}$$

Applying the Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\|\nabla c\|^2_{L^4(\Omega)} \leq C_{GN}(\|\nabla c\|_{L^2(\Omega)}\|D^2 c\|_{L^2(\Omega)} + \|\nabla c\|^2_{L^2(\Omega)}) \tag{3.22}$$

and

$$\|\nabla w\|^2_{L^4(\Omega)} \leq C_{GN}(\|\nabla w\|_{L^2(\Omega)}\|D^2 w\|_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}). \tag{3.23}$$

We plug (3.22) into (3.21) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx &= \int_\Omega (u\cdot\nabla c)\Delta c dx - \mu_3\int_\Omega c^\alpha w^{1-\alpha}\Delta c dx\\ &\leq \frac{1}{4}\int_\Omega |\Delta c|^2 dx + 2\|u\|^2_{L^4(\Omega)}\|\nabla c\|^2_{L^4(\Omega)} + 2\mu_3^2\|c\|^{2\alpha}_{L^2(\Omega)}\|w\|^{2(1-\alpha)}_{L^2(\Omega)}\\ &\leq \frac{1}{4}\int_\Omega |\Delta c|^2 dx + 2C_{GN}\|u\|^2_{L^4(\Omega)}\|\nabla c\|_{L^2(\Omega)}\|D^2 c\|_{L^2(\Omega)}\\ &\quad + 2C_{GN}\|u\|^2_{L^4(\Omega)}\|\nabla c\|^2_{L^2(\Omega)} + \|c\|^2_{L^2(\Omega)} + C_3\|w\|^2_{L^2(\Omega)}\\ &\leq \frac{1}{4}\int_\Omega |\Delta c|^2 dx + \frac{3}{16}\|D^2 c\|^2_{L^2(\Omega)} + C_{41}\|u\|^4_{L^4(\Omega)}\|\nabla c\|^2_{L^2(\Omega)}\\ &\quad + 2C_{GN}\|u\|^2_{L^4(\Omega)}\|\nabla c\|^2_{L^2(\Omega)} + \|c\|^2_{L^2(\Omega)} + C_3\|w\|^2_{L^2(\Omega)},\end{aligned} \tag{3.24}$$

where $C_{41} > 0$ is a constant.
On the other hand, note that the identities $|\Delta c|^2 = \nabla\cdot(\Delta c\nabla c) - \nabla c\cdot\nabla\Delta c$ and $\Delta|\nabla c|^2 = 2\nabla c\cdot\nabla\Delta c + 2|D^2c|^2$, we deduce that

$$\begin{aligned}\int_\Omega |\Delta c|^2 dx &= \int_\Omega \nabla\cdot(\Delta c\nabla c)dx - \int_\Omega \nabla c\cdot\nabla\Delta c dx\\ &= \int_{\partial\Omega} \Delta c\frac{\partial c}{\partial\nu}dS - \int_\Omega \nabla c\cdot\nabla\Delta c dx\\ &= -\int_\Omega \nabla c\cdot\nabla\Delta c dx\\ &= \int_\Omega |D^2c|^2dx - \frac{1}{2}\int_\Omega \Delta|\nabla c|^2dx\\ &= \int_\Omega |D^2c|^2dx - \frac{1}{2}\int_{\partial\Omega}\frac{\partial|\nabla c|^2}{\partial\nu}dS.\end{aligned} \tag{3.25}$$

Thanks to the fact $\frac{\partial|\nabla c|^2}{\partial\nu} \le 2\kappa_2|\nabla c|^2$, where $\kappa_2 := \kappa_2(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$ in ([35], Lemma 4.2), the trace theorem and (3.25), we can see that

$$\begin{aligned}\int_\Omega |D^2c|^2dx &\le \int_\Omega |\Delta c|^2dx + \kappa_2\int_{\partial\Omega}|\nabla c|^2dS\\ &\le \int_\Omega |\Delta c|^2dx + \kappa_2\tilde{C}_{41}(\Omega,s)\|c\|^2_{H^{\frac{3+s}{2}}(\Omega)}\\ &\le \int_\Omega |\Delta c|^2dx + \tilde{C}_{42}\Big(\|D^2c\|^{\frac{3+s}{2}}_{L^2(\Omega)}\|c\|^{\frac{1-s}{2}}_{L^2(\Omega)} + \|c\|^2_{L^2(\Omega)}\Big)\\ &\le \int_\Omega |\Delta c|^2dx + \frac{1}{4}\int_\Omega |D^2c|^2dx + \tilde{C}_{43},\end{aligned}$$

where $\tilde{C}_{41}, \tilde{C}_{42}, \tilde{C}_{43}$ and $s \in (0,1)$ are positive constants.
That is

$$\int_\Omega |D^2c|^2dx \le \frac{4}{3}\int_\Omega |\Delta c|^2dx + \frac{4}{3}\tilde{C}_{43}. \tag{3.26}$$

Similarly, we have

$$\int_\Omega |D^2w|^2dx \le \frac{4}{3}\int_\Omega |\Delta w|^2dx + \frac{4}{3}\tilde{C}_{43}. \tag{3.27}$$

Then, we apply Gagliardo-Nirenberg inequality, Lemma 3.5 and Poincaré's inequality to get

$$\begin{aligned}\|u\|^4_{L^4(\Omega)} &\le C_{GN}\Big(\|u\|^2_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^4_{L^2(\Omega)}\Big)\\ &\le \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + C_{42}\|u\|^4_{L^2(\Omega)} \le C_{43}\|\nabla u\|^2_{L^2(\Omega)},\end{aligned} \tag{3.28}$$

where C_{42}, C_{43} are two positive constants.
Therefore, (3.24) together with (3.26) and (3.28) shows that

$$\frac{d}{dt}\int_\Omega |\nabla c|^2dx + \int_\Omega |\Delta c|^2dx \le 2\|c\|^2_{L^2(\Omega)} + 2C_3\|w\|^2_{L^2(\Omega)} + C_4\Big(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big)\|\nabla c\|^2_{L^2(\Omega)}, \tag{3.29}$$

where $C_4 = \max\Big\{C_{41}C_{43},\ 2C_{GN}\sqrt{C_{43}}\Big\}$.
So, we use Gronwall inequality, and use Lemmas 3.4 and 3.5 and Hölder's inequality to arrive at the Lemma 3.6. □

Lemma 3.7. *Suppose that* (1.3) *holds and that* $T \in (0, T_{\max})$. *Then there exists* $C(T) > 0$ *such that*

$$\int_\Omega |\nabla w(x,t)|^2 dx \le C(T)$$

and

$$\int_0^T \int_\Omega \left(|\Delta w(x,t)|^2 + |\nabla w(x,t)|^4\right) dxdt \le C(T).$$

