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Research article

Decay rate of the solutions to the Bresse-Cattaneo system with distributed delay

Abdelbaki Choucha[1], **Asma Alharbi**[2], **Bahri Cherif**[2], **Rashid Jan**[3] **and Salah Boulaaras**[2,*]

[1] Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Algeria
[2] Department of Mathematics, College of Sciences and Arts, ArRas, Qassim University, Saudi Arabia
[3] Department of Mathematics, University of Swabi, Swabi 23561, KPK Pakistan

* **Correspondence:** Email: s.boularas@qu.edu.sa.

Abstract: This study examines the pace at which solutions to a Bresse system in combination with the Cattaneo law of heat conduction and the dispersed delay term degradation. We establish our major finding utilizing the energy approach in the Fourier space.

Keywords: partial differential equations; mathematical operators; decay rate; Bresse system; Cattaneo's law; Fourier transform; distributed delay
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1. Introduction and preliminaries

In this paper, we are particularly interested in examining the pace at which the following problem solution degrades

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) + m\theta_x = 0, \\ \omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma_1\omega_t + \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\omega_t(x, t-s)\, ds = 0, \\ \theta_t + q_x + m\psi_{tx} = 0, \\ \tau_q q_t + \beta q + \theta_x = 0. \end{cases} \tag{1.1}$$

where

$$(x, s, t) \in \mathbb{R} \times (\tau_1, \tau_2) \times \mathbb{R}_+,$$

under the initial

$$\begin{aligned} &(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta, q)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_0, q_0), x \in \mathbb{R}, \\ &\omega_t(x, -t) = f_0(x, t), \quad (x, t) \in \mathbb{R} \times (0, \tau_2), \end{aligned} \tag{1.2}$$

where the parameters a, l, m, k_0, γ_1 and β are considered to be positive constants, the function θ stands for temperature gradient, the functions φ, ψ and ω stand for the vertical displacements of the girder, the tilt angle of the linear filament substance and the longitudinal displacements, respectively, the integral embodies the dispersed delay terms with $\tau_1, \tau_2 > 0$ being a time delay, and γ_2 is a L^∞ function.

There are several consequences from the coupling of the Cattaneo law of heat conduction with various systems, which has been discussed by several writers. For instance, see (Bresse-Cattaneo) in [6, 14], the Bresse concept and the Fourier law of heat conduction (Bresse-Fourier) have both been addressed in [13], Timoshenko system with historical data in [1, 7, 9] and Moore-Gibson-Thompson problem in [4]. The following papers are recommended to the reader for further information [2, 3, 5, 8, 16].

In the absence of distributed delay term. The researchers briefly looked into the decay rate of system (1.1) in [14], and they presented the results as follows:

- For $\alpha = 0$

$$\|\partial_x^k U(t)\|_2 \leq C\|U_0\|_1(1+t)^{-\frac{1}{12}-\frac{k}{6}} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} U_0\|_2. \tag{1.3}$$

- For $\alpha \neq 0$

$$\|\partial_x^k U(t)\|_2 \leq C\|U_0\|_1(1+t)^{-\frac{1}{12}-\frac{k}{6}} + C(1+t)^{-\frac{\ell}{10}}\|\partial_x^{k+\ell} U_0\|_2, \tag{1.4}$$

where

$$\alpha = \alpha(\tau_q) = (\tau_q - 1)(1 - a^2) - \tau_q m^2. \tag{1.5}$$

The Bresse-Cattaneo system (1.1) optimality decay rates were subsequently displayed by the authors in [6]. Alternatively, they enhanced the approximations (1.3) and (1.4) obtained by incorporating new, extremely sensitive Lyapunov functionals.

- For $\alpha = 0$

$$\|\partial_x^k U(t)\|_2 \leq C\|U_0\|_1(1+t)^{-\frac{1}{4}-\frac{k}{2}} + Ce^{-ct}\|\partial_x^{k+\ell} U_0\|_2. \tag{1.6}$$

- For $\alpha \neq 0$

$$\|\partial_x^k U(t)\|_2 \leq C\|U_0\|_1(1+t)^{-\frac{1}{4}-\frac{k}{2}} + C(1+t)^{-\ell}\|\partial_x^{k+\ell} U_0\|_2, \tag{1.7}$$

and they demonstrated that the estimations (1.6) and (1.7) depending on the parameter δ and under the following supposition

$$\delta = k_0^2 l^2 - l^2 - 1 \neq 0, \tag{1.8}$$

are accurate. After a thorough examination of the concept of dispersed postponement, the following questions seem intuitive: What sort of phrase has systemic suppressive activities? How should one determine the complexities that will enable them to "predict" devaluation? Is the concept of amortization always useful? Could the inclusion of the dispersed delay term have somehow made this type of issue more difficult to solve? This work is an attempt to comprehend the Bresse-Cattaneo framework and the dispersed delay term. The distributed delay term that is shown below, notably in Fourier space, does not apply to the Bresse-Cattaneo with friction attenuation solutions if they are relatively simple.

We aim to demonstrate the decay properties of the solution using the energy method in the Fourier space for the problems (1.1) and (1.2) relying on all recent publications, particularly [6, 14]. This is one of the first studies we are aware of that looks at the Bresse-Cattaneo system with the dispersed delay factor in the Fourier space.

The sections of this manuscript are as follows: Here, we apply our presumptions and preliminary findings to the major decay conclusion. We build the Lyapunov component and determine the estimation for the Fourier image in the subsequent portion by employing the energy approach in Fourier space.

First, as in [11], we introduce the new variable

$$\mathcal{Y}(x,\rho,s,t) = \omega_t(x, t - s\rho),$$

then we get

$$\begin{cases} s\mathcal{Y}_t(x,\rho,s,t) + \mathcal{Y}_\rho(x,\rho,s,t) = 0, \\ \mathcal{Y}(x,0,s,t) = \omega_t(x,t). \end{cases}$$

Therefore, our problem is expressed as follows

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) + m\theta_x = 0, \\ \omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma_1\omega_t + \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\mathcal{Y}(x,1,s,t)\,ds = 0, \\ \theta_t + q_x + m\psi_{tx} = 0, \\ \tau_q q_t + \beta q + \theta_x = 0, \\ s\mathcal{Y}_t(x,\rho,s,t) + \mathcal{Y}_\rho(x,\rho,s,t) = 0, \end{cases} \tag{1.9}$$

where

$$(x,\rho,s,t) \in \mathbb{R} \times (0,1) \times (\tau_1,\tau_2) \times \mathbb{R}_+,$$

using the initial data

$$\begin{cases} (\varphi,\varphi_t,\psi,\psi_t,\omega,\omega_t,\theta,q)(x,0) = (\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1,\theta_0,q_0), \\ \mathcal{Y}(x,\rho,s,0) = f_0(x,s\rho), \quad (x,\rho,s) \in \mathbb{R}\times(0,1)\times(0,\tau_2). \end{cases} \tag{1.10}$$

Regarding the significance of the delay, we only presumptively determine that
(**H1**) $\gamma_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a limited function considering

$$\int_{\tau_1}^{\tau_2} |\gamma_2(s)|ds < \gamma_1. \tag{1.11}$$

To support our primary finding, we require the Hausdorff-Young inequality in the following Lemma

Lemma 1.1. *[10] There is a constant $C > 0$ such that, for each $k, \varsigma \geq 0, c > 0$, guarantees that the estimation given below is true $\forall t \geq 0$:*

$$\int_{|\xi|\leq 1} |\xi|^k e^{-c|\xi|^\varsigma t} d\xi \leq C(1+t)^{-(k+n)/\varsigma}, \quad \xi \in \mathbb{R}^n. \tag{1.12}$$

Also, we recall Plancherel's theorem.

Theorem 1.1. *([15] Plancherel theorem)*
The integral of a function's squared modulus is equal to the integral of the squared modulus of its frequency spectrum. That is, if $f(x)$ is a function on the real line, and $\widehat{f}(\xi)$ is its frequency spectrum, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\,d\xi.$$

2. Energy method and decay estimates

In this section, we obtain a degradation estimation for the Fourier image of the remedy to problems (1.9) and (1.10). This approach enables us to provide the decay rate of the solution in the energy space by utilizing Plancherel's theorem together with some integral estimations, such as Lemma (1.1). By utilizing the energy approach in Fourier space, we create the proper Lyapunov functionals for this problem. Ultimately, we substantiate our major finding.

2.1. The energy method in the Fourier space

Allow us to now incorporate the control values to construct the Lyapunov functional in the Fourier space and for convenience

$$v = (\varphi_x - \psi - l\omega), \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t \quad \phi = k_0(\omega_x - l\varphi), \quad \eta = \omega_t. \tag{2.1}$$

The system (1.9) then adopts the following structure

$$\begin{cases} v_t - u_x + y + l\eta = 0 \\ u_t - v_x - k_0 l\phi = 0 \\ z_t - ay_x = 0 \\ y_t - az_x - v + m\theta_x = 0 \\ \phi_t - k_0\eta_x + k_0 lu = 0 \\ \eta_t - k_0\phi_x - lv + \gamma_1\eta + \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\mathcal{Y}(x,1,s,t)\,ds = 0 \\ \theta_t + q_x + my_x = 0 \\ \tau_q q_t + \beta q + \theta_x = 0 \\ s\mathcal{Y}_t + \mathcal{Y}_\rho = 0, \end{cases} \tag{2.2}$$

with the initial data

$$(v,u,z,y,\phi,\eta,\theta,q,\mathcal{Y})(x,0) = (v_0,u_0,z_0,y_0,\phi_0,\eta_0,\theta_0,q_0,f_0), x \in \mathbb{R}, \tag{2.3}$$

where

$$v_0 = (\varphi_{0,x} - \psi_0 - l\omega_0), u_0 = \varphi_1, z_0 = a\psi_{0,x}, y_0 = \psi_1, \phi_0 = k_0(\omega_{0,x} - l\varphi_0), \eta_0 = \omega_1.$$

Hence, for ($\tau_q \neq 0$) the problem (2.2) and (2.3) is written as

$$\begin{cases} U_t + \mathcal{A}U_x + \mathcal{L}U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{2.4}$$

with $U = (v,u,z,y,\phi,\eta,\vartheta,q,\mathcal{Y})^T$, $\;U_0 = (v_0,u_0,z_0,y_0,\phi_0,\eta_0,\vartheta_0,q_0,f_0)$ and

$$\mathcal{A}U = \begin{pmatrix} -u \\ -v \\ -ay \\ -az + m\theta \\ -k_0\eta \\ -k_0\phi \\ +q + my \\ +\frac{1}{\tau_q}\theta \\ 0 \end{pmatrix}, \mathcal{L}U = \begin{pmatrix} y + l\eta \\ -k_0 l\phi \\ 0 \\ v \\ k_0 lu \\ -lv + \gamma_1\eta + \int_{\tau_1}^{\tau_2} \gamma_2(s)\mathcal{Y}(x,1,s,t)\,ds \\ 0 \\ \frac{\beta}{\tau_q}q \\ \frac{1}{s}\mathcal{Y}_\rho \end{pmatrix}.$$

Now, we will state the well-posedness result of system (2.4).

