AIMS Mathematics, 8(8): 17860–17889.
DOI: 10.3934/math.2023910
Received: 11 April 2023
Revised: 09 May 2023
Accepted: 14 May 2023
Published: 24 May 2023

Mathematics
http://www.aimspress.com/journal/Math

Research article

Lifespan estimate of solution to the semilinear wave equation with damping term and mass term

Xiongmei Fan[1], **Sen Ming**[2,*], **Wei Han**[2] **and Zikun Liang**[3]

[1] Data Science And Technology, North University of China, Taiyuan 030051, China

[2] Department of Mathematics, North University of China, Taiyuan 030051, China

[3] Department of Mechanics and Aerospace Engineering, Southern University of Science and Technology, Shenzhen 518055, China

* **Correspondence:** Email: senming1987@163.com.

Abstract: This paper is mainly concerned with the initial boundary value problems of semilinear wave equations with damping term and mass term as well as Neumann boundary conditions on exterior domain in three dimensions. Blow-up and upper bound lifespan estimates of solutions to the problem with damping term and mass term are derived by applying test function technique and iterative method, where nonlinear terms are power nonlinearity $|u|^p$, derivative nonlinearity $|u_t|^p$, combined nonlinearities $|u_t|^p + |u|^q$, respectively. Moreover, upper bound lifespan estimate of solution to the problem with scale invariant damping term, non-negative mass term and combined nonlinearities $|u_t|^p + |u|^q$ is obtained. The proofs are based on the test function method and iterative approach. The main new contribution is that upper bound lifespan estimates of solutions are associated with the Strauss exponent and Glassey exponent. In addition, the variation trend of wave is achieved by taking advantage of numerical simulation.

Keywords: semilinear wave equations; mass term; lifespan estimates; test function technique; iterative method
Mathematics Subject Classification: 35L70, 58J45

1. Introduction

Our main goal of the present work is to investigate the following semilinear wave equations with damping term and mass term, namely

$$\begin{cases} u_{tt} - \Delta u + b_1(t)u_t - b_2(t)u = f(u, u_t), & x \in \Omega^c,\ t > 0, \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \Omega^c, \\ \dfrac{\partial u}{\partial n}|_{\partial B_1(0)} = 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t}u_t + \dfrac{\nu^2}{(1+t)^2}u = f(u, u_t), & x \in \Omega^c,\ t > 0, \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon g(x), & x \in \Omega^c, \\ \dfrac{\partial u}{\partial n}|_{\partial B_1(0)} = 0, \end{cases} \tag{1.2}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The coefficients $b_1(t) \in C([0,\infty)) \cap L^1([0,\infty))$, $b_2(t) = \frac{\nu_0}{(1+t)^{\beta+1}}$ ($\nu_0 > 0, \beta > 1$) are non-negative functions. $\mu,\ \nu \geq 0$. We set $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$ in problem (1.1) and $f(u, u_t) = |u_t|^p + |u|^q$ in problem (1.2), respectively. The exponents of nonlinear terms satisfy $1 < p, q < \infty$. Let $\Omega = B_1(0) = \{x \in \mathbb{R}^3 \big| |x| \leq 1\}$ and $\Omega^c = \mathbb{R}^3 \backslash B_1(0)$. Ω^c and $\partial\Omega^c$ are smooth and compact. Initial values satisfy $f(x), g(x) \in C^\infty(\Omega^c)$ and $\operatorname{supp}(f(x), g(x)) \subset \Omega^c \cap B_R(0)$, where $B_R(0) = \{x \big| |x| \leq R\}$, $R > 2$. The small parameter $\varepsilon > 0$ describes the size of initial values. $\frac{\partial u}{\partial n}$ stands for the derivative of external normal direction. It is well known that a solution u has compact support when the initial values have compact supports. As a consequence, we directly suppose that the solution has compact support set.