Proof. Multiplying the Eq $(1.1)_3$ with $-\Delta w$ and using Hölder's inequality, (3.23), (3.27) and (3.28), one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla w|^2 dx + \int_\Omega |\Delta w|^2 dx \\
&= \int_\Omega (u\cdot\nabla w)\Delta w dx - \int_\Omega n\Delta w dx \\
&\le \frac{1}{4}\int_\Omega |\Delta w|^2 dx + 2\|u\|^2_{L^4(\Omega)}\|\nabla w\|^2_{L^4(\Omega)} + 2\|n\|^2_{L^2(\Omega)} \\
&\le \frac{1}{4}\int_\Omega |\Delta w|^2 dx + 2\|u\|^2_{L^4(\Omega)}\left(\|\nabla w\|_{L^2(\Omega)}\|D^2 w\|_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}\right) + 2\|n\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\Delta w|^2 dx + C_5\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)\|\nabla w\|_{L^2(\Omega)} + 2\|n\|^2_{L^2(\Omega)},
\end{aligned} \tag{3.30}$$

where $C_5 > 0$ is a constant.

For the term of $\|n\|^2_{L^2(\Omega)}$, we apply the Gagliardo-Nirenberg inequality and the mass conservation of $\|n\|_{L^1(\Omega)}$ to deduce that

$$\|n\|^2_{L^2(\Omega)} = \|\sqrt{n}\|^4_{L^4(\Omega)} \le C_{GN}\left(\|\nabla\sqrt{n}\|^2_{L^2(\Omega)}\|\sqrt{n}\|^2_{L^2(\Omega)} + \|\sqrt{n}\|^4_{L^2(\Omega)}\right) \le C_5(\|\nabla\sqrt{n}\|^2_{L^2(\Omega)} + 1). \tag{3.31}$$

Multiplying the Eq $(1.1)_1$ with $(1+\ln n)$ and using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n\ln n dx &= \int (\Delta n - \chi\nabla\cdot(n\nabla c))(1+\ln n)dx \\
&\le -\int_\Omega \frac{|\nabla n|^2}{n}dx + \chi\int_\Omega \nabla n\nabla c \\
&= -\int_\Omega \frac{|\nabla n|^2}{n}dx + \chi\int_\Omega \frac{\nabla n}{\sqrt{n}}\sqrt{n}\nabla c dx \\
&\le -\frac{1}{2}\int_\Omega \frac{|\nabla n|^2}{n}dx + \frac{\chi^2}{2}\int_\Omega n|\nabla c|^2 dx \\
&\le -2\|\nabla\sqrt{n}\|^2_{L^2(\Omega)} + \frac{1}{C_5}\|n\|^2_{L^2(\Omega)} + \frac{\chi^4 C_5}{8}\|\nabla c\|^4_{L^4(\Omega)}.
\end{aligned} \tag{3.32}$$

Then, we add (3.31) into (3.32) to obtain

$$\frac{d}{dt}\int_\Omega n\ln n dx + \|\nabla\sqrt{n}\|^2_{L^2(\Omega)} \le 1 + \frac{\chi^4 C_5}{8}\|\nabla c\|^4_{L^4(\Omega)}. \tag{3.33}$$

We integrate the two ends of (3.33) with respect to t, and use Lemma 3.6 to get

$$\int_\Omega n\ln n dx + \int_0^T \|\nabla\sqrt{n}\|^2_{L^2(\Omega)}dt \le T + \frac{\chi^4 C_5}{8}\int_0^T \|\nabla c\|^4_{L^4(\Omega)}dt \le C(T) \quad \text{for all } T \in (0, T_{\max}). \tag{3.34}$$

Finally, we use Gronwall's inequality to (3.30) and note that $n\ln n \geq -e^{-1}$ and (3.34) to complete the Lemma 3.7. □

Lemma 3.8. *Assume* (1.3)*, and let* $T \in (0, T_{\max})$*. Then there exists* $C(T) > 0$ *such that*

$$\int_\Omega |n(\cdot,t)|^2 dx \leq C(T).$$

Proof. Testing the first equation in (1.1) against n and integrating by parts show that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 dx + \int_\Omega |\nabla n|^2 dx = -\chi \int_\Omega n\nabla\cdot(n\nabla c)dx = \chi\int_\Omega n\nabla n\cdot\nabla c dx.$$

Applying the identity $n\nabla\cdot(n\nabla c) = n\nabla n\cdot\nabla c + n^2\Delta c$, we show that

$$\frac{d}{dt}\int_\Omega n^2 dx + 2\int_\Omega |\nabla n|^2 dx = -\chi\int_\Omega n^2\Delta c dx \leq \chi\|n^2\|_{L^2(\Omega)}\|\Delta c\|_{L^2(\Omega)} = \chi\|n\|^2_{L^4(\Omega)}\|\Delta c\|_{L^2(\Omega)}. \tag{3.35}$$

Using the Gagliardo-Nirenberg inequality again, we have

$$\|n\|^2_{L^4(\Omega)} \leq C_{GN}\Big(\|\nabla n\|_{L^2(\Omega)}\|n\|_{L^2(\Omega)} + m_0^2\Big). \tag{3.36}$$

Combining (3.35) with (3.36) and using the Young' s inequality, one has

$$\begin{aligned}\frac{d}{dt}\int_\Omega n^2 dx + 2\int_\Omega |\nabla n|^2 dx &\leq C_{GN}\chi\|\Delta c\|_{L^2(\Omega)}\|\nabla n\|_{L^2(\Omega)}\|n\|_{L^2(\Omega)} + C_{GN}\chi m_0^2\|\Delta c\|_{L^2(\Omega)}\\ &\leq \|\nabla n\|^2_{L^2(\Omega)} + C_6\|\Delta c\|^2_{L^2(\Omega)}\|n\|^2_{L^2(\Omega)} + C_6(\|\Delta c\|^2_{L^2(\Omega)} + 1).\end{aligned}$$

Applying Gronwall's inequality and the Lemma 3.7, we can obtain

$$\begin{aligned}\int_\Omega n^2 dx &\leq \|n_0\|^2_{L^2(\Omega)} e^{C_6\int_0^t \|\Delta c(\cdot,s)\|^2_{L^2(\Omega)}ds} + C_6 e^{C_6\int_0^t \|\Delta c(\cdot,s)\|^2_{L^2(\Omega)}ds}\int_0^t\Big(\|\Delta c(\cdot,s)\|^2_{L^2(\Omega)} + 1\Big)e^{-C_6\int_0^s\|\Delta c(\cdot,\tau)\|^2_{L^2(\Omega)}d\tau}ds\\ &\leq C(T) \quad \text{for all} \;\; t\in(0,T_{\max}).\end{aligned}$$