Theorem 2.1. *Suppose that (1.11). Let $U_0 \in H^s(\mathbb{R})$, $s \in \mathbb{N}$ and $s \geq 2$, then problem (2.4) has a unique solution U such that*

$$U \in C^0([0;\infty); H^s(\mathbb{R})) \cap C^1([0;\infty); H^{s-1}(\mathbb{R})).$$

For a complete proof and more information, see [12].

When we execute the Fourier transform to (2.4), the underneath respective problem arises:

$$\begin{cases} \widehat{U}_t + i\xi\mathcal{A}\widehat{U} + \mathcal{L}\widehat{U} = 0, \\ \widehat{U}(\xi,0) = \widehat{U}_0(\xi), \end{cases} \tag{2.5}$$

where the solution $\widehat{U}(\xi,t) = (\widehat{v},\widehat{u},\widehat{z},\widehat{y},\widehat{\phi},\widehat{\eta},\widehat{\theta},\widehat{q},\widehat{\mathcal{Y}})^T(\xi,t)$ is given by

$$\widehat{U}(\xi,t) = e^{\Psi(\xi)t}\widehat{U}(\xi,0),$$

with

$$\Psi(\xi) := -(i\xi\mathcal{A} + \mathcal{L}).$$

Hence, in order to prove the asymptotic behavior of the solution, it suffices to get a function $\rho(\xi)$ such that

$$|e^{\Psi(\xi)t}| \leq Ce^{-c\rho(\xi)t},$$

where C and c positive constants. Thus, the behavior of the solution depends on a critical way on the behavior of the function $\rho(\xi)$.

To arrive at this result, we start with $(2.5)_1$ where it is rewritten as:

$$\begin{cases} \widehat{v}_t - i\xi\widehat{u} + \widehat{y} + l\widehat{\eta} = 0 \\ \widehat{u}_t - i\xi\widehat{v} - k_0 l\widehat{\phi} = 0 \\ \widehat{z}_t - ai\xi\widehat{y} = 0 \\ \widehat{y}_t - ai\xi\widehat{z} - \widehat{v} + mi\xi\widehat{\theta} = 0 \\ \widehat{\phi}_t - k_0 i\xi\widehat{\eta} + k_0 l\widehat{u} = 0 \\ \widehat{\eta}_t - k_0 i\xi\widehat{\phi} - l\widehat{v} + \gamma_1\widehat{\eta} + \displaystyle\int_{\tau_1}^{\tau_2} \gamma_2(s)\widehat{\mathcal{Y}}(\xi,1,s,t)\,ds = 0 \\ \widehat{\theta}_t + i\xi\widehat{q} + mi\xi\widehat{y} = 0 \\ \widehat{q}_t + \frac{\beta}{\tau_q}\widehat{q} + \frac{1}{\tau_q}i\xi\widehat{\theta} = 0 \\ s\widehat{\mathcal{Y}}_t + \widehat{\mathcal{Y}}_\rho = 0. \end{cases} \tag{2.6}$$

Lemma 2.1. *Assume that (1.11) is accurate. Let $\widehat{U}(\xi,t)$ be the solution of (2.5). Then the energy functional $\widehat{E}(\xi,t)$ is thus given by*

$$\begin{aligned}\widehat{E}(\xi,t) &= \frac{1}{2}\Big\{|\widehat{v}|^2+|\widehat{u}|^2+|\widehat{z}|^2+|\widehat{y}|^2+|\widehat{\phi}|^2+|\widehat{\eta}|^2+|\widehat{\theta}|^2+\tau_q|\widehat{q}|^2\Big\}\\ &+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho,\end{aligned} \tag{2.7}$$

satisfies

$$\frac{d\widehat{E}(\xi,t)}{dt} \le -C_1|\widehat{\eta}|^2-\beta|\widehat{q}|^2\le 0, \tag{2.8}$$

where $C_1=\Big(\gamma_1-\int_{\tau_1}^{\tau_2}|\gamma_2(s)|ds\Big)>0.$

Proof. Firstly, multiplying $(2.6)_{1,2,3,4,5,6}$ by $\overline{\widehat{v}},\overline{\widehat{u}},\overline{\widehat{z}},\overline{\widehat{y}},\overline{\widehat{\phi}}$ and $\overline{\widehat{\eta}}$ respectively, and multiplying $(2.6)_{7,8}$ by $\widehat{\theta},\tau_q\overline{\widehat{q}}$, adding these equalities and taking the real part, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[|\widehat{v}|^2+|\widehat{u}|^2+|\widehat{z}|^2+|\widehat{y}|^2+|\widehat{\phi}|^2+|\widehat{\eta}|^2+|\widehat{\theta}|^2+\tau_q|\widehat{q}|^2\Big]dx\\ &+\beta|\widehat{q}|^2+\gamma_1|\widehat{\eta}|^2+\Re e\Big\{\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\eta}}\widehat{\mathcal{Y}}(\xi,1,s,t)\,ds\Big\}=0.\end{aligned} \tag{2.9}$$

Secondly, multiplying the Eq $(2.6)_9$ by $\overline{\widehat{\mathcal{Y}}}|\gamma_2(s)|$, and integrating the findings with $(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho\\ =\;& -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\frac{d}{d\rho}|\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho\\ =\;& \frac{1}{2}\int_{\tau_1}^{\tau_2}|\gamma_2(s)|\Big(|\widehat{\mathcal{Y}}(\xi,0,s,t)|^2-|\widehat{\mathcal{Y}}(\xi,1,s,t)|^2\Big)ds\\ =\;& \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\gamma_2(s)|ds\Big)|\widehat{\eta}|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds,\end{aligned} \tag{2.10}$$

as well as utilizing Young's inequality, we have

$$\begin{aligned}&\Re e\Big\{\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\eta}}\widehat{\mathcal{Y}}(\xi,1,s,t)\,ds\Big\}\\ \le\;& \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\gamma_2(s)|ds\Big)|\widehat{\eta}|^2+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds,\end{aligned} \tag{2.11}$$

by substituting (2.10) and (2.11) into (2.9), we find

$$\frac{d\widehat{E}(\xi,t)}{dt} \le -\Big(\gamma_1-\int_{\tau_1}^{\tau_2}|\gamma_2(s)|ds\Big)|\widehat{\eta}|^2-\beta|\widehat{q}|^2,$$

then, by (1.11), $\exists C_1 = (\gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)|ds) > 0$ so that

$$\frac{d\widehat{E}(\xi,t)}{dt} \leq -C_1|\widehat{\eta}|^2 - \beta|\widehat{q}|^2 \leq 0. \tag{2.12}$$

Hence, we obtain (2.7) ($\widehat{E}$ is a non-increasing function). □

We now require the following lemmas in order to accomplish our objectives.

Lemma 2.2. *The functional*

$$\mathcal{D}_1(\xi,t) := \Re e\Big\{i\xi\tau_q\Big(\widehat{\theta}\overline{\widehat{q}}\Big)\Big\}, \tag{2.13}$$

satisfies, for any $\varepsilon_1 > 0$

$$\frac{d\mathcal{D}_1(\xi,t)}{dt} \leq -\frac{1}{2}\xi^2|\widehat{\theta}|^2 + \varepsilon_1\xi^2|\widehat{y}|^2 + c(\varepsilon_1)(1+\xi^2)|\widehat{q}|^2. \tag{2.14}$$

Proof. Differentiating $\mathcal{D}_1$ and by using (2.6), we get

$$\begin{aligned}\frac{d\mathcal{D}_1(\xi,t)}{dt} &= \Re e\Big\{i\xi\tau_q\widehat{\theta}_t\overline{\widehat{q}} - i\xi\tau_q\widehat{q}_t\overline{\widehat{\theta}}\Big\}\\ &= -\xi^2|\widehat{\theta}|^2 + \tau_q\xi^2|\widehat{q}|^2 + \Re e\Big\{m\tau_q\xi^2\widehat{y}\overline{\widehat{q}}\Big\} + \Re e\Big\{i\beta\xi\widehat{q}\overline{\widehat{\theta}}\Big\}.\end{aligned} \tag{2.15}$$

With the help of Young's inequality, we estimate the terms in the RHS of (2.15) and obtain for every $\varepsilon_1, \delta_1 > 0$

$$\begin{aligned}+\Re e\Big\{i\beta\xi\widehat{q}\overline{\widehat{\theta}}\Big\} &\leq \delta_1\xi^2|\widehat{\theta}|^2 + c(\delta_1)|\widehat{q}|^2,\\ +\Re e\Big\{m\tau_q\xi^2\widehat{y}\overline{\widehat{q}}\Big\} &\leq \varepsilon_1\xi^2|\widehat{y}|^2 + c(\varepsilon_1)\xi^2|\widehat{q}|^2.\end{aligned} \tag{2.16}$$

By adding the aforementioned estimations (2.16) to (2.15) and setting $\delta_1 = \frac{1}{2}$, we find (2.14). □

Lemma 2.3. *The functional*

$$\mathcal{D}_2(\xi,t) := ld_1\delta^2\mathcal{F}_1(\xi,t) + d_1m\delta^2\mathcal{F}_2(\xi,t) + m^2\delta\mathcal{F}_3(\xi,t) + m^2\delta lk_0\mathcal{F}_4(\xi,t), \tag{2.17}$$

where

$$\begin{aligned}\mathcal{F}_1(\xi,t) &:= \Re e\Big\{-m^2\xi^2\widehat{y}\overline{\widehat{v}} - m^2a\xi^2\widehat{u}\overline{\widehat{z}} + (1-a^2)m\xi^2\widehat{\theta}\overline{\widehat{u}} + \frac{d_2\tau_q}{l}\xi^2\widehat{v}\overline{\widehat{q}}\Big\},\\ \mathcal{F}_2(\xi,t) &:= \Re e\Big\{il\xi\widehat{y}\overline{\widehat{\theta}} - i\xi\widehat{\eta}\overline{\widehat{\theta}}\Big\},\\ \mathcal{F}_3(\xi,t) &:= \Re e\Big\{-l(\delta-\xi^2)\widehat{\eta}\overline{\widehat{v}} + lk_0\xi^2\widehat{\phi}\overline{\widehat{u}} + il^2k_0\xi\widehat{\eta}\overline{\widehat{\phi}}\Big\},\\ \mathcal{F}_4(\xi,t) &:= \Re e\Big\{-ilk_0\xi\widehat{v}\overline{\widehat{u}} + i\xi\widehat{y}\overline{\widehat{\phi}}\Big\}\end{aligned} \tag{2.18}$$

and

$$d_1 = ml(a^2+m^2-1), \quad d_2 = l(1-\frac{1}{\delta l^2}), \tag{2.19}$$

satisfies, for any $\varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$

$$
\begin{aligned}
\frac{d\mathcal{D}_2(\xi,t)}{dt} \leq\ & -\frac{m^2\delta^2}{2}\xi^2|\widehat{v}|^2 - \frac{m^2\delta^2 l^2}{2}(1+\xi^2)|\widehat{v}|^2 + 2\varepsilon_2|\widehat{y}|^2 \\
& +(3+\gamma_1)\varepsilon_3\xi^2|\widehat{\phi}|^2 + 3\varepsilon_4\xi^2|\widehat{u}|^2 + c(\varepsilon_3)\xi^2|\widehat{\theta}|^2 + c(\varepsilon_3)\xi^2|\widehat{z}|^2 \\
& +c(\varepsilon_2,\varepsilon_4)\xi^2(1+\xi^2)|\widehat{q}|^2 + c(\varepsilon_2,\varepsilon_3,\varepsilon_4)(1+\xi^2+\xi^4)|\widehat{\eta}|^2 \\
& +c(\varepsilon_3)(1+\xi^2)\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds.
\end{aligned} \tag{2.20}
$$