We briefly review several previous results concerning problem (1.1) with $b_1(t) = b_2(t) = 0$. It is worth pointing out that the Cauchy problem with $f(u, u_t) = |u|^p$ asserts the Strauss exponent $p_c(n)$ (see [31, 40–42]), which is the positive root of quadratic equation

$$r(n, p) = -(n-1)p^2 + (n+1)p + 2 = 0.$$

The Cauchy problem with $f(u, u_t) = |u_t|^p$ admits the Glassey exponent $p_G(n) = \frac{n+1}{n-1}$, which has been investigated in [14, 19]. Ikeda et al. [15] establish blow-up dynamic and lifespan estimate of solution to the semilinear wave equation and related weakly coupled system by using a framework of test function approach. The Cauchy problem with $f(u, u_t) = |u_t|^p + |u|^q$ is discussed in Han et al. [13]. Upper bound lifespan estimate of solution is illustrated by making use of test function method and the Kato lemma.

Recently, many researchers have been devoted to the study of Cauchy problem for semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = f(u, u_t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = \varepsilon u_0(x),\ u_t(x,0) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$. Problem (1.3) with damping term $g(u_t) = u_t, \frac{\mu}{1+t}u_t, \frac{\mu}{(1+t)^\beta}u_t$ ($\beta > 1$), $(-\Delta)^\delta u_t$ ($\delta \in (0, \frac{1}{2}]$), $a(x)u_t$ ($a(x) \in C(\mathbb{R}^n)$) and power nonlinear term $f(u, u_t) = |u|^p$ is considered in [6, 9, 18, 24, 27, 30, 38]. Lai et al. [27] derive upper bound lifespan estimate of solution to problem (1.3) with damping term $g(u_t) = \frac{\mu}{1+t}u_t$ by exploiting the Kato lemma. Imai et al. [18] investigate problem (1.3) with scale invariant damping in two dimensions. Blow-up result and lifespan estimate of solution are discussed under certain restriction on the constant μ. Applying test function approach and imposing certain integral sign conditions on the initial values, Georgiev et al. [9] illustrate blow-up result of solution to problem (1.3) with $g(u_t) = u_t$ on the Heisenberg group when $1 < p < p_F(n)$. Wakasa et al. [38] consider formation of singularity of solution to problem (1.3) with scattering damping $\frac{\mu}{(1+t)^\beta}u_t\,(\beta > 1)$. Lifespan estimate of solution to the variable coefficient wave equation in the critical case is analyzed by employing rescaled test function method and iteration technique, which has been utilized in [39]. Problem (1.3) with damping term $g(u_t) = \frac{\mu}{(1+t)^\beta}u_t\,(\beta > 1)$, $\frac{\mu}{(1+|x|)^\beta}u_t\,(\beta > 2)$, $\mu(-\Delta)^{\frac{\sigma}{2}}u_t\,(\mu > 0,\, 0 < \sigma \leq 2)$ and derivative type nonlinear term $f(u, u_t) = |u_t|^p$ is considered in [7, 25, 28]. Lai et al. [25] derive upper bound lifespan estimate of solution to problem (1.3) with scattering damping term $g(u_t) = \frac{\mu}{(1+t)^\beta}u_t\,(\beta > 1)$ in the sub-critical and critical cases by introducing a bounded multiplier. Lai et al. [28] verify blow-up and lifespan estimate of solutions to problem (1.3) with space dependent damping term $g(u_t) = \frac{\mu}{(1+|x|)^\beta}u_t\,(\beta > 2)$ in the case $1 < p \leq p_G(n) = \frac{n+1}{n-1}$ by utilizing test function method ($\Psi = \partial_t\psi = \partial_t(-\eta_M^{2p'}(t)e^{-t}\phi_1(x))$). Dao et al. [7] investigate formation of singularity of solution to problem (1.3) with structural damping term $g(u_t) = \mu(-\Delta)^{\frac{\sigma}{2}}u_t\,(\mu > 0,\, 0 < \sigma \leq 2)$ and derivative nonlinearity. Problem (1.3) with damping term $g(u_t) = \frac{\mu}{1+t}u_t$, $\frac{\mu}{(1+t)^\beta}u_t\,(\beta > 1)$ and combined nonlinearities $f(u, u_t) = |u_t|^p + |u|^q$ is illustrated in [12, 26, 32, 33]. Applying the rescaled test function approach and iterative method, Ming et al. [33] establish upper bound lifespan estimate of solution to problem (1.3) with scattering damping and divergence form nonlinearity in the sub-critical and critical cases. Hamouda et al. [12] illustrate influence of scale invariant damping on the formation of singularity of solution. Lifespan estimate of solution is derived by imposing certain assumptions on the parameter μ. Liu and Wang [32] consider blow-up of solution to the semilinear wave equation with combined nonlinearities on asymptotically Euclidean manifolds in the case $n = 2$, $\mu = 0$.