Thus, we complete the proof of the Lemma 3.8. □

Lemma 3.9. *Suppose that* (1.3) *holds and that* $T \in (0, T_{\max})$*. Then there exists* $C(T) > 0$ *such that*

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq C(T)$$

and

$$\int_0^T\int_\Omega |Au(x,t)|^2 dxdt \leq C(T).$$

Proof. Testing $(1.1)_4$ by Au and using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality and (3.16), one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2dx + \int_\Omega |Au|^2dx &= \int_\Omega (n\nabla\Phi)Audx - \int_\Omega (u\cdot\nabla u)Audx\\ &\le \frac{1}{2}\|Au\|_{L^2(\Omega)} + \|\nabla\Phi\|_{L^\infty(\Omega)}\|n\|^2_{L^2(\Omega)} + \|u\cdot\nabla u\|^2_{L^2(\Omega)}\\ &\le \frac{1}{2}\|Au\|_{L^2(\Omega)} + \|\nabla\Phi\|_{L^\infty(\Omega)}\|n\|^2_{L^2(\Omega)} + \|u\|^2_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\ &\le \frac{1}{2}\|Au\|_{L^2(\Omega)} + \|\nabla\Phi\|_{L^\infty(\Omega)}\|n\|^2_{L^2(\Omega)} + C_{GN}\|u\|_{L^2(\Omega)}\|u\|_{W^{2,2}(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\ &\le \frac{3}{4}\|Au\|_{L^2(\Omega)} + C_7 + C_7\|\nabla u\|^4_{L^2(\Omega)}.\end{aligned} \tag{3.37}$$

Applying the variation of constant formula and (3.17), we have

$$\int_\Omega |\nabla u|^2dx \le \|\nabla u_0\|_{L^2(\Omega)}e^{2C_7\int_0^t\|\nabla u(\cdot,s)\|_{L^2(\Omega)}ds} + 2C_7e^{2C_7\int_0^t\|\nabla u(\cdot,s)\|_{L^2(\Omega)}ds}\int_0^t e^{-2C_7\int_0^\tau\|\nabla u(\cdot,s)\|_{L^2(\Omega)}ds}d\tau \le C_8 \tag{3.38}$$

for all $t\in(0,T_{\max})$.
Integrating the two sides of (3.37) and applying (3.38), we complete the proof. □

Lemma 3.10. *Assume that* (1.3) *holds and let* $\gamma_0\in(\frac{1}{2},\gamma]\subset(\frac{1}{2},1)$. *Then for all* $T\in(0,T_{\max})$. *there exists* $C(T)>0$ *such that*

$$\int_\Omega |A^{\gamma_0}u(\cdot,t)|^2dx \le C(T) \tag{3.39}$$

and

$$\|u(\cdot,t)\|_{C^\theta(\Omega)} \le C(T).$$

Proof. We fix γ_0 and let $p>\frac{1}{1-\gamma_0}$, then use the Helmholtz projection operator to the fourth equation of (1.1) and the variation of constant formula to deduce that

$$\begin{aligned}\|A^{\gamma_0}u(\cdot,t)\|_{L^2(\Omega)} &= \Big\|A^{\gamma_0}\Big(e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\Phi - u(\cdot,s)\cdot\nabla u(\cdot,s))ds\Big)\Big\|_{L^2(\Omega)}\\ &\le C_9 + C_9\int_0^t (t-s)^{-\gamma_0}\|u(\cdot,s)\cdot\nabla u(\cdot,s)\|_{L^2(\Omega)}ds\\ &\le C_9 + C_9\Big(\int_0^t (t-s)^{-\frac{p\gamma_0}{p-1}}ds\Big)^{\frac{p-1}{p}}\Big(\int_0^t \|u(\cdot,s)\cdot\nabla u(\cdot,s)\|^p_{L^2(\Omega)}ds\Big)^{\frac{1}{p}}\\ &:= C_9 + C_9J_1^{\frac{p}{p-1}}J_2^{\frac{1}{p}}.\end{aligned}$$

Due to $p>\frac{1}{1-\gamma_0}$, we have $\frac{p\gamma_0}{p-1}\in(0,1)$. So, $J_1\in(0,\infty)$.
For J_2, we apply the Hölder's inequality, Sobolev embedding, Poincaré's inequality and Gagliardo-

Nirenberg inequality to obtain

$$\begin{aligned}
J_2 &= \int_0^t \|u(\cdot,s)\cdot\nabla u(\cdot,s)\|^p_{L^2(\Omega)}ds\\
&\le \int_0^t \|u(\cdot,s)\|^p_{L^q(\Omega)}\|\nabla u(\cdot,s)\|^p_{L^{\frac{2q}{q-2}}(\Omega)}ds\\
&\le \int_0^t \|u(\cdot,s)\|^p_{W^{1,2}(\Omega)}\|\nabla u(\cdot,s)\|^p_{L^{\frac{2q}{q-2}}(\Omega)}ds\\
&\le C_{10}\int_0^t \|\nabla u(\cdot,s)\|^{2p-2}_{L^2(\Omega)}\|\Delta u(\cdot,s)\|^2_{L^2(\Omega)}\\
&\le C_{10}\sup_{t\in(0,T)}\|\nabla u(\cdot,s)\|^{2p-2}_{L^2(\Omega)}\int_0^T \|Au(\cdot,s)\|^2_{L^2(\Omega)}ds.
\end{aligned}$$

Applying Lemma 3.9, we can get (3.39). Then we apply the embedding of $D(A^{\gamma_0}) \hookrightarrow C^{\theta}(\Omega)$ for all $\theta\in(0,2\gamma_0-1)$ to complete the proof of Lemma 3.10. □

Lemma 3.11. *If* (1.3) *holds, there for all* $T\in(0,T_{\max})$. *there exists* $C(T)>0$ *such that*

$$\|c(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(T) \quad \text{for all q} > 1.$$