Proof. Firstly, differentiating $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ and by using (2.6), we get

$$
\begin{aligned}
\frac{d\mathcal{F}_1(\xi,t)}{dt} =\ & -m^2\xi^2|\widehat{v}|^2 + m^2\xi^2|\widehat{y}|^2 + \Re e\Big\{lm^2\xi^2\widehat{\eta}\overline{\widehat{y}}\Big\} - \Re e\Big\{am^2lk_0\xi^2\widehat{\phi}\overline{\widehat{z}}\Big\} \\
& +\Re e\Big\{im\alpha\xi^3\widehat{q}\overline{\widehat{u}}\Big\} + \Re e\Big\{(1-a^2)mlk_0\xi^2\widehat{\phi}\overline{\widehat{\theta}}\Big\} \\
& -\Re e\Big\{d_2\tau_q\xi^2\widehat{\eta}\overline{\widehat{q}}\Big\} - \frac{d_2\tau_q}{l}\Re e\Big\{\xi^2\widehat{y}\overline{\widehat{q}}\Big\} - \frac{d_2\beta}{l}\Re e\Big\{\xi^2\widehat{q}\overline{\widehat{v}}\Big\},
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
\frac{d\mathcal{F}_2(\xi,t)}{dt} =\ & -lm\xi^2|\widehat{y}|^2 + ml\xi^2|\widehat{\theta}|^2 - \Re e\Big\{al\xi^2\widehat{z}\overline{\widehat{\theta}}\Big\} - \Re e\Big\{l\xi^2\widehat{q}\overline{\widehat{y}}\Big\} \\
& +\Re e\Big\{k_0\xi^2\widehat{\phi}\overline{\widehat{\theta}}\Big\} + \Re e\Big\{i\gamma_1\xi\widehat{\eta}\overline{\widehat{\theta}}\Big\} + \Re e\Big\{m\xi^2\widehat{y}\overline{\widehat{\eta}}\Big\} \\
& +\Re e\Big\{\xi^2\widehat{q}\overline{\widehat{\eta}}\Big\} + \Re e\Big\{i\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\theta}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\},
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
\frac{d\mathcal{F}_3(\xi,t)}{dt} =\ & -k_0^2l^2\xi^2|\widehat{u}|^2 - l^2(\delta-\xi^2)|\widehat{v}|^2 + (l^2(\delta-\xi^2) + l^2k_0^2\xi^2)|\widehat{\eta}|^2 \\
& -\Re e\Big\{ilk_0\xi(l^2k_0^2-1)\widehat{\phi}\overline{\widehat{v}}\Big\} + \Re e\Big\{l\gamma_1(\delta-\xi^2)\widehat{\eta}\overline{\widehat{v}}\Big\} \\
& +\Re e\Big\{il\xi\Big(\xi^2(k_0^2-1)-(l^2+1)\Big)\widehat{\eta}\overline{\widehat{u}}\Big\} + \Re e\Big\{l(\delta-\xi^2)\widehat{y}\overline{\widehat{\eta}}\Big\} \\
& -\Re e\Big\{il^2k_0\gamma_1\xi\widehat{\eta}\overline{\widehat{\phi}}\Big\} + \Re e\Big\{l(\delta-\xi^2)\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{v}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\
& -\Re e\Big\{il^2k_0\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\phi}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\},
\end{aligned} \tag{2.23}
$$

and

$$
\begin{aligned}
\frac{d\mathcal{F}_4(\xi,t)}{dt} =\ & -k_0l\xi^2|\widehat{v}|^2 + k_0l\xi^2|\widehat{u}|^2 + \Re e\Big\{k_0\xi^2\widehat{\eta}\overline{\widehat{y}}\Big\} - \Re e\Big\{a\xi^2\widehat{z}\overline{\widehat{\phi}}\Big\} \\
& +\Re e\Big\{il^2k_0\xi\widehat{\eta}\overline{\widehat{u}}\Big\} + \Re e\Big\{m\xi^2\widehat{\theta}\overline{\widehat{\phi}}\Big\} + \Re e\Big\{i(k_0^2l^2-1)\xi\widehat{\phi}\overline{\widehat{v}}\Big\}.
\end{aligned} \tag{2.24}
$$

Now, differentiating $\mathcal{D}_2$ and by exploiting (2.21)–(2.24), gives

$$
\begin{aligned}
\frac{d\mathcal{D}_2(\xi,t)}{dt} &= -m^2\delta^2\xi^2|\widehat{v}|^2 - m^2l^2\delta^2(1+\xi^2)|\widehat{v}|^2 + d_1m^2\delta^2l\xi^2|\widehat{\theta}|^2 \\
&+m^2\delta[l^2(\delta-\xi^2)+l^2k_0\xi^2]|\widehat{\eta}|^2 + \Re e\Big\{\delta d_3\xi^2\widehat{\eta}\bar{\widehat{y}}\Big\} + \Re e\Big\{lm^2\delta^2\widehat{\eta}\bar{\widehat{y}}\Big\} \\
&+\Re e\Big\{im\alpha ld_1\delta^2\xi^3\widehat{q}\bar{\widehat{u}}\Big\} - \Re e\Big\{d_4\xi^2\widehat{\phi}\bar{\widehat{z}}\Big\} + \Re e\Big\{d_5\xi^2\widehat{\phi}\bar{\widehat{\theta}}\Big\} \\
&-\Re e\Big\{d_1\delta^2d_2\beta\xi^2\widehat{q}\bar{\widehat{v}}\Big\} + \Re e\Big\{id_1m\gamma_1\delta^2\xi\widehat{\eta}\bar{\widehat{\theta}}\Big\} + \Re e\Big\{d_6\xi^2\widehat{\eta}\bar{\widehat{q}}\Big\} \\
&-\Re e\Big\{d_1ma\delta^2l\xi^2\widehat{z}\bar{\widehat{\theta}}\Big\} + \Re e\Big\{m^2\delta l\gamma_1(\delta-\xi^2)\widehat{\eta}\bar{\widehat{v}}\Big\} + \Re e\Big\{d_7\xi^2\widehat{y}\bar{\widehat{q}}\Big\} \\
&-\Re e\Big\{im^2\delta l^2k_0\gamma_1\xi\widehat{\eta}\bar{\widehat{\phi}}\Big\} + \Re e\Big\{il(k_0^2-1)m^2\delta\xi^3\widehat{\eta}\bar{\widehat{u}}\Big\} \\
&+\Re e\Big\{ilm^2\delta^2\xi\widehat{\eta}\bar{\widehat{u}}\Big\} + \Re e\Big\{id_1m\delta^2\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\bar{\widehat{\theta}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\
&+\Re e\Big\{m^2l\delta(\delta-\xi^2)\int_{\tau_1}^{\tau_2}\gamma_2(s)\bar{\widehat{v}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\
&-\Re e\Big\{im^2\delta l^2k_0\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\bar{\widehat{\phi}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\}, \qquad (2.25)
\end{aligned}
$$

where

$$
\alpha := (a^2-1)(1-\tau_q) - m^2\tau_q, \qquad (2.26)
$$

and

$$
\begin{aligned}
d_3 &= m^2\delta d_1(l^2+1) + m^2l(k_0^2-1), \\
d_4 &= l^2d_1\delta^2am^2k_0 + m^2\delta alk_0, \\
d_5 &= l^2k_0md_1\delta^2(1-a^2) + d_1m\delta^2k_0 + m^3\delta lk_0, \\
d_6 &= d_1m\delta^2 - ld_1\delta^2d_2\tau_q, \\
d_7 &= d_1\delta^2d_2\tau_q + d_1m\delta^2l.
\end{aligned}
$$

By applying the Young's inequality to the terms on the RHS of (2.25), we obtain for any $\varepsilon_2,\varepsilon_3,\varepsilon_4,\delta_2,\delta_3,\delta_4 > 0$

$$
\begin{aligned}
\frac{d\mathcal{D}_2(\xi,t)}{dt} &\le -(m^2\delta^2-\delta_2)\xi^2|\widehat{v}|^2 - (m^2l^2\delta^2-\delta_3-\gamma_1\delta_4)(1+\xi^2)|\widehat{v}|^2 + 2\varepsilon_2|\widehat{y}|^2 \\
&+(3+\gamma_1)\varepsilon_3\xi^2|\widehat{\phi}|^2 + 3\varepsilon_4\xi^2|\widehat{u}|^2 + c(\varepsilon_3)\xi^2|\widehat{\theta}|^2 \\
&+c(\varepsilon_2,\varepsilon_4,\delta_2)\xi^2(1+\xi^2)|\widehat{q}|^2 + c(\varepsilon_2,\varepsilon_3,\varepsilon_4,\delta_3)(1+\xi^2+\xi^4)|\widehat{\eta}|^2 \\
&+c(\varepsilon_3)\xi^2|\widehat{z}|^2 + c(\varepsilon_3,\delta_4)(1+\xi^2)\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds. \qquad (2.27)
\end{aligned}
$$

By letting $\delta_2 = \frac{m^2\delta^2}{2}, \delta_3 = \frac{m^2\delta^2l^2}{4}, \delta_4 = \frac{m^2\delta^2l^2}{4\gamma_1}$, we obtain (2.20). □

Lemma 2.4. *Assume that (1.8) holds. The functional*

$$
\mathcal{D}_3(\xi,t) := \tau_q lk_0\mathcal{F}_1(\xi,t) + \tau_q mk_0\mathcal{F}_2(\xi,t) - m^2\tau_q\mathcal{F}_4(\xi,t) + \mathcal{F}_5(\xi,t) + \mathcal{F}_6(\xi,t), \qquad (2.28)
$$

where

$$
\begin{aligned}
\mathcal{F}_5(\xi,t) &:= \Re e\Big\{-i\beta k_0 d_1\tau_q\xi\widehat{u}\overline{\widehat{q}}+i\tau_q m^2 l\xi\widehat{\eta}\overline{\widehat{\phi}}\Big\},\\
\mathcal{F}_6(\xi,t) &:= \Re e\Big\{ak_0 l\widehat{y}\overline{\widehat{v}}+ilk_0\xi\widehat{z}\overline{\widehat{y}}-ik_0\xi\widehat{z}\overline{\widehat{\eta}}-ia\xi\widehat{y}\overline{\widehat{\phi}}\Big\},
\end{aligned}
\tag{2.29}
$$

satisfies

(1) For $\alpha=0$. *Then,*

$$
\begin{aligned}
\frac{d\mathcal{D}_3(\xi,t)}{dt} \le\; & -\frac{ak_0 l}{2}(1+\xi^2)|\widehat{y}|^2+c|\widehat{v}|^2-\frac{\tau_q m^2 lk_0}{2}\xi^2|\widehat{\phi}|^2+c(1+\xi^2)|\widehat{\eta}|^2\\
&-\frac{\tau_q m^2 lk_0}{2}\xi^2|\widehat{u}|^2+c\xi^2|\widehat{z}|^2+c(1+\xi^2)|\widehat{q}|^2+c\xi^2|\widehat{\theta}|^2\\
&+c\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds.
\end{aligned}
\tag{2.30}
$$