Scholars focus widespread attention on the Cauchy problem for semilinear wave equation with damping term and mass term (see detailed illustrations in [1, 4, 11, 17, 22, 36, 37]). Taking advantage of the iteration method, Lai et al. [22] establish blow-up result of solution to the semilinear wave equation with scattering damping term and negative mass term, where the nonlinearity is $|u|^p$. Ikeda et al. [17] investigate lifespan estimate of solution to the semilinear wave equation with damping term, mass term as well as power nonlinearity in the sub-critical and critical cases by utilizing test function approach ($\psi(x, t) = \rho(t)\phi_1(x)$), which is inspired by [36]. Lai et al. [23] derive upper bound lifespan estimate of solution to the semilinear wave equation with damping term and mass term by employing the Kato lemma and iteration approach. Blow-up phenomenon and lifespan estimate of solution to the semilinear wave equation with scale invariant damping, non-negative mass term and power type of nonlinear term are documented in [36], where the iteration method is performed. Hamouda et al. [11] show blow-up dynamic of solution to the semilinear wave equation with scale invariant damping, mass term and combined nonlinearities. The proof is based on the multiplier technique and solving the ordinary differential inequality. We refer readers to the works in [2, 3, 5, 8, 10, 16, 20, 21, 29, 34, 35] for more details.

Enlightened by the works in [11, 17, 22, 24–26, 36], our interest is to show blow-up results of solutions to problems (1.1) and (1.2) with Neumann boundary conditions on exterior domain in three dimensions. It is worth pointing out that upper bound lifespan estimates of solutions to the Cauchy problem of semilinear wave equation with scattering damping term $\frac{\mu}{(1+t)^\beta}u_t\,(\mu > 0, \beta > 1)$ and nonlinear terms $|u|^p$, $|u_t|^p$, $|u_t|^p + |u|^q$ are discussed in [24–26]. Lai et al. [22] derive blow-up and lifespan estimate of solution to the semilinear wave equation with scattering damping and negative mass term by exploiting the test function technique and iterative approach, where the nonlinear term is $|u|^p$. However, there is no related result about blow-up dynamic of solution to problem (1.1). Thus, we extend the Cauchy problem studied in [24–26] to problem (1.1) with damping term, negative mass term and Neumann boundary condition on exterior domain in three dimensions. Upper bound lifespan estimate of solution to problem (1.1) is established by making use of a radial symmetry test function $\psi(x,t) = e^{-t}\frac{1}{r}e^{r}$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (see Theorems 1.1, 1.3–1.5). The Cauchy problem investigated in [23] is extended to problem (1.1) by utilizing the test function method ($\psi(x,t) = e^{-t}\frac{1}{r}e^{r}$) and the Kato lemma (see Theorem 1.2). We derive lifespan estimate of solution to problem (1.1) with $f(u,\ u_t) = |u|^p$ (see Theorem 1.6) by taking advantage of the test function approach ($\psi_1(x,t) = \rho(t)\frac{1}{r}e^{r}$), which is inspired by the work [17]. Making use of a multiplier, Hamouda et al. [11] verify blow-up phenomenon of solution to the semilinear wave equation with scale invariant damping and mass term as well as combined nonlinearities. We extend the problem discussed in [11] to problem (1.2). Upper bound lifespan estimate of solution to problem (1.2) with combined nonlinearities $f(u,\ u_t) = |u_t|^p + |u|^q$ is acquired by applying the test function technique ($\psi_2(x,t) = \rho_1(t)\frac{1}{r}e^{r}$) and iterative method (see Theorem 1.7). To the best of our knowledge, the results in Theorems 1.1–1.7 are new. Moreover, we characterize the variation of wave by utilizing numerical simulation.