Proof. Without loss of generality, we assume that $q>2$. Using the Duhamel principle for c and using standard semigroup estimates for the Neumann heat semigroup in ([61], Lemma 1.3) and embedding in ([19], Lemma 1.6.1) and the estimate in ([20] Lemma 2.1 or [15], Lemma 2.2), and using the Lemmas 3.4, 3.6 and 3.7, we can see that

$$\begin{aligned}
\|c(\cdot,t)\|_{W^{1,q}(\Omega)} &\le \|e^{t(\Delta-1)}c_0\|_{W^{1,q}(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}\Big(\mu_3 c^{\alpha}(\cdot,s)w^{1-\alpha}(\cdot,s)+u(\cdot,s)\cdot\nabla c(\cdot,s)\Big)\right\|_{W^{1,q}(\Omega)}ds\\
&\le C_{11}+\mu_3\int_0^t \left\|e^{(t-s)(\Delta-1)}c^{\alpha}(\cdot,s)w^{1-\alpha}(\cdot,s)\right\|_{W^{1,q}(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}\nabla\cdot\Big(u(\cdot,s)\cdot c(\cdot,s)\Big)\right\|_{W^{1,q}(\Omega)}ds\\
&\le C_{11}+C_{12}\int_0^t (1+(t-s)^{-\frac34+\frac1q})e^{-\lambda_1(t-s)}\|c^{\alpha}(\cdot,s)w^{1-\alpha}(\cdot,s)\|_{L^4(\Omega)}ds\\
&\quad + C_{12}\int_0^t \left\|(-\Delta+1)^{\kappa_3}e^{(t-s)(\Delta-1)}\nabla\cdot\Big(u(\cdot,s)c(\cdot,s)\Big)\right\|_{L^{2q}(\Omega)}ds\\
&\le C_{11}+C_{12}\|c(\cdot,s)\|^{\alpha}_{L^4(\Omega)}\|w(\cdot,s)\|^{1-\alpha}_{L^4(\Omega)}\int_0^t (1+(t-s)^{-\frac34+\frac1q})e^{-\lambda_1(t-s)}ds\\
&\quad + C_{13}\int_0^t (t-s)^{-\kappa_3-\frac12-\delta_1}e^{-\lambda_1(t-s)}\|u(\cdot,s)c(\cdot,s)\|_{L^{2q}(\Omega)}\\
&\le C_{11}+C_{13}\Big(\|c\|^{\alpha}_{W^{1,2}(\Omega)}\|w\|^{1-\alpha}_{W^{1,2}(\Omega)}+\|u(\cdot,s)\|_{L^{\infty}(\Omega)}\|c(\cdot,s)\|_{W^{1,2}(\Omega)}\int_0^t (t-s)^{-\kappa_3-\frac12-\delta_1}e^{-\lambda_1(t-s)}ds\Big)\\
&\le C_{14} \quad \text{for all } \kappa_3 > \frac12-\frac{1}{2q} \text{ and } 0<\kappa_3+\delta_1<\frac12.
\end{aligned}$$

□

Lemma 3.12. *Suppose that* (1.3) *holds and that* $T\in(0,T_{\max})$. *Then there exists* $C(T)>0$ *such that*

$$\|n(\cdot,t)\|_{L^{\infty}(\Omega)} \le C(T).$$

Proof. Let $M(T^\star) := \sup_{t\in(0,T^\star)} \|n(\cdot,t)\|_{L^\infty(\Omega)}$ for all $T^\star \in (0,T)$ and let $t_0 = (t-1)_+$. We use the Duhamel principle for n and use the semigroup estimate, Interpolation inequality and Young's inequality to deduce that

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &= \left\|e^{(t-t_0)\Delta}n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(\chi n(\cdot,s)\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s))ds\right\|_{L^\infty(\Omega)} \\
&\le C_{15} + \int_0^1 (1+s^{-\frac{5}{6}})\|\chi n(\cdot,s)\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\|_{L^3(\Omega)}ds \\
&\le C_{15} + C_{16}\int_0^1 (1+s^{-\frac{5}{6}})\|n(\cdot,s)\|_{L^4(\Omega)}ds \\
&\le C_{15} + C_{16}\int_0^1 (1+s^{-\frac{5}{6}})\|n(\cdot,s)\|^{\frac{1}{4}}_{L^1(\Omega)}\|n\|^{\frac{3}{4}}_{L^\infty(\Omega)}ds \\
&\le C_{15} + C_{16}m_0^{\frac{1}{4}}M^{\frac{3}{4}}(T^\star)\int_0^1 (1+s^{-\frac{5}{6}})ds \\
&\le C_{17} + \frac{1}{2}M(T^\star) + C_{17} \quad \text{for all } t\in(0,T^\star).
\end{aligned} \tag{3.40}$$

We take the supremum of time for both sides of (3.40) to obtain the Lemma 3.12. □

Lemma 3.13. *Assume* (1.3)*, and let* $T \in (0, T_{\max})$*. Then there exists* $C(T) > 0$ *such that*

$$\|w(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(T).$$

Proof. Since the estimate of $\|n\|_{L^\infty(\Omega)}$ in Lemma 3.12 has been obtained, we only need to use the Duhamel principle and the processing techniques similar to Lemma 3.11. □

Proof of Theorem 1.1. For the two-dimensional Navier-Stokes case, applying the Lemmas 2.1 and 3.10–3.13, if T is finite, then using the extendability criterion, we can see that n, c, w and u are unbounded of their respective norms, which contradict the boundedness of our a prior estimates. Next, we will give the asymptotic behavior of the system (1.1) with logistic source. Finally, we give a priori estimates of the corresponding solution in the three-dimensional case.

4. Boundedness and asymptotic behavior of $n.c, w$ and u with logistic source

For $\mu_1 < 0$, we can obtain the decay estimates of the following.

4.1. $\mu_1 < 0$ *and* $\mu_2 \ge 0$

Since $\mu_1 < 0$ and μ_2 are nonnegative, we can easily obtain the corresponding global boundedness results of the system (1.1) by using the previous processing ways. Next, we give the corresponding large time behavior.

Lemma 4.1. *Under the assumption of Lemma 3.10, there exist* $\theta \in (0,1)$ *and* $C = C(\chi,\mu_1,\mu_2,\mu_3,\alpha) > 0$*, independent of t, such that*

$$\|u(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times(0,\infty))} \le C.$$

Proof. Applying the estimates obtained by Lemmas 3.10 and 3.12, and then combining with the standard Schauder estimate in [45], we arrive the proof. □

Lemma 4.2. *Under the assumption of Lemma 3.12, there is an C, independent of time t such that*

$$\|n(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C.$$

Proof. Let $\mathbf{p} := \nabla n$, $\mathbf{q} := \nabla c$. We rewrite the first equation of (1.1) to obtain

$$\frac{d}{dt}n(x,t) = \nabla\cdot(\nabla n - \chi n\nabla c - nu) + \mu_1 n - \mu_2 n^k := \nabla\cdot a(x,t,\mathbf{p}) + b(x,t) \quad (x,t,\mathbf{p}) \in \Omega\times(0,+\infty)\times\mathbb{R}^N,$$

where $a(x,t,\mathbf{p}) = \mathbf{p} - n(\chi\,\mathbf{q} - u)$ and $b = \mu_1 n - \mu_2 n^k$.
Using Lemmas 3.10–3.12 and 4.1, there exists $C_{18} > 0$ satisfying

$$a(x,t,\mathbf{p})\cdot\mathbf{p} = |\mathbf{p}|^2 - \chi n\mathbf{p}\cdot\mathbf{q} - nu\cdot\mathbf{p} \geq \frac{1}{2}|\mathbf{p}|^2 - C_{18}|\mathbf{q}|^2 - C_{18}$$