(2) For $\alpha\neq 0$. *Then,*

$$
\begin{aligned}
\frac{d\mathcal{D}_3(\xi,t)}{dt} \le\; & -\frac{ak_0 l}{2}(1+\xi^2)|\widehat{y}|^2+c|\widehat{v}|^2-\frac{\tau_q m^2 lk_0}{2}\xi^2|\widehat{\phi}|^2+c(1+\xi^2)|\widehat{\eta}|^2\\
&-\frac{\tau_q m^2 lk_0}{2}\xi^2|\widehat{u}|^2+c\xi^2|\widehat{z}|^2+c(1+\xi^2+\xi^4)|\widehat{q}|^2+c\xi^2|\widehat{\theta}|^2\\
&+c\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds.
\end{aligned}
\tag{2.31}
$$

Proof. Firstly, a direct differentiation of $\mathcal{F}_5,\mathcal{F}_6$ and by using (2.6), we get

$$
\begin{aligned}
\frac{d\mathcal{F}_5(\xi,t)}{dt} =\; & -\tau_q m^2 lk_0\xi^2|\widehat{\phi}|^2+\tau_q m^2 lk_0\xi^2|\widehat{\eta}|^2+\Re e\Big\{\tau_q\beta k_0 ld_1\xi^2\widehat{v}\overline{\widehat{q}}\Big\}\\
&-\Re e\Big\{i\beta^2 d_1k_0\xi\widehat{q}\overline{\widehat{u}}\Big\}+\Re e\Big\{\beta k_0 d_1\xi^2\widehat{\theta}\overline{\widehat{u}}\Big\}-\Re e\Big\{\tau_q\beta lk_0^2 d_1\xi\widehat{\phi}\overline{\widehat{q}}\Big\}\\
&+\Re e\Big\{i\tau_q m^2 l^2\xi\widehat{v}\overline{\widehat{\phi}}\Big\}-\Re e\Big\{i\tau_q m^2 l\gamma_1\xi\widehat{\eta}\overline{\widehat{\phi}}\Big\}+\Re e\Big\{i\tau_q m^2 l^2k_0\xi\widehat{u}\overline{\widehat{\eta}}\Big\}\\
&+\Re e\Big\{i\tau_q m^2 l\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\phi}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\},
\end{aligned}
\tag{2.32}
$$

and

$$
\begin{aligned}
\frac{d\mathcal{F}_6(\xi,t)}{dt} =\; & -ak_0 l(1+\xi^2)|\widehat{y}|^2+ak_0 l|\widehat{v}|^2+ak_0 l\xi^2|\widehat{z}|^2+\Re e\Big\{ia^2k_0 l\xi\widehat{z}\overline{\widehat{v}}\Big\}\\
&+\Re e\Big\{(a^2-k_0^2)\xi^2\widehat{\phi}\overline{\widehat{z}}\Big\}-\Re e\Big\{ik_0\gamma_1\xi\widehat{\eta}\overline{\widehat{z}}\Big\}-\Re e\Big\{ialk_0 m\xi\widehat{\theta}\overline{\widehat{v}}\Big\}\\
&-\Re e\Big\{k_0 lm\xi^2\widehat{\theta}\overline{\widehat{z}}\Big\}-\Re e\Big\{ia\xi\widehat{v}\overline{\widehat{\phi}}\Big\}-\Re e\Big\{am\xi^2\widehat{\theta}\overline{\widehat{\phi}}\Big\}\\
&-\Re e\Big\{ak_0 l^2\widehat{\eta}\overline{\widehat{y}}\Big\}-\Re e\Big\{ik_0\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{z}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\}.
\end{aligned}
\tag{2.33}
$$

Now, by differentiating $\mathcal{D}_3$ and exploiting (2.32), (2.33), (2.21), (2.22) and (2.24), we find

$$\begin{aligned}\frac{d\mathcal{D}_3(\xi,t)}{dt} &= -ak_0l(1+\xi^2)|\widehat{y}|^2 - \tau_q m^2 lk_0\xi^2|\widehat{\phi}|^2 + \tau_q m^2 lk_0\xi^2|\widehat{\eta}|^2 + ak_0l|\widehat{v}|^2 \\ &\quad + ak_0l\xi^2|\widehat{z}|^2 - \tau_q m^2 lk_0\xi^2|\widehat{u}|^2 + \tau_q m^2 lk_0\xi^2|\widehat{\theta}|^2 + \Re e\Big\{ia^2k_0l\xi\widehat{z}\overline{\widehat{v}}\Big\} \\ &\quad + \Re e\Big\{d_8\xi^2\widehat{\phi}\overline{\widehat{z}}\Big\} - \Re e\Big\{ik_0\gamma_1\xi\widehat{\eta}\overline{\widehat{z}}\Big\} - \Re e\Big\{ialk_0m\xi\widehat{\theta}\overline{\widehat{v}}\Big\} \\ &\quad - \Re e\Big\{k_0lm(1+a\tau_q)\xi^2\widehat{\theta}\overline{\widehat{z}}\Big\} - \Re e\Big\{id_9\xi\widehat{v}\overline{\widehat{\phi}}\Big\} - \Re e\Big\{d_{10}\xi^2\widehat{\theta}\overline{\widehat{\phi}}\Big\} \\ &\quad - \Re e\Big\{ak_0l^2\widehat{\eta}\overline{\widehat{y}}\Big\} + \Re e\Big\{l^2m^2k_0\tau_q\xi^2\widehat{\eta}\overline{\widehat{y}}\Big\} - \Re e\Big\{i\tau_q m^2 l\gamma_1\xi\widehat{\eta}\overline{\widehat{\phi}}\Big\} \\ &\quad - \Re e\Big\{d_{11}\xi^2\widehat{q}\overline{\widehat{y}}\Big\} - \Re e\Big\{ik_0\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{z}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\ &\quad - \Re e\Big\{i\beta^2 d_1k_0\xi\widehat{q}\overline{\widehat{u}}\Big\} + \Re e\Big\{\beta k_0d_1\xi^2\widehat{\theta}\overline{\widehat{u}}\Big\} - \Re e\Big\{\tau_q\beta lk_0^2d_1\xi\widehat{\phi}\overline{\widehat{q}}\Big\} \\ &\quad + \Re e\Big\{2i\tau_q m^2l^2k_0\xi\widehat{u}\overline{\widehat{\eta}}\Big\} - \Re e\Big\{i\tau_q m^2 l\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\phi}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\ &\quad + \Re e\Big\{im\gamma_1\tau_q k_0\xi\widehat{\eta}\overline{\widehat{\theta}}\Big\} + \Re e\Big\{i\tau_q mk_0\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\theta}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\} \\ &\quad + \Re e\Big\{\tau_q l(m-d_2\tau_qk_0)\xi^2\widehat{q}\overline{\widehat{\eta}}\Big\} + \Re e\Big\{im\alpha\tau_q lk_0\xi^3\widehat{q}\overline{\widehat{u}}\Big\}, \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} d_8 &= a^2 - k_0^2 - \tau_q am^2l^2k_0^2 + m^2\tau_q, \\ d_9 &= a - \tau_q m^2(l^2k_0^2 - 1) - m^2l^2\tau_q, \\ d_{10} &= am - \tau_q mk_0^2 + \tau_q(a^2-1)ml^2k_0^2 + m^3\tau_q, \\ d_{11} &= \tau_q k_0(lm + d_2\tau_q). \end{aligned}$$

At this point, we distinguish two cases according to the values of α:

Case 1. ($\alpha = 0$).

In this case, the last term in the RHS of (2.34) is zero. Then, by applying the Young's inequality we obtain for any $\delta_i, i = 5, ..., 9 > 0$

$$\begin{aligned}\frac{d\mathcal{D}_3(\xi,t)}{dt} &\leq -(ak_0l - \delta_7)|\widehat{y}|^2 - (ak_0l - 2\delta_8)\xi^2)|\widehat{y}|^2 + c(\delta_5)|\widehat{v}|^2 \\ &\quad -(\tau_q m^2lk_0 - 5\delta_5 - \delta_6\gamma_1)\xi^2|\widehat{\phi}|^2 + c(\delta_5,\delta_7,\delta_8,\delta_9)(1+\xi^2)|\widehat{\eta}|^2 \\ &\quad -(\tau_q m^2lk_0 - 3\delta_9)\xi^2|\widehat{u}|^2 + c(\delta_5)\xi^2|\widehat{z}|^2 + c(\delta_5,\delta_8,\delta_9)(1+\xi^2)|\widehat{q}|^2 \\ &\quad +c(\delta_5,\delta_9)\xi^2|\widehat{\theta}|^2 + c(\delta_6)\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds. \end{aligned} \tag{2.35}$$

By letting $\delta_5 = \frac{\tau_q m^2 lk_0}{20}, \delta_6 = \frac{\tau_q m^2 lk_0}{4\gamma_1}, \delta_7 = \frac{alk_0}{2}, \delta_8 = \frac{alk_0}{4}$ and $\delta_9 = \frac{\tau_q m^2 lk_0}{6}$, we find (2.30).

Case 2. ($\alpha \neq 0$).

The last term of the RHS in this situation is estimated as follows in (2.34)

$$+\Re e\Big\{im\alpha\tau_q lk_0\xi^3\widehat{q}\overline{\widehat{u}}\Big\} \;\leq\; \delta_9\xi^2|\widehat{u}|^2 + c(\delta_9)\xi^4|\widehat{q}|^2. \tag{2.36}$$

Similarly to (2.35) and by Young's inequality, we get

$$\begin{aligned}\frac{d\mathcal{D}_3(\xi,t)}{dt} \;\leq\; & -(ak_0l-\delta_7)|\widehat{y}|^2 - (ak_0l - 2\delta_8)\xi^2)|\widehat{y}|^2 + c(\delta_5)|\widehat{v}|^2 \\ & -(\tau_q m^2lk_0 - 5\delta_5 - \delta_6\gamma_1)\xi^2|\widehat{\phi}|^2 + c(\delta_5,\delta_7,\delta_8,\delta_9)(1+\xi^2)|\widehat{\eta}|^2 \\ & -(\tau_q m^2lk_0 - 4\delta_9)\xi^2|\widehat{u}|^2 + c(\delta_5)\xi^2|\widehat{z}|^2 + c(\delta_5,\delta_8,\delta_9)(1+\xi^2+\xi^4)|\widehat{q}|^2 \\ & +c(\delta_5,\delta_9)\xi^2|\widehat{\theta}|^2 + c(\delta_6)\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds. \end{aligned} \tag{2.37}$$

By letting $\delta_5 = \frac{\tau_q m^2 lk_0}{20}, \delta_6 = \frac{\tau_q m^2 lk_0}{4\gamma_1}, \delta_7 = \frac{alk_0}{2}, \delta_8 = \frac{alk_0}{4}$ and $\delta_9 = \frac{\tau_q m^2 lk_0}{8}$, we find (2.31). Lemma 2.4 has been successfully proved. □

Next, we have the following lemma.