Definitions of weak solutions and the main results in this paper are illustrated as follows.

Definition 1.1. *A function u is called a weak solution of problem* (1.1) *on* $[0,T)$ *if* $u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap L^p_{loc}((0,T)\times\Omega^c)$ *when* $f(u,\ u_t) = |u|^p$, $u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap C^1((0,T), L^p(\Omega^c))$ *when* $f(u,\ u_t) = |u_t|^p$, $u \in \cap_{i=0}^{1} C^i([0,T), H^{1-i}(\Omega^c)) \cap C^1((0,T), L^p(\Omega^c)) \cap L^q_{loc}((0,T)\times\Omega^c)$ *when* $f(u,\ u_t) = |u_t|^p + |u|^q$ *and*

$$
\begin{aligned}
&\int_{\Omega^c} u_t(x,t)\phi(x,t)dx - \int_{\Omega^c} \varepsilon g(x)\phi(x,0)dx \\
&+ \int_0^t ds \int_{\Omega^c} \{-u_t(x,s)\phi_t(x,s) - \Delta u(x,s)\phi(x,s)\}dx \\
&+ \int_0^t ds \int_{\Omega^c} b_1(s)u_t(x,s)\phi(x,s)dx - \int_0^t ds \int_{\Omega^c} b_2(s)u(x,s)\phi(x,s)dx \\
&= \int_0^t ds \int_{\Omega^c} f(u,\ u_t)(x,s)\phi(x,s)dx,
\end{aligned}
\tag{1.4}
$$

where $\phi \in C_0^\infty([0,T)\times\Omega^c)$ and $t \in [0,T)$.

Definition 1.2. *A function u is called a weak solution of problem* (1.2) *on* $[0,T)$ *if* $u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c))$, $u \in L^q_{loc}((0,T)\times\Omega^c)$, $u_t \in L^p_{loc}((0,T)\times\Omega^c)$ *when* $f(u,\ u_t) = |u_t|^p + |u|^q$ *and*

$$
\begin{aligned}
&\int_{\Omega^c} u_t(x,t)\phi(x,t)dx - \int_{\Omega^c} u_t(x,0)\phi(x,0)dx - \int_0^t \int_{\Omega^c} u_t(x,s)\phi_t(x,s)dxds \\
&+ \int_0^t \int_{\Omega^c} \nabla u(x,s)\nabla\phi(x,s)dxds + \int_0^t \int_{\Omega^c} \frac{\mu}{1+s}u_t(x,s)\phi(x,s)dxds \\
&+ \int_0^t \int_{\Omega^c} \frac{\nu^2}{(1+s)^2}u(x,s)\phi(x,s)dxds \\
&= \int_0^t \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)\phi(x,s)dxds,
\end{aligned}
\tag{1.5}
$$

where $\phi \in C_0^\infty([0,T)\times\Omega^c)$ and $t \in [0,T)$.

Setting

$$m(t) = (1+t)^{\mu},$$

we rewrite Definition 1.2 by choosing $m(t)\phi(x,t)$ as a test function.