and

$$|a(x,t,\mathbf{p})| = |\mathbf{p} - \chi n\mathbf{q} - nu| \leq |\mathbf{p}| + C_{18}|\mathbf{q}| + C_{18}$$

as well as

$$|b(x,t)| = |\mu_1 n - \mu_2 n^k| \leq C_{18}.$$

Thanks to $\mathbf{q} \in L^\infty(0,T;L^\infty(\Omega))$, it evident that $\frac{1}{\infty} + \frac{N}{2\cdot\infty} = 0 < 1$. Apply the standard result on Hölder's regularity in scalar parabolic equation in ([40], Theorem 1.3) to get $\|n\|_{C^{\theta,\frac{\theta}{2}}(\Omega\times(0,T))}$ bounded. Then the Lemma 4.2 now follows from ([23], Theorem IV. 5.3). □
Next, we adapt the similar methods to obtain the following:

Lemma 4.3. *Under the assumption of Lemmas 3.11 and 3.13, there is an C, independent of time t such that*

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C.$$

Lemma 4.4. *Assume that* (1.3) *holds. If* $\mu_1 < 0$, $\mu_2 \geq 0$*, then there exist a constant* c_1*, independent of time t such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 e^{\frac{\mu_1}{3}t}.$$

Proof. We integrate the first equation of (1.1) to obtain

$$\frac{d}{dt}\int_\Omega n(\cdot,t)dx - \mu_1\int_\Omega n(\cdot,t)dx \leq 0. \tag{4.1}$$

Using the Gronwall's inequality for the Eq (4.1), we can see that

$$\|n\|_{L^1(\Omega)} \leq m_0 e^{\mu_1 t}. \tag{4.2}$$

Applying the Gagliardo-Nirenberg inequality, the Lemma 4.2 and the estimate (4.2), we have

$$\|n\|_{L^\infty(\Omega)} \leq C_{GN}\big(\|n\|^{\frac{1}{3}}_{L^1(\Omega)}\|\nabla n\|^{\frac{2}{3}}_{L^\infty(\Omega)} + \|n\|_{L^1(\Omega)}\big) \leq C_{19}e^{\frac{\mu_1}{3}t}. \tag{4.3}$$

Thus, we complete the proof of the Lemma 4.4. □

Lemma 4.5. *Suppose that* (1.3) *holds. If* $\mu_1 < 0$, $\mu_2 \geq 0$*, then there exist a constant* c_2*, independent of time t such that*

$$\|c(\cdot,t)\|_{W^{1,q}(\Omega)} \leq c_2 e^{\max\{\delta_2-1,\mu_1\}\cdot\frac{2}{3q}t} \quad \text{and} \quad \|w(\cdot,t)\|_{W^{1,q}(\Omega)} \leq c_2 e^{\max\{-1,\mu_1\}\cdot\frac{2}{3q}t}.$$

Proof. We integrate the first equation of (1.1) and (4.2) to deduce that

$$\frac{d}{dt}\int_\Omega w dx + \int_\Omega w dx = \int_\Omega n dx \leq m_0 e^{\mu_1 t}.$$

Thus, using the Gronwall's inequality, we can obtain

$$\int_\Omega w(\cdot,t)dx \leq \|w_0\|_{L^1(\Omega)}e^{-t} + \frac{m}{\mu_1+1}e^{\mu_1 t} \leq C_{20}e^{\max\{-1,\mu_1\}t}. \tag{4.4}$$

Similarly, using Hölder's inequality and Young's inequality, there exist a suitable small $0 < \delta_2 \ll 1$ such that

$$\frac{d}{dt}\int_\Omega c dx + \int_\Omega c dx \leq \mu_3\|c\|^{\alpha}_{L^1(\Omega)}\|w\|^{1-\alpha}_{L^1(\Omega)} \leq \delta_2\|c\|_{L^1(\Omega)} + C_{21}\|w\|_{L^1(\Omega)}.$$

Thus, we use ODE argument to get

$$\|c\|_{L^1(\Omega)} \leq C_{22}e^{\max\{\delta_2-1,\mu_1\}t}. \tag{4.5}$$

Then, for all $q > 1$ we apply the Gagliardo-Nirenberg inequality to see that

$$\|c\|_{W^{1,q}(\Omega)} \leq C_{GN}\big(\|c\|^{\frac{2}{3q}}_{L^1(\Omega)}\|\nabla c\|^{\frac{3q-2}{3q}}_{L^\infty(\Omega)} + \|c\|_{L^1(\Omega)}\big)$$

and

$$\|w\|_{W^{1,q}(\Omega)} \leq C_{GN}\big(\|w\|^{\frac{2}{3q}}_{L^1(\Omega)}\|\nabla w\|^{\frac{3q-2}{3q}}_{L^\infty(\Omega)} + \|w\|_{L^1(\Omega)}\big).$$

Using the above two estimates and (4.4), (4.5) proves that the Lemma 4.5. □

Lemma 4.6. *Suppose* (1.3) *and* $\mu_1 < 0$, $\mu_2 \geq 0$ *hold, then there exist a constant* c_3*, independent of time t such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq c_3 e^{-\delta_3 t}.$$

Proof. Testing the Eq $(1.1)_4$ with u and using Poincaré's inequality and Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx &= \int_\Omega n\nabla\Phi\cdot u \\ &\leq \|\nabla\Phi\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|n\|_{L^2(\Omega)} \\ &\leq C_{23}\|\nabla u\|_{L^2(\Omega)}\|n\|_{L^2(\Omega)} \\ &\leq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{C_{23}^2}{2}\|n\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.6}$$

And using Poincaré's inequality once more, there is a constant $\tilde{C}_{23} > 0$ such that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \tilde{C}_{23}\|u\|^2_{L^2(\Omega)} \leq C_{23}^2\|n\|^2_{L^2(\Omega)}.$$

Using Gronwall's inequality and the Lemma 4.4, there exists a constant $C_{24} > 0$ fulfilling

$$\|u\|_{L^2(\Omega)} \leq C_{24}e^{\max\left\{-\tilde{C}_{23},\frac{\mu_1}{3}\right\}t}.$$

Then, applying the Gagliardo-Nirenberg inequality, this shows that

$$\|u\|_{W^{1,\infty}(\Omega)} \leq C_{GN}\big(\|u\|^{\frac{1}{3}}_{L^2(\Omega)}\|u\|^{\frac{2}{3}}_{W^{2,\infty}(\Omega)} + \|u\|_{L^2(\Omega)}\big) \leq C_{25}e^{\max\left\{-\frac{\tilde{C}_{23}}{3},\frac{\mu_1}{9}\right\}t}. \quad \square$$