Lemma 2.5. *The functional*

$$\mathcal{D}_4(\xi,t) := l\mathcal{F}_1(\xi,t) + d_{12}\mathcal{F}_2(\xi,t) + l\mathcal{F}_7(\xi,t), \tag{2.38}$$

where

$$\mathcal{F}_7(\xi,t) := alm^2\Re e\Big\{i\xi\Big(-l\widehat{z}\overline{\widehat{y}} + \widehat{z}\overline{\widehat{\eta}}\Big)\Big\} \quad \textit{and} \quad d_{12} = m(1+a^2l^2), \tag{2.39}$$

satisfies, for any $\varepsilon_5, \varepsilon_6, \varepsilon_7 > 0$.

(1) For $\alpha = 0$. *Then,*

$$\begin{aligned}\frac{d\mathcal{D}_4(\xi,t)}{dt} \;\leq\; & -\frac{a^2l^3m^2}{2}\xi^2|\widehat{z}|^2 - \frac{lm^2}{2}\xi^2|\widehat{v}|^2 + \varepsilon_5\xi^2|\widehat{\phi}|^2 + 2\varepsilon_6\xi^2|\widehat{y}|^2 \\ & +c(\varepsilon_5)\xi^2|\widehat{\theta}|^2 + c(\varepsilon_6)(1+\xi^2)|\widehat{\eta}|^2 + c(\varepsilon_6)\xi^2|\widehat{q}|^2 \\ & +c\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds. \end{aligned} \tag{2.40}$$

(2) For $\alpha \neq 0$. *Then,*

$$\begin{aligned}\frac{d\mathcal{D}_4(\xi,t)}{dt} \;\leq\; & -\frac{a^2l^3m^2}{2}\xi^2|\widehat{z}|^2 - \frac{lm^2}{2}\xi^2|\widehat{v}|^2 + \varepsilon_5\xi^2|\widehat{\phi}|^2 + 2\varepsilon_6\xi^2|\widehat{y}|^2 + \varepsilon_7\xi^2|\widehat{u}|^2 \\ & +c(\varepsilon_5)\xi^2|\widehat{\theta}|^2 + c(\varepsilon_6)(1+\xi^2)|\widehat{\eta}|^2 + c(\varepsilon_6,\varepsilon_7)(\xi^2+\xi^4)|\widehat{q}|^2 \\ & +c\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds. \end{aligned} \tag{2.41}$$

Proof. Firstly, differentiating $\mathcal{F}_7$ and by using (2.6), we get

$$\frac{d\mathcal{F}_7(\xi,t)}{dt} \;=\; a^2l^2m^2\xi^2|\widehat{y}|^2 - a^2l^2m^2\xi^2|\widehat{z}|^2 + \Re e\Big\{i\gamma_1 alm^2\xi\widehat{\eta}\overline{\widehat{z}}\Big\}$$

$$-\Re e\Big\{a^2lm^2\xi^2\widehat{y}\overline{\widehat{\eta}}\Big\}+\Re e\Big\{alm^2k_0\xi^2\widehat{\phi}\overline{\widehat{z}}\Big\}+\Re e\Big\{am^3l^2\xi^2\widehat{\theta}\overline{\widehat{z}}\Big\}$$
$$-\Re e\Big\{ialm^2\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{z}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\}. \tag{2.42}$$

Now, differentiating $\mathcal{D}_4$ and by exploiting (2.42), (2.21) and (2.22), gives

$$\begin{aligned}\frac{d\mathcal{D}_4(\xi,t)}{dt} &= -a^2l^3m^2\xi^2|\widehat{z}|^2-lm^2\xi^2|\widehat{v}|^2+lmd_{12}\xi^2|\widehat{\theta}|^2+\Re e\Big\{ial^2m^2\gamma_1\xi\widehat{\eta}\overline{\widehat{z}}\Big\}\\ &+\Re e\Big\{d_{13}\xi^2\widehat{y}\overline{\widehat{\eta}}\Big\}+\Re e\Big\{(al^3m^3-ald_{12})\xi^2\widehat{\theta}\overline{\widehat{z}}\Big\}+\Re e\Big\{i\gamma_1d_{12}\xi^2\widehat{\eta}\overline{\widehat{\theta}}\Big\}\\ &+\Re e\Big\{d_{14}\xi^2\widehat{\phi}\overline{\widehat{\theta}}\Big\}-\Re e\Big\{(d_2\tau_ql-d_{12})\xi^2\widehat{\eta}\overline{\widehat{q}}\Big\}-\Re e\Big\{(d_2\tau_q-ld_{12})\xi^2\widehat{y}\overline{\widehat{q}}\Big\}\\ &-\Re e\Big\{d_2\beta\xi^2\widehat{q}\overline{\widehat{v}}\Big\}+\Re e\Big\{ial^2m^2\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{z}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\}\\ &+\Re e\Big\{i\xi\int_{\tau_1}^{\tau_2}\gamma_2(s)\overline{\widehat{\theta}}\widehat{\mathcal{Y}}(\xi,1,s,t)ds\Big\}+\Re e\Big\{imla\xi^3\widehat{q}\overline{\widehat{u}}\Big\},\end{aligned} \tag{2.43}$$

where

$$\begin{aligned}d_{13} &= l^2m^2-a^2l^2m^2+md_{12}\\ d_{14} &= (1-a^2)ml^2k_0+k_0d_{12}.\end{aligned}$$

At this point, we distinguish two cases:
Case 1. ($\alpha = 0$).

In this instance, we obtain for any $\delta_{10},\delta_{11},\delta_{12}>0$ and $\varepsilon_5,\varepsilon_6>0$ by applying the Young's inequality to the elements on the RHS of (2.43).

$$\begin{aligned}\frac{d\mathcal{D}_4(\xi,t)}{dt} &\le -(a^2l^3m^2-2\delta_{10}-\gamma_1\delta_{11})\xi^2|\widehat{z}|^2-(lm^2-\delta_{12})\xi^2|\widehat{v}|^2\\ &+\varepsilon_5\xi^2|\widehat{\phi}|^2+2\varepsilon_6\xi^2|\widehat{y}|^2+c(\delta_{10},\varepsilon_5)\xi^2|\widehat{\theta}|^2\\ &+c(\delta_{10},\varepsilon_6)(1+\xi^2)|\widehat{\eta}|^2+c(\delta_{12},\varepsilon_6)\xi^2|\widehat{q}|^2\\ &+c(\delta_{11})\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2ds,\end{aligned} \tag{2.44}$$

by letting $\delta_{10}=\frac{a^2l^3m^2}{8}$, $\delta_{11}=\frac{a^2l^3m^2}{4\gamma_1}$, $\delta_8=\frac{m^2l}{2}$, we find (2.40).
Case 2. ($\alpha = 0$).

The last term of the RHS in this situation is estimated as follows in (2.43), for any $\varepsilon_7>0$

$$\Re e\Big\{imla\xi^3\widehat{q}\overline{\widehat{u}}\Big\} \le \varepsilon_7\xi^2|\widehat{u}|^2+c(\varepsilon_7)\xi^4|\widehat{q}|^2. \tag{2.45}$$

Substituting the inequality (2.45) into the statement (2.43), we find (2.41). Lemma 2.5 has been successfully proved. □

After that, we have the following lemma.

Lemma 2.6. *The functional*

$$\mathcal{D}_5(\xi,t) := \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho,$$

satisfies,

$$\begin{aligned} \frac{d\mathcal{D}_5(\xi,t)}{dt} &\leq -\zeta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho + \gamma_1|\widehat{\eta}|^2 \\ &\quad -\zeta_1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds, \end{aligned} \tag{2.46}$$

where $\zeta_1 > 0$.

Proof. By differentiating $\mathcal{D}_5$, with respect to t and we use $(2.6)_9$, we have

$$\begin{aligned} \frac{d\mathcal{D}_5(\xi,t)}{dt} &= -\int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho \\ &\quad -\int_{\tau_1}^{\tau_2} |\gamma_2(s)|[e^{-s}|\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 - |\widehat{\mathcal{Y}}(\xi,0,s,t)|^2] ds. \end{aligned}$$

Using the fact that $\mathcal{Y}(\xi,0,s,t) = \omega_t(\xi,t) = \eta$, and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} \frac{d\mathcal{D}_5(\xi,t)}{dt} &\leq -\int_0^1 \int_{\tau_1}^{\tau_2} se^{-s}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho \\ &\quad -\int_{\tau_1}^{\tau_2} e^{-s}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds + (\int_{\tau_1}^{\tau_2} |\gamma_2(s)| ds)|\widehat{\eta}|^2. \end{aligned}$$

□

We have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1,\tau_2]$. Since $-e^{-s}$ is an increasing function. Lastly, by setting $\zeta_1 = e^{-\tau_2}$ and remembering (1.11), we obtain (2.46).

At this stage, we define the Lyapunov functionals

- For $\alpha = 0$:

$$\begin{aligned} \mathcal{K}_1(\xi,t) &:= N(1+\xi^2)\widehat{E}(\xi,t) + N_1\mathcal{D}_1(\xi,t) + N_2\frac{1}{(1+\xi^2)}\mathcal{D}_2(\xi,t) \\ &\quad +N_3\mathcal{D}_3(\xi,t) + N_4\mathcal{D}_4(\xi,t) + N_5(1+\xi^2)\mathcal{D}_5(\xi,t). \end{aligned} \tag{2.47}$$

- For $\alpha \neq 0$:

$$\begin{aligned} \mathcal{K}_2(\xi,t) &:= M(1+\xi^2)^2\widehat{E}(\xi,t) + M_1\mathcal{D}_1(\xi,t) + M_2\frac{1}{(1+\xi^2)}\mathcal{D}_2(\xi,t) \\ &\quad +M_3\mathcal{D}_3(\xi,t) + M_4\mathcal{D}_4(\xi,t) + M_5(1+\xi^2)\mathcal{D}_5(\xi,t), \end{aligned} \tag{2.48}$$

positive constants with values of $N, M, N_i, M_i, i = 1, ..., 5$ are to be carefully selected subsequently.