4.2. $\mu_1 = 0$ and $\mu_2 > 0$

Lemma 4.7. *Assume that* (1.3) *holds. If $\mu_1 = 0$, $\mu_2 > 0$, then there exist a constant c_4, independent of time t such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le c_4 e^{-\frac{1}{3}\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|_{L^1(\Omega)}^{k-1}ds}.$$

Proof. We integrate the first equation of (1.1) to obtain

$$\frac{d}{dt}\int_\Omega n(\cdot,t)dx + \mu_2\int_\Omega n^k(\cdot,t)dx = 0.$$

We use Hölder's inequality to deduce that

$$\frac{d}{dt}\int_\Omega n(\cdot,t)dx + \mu_2|\Omega|^{\frac{1}{k-1}}(\int_\Omega n(\cdot,t)dx)^k \le 0.$$

We apply ODE argument to get

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)} e^{-\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|_{L^1(\Omega)}^{k-1}ds}.$$

Similarly, using the inequality (4.3), we complete the proof of the Lemma 4.5. □

Lemma 4.8. *Suppose that* (1.3) *holds. If $\mu_1 = 0$, $\mu_2 > 0$, then there exist a constant c_5, independent of time t such that*

$$\|c(\cdot,t)\|_{W^{1,q}(\Omega)} \le c_5 e^{\max\left\{\delta_2-1,\, -\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|_{L^1(\Omega)}^{k-1}ds\right\}\cdot\frac{2}{3q}t}$$

and

$$\|w(\cdot,t)\|_{W^{1,q}(\Omega)} \le c_5 e^{\max\left\{-1,\, -\mu_2|\Omega|^{\frac{1}{k-1}}\int_0^t \|n(\cdot,s)\|_{L^1(\Omega)}^{k-1}ds\right\}\cdot\frac{2}{3q}t}$$

as well as

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_5 e^{-\delta_4 t}.$$

Proof. The proof is completely similar to Lemmas 4.5 and 4.6, so we omit the details. □

4.3. $\mu_1 > 0$ and $\mu_2 > 0$

Next, we will give a priori estimates when $\mu_1 > 0$, $\mu_2 > 0$.

Lemma 4.9. *Assume that* (1.3) *holds. Then for all $T > 0$ there exist $C(T) > 0$ such that*

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le \max\left\{\|n_0\|_{L^1(\Omega)}, \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{k-1}}|\Omega|\right\}. \tag{4.7}$$

and

$$\int_0^T \|n(\cdot,t)\|_{L^k(\Omega)}^k dt \le C(T). \tag{4.8}$$

Proof. We integrate the first equation of (1.1) to get

$$\frac{d}{dt}\int_\Omega n(\cdot,t)dx = \mu_1\int_\Omega n(\cdot,t)dx - \mu_2\int_\Omega n^k(\cdot,t)dx. \tag{4.9}$$

Applying ODE comparison, we have

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)} \tag{4.10}$$

or

$$\mu_1\int_\Omega n(\cdot,t)dx > \mu_2\int_\Omega n^k(\cdot,t)dx \ge \mu_2|\Omega|^{1-k}\cdot\|n(\cdot,t)\|^k_{L^1(\Omega)}. \tag{4.11}$$

Combining (4.10) with (4.11), this entails (4.7). Then, we integrate the two sides of Eq (4.9) to get (4.8). □

Lemma 4.10. *Suppose that* (1.3) *holds. Then for all* $T > 0$ *there exist* $C(T) > 0$ *such that*

$$\int_\Omega \left(c^2(x,t) + w^2(x,t)\right)dx \le C(T) \tag{4.12}$$

and

$$\int_0^T\int_\Omega \left(|\nabla c(x,t)|^2 + |\nabla w(x,t)|^2\right)dxdt \le C(T). \tag{4.13}$$

Proof. Using the inequality (3.8) and (3.11), and using Hölder's inequality and Young's inequality we have

$$\frac{d}{dt}\int_\Omega c^2dx + 2\int_\Omega|\nabla c|^2dx + \int_\Omega c^2dx \le 2C_1\|w\|^2_{L^2(\Omega)} \tag{4.14}$$

and

$$\frac{d}{dt}\int_\Omega w^2dx + 2\int_\Omega|\nabla w|^2dx + \int_\Omega w^2dx \le \int_\Omega n^2dx \le |\Omega| + \int_\Omega n^kdx. \tag{4.15}$$

We can get (4.12) and (4.13) by integrating (4.14) and (4.15) and using Lemma 4.9. □

Lemma 4.11. *If* (1.3) *holds, then for all* T *there exist* $C(T) > 0$ *such that*

$$\int_\Omega |u(\cdot,t)|^2dx \le C(T) \tag{4.16}$$

and

$$\int_0^T\int_\Omega |\nabla u(\cdot,t)|^2dxdt \le C(T). \tag{4.17}$$

Proof. Applying the estimate of (4.6), we have

$$\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega|\nabla u|^2dx \le C_{23}^2\|n\|^2_{L^2(\Omega)}. \tag{4.18}$$

Integrating both sides of (4.18) and applying the estimate of (4.8), we obtain (4.16) and (4.17). □

4.4. For the two-dimensional Navier-Stokes case

The proof of the remaining part is completely similar to the processing of Lemmas 3.6–3.13, so we omit the details.

4.5. For the three-dimensional Stokes case

Next, we can use semigroup estimation to obtain the following prior estimates for the three-dimensional case.

Lemma 4.12. *Suppose that* (1.3) *holds and let* $\gamma_0 \in (\frac{1}{2}, \gamma] \subset (\frac{1}{2}, 1)$. *Then for all* $T \in (0, T_{\max})$ *there exist* $C(T) > 0$ *and* $\theta > 0$ *such that*

$$\int_\Omega |A^{\gamma_0} u(\cdot, t)|^{\frac{22}{5}} dx \le C(T)$$

and

$$\|u(\cdot, t)\|_{C^\theta(\Omega)} \le C(T).$$

Proof. Let $\delta_0 = 0.1$, $\gamma_0 = 0.501$, $r_0 = 3$, $r_1 = 3.7$, $r_2 = 4.4$. We have $2\delta_0 > \frac{3}{2}(\frac{1}{r_0} - \frac{1}{r_1})$ and $\gamma_1 := \gamma_0 + \delta_0 + \frac{3}{2}(\frac{1}{r_1} - \frac{1}{r_2}) < \frac{2}{3}$. Therefore, we use standard semigroup estimates, Hölder's inequality and (4.8) to deduce that