Lemma 2.7. *There exist $\mu_i > 0, i = 1, ..., 6$ such that the functionals $\mathcal{K}_1(\xi, t)$ and $\mathcal{K}_2(\xi, t)$ given by (2.47) and (2.48) satisfies*

- *For $\alpha = 0$:*

$$\begin{cases} \mu_1(1+\xi^2)\widehat{E}(\xi,t) \le \mathcal{K}_1(\xi,t) \le \mu_2(1+\xi^2)\widehat{E}(\xi,t), \\ \mathcal{K}_1'(\xi,t) \le -\mu_3\rho_1(\xi)\mathcal{K}_1(\xi,t), \quad \forall t > 0. \end{cases} \tag{2.49}$$

- *For $\alpha \neq 0$:*

$$\begin{cases} \mu_4(1+\xi^2)^2\widehat{E}(\xi,t) \le \mathcal{K}_2(\xi,t) \le \mu_5(1+\xi^2)^2\widehat{E}(\xi,t), \\ \mathcal{K}_2'(\xi,t) \le -\mu_6\rho_2(\xi)\mathcal{K}_2(\xi,t), \quad \forall t > 0, \end{cases} \tag{2.50}$$

where

$$\rho_1(\xi) = \frac{\xi^2}{(1+\xi^2)}, \quad \textit{and} \quad \rho_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}. \tag{2.51}$$

Proof. Firstly, by differentiating (2.47) and using (2.8), (2.14), (2.20), (2.30), (2.40) and (2.46), with the fact that $\frac{\xi^2}{1+\xi^2} \le \min\{1, \xi^2\}$ and $\frac{1}{1+\xi^2} \le 1$, we find

$$\begin{aligned} \mathcal{K}_1'(\xi,t) \le\; & -\xi^2\Big[\frac{\tau_q m^2 k_0 l}{2}N_3 - 3\varepsilon_4 N_2\Big]|\widehat{u}|^2 - \Big[\frac{m^2\delta^2 l^2}{2}N_2 - cN_3\Big]|\widehat{v}|^2 \\ & -\xi^2\Big[\frac{m^2 l}{2}N_4\Big]|\widehat{v}|^2 - \frac{\xi^2}{(1+\xi^2)}\Big[\frac{m^2\delta^2}{2}N_2\Big]|\widehat{v}|^2 \\ & -(1+\xi^2)\Big[\frac{alk_0}{2}N_3 - \varepsilon_1 N_1 - 2\varepsilon_2 N_2 - 2\varepsilon_6 N_4\Big]|\widehat{y}|^2 \\ & -\xi^2\Big[\frac{a^2 l^3 m^2}{2}N_4 - cN_2 - cN_3\Big]|\widehat{z}|^2 \\ & -\xi^2\Big[\frac{\tau_q m^2 l k_0}{2}N_3 - (3+\gamma_1)\varepsilon_3 N_2 - \varepsilon_5 N_4\Big]|\widehat{\phi}|^2 \\ & -\xi^2\Big[\frac{1}{2}N_1 - c(\varepsilon_3)N_2 - cN_3 - c(\varepsilon_5)N_4\Big]|\widehat{\theta}|^2 \\ & -(1+\xi^2)\Big[\beta N - c(\varepsilon_1)N_1 - c(\varepsilon_2,\varepsilon_4)N_2 - cN_3 - c(\varepsilon_6)N_4\Big]|\widehat{q}|^2 \\ & -(1+\xi^2)\Big[C_1 N - c(\varepsilon_2,\varepsilon_3,\varepsilon_4)N_2 - cN_3 - c(\varepsilon_6)N_4 - \gamma_1 N_5\Big]|\widehat{\eta}|^2 \\ & -(1+\xi^2)\Big[\zeta_1 N_5 - cN_2 - cN_3 - cN_4\Big]\int_{\tau_1}^{\tau_2} |\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds \\ & -\zeta_1 N_5(1+\xi^2)\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho. \end{aligned} \tag{2.52}$$

By setting

$$\begin{aligned} &\varepsilon_1 = \frac{alk_0N_3}{12N_1}, \quad \varepsilon_2 = \frac{alk_0N_3}{24N_2}, \quad \varepsilon_3 = \frac{\tau_q m^2 lk_0 N_3}{8(3+\gamma_1)N_2}, \quad \varepsilon_4 = \frac{\tau_q m^2 lk_0N_3}{12N_2}, \\ &\varepsilon_5 = \frac{\tau_q m^2 lk_0 N_3}{8N_4}, \quad \varepsilon_6 = \frac{alk_0N_3}{24N_4}, \end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{K}_1'(\xi,t) \leq\ & -\xi^2\Big[\frac{\tau_q m^2 k_0 l}{4}N_3\Big]|\widehat{u}|^2 - \Big[\frac{m^2\delta^2 l^2}{2}N_2 - cN_3\Big]|\widehat{v}|^2 - (1+\xi^2)\Big[\frac{alk_0}{4}N_3\Big]|\widehat{y}|^2\\
& -\xi^2\Big[\frac{m^2 l}{2}N_4\Big]|\widehat{v}|^2 - \frac{\xi^2}{(1+\xi^2)}\Big[\frac{m^2\delta^2}{2}N_2\Big]|\widehat{v}|^2\\
& -\xi^2\Big[\frac{a^2 l^3 m^2}{2}N_4 - cN_2 - cN_3\Big]|\widehat{z}|^2 - \xi^2\Big[\frac{\tau_q m^2 l k_0}{4}N_3\Big]|\widehat{\phi}|^2\\
& -\xi^2\Big[\frac{1}{2}N_1 - c(N_2,N_3)N_2 - cN_3 - c(N_3,N_4)N_4\Big]|\widehat{\theta}|^2\\
& -(1+\xi^2)\Big[\beta N - c(N_1,N_3)N_1 - c(N_2,N_3)N_2 - cN_3 - c(N_3,N_4)N_4\Big]|\widehat{q}|^2\\
& -(1+\xi^2)\Big[C_1 N - c(N_2,N_3)N_2 - cN_3 - c(N_3,N_4)N_4 - \gamma_1 N_5\Big]|\widehat{\eta}|^2\\
& -(1+\xi^2)\Big[\zeta_1 N_5 - cN_2 - cN_3 - cN_4\Big]\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds\\
& -(1+\xi^2)\zeta_1 N_5\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho. \end{aligned} \tag{2.53}$$

Now, we fixed N_3 and choosing N_2 large enough such that

$$\frac{m^2\delta^2 l^2}{2}N_2 - cN_3 > 0,$$

then we select N_4 in a size big enough that

$$\alpha_2 = \frac{a^2 l^3 m^2}{2}N_4 - cN_2 - cN_3 > 0,$$

then we select N_1, N_5 in a size big enough that

$$\begin{aligned}&\alpha_3 = \frac{1}{2}N_1 - c(N_2,N_3)N_2 - cN_3 - c(N_3,N_4)N_4 > 0,\\ &\zeta_1 N_5 - cN_2 - cN_3 - cN_4 > 0.\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}\mathcal{K}_1'(\xi,t) \leq\ & -\alpha_0\xi^2|\widehat{u}|^2 - \alpha_5\xi^2|\widehat{\phi}|^2 - (1+\xi^2)\Big[\beta N - c\Big]|\widehat{q}|^2 - \alpha_1\xi^2|\widehat{v}|^2\\
& -\alpha_4(1+\xi^2)|\widehat{y}|^2 - \alpha_2\xi^2|\widehat{z}|^2 - \alpha_3\xi^2|\widehat{\theta}|^2 - (1+\xi^2)\Big[C_1 N - c\Big]|\widehat{\eta}|^2\\
& -\alpha_6(1+\xi^2)\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho, \end{aligned} \tag{2.54}$$

where $\alpha_1 = \frac{m^2 l}{2}N_4, \alpha_6 = \zeta_1 N_5$.

Secondly, we have

$$\begin{aligned}\left|\mathcal{K}_1(\xi,t)-N(1+\xi^2)\widehat{E}(\xi,t)\right| &= N_1\left|\mathcal{D}_1(\xi,t)\right|+N_2\frac{1}{(1+\xi^2)}\left|\mathcal{D}_2(\xi,t)\right|\\ &\quad+N_3\left|\mathcal{D}_3(\xi,t)\right|+N_4\left|\mathcal{D}_4(\xi,t)\right|+N_5(1+\xi^2)\left|\mathcal{D}_5(\xi,t)\right|.\end{aligned}$$

Using Young's inequality, the fact that $\frac{\xi^2}{1+\xi^2}\le\min\{1,\xi^2\}$ and $\frac{1}{1+\xi^2}\le 1$, we find

$$\left|\mathcal{K}_1(\xi,t)-N(1+\xi^2)\widehat{E}(\xi,t)\right|\le c(1+\xi^2)\widehat{E}(\xi,t).$$

Hence, we get

$$(N-c)(1+\xi^2)\widehat{E}(\xi,t)\le\mathcal{K}_1(\xi,t)\le(N+c)(1+\xi^2)\widehat{E}(\xi,t). \tag{2.55}$$

Now, we select N in a size big enough that

$$N-c>0,\ \ C_1N-c>0,\ \ \beta N-c>0,$$

and using the estimations (2.7), (2.54) and (2.55), there is a positive constant $\kappa>0$, for all $t>0$ and for all $\xi\in\mathbb{R}$, we have

$$\mu_1(1+\xi^2)\widehat{E}(\xi,t)\le\mathcal{K}_1(\xi,t)\le\mu_2(1+\xi^2)\widehat{E}(\xi,t), \tag{2.56}$$

and

$$\begin{aligned}\mathcal{K}_1'(\xi,t) &\le -\kappa\xi^2\Big\{|\widehat{u}|^2+|\widehat{\phi}|^2+|\widehat{\theta}|^2+|\widehat{v}|^2+|\widehat{y}|^2+|\widehat{z}|^2+|\widehat{q}|^2+|\widehat{\eta}|^2\\ &\quad+\int_0^1\int_{\tau_1}^{\tau_2}s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2dsd\rho\Big\},\end{aligned} \tag{2.57}$$

then

$$\mathcal{K}_1'(\xi,t)\le-\lambda_1\rho_1(\xi)\widehat{E}(\xi,t),\ \ \forall t\ge 0. \tag{2.58}$$

Furthermore, we derive the following for any positive constant $\mu_3=\frac{\lambda_1}{\mu_2}>0$

$$\mathcal{K}_1'(\xi,t)\le-\mu_3\rho_1(\xi)\mathcal{K}_1(\xi,t),\ \ \forall t\ge 0, \tag{2.59}$$

where $\rho_1(\xi)=\frac{\xi^2}{(1+\xi^2)}$, for some $\lambda_1,\mu_i>0, i=1,2,3$. The proof of the first result (2.49) is finished.