$$\begin{aligned}
\|A^{\gamma_0} u(\cdot, t)\|_{L^{r_2}(\Omega)} &= \Big\| A^{\gamma_0}\Big(e^{-tA} u_0 + \int_0^t e^{-(t-s)A} \mathcal{P}(n(\cdot, s)\nabla\Phi) ds\Big)\Big\|_{L^{r_2}(\Omega)} \\
&\le \|e^{-tA} A^{\gamma_0} u_0\|_{L^{r_2}(\Omega)} + \int_0^t \|A^{\gamma_0+\delta_0} e^{-(t-s)A} A^{-\delta_0}(n(\cdot, s)\nabla\Phi)\|_{L^{r_2}(\Omega)} ds \\
&\le \|A^{\gamma_0} u_0\|_{L^{r_2}(\Omega)} + C_{26} \int_0^t (t-s)^{-\gamma_0-\delta_0-\frac{3}{2}\times(\frac{1}{r_1}-\frac{1}{r_2})} e^{-\lambda_1(t-s)} \|A^{-\delta_0} n(\cdot, s)\|_{L^{r_1}(\Omega)} ds \\
&\le C_{27} + C_{27} \int_0^t (t-s)^{-\gamma_1} \times e^{-\lambda_1(t-s)} \|n(\cdot, s)\|_{L^3(\Omega)} ds \\
&\le C_{27} + C_{27} \int_0^t \|n(\cdot, s)\|^3_{L^3(\Omega)} ds \cdot \int_0^t (t-s)^{-\frac{3}{2}\gamma_1} \times e^{-\lambda_1(t-s)} ds \\
&\le C_{28} \quad \text{for all } t \in (0, T).
\end{aligned}$$

Then, we apply the embedding $D(A^{\gamma_0}_{r_2}) \hookrightarrow C^\theta$, $0 < \theta < 2\gamma_0 - \frac{3}{r_2}$ to obtain the Lemma 4.12. □

Lemma 4.13. *Assume that* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there exist* $C(T) > 0$ *such that*

$$\int_\Omega |\nabla c(x, t)|^2 dx \le C(T)$$

and

$$\int_0^T \int_\Omega |\Delta c(x, t)|^2 dx dt \le C(T).$$

Proof. We multiply the Eq $(1.1)_2$ with $-\Delta c$ and use the integration by parts and Hölder's inequality to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx &= \int_\Omega (u \cdot \nabla c)\Delta c dx - \mu_3 \int_\Omega c^\alpha w^{1-\alpha} \Delta c dx \\
&\le \frac{1}{2}\int_\Omega |\Delta c|^2 dx + \|u\|^2_{L^\infty(\Omega)} \|\nabla c\|^2_{L^2(\Omega)} + \mu_3^2 \|c\|^{2\alpha}_{L^2(\Omega)} \|w\|^{2(1-\alpha)}_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\Delta c|^2 dx + \|u\|^2_{L^\infty(\Omega)} \|\nabla c\|^2_{L^2(\Omega)} + \frac{\mu_2^3}{2}(\|c\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}).
\end{aligned}$$

That is

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + 2\int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx \le 2\|u\|^2_{L^\infty(\Omega)}\|\nabla c\|^2_{L^2(\Omega)} + \mu_2^3(\|c\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}). \quad (4.19)$$

Integrating the two sides of the inequality (4.19) and applying the Lemmas 4.10 and 4.12, we completely the proof of the Lemma 4.13. □

Lemma 4.14. *If* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there exist* $C(T) > 0$ *such that*

$$\int_\Omega |n(\cdot, t)|^2 dx \le C(T). \quad (4.20)$$

Proof. We integrate the first equation of (1.1) and use the Höder's inequality and Young's inequality to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega n^2 dx + 2\int_\Omega |\nabla n|^2 dx &= -\chi\int_\Omega n^2\Delta c dx + \mu_1\int_\Omega n^2 dx - \mu_2\int_\Omega n^{k+1}dx \\ &\le \frac{\chi^2}{4}\int_\Omega |\Delta c|^2 dx + \int_\Omega n^4 dx + \frac{\mu_2}{2}\int_\Omega n^{k+1}dx + C_{29} - \mu_2\int_\Omega n^{k+1}dx \quad (4.21)\\ &\le \frac{\chi^2}{4}\int_\Omega |\Delta c|^2 dx + C_{30} \quad \text{for all } k > 3.\end{aligned}$$

For $k = 3$, using the same method, we can get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n^2 dx + 2\int_\Omega |\nabla n|^2 dx \\ &\le \frac{\mu_2}{2}\int_\Omega n^4 dx + \frac{\chi^2}{2\mu_2}\int_\Omega |\Delta c|^2 dx + \frac{\mu_2}{2}\int_\Omega n^4 dx + \frac{\mu_1^2}{2\mu_2}|\Omega| - \mu_2\int_\Omega n^4 dx \quad (4.22)\\ &\le \frac{\chi^2}{2\mu_2}\int_\Omega |\Delta c|^2 dx + C_{31}.\end{aligned}$$

By integrating the expressions of (4.21) or (4.22) and using the Lemma 4.13, the proof is complete. □

Lemma 4.15. *Assume that* (1.3) *holds. Then for all* $T > 0$ *there exist* $C(T) > 0$ *such that*

$$\int_\Omega |\nabla w(\cdot, s)|^2 dx \le C(T)$$

and

$$\int_0^t\int_\Omega |\Delta w(\cdot, s)|^2 dxdt \le C(T).$$

Proof. Multiplying the Eq $(1.1)_3$ with $-\Delta w$ and using Hölder's inequality, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \int_\Omega |\nabla w|^2 dx + \int_\Omega |\Delta w|^2 dx &= \int_\Omega (u\cdot\nabla w)\Delta w dx - \int_\Omega n\Delta w dx \\ &\le \frac{1}{2}\int_\Omega |\Delta w|^2 dx + \|u\|^2_{L^\infty(\Omega)}\|\nabla w\|^2_{L^2(\Omega)} + \|n\|^2_{L^2(\Omega)}.\end{aligned} \quad (4.23)$$

Integrating the two sides of (4.23) and applying the estimates (4.12) and (4.20), we complete the proof of the Lemma 4.15. □

Lemma 4.16. *If* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there exist* $C(T) > 0$ *such that*

$$\|c(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(T).$$

Proof. Applying the variation of constant formula of n, we have

$$\begin{aligned}\|c(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{t(\Delta-1)}c_0\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}\Big(\mu_3 c^\alpha(\cdot,s)w^{1-\alpha}(\cdot,s) + \nabla\cdot\big(u(\cdot,s)c(\cdot,s)\big)\Big)\right\|_{L^\infty(\Omega)}ds\\
&\le C_{32} + C_{32}\int_0^t (1+(t-s)^{-\frac{3}{4}})\|c^\alpha(\cdot,s)w^{1-\alpha}(\cdot,s)\|_{L^2(\Omega)}ds\\
&\quad + C_{32}\int_0^t (1+(t-s)^{-\frac{7}{8}})\|c(\cdot,s)u(\cdot,s)\|_{L^4(\Omega)}ds\\
&\le C_{32} + C_{32}\big(\|c\|^\alpha_{L^2(\Omega)}\|w\|^{1-\alpha}_{L^2(\Omega)} + \|c\|_{W^{1,2}(\Omega)}\big) \le C_{33}.\end{aligned}$$