We demonstrate the second result similarly to the first proof. So we derive (2.48) and using (2.8), (2.14), (2.20), (2.31), (2.41) and (2.46), with the fact that $\frac{\xi^2}{1+\xi^2}\le\min\{1,\xi^2\}$ and $\frac{1}{1+\xi^2}\le 1$, we get

$$\begin{aligned}\mathcal{K}_2'(\xi,t) &\le -\xi^2\Big[\frac{\tau_q m^2k_0l}{2}M_3-3\varepsilon_4M_2-\varepsilon_7M_4\Big]|\widehat{u}|^2-\Big[\frac{m^2\delta^2l^2}{2}M_2-cM_3\Big]|\widehat{v}|^2\\ &\quad-\xi^2\Big[\frac{m^2l}{2}M_4\Big]|\widehat{v}|^2-\frac{\xi^2}{(1+\xi^2)}\Big[\frac{m^2\delta^2}{2}M_2\Big]|\widehat{v}|^2\\ &\quad-(1+\xi^2)\Big[\frac{alk_0}{2}M_3-\varepsilon_1M_1-2\varepsilon_2M_2-2\varepsilon_6M_4\Big]|\widehat{y}|^2\end{aligned}$$

$$
\begin{aligned}
&-\xi^2\Big[\frac{a^2l^3m^2}{2}M_4 - cM_2 - cM_3\Big]|\widehat{z}|^2\\
&-\xi^2\Big[\frac{\tau_q m^2 l k_0}{2}M_3 - (3+\gamma_1)\varepsilon_3 M_2 - \varepsilon_5 M_4\Big]|\widehat{\phi}|^2\\
&-\xi^2\Big[\frac{1}{2}M_1 - c(\varepsilon_3)M_2 - cM_3 - c(\varepsilon_5)M_4\Big]|\widehat{\theta}|^2\\
&-(1+\xi^2)^2\Big[\beta M - c(\varepsilon_1)M_1 - c(\varepsilon_2,\varepsilon_4)M_2 - cN_3 - c(\varepsilon_6,\varepsilon_7)M_4\Big]|\widehat{q}|^2\\
&-(1+\xi^2)^2\Big[C_1 M - c(\varepsilon_2,\varepsilon_3,\varepsilon_4)M_2 - cM_3 - c(\varepsilon_6)M_4 - \gamma_1 M_5\Big]|\widehat{\eta}|^2\\
&-(1+\xi^2)\Big[\zeta_1 M_5 - cM_2 - cM_3 - cM_4\Big]\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds\\
&-\zeta_1 M_5(1+\xi^2)\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho. \qquad (2.60)
\end{aligned}
$$

By setting

$$
\varepsilon_1 = \frac{alk_0M_3}{12M_1},\quad \varepsilon_2 = \frac{alk_0M_3}{24M_2},\quad \varepsilon_3 = \frac{\tau_q m^2 lk_0 M_3}{8(3+\gamma_1)M_2},\quad \varepsilon_4 = \frac{\tau_q m^2 lk_0 M_3}{24M_2},
$$
$$
\varepsilon_5 = \frac{\tau_q m^2 lk_0 M_3}{8M_4},\quad \varepsilon_6 = \frac{alk_0M_3}{24M_4},\quad \varepsilon_7 = \frac{\tau_q m^2 lk_0 M_3}{8M_2}.
$$

We obtain

$$
\begin{aligned}
\mathcal{K}_2'(\xi,t) \leq\ & -\xi^2\Big[\frac{\tau_q m^2 k_0 l}{4}M_3\Big]|\widehat{u}|^2 - \Big[\frac{m^2\delta^2 l^2}{2}M_2 - cM_3\Big]|\widehat{v}|^2 - (1+\xi^2)\Big[\frac{alk_0}{4}M_3\Big]|\widehat{y}|^2\\
&-\xi^2\Big[\frac{m^2 l}{2}M_4\Big]|\widehat{v}|^2 - \frac{\xi^2}{(1+\xi^2)}\Big[\frac{m^2\delta^2}{2}M_2\Big]|\widehat{v}|^2\\
&-\xi^2\Big[\frac{a^2l^3m^2}{2}M_4 - cM_2 - cM_3\Big]|\widehat{z}|^2 - \xi^2\Big[\frac{\tau_q m^2 lk_0}{4}M_3\Big]|\widehat{\phi}|^2\\
&-\xi^2\Big[\frac{1}{2}M_1 - c(M_2,M_3)M_2 - cM_3 - c(M_3,M_4)M_4\Big]|\widehat{\theta}|^2\\
&-(1+\xi^2)\Big[\beta M - c(M_1,M_3)M_1 - c(M_2,M_3)M_2 - cM_3 - c(M_2,M_3,M_4)M_4\Big]|\widehat{q}|^2\\
&-(1+\xi^2)\Big[C_1 M - c(M_2,M_3)M_2 - cM_3 - c(M_3,M_4)M_4 - \gamma_1 M_5\Big]|\widehat{\eta}|^2\\
&-(1+\xi^2)\Big[\zeta_1 M_5 - cM_2 - cM_3 - cM_4\Big]\int_{\tau_1}^{\tau_2}|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,1,s,t)|^2 ds\\
&-\zeta_1 M_5(1+\xi^2)\int_0^1\int_{\tau_1}^{\tau_2} s|\gamma_2(s)||\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho. \qquad (2.61)
\end{aligned}
$$

Now, we fixed M_3 and choosing M_2 large enough such that

$$
\frac{m^2\delta^2 l^2}{2}M_2 - cM_3 > 0,
$$

then we select M_4 in a size big enough that

$$\kappa_2 = \frac{a^2 l^3 m^2}{2} M_4 - cM_2 - cM_3 > 0.$$

Likewise we select M_1, M_5 in a size big enough that

$$\begin{aligned} &\kappa_3 = \frac{1}{2} M_1 - c(M_2, M_3) M_2 - cM_3 - c(M_2, M_3, M_4) M_4 > 0, \\ &\zeta_1 M_5 - cM_2 - cM_3 - cM_4 > 0. \end{aligned}$$

Hence, we arrive at

$$\begin{aligned} \mathcal{K}_1'(\xi,t) \quad \leq \quad & -\kappa_0 \xi^2 |\widehat{u}|^2 - \kappa_5 \xi^2 |\widehat{\phi}|^2 - (1+\xi^2)^2 \Big[\beta M - c\Big] |\widehat{q}|^2 - \kappa_1 \xi^2 |\widehat{v}|^2 \\ & -\kappa_4 (1+\xi^2) |\widehat{y}|^2 - \kappa_2 \xi^2 |\widehat{z}|^2 - \kappa_3 \xi^2 |\widehat{\theta}|^2 - (1+\xi^2)^2 \Big[C_1 M - c\Big] |\widehat{\eta}|^2 \\ & -\kappa_6 (1+\xi^2) \int_0^1 \int_{\tau_1}^{\tau_2} s|\gamma_2(s)| |\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho, \end{aligned} \tag{2.62}$$

where $\kappa_1 = \frac{m^2 l}{2} M_4, \kappa_6 = \zeta_1 M_5$.

As an alternative, we have

$$\begin{aligned} \Big|\mathcal{K}_2(\xi,t) - M(1+\xi^2)^2 \widehat{E}(\xi,t)\Big| \quad = \quad & M_1 \Big|\mathcal{D}_1(\xi,t)\Big| + M_2 \frac{1}{(1+\xi^2)} \Big|\mathcal{D}_2(\xi,t)\Big| \\ & + M_3 \Big|\mathcal{D}_3(\xi,t)\Big| + M_4 \Big|\mathcal{D}_4(\xi,t)\Big| + M_5 (1+\xi^2) \Big|\mathcal{D}_5(\xi,t)\Big|. \end{aligned}$$

Using Young's inequality, the fact that $\frac{\xi^2}{1+\xi^2} \leq \min\{1, \xi^2\}$ and $\frac{1}{1+\xi^2} \leq 1$, we find

$$\Big|\mathcal{K}_2(\xi,t) - M(1+\xi^2)^2 \widehat{E}(\xi,t)\Big| \leq c(1+\xi^2)^2 \widehat{E}(\xi,t).$$

Hence, we get

$$(M-c)(1+\xi^2)^2 \widehat{E}(\xi,t) \leq \mathcal{K}_2(\xi,t) \leq (M+c)(1+\xi^2)^2 \widehat{E}(\xi,t). \tag{2.63}$$

Now we select M in a size big enough that

$$M - c > 0, \;\; C_1 M - c > 0, \;\; \beta M - c > 0,$$

and using the estimations (2.7), (2.62) and (2.63), there is a positive constant $\widehat{\kappa} > 0$, for all $t > 0$ and for all $\xi \in \mathbb{R}$, we have

$$\mu_4 (1+\xi^2)^2 \widehat{E}(\xi,t) \leq \mathcal{K}_2(\xi,t) \leq \mu_5 (1+\xi^2)^2 \widehat{E}(\xi,t), \tag{2.64}$$

and

$$\begin{aligned} \mathcal{K}_2'(\xi,t) \quad \leq \quad & -\widehat{\kappa} \xi^2 \Big\{ |\widehat{u}|^2 + |\widehat{\phi}|^2 + |\widehat{\theta}|^2 + |\widehat{v}|^2 + |\widehat{y}|^2 + |\widehat{z}|^2 + |\widehat{q}|^2 + |\widehat{\eta}|^2 \\ & + \int_0^1 \int_{\tau_1}^{\tau_2} s|\gamma_2(s)| |\widehat{\mathcal{Y}}(\xi,\rho,s,t)|^2 ds d\rho \Big\}, \end{aligned} \tag{2.65}$$

then

$$\mathcal{K}_2'(\xi,t) \le -\lambda_2\rho_2(\xi)\widehat{E}(\xi,t), \quad \forall t \ge 0. \tag{2.66}$$

Furthermore, we derive the following for any positive constant $\mu_6 = \frac{\lambda_2}{\mu_5} > 0$

$$\mathcal{K}_2'(\xi,t) \le -\mu_6\rho_2(\xi)\mathcal{K}_2(\xi,t), \quad \forall t \ge 0, \tag{2.67}$$

where $\rho_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, for some $\lambda_2, \mu_i > 0, i = 4,5,6$. The proof of the second result (2.50) is finished. □

The pointwise estimations of the functional $\widehat{E}(\xi,t)$ provided by the next proposition.

Proposition 2.1. *Suppose (1.8) and (1.11) hold. Then, for every $t \ge 0$ and $\xi \in \mathbb{R}$, positive constants $k_1, k_2 > 0$ exists such that the energy functional provided by (2.7) meets the following conditions*

$$\begin{cases} \widehat{E}(\xi,t) \le k_1\widehat{E}(\xi,0)e^{-\mu_3\rho_1(\xi)t}, & \text{if } \alpha = 0, \\ \widehat{E}(\xi,t) \le k_2\widehat{E}(\xi,0)e^{-\mu_6\rho_2(\xi)t}, & \text{if } \alpha \ne 0, \end{cases} \tag{2.68}$$

where $\rho_1(\xi) = \dfrac{\xi^2}{(1+\xi^2)}, \ \rho_2(\xi) = \dfrac{\xi^2}{(1+\xi^2)^2}.$

Proof. From $(2.49)_2$ and $(2.50)_2$, we have by integration over $(0,t)$

$$\begin{aligned} \mathcal{K}_1(\xi,t) &\le \mathcal{K}_1(\xi,0)e^{-\mu_3\rho_1(\xi)t}, \quad \forall t \ge 0, \quad \text{if } \alpha = 0 \qquad (2.69)\\ \mathcal{K}_2(\xi,t) &\le \mathcal{K}_2(\xi,0)e^{-\mu_6\rho_2(\xi)t}, \quad \forall t \ge 0, \quad \text{if } \alpha \ne 0. \qquad (2.70) \end{aligned}$$

Hence, by according of $(2.49)_1$, $(2.50)_1$ and (2.69), (2.70), we establish (2.68) . □

2.2. Decay estimates

Now, we declare and support the following finding

Theorem 2.2. *Suppose that s be a nonnegative integer, and let $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Consequently, for any $t \ge 0$, the following decay estimates are satisfied by the solution U to problems (2.2) and (2.3)*

- *When $\alpha = 0$*

$$\|\partial_x^k U(t)\|_2 \le C\|U_0\|_1(1+t)^{-\frac{1}{4}-\frac{k}{2}} + Ce^{-\frac{\mu_3}{4}t}\|\partial_x^k U_0\|_2. \tag{2.71}$$

- *When $\alpha \ne 0$*

$$\|\partial_x^k U(t)\|_2 \le C\|U_0\|_1(1+t)^{-\frac{1}{4}-\frac{k}{2}} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell} U_0\|_2, \tag{2.72}$$

where k and ℓ are positive integers that meet the equation $k + \ell \le s$ and $C > 0$.