Then, we use the similar method of Lemma 3.11 to deduce that

$$\begin{aligned}\|c(\cdot,t)\|_{W^{1,q}(\Omega)} &\le \|e^{t(\Delta-1)}c_0\|_{W^{1,q}(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}\Big(\mu_3 c^\alpha(\cdot,s)w^{1-\alpha}(\cdot,s) + u(\cdot,s)\cdot\nabla c(\cdot,s)\Big)\right\|_{W^{1,q}(\Omega)}ds\\
&\le C_{34} + C_{35}\int_0^t (1+(t-s)^{-\frac{3}{4}+\frac{3}{2q}})e^{-\lambda_1(t-s)}\|c^\alpha(\cdot,s)w^{1-\alpha}(\cdot,s)\|_{L^6(\Omega)}ds\\
&\quad + C_{35}\int_0^t \left\|(-\Delta+1)^{\kappa_4}e^{(t-s)(\Delta-1)}\nabla\cdot\Big(u(\cdot,s)c(\cdot,s)\Big)\right\|_{L^\infty(\Omega)}ds\\
&\le C_{34} + C_{35}\|c(\cdot,s)\|^\alpha_{L^6(\Omega)}\|w(\cdot,s)\|^{1-\alpha}_{L^6(\Omega)}\int_0^t (1+(t-s)^{-\frac{3}{4}+\frac{3}{2q}})e^{-\lambda_1(t-s)}ds\\
&\quad + C_{36}\int_0^t (t-s)^{-\kappa_4-\frac{1}{2}-\delta_5}e^{-\lambda_1(t-s)}\|u(\cdot,s)c(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le C_{34} + C_{36}\|c\|^\alpha_{W^{1,2}(\Omega)}\|w\|^{1-\alpha}_{W^{1,2}(\Omega)} + C_{36}\int_0^t (t-s)^{-\kappa_4-\frac{1}{2}-\delta_5}e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|c(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le C_{37} \quad \text{for all } q>1,\ \kappa_4 > \frac{1}{2}-\frac{3}{2q},\ 0<\kappa_4+\delta_5<\frac{1}{2}. \qquad \square\end{aligned}$$

Next, we give the estimates of n, and then apply them to obtain the estimate of w.

Lemma 4.17. *Suppose that* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there is* $C(T) > 0$ *such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C(T).$$

Proof. Let $M(T^\star) := \sup_{t\in(0,T^\star)} \|n(\cdot,t)\|_{L^\infty(\Omega)}$ for all $T^\star \in (0,T)$ and let $t_0 = (t-1)_+$. Applying the variation of constant formula of n, we can see that

$$\begin{aligned}\|n(\cdot,t)\|_{L^\infty(\Omega)} &\le \left\|e^{(t-t_0)\Delta}n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\Big(\nabla\cdot(\chi n(\cdot,s)\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s))ds + \mu_1 n\Big)ds\right\|_{L^\infty(\Omega)}\\
&\le C_{38} + \int_0^1 (1+s^{-\frac{7}{8}})\|\chi n(\cdot,s)\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\|_{L^4(\Omega)}ds + \mu_1\int_0^1 (1+s^{-\frac{3}{8}})\|n\|_{L^4(\Omega)}ds\end{aligned}$$

$$
\begin{aligned}
&\le C_{38} + \int_0^1 (1 + s^{-\frac{7}{8}})\Big(\chi \|n(\cdot, s)\|_{L^{20}(\Omega)} \|\nabla c(\cdot, s)\|_{L^5(\Omega)} + \|u(\cdot, s)\|_{L^\infty(\Omega)} \|n(\cdot, s)\|_{L^4(\Omega)}\Big) ds \\
&\quad + \mu_1 \int_0^t (1 + s^{-\frac{3}{8}}) \|n(\cdot, s)\|^{\frac{1}{2}}_{L^2(\Omega)} \|n(\cdot, s)\|^{\frac{1}{2}}_{L^\infty(\Omega)} ds \\
&\le C_{38} + C_{39}\Big(M^{\frac{9}{10}}(T^\star) + M^{\frac{1}{2}}(T^\star)\Big).
\end{aligned}
$$

Thus, using the Young's inequality, we obtain the result. □

Lemma 4.18. *Assume that* (1.3) *holds. Then for all* $T \in (0, T_{\max})$ *there exist* $C(T) > 0$ *such that*

$$
\|w(\cdot, t)\|_{W^{1,q}(\Omega)} \le C(T).
$$

Proof. Using the variation of constant formula of w and taking $\delta_6 > 0$ suitable small, we have

$$
\begin{aligned}
\|w(\cdot, t)\|_{W^{1,q}(\Omega)} &= \|e^{t(\Delta-1)} w_0\|_{W^{1,q}(\Omega)} + \int_0^t \Big\|e^{(t-s)(\Delta-1)}\Big(n + \nabla \cdot (u(\cdot, s) w(\cdot, s))\Big)\Big\|_{W^{1,q}(\Omega)} ds \\
&\le C_{40} + C_{40} \int_0^t (1 + (t - s)^{-\frac{1}{2}+\frac{3}{2q}}) e^{-\lambda_1 (t-s)} \|n(\cdot, s)\|_{L^\infty(\Omega)} ds \\
&\quad + C_{40} \int_0^t \|(-\Delta + 1)^{\kappa_5} \nabla \cdot (u(\cdot, s) w(\cdot, s))\|_{L^\infty(\Omega)} \\
&\quad + C_{40} \int_0^t (1 + (t - s)^{-1+\frac{3}{2q}-\delta_6}) e^{-\lambda_1 (t-s)} \|u(\cdot, s) w(\cdot, s)\|_{L^\infty(\Omega)} ds
\end{aligned}
$$

for all $q > 1$, $\kappa_5 > \frac{1}{2} - \frac{3}{2q}$, $\delta_6 < \frac{3}{2q}$.
Similar to Lemma 4.16, we get the proof of Lemma 4.18. □

Proof of Theorem 1.1 for the three-dimensional case. Finally, we arrive at the proof of Theorem 1.1, using the estimates we obtained in Lemmas 4.16–4.18 and then using the extendability criterion. □

Proof of Theorem 1.2. Based on the estimates collected in Lemmas 4.4–4.8, and the three-dimensional case is similar. We finish the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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