Proof. From (2.7), we have $|\widehat{U}(\xi,t)|^2 \equiv \widehat{E}(\xi,t)$.

- By using the Plancherel theorem 1.1 and making use of $(2.68)_1$, If $\alpha = 0$, we can determine

$$\begin{aligned} \|\partial_x^k U(t)\|_2^2 &= \int_{\mathbb{R}} |\xi|^{2k} |\widehat{U}(\xi,t)|^2 d\xi \\ &\leq c\int_{\mathbb{R}} |\xi|^{2k} e^{-\mu_3\rho_1(\xi)t} |\widehat{U}(\xi,0)|^2 d\xi \\ &\leq \underbrace{c\int_{|\xi|\leq 1} |\xi|^{2k} e^{-\mu_3\rho_1(\xi)t} |\widehat{U}(\xi,0)|^2 d\xi}_{R_1} \\ &\quad + \underbrace{c\int_{|\xi|\geq 1} |\xi|^{2k} e^{-\mu_3\rho_1(\xi)t} |\widehat{U}(\xi,0)|^2 d\xi}_{R_2}. \end{aligned} \tag{2.73}$$

We now determine R_1, R_2, the low-frequency component$|\xi| \leq 1$, and the high-frequency component $|\xi| \geq 1$, separately.
Firstly, we have $\rho_1(\xi) \geq \frac{1}{2}\xi^2$, for $|\xi| \leq 1$. Then

$$\begin{aligned} R_1 &\leq c\int_{|\xi|\leq 1} |\xi|^{2k} e^{-\frac{\mu_3}{2}|\xi|^2 t} |\widehat{U}(\xi,0)|^2 d\xi \\ &\leq c\sup_{|\xi|\leq 1}\{|\widehat{U}(\xi,0)|^2\} \int_{|\xi|\leq 1} |\xi|^{2k} e^{-\frac{\mu_3}{2}|\xi|^2 t} d\xi, \end{aligned} \tag{2.74}$$

Lemma 1.1 allows us to acquire

$$\begin{aligned} R_1 &\leq c\sup_{|\xi|\leq 1}\{|\widehat{U}(\xi,0)|^2\}(1+t)^{-k-\frac{1}{2}} \\ &\leq c\|U_0\|_1^2 (1+t)^{-k-\frac{1}{2}}. \end{aligned} \tag{2.75}$$

Secondly, we have $\rho_1(\xi) \geq \frac{1}{2}$, for $|\xi| \geq 1$. Then

$$R_2 \leq c\int_{|\xi|\geq 1} |\xi|^{2k} e^{-\frac{\mu_3}{2}t} |\widehat{U}(\xi,0)|^2 d\xi, \quad \forall t \geq 0. \tag{2.76}$$

$$\begin{aligned} &\leq ce^{-\frac{\mu_3}{2}t} \int_{|\xi|\geq 1} |\xi|^{2k} |\widehat{U}(\xi,0)|^2 d\xi \\ &\leq ce^{-\frac{\mu_3}{2}t} \|\partial_x^k U(x,0)\|_2^2, \quad \forall t \geq 0. \end{aligned} \tag{2.77}$$

Substituting (2.75) and (2.77) into (2.73), we find (2.71).

- If $\alpha \neq 0$, similar to the first estimate, we apply the Plancherel theorem 1.1 and exploiting $(2.68)_2$, we find

$$\begin{aligned} \|\partial_x^k U(t)\|_2^2 &= \int_{\mathbb{R}} |\xi|^{2k} |\widehat{U}(\xi,t)|^2 d\xi \\ &\leq c\int_{\mathbb{R}} |\xi|^{2k} e^{-\mu_6\rho_2(\xi)t} |\widehat{U}(\xi,0)|^2 d\xi \end{aligned}$$

$$
\begin{aligned}
&\leq \underbrace{c\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\mu_6\rho_2(\xi)t}|\widehat{U}(\xi,0)|^2d\xi}_{R_3} \\
&\quad +\underbrace{c\int_{|\xi|\geq 1}|\xi|^{2k}e^{-\mu_6\rho_2(\xi)t}|\widehat{U}(\xi,0)|^2d\xi}_{R_4}. \qquad (2.78)
\end{aligned}
$$

Now, we determine R_3, R_4, the low-frequency component $|\xi| \leq 1$ and the high-frequency component $|\xi| \geq 1$ separately.
Firstly, we have $\rho_2(\xi) \geq \frac{1}{4}\xi^2$, for $|\xi| \leq 1$. Then

$$
\begin{aligned}
R_3 &\leq c\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\frac{\mu_6}{4}|\xi|^2t}|\widehat{U}(\xi,0)|^2d\xi \\
&\leq c\sup_{|\xi|\leq 1}\{|\widehat{U}(\xi,0)|^2\}\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\frac{\mu_6}{4}|\xi|^2t}d\xi, \qquad (2.79)
\end{aligned}
$$

by using Lemma 1.1, we obtain

$$
\begin{aligned}
R_3 &\leq c\sup_{|\xi|\leq 1}\{|\widehat{U}(\xi,0)|^2\}(1+t)^{-k-\frac{1}{2}} \\
&\leq c\|U_0\|_1^2(1+t)^{-k-\frac{1}{2}}. \qquad (2.80)
\end{aligned}
$$

Secondly, we have $\rho_2(\xi) \geq \frac{1}{4}\xi^{-2}$, for $|\xi| \geq 1$. Then

$$
R_4 \leq c\int_{|\xi|\geq 1}|\xi|^{2k}e^{-\frac{\mu_6}{4}|\xi|^{-2}t}|\widehat{U}(\xi,0)|^2d\xi, \quad \forall t \geq 0. \qquad (2.81)
$$

Expoiting the inequality

$$
\sup_{|\xi|\geq 1}\left\{|\xi|^{-2\ell}e^{-c\frac{1}{4}|\xi|^{-2}t}\right\} \leq C(1+t)^{-\ell}, \qquad (2.82)
$$

we get that

$$
\begin{aligned}
R_4 &\leq c\sup_{|\xi|\geq 1}\left\{|\xi|^{-2\ell}e^{-\frac{\mu_6}{4}|\xi|^{-2}t}\right\}\int_{|\xi|\geq 1}|\xi|^{2(k+\ell)}|\widehat{U}(\xi,0)|^2d\xi \\
&\leq c(1+t)^{-\ell}\|\partial_x^{k+\ell}U(x,0)\|_2^2, \quad \forall t \geq 0. \qquad (2.83)
\end{aligned}
$$

Substituting (2.80) and (2.83) into (2.78), we find (2.72).

□

3. Conclusions

The investigation of the generalized degradation assessment of Bresse-Cattaneo system integration about the distributed delay term is the goal of this research, which employs the energy technique in Fourier space.

The distinct process that emerges from the distributed delay, which determines the system's development of this feature in Fourier space, is what we are interested in with this present work.

The same strategy will be used in the same systems in the upcoming works, but we will use various types of memory because we anticipate getting comparable outcomes.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Afilal, B. Feng, A. Soufyane, New decay rates for Cauchy problem of Timoshenko thermoelastic systems with past history: Cattaneo and Fourier law, *Math. Methods Appl. Sci.*, **44** (2021), 11873–11894. https://doi.org/10.1002/mma.6579
2. P. R. Agarwal, Q. Bazighifan, M. A. Ragusa, Nonlinear neutral delay differential equations of fourth-order: oscillation of solutions, *Entropy*, **23** (2021), 129. https://doi.org/10.3390/e23020129
3. H. Bounadja, B. Said-Houari, Decay rates for the Moore-Gibson-Thompson equation with memory, *Evol. Equ. Control Theory.*, **10** (2021), 431–460.
4. S. Boulaaras, A. Choucha, A. Scapellato, General decay of the Moore-Gibson-Thompson equation with viscoelastic memory of Type II, *J. Funct. Spaces*, **2022** (2022), 9015775. https://doi.org/10.1155/2022/9015775
5. A. Choucha, S. Boulaaras, D. Ouchenane, Exponential decay and global existence Of solutions of a singular nonlocal viscoelastic system with distributed delay and damping terms, *Filomat*, **35** (2021), 795–826.
6. L. Djouamai, B. Said-Houari, A new stability number of the Bresse-Cattaneo system, *Math. Methods Appl. Sci.*, **41** (2018), 2827–2847. https://doi.org/10.1002/mma.4784
7. C. D. Enyi, Timoshenko systems with Cattaneo law and partial Kelvin-Voigt damping:well-posedness and stability, *Appl. Anal.*, 2022. https://doi.org/10.1080/00036811.2022.2152802
8. M. E. Gurtin, A. S. Pipkin, A general decay of a heat condition with finite wave speeds, *Arch. Rational. Mech. Anal.*, **31** (1968), 113–126.
9. M. Khader, B. Said-Houari, Decay rate of solution for the Cauchy problem in Timoshenko system with past history, *Appl. Math. Optim.*, **75** (2017), 403–428. https://doi.org/10.1007/s00245-016-9336-6
10. N. Mori, S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, *J. Hyperbolic Differ. Equ.*, **11** (2014), 135–157. https://doi.org/10.1142/S0219891614500039
11. A. S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Diff. Int. Equs.*, **21** (2008), 935–958.
12. R. Racke, *Lectures on nonlinear evolution equations*, Berlin: Springer, 1992.
13. B. Said-Houari, A. Soufyane, The Bresse system in thermoelasticity, *Math. Methods. Appl. Sci.*, **38** (2015), 3642–3652. https://doi.org/10.1002/mma.3305

14. B. Said-Houari, T. Hamadouche, The asymptotic behavior of the Bresse-Cattanao system, *Commun. Contemp. Math.*, **18** (2016), 1550045. https://doi.org/10.1142/S0219199715500455

15. C. C. Tannoudji, J. Dupont-Roc, G. Grynberg, *Photons and atoms introduction to quantum electrodynamics. Photons et atomes. Introduction a l'electrodynamique quantique*, Hoboken: Wiley, 1997.

16. J. B. Zuo, A. Rahmoune, Y. J. Li, General decay of a nonlinear viscoelastic wave equation with Balakrishnan-Taylor damping and a delay involving variable exponents, *J. Funct. Spaces*, **2022** (2022), 9801331. https://doi.org/10.1155/2022/9801331