Definition 1.3. *A function u is called a weak solution of problem* (1.2) *on $[0,T)$ if $u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c))$, $u \in L^q_{loc}((0,T)\times\Omega^c)$, $u_t \in L^p_{loc}((0,T)\times\Omega^c)$ when $f(u,\, u_t) = |u_t|^p + |u|^q$ and*

$$\begin{aligned}
&m(t)\int_{\Omega^c} u_t(x,t)\phi(x,t)dx - \int_{\Omega^c} u_t(x,0)\phi(x,0)dx \\
&-\int_0^t m(s)\int_{\Omega^c} u_t(x,s)\phi_t(x,s)dxds + \int_0^t m(s)\int_{\Omega^c} \nabla u(x,s)\nabla\phi(x,s)dxds \\
&+\int_0^t\int_{\Omega^c} \frac{\nu^2 m(s)}{(1+s)^2}u(x,s)\phi(x,s)dxds \\
&=\int_0^t m(s)\int_{\Omega^c}(|u_t(x,s)|^p + |u(x,s)|^q)\phi(x,s)dxds,
\end{aligned} \tag{1.6}$$

where $\phi \in C_0^\infty([0,T)\times\Omega^c)$ and $t\in[0,T)$.

Theorem 1.1. *Let $1 < p < p_c(3)$. Assume that $(f,g)\in H^1(\Omega^c)\times L^2(\Omega^c)$ are non-negative functions and f does not vanish identically. If a solution u to problem* (1.1) *with $f(u,\, u_t) = |u|^p$ satisfies supp$\,(u,\, u_t) \subset \{(x,t)\in\Omega^c\times[0,T)\big|\,|x|\le t+R\}$, then u blows up in finite time. Moreover, there exists a constant $\varepsilon_0 = \varepsilon_0(f,\, g,\, R,\, p,\, b_1(t),\, b_2(t)) > 0$ such that the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon)\le C\varepsilon^{\frac{-2p(p-1)}{r(p,3)}}, \tag{1.7}$$

where $0<\varepsilon\le\varepsilon_0$, $C>0$ is independent of ε.

Theorem 1.2. *Assume $b_1(t) = \frac{\nu_1}{(1+t)^{\beta}}$, $b_2(t) = \frac{\nu_2}{(1+t)^2}$, $\nu_1\ge 0$, $\beta>1$, $\nu_2>0$. Let $\delta = 1+4\nu_2 e^{\frac{\nu_1}{1-\beta}} > 1$, $d_*(3) = 2\sqrt{2}-2\in[0,2)$, $1<p<p_\delta(3)$ and*

$$p_\delta(3) = \max\{p_F(\frac{5-\sqrt{\delta}}{2}),\ p_c(3)\} = \begin{cases} p_c(3), & \sqrt{\delta}\le 3-d_*(3),\\ p_F(\frac{5-\sqrt{\delta}}{2}), & 3-d_*(3)<\sqrt{\delta}<5,\\ +\infty, & \sqrt{\delta}\ge 5.\end{cases}$$

Here, $p_F(n) = 1+\frac{2}{n}$ is the solution of equation $r_F(p,n) = 2-n(p-1) = 0$. Suppose that $(f,g)\in H^1(\Omega^c)\times L^2(\Omega^c)$ are non-negative functions and do not vanish identically. If a solution u to problem (1.1) *with $f(u,\, u_t) = |u|^p$ satisfies supp$\,(u,\, u_t)\subset\{(x,t)\in\Omega^c\times[0,T)\big|\,|x|\le t+R\}$, then u blows up in finite time. Moreover, the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon)\le\begin{cases} C\varepsilon^{\frac{-2p(p-1)}{r(p,3)}}, & \sqrt{\delta}\le 1,\\ C\varepsilon^{\frac{-(p-1)}{r_F(p,3-\frac{1+\sqrt{\delta}}{2})}}, & 1<\sqrt{\delta}<3-d_*(3),\ 1<p\le\frac{2}{3-\sqrt{\delta}},\\ C\varepsilon^{\frac{-2p(p-1)}{r(p,3)}}, & 1<\sqrt{\delta}<3-d_*(3),\ \frac{2}{3-\sqrt{\delta}}<p<p_\delta(3),\\ C\varepsilon^{-(\frac{2}{p-1}-3+\frac{1+\sqrt{\delta}}{2})^{-1}}, & \sqrt{\delta}\ge 3-d_*(3),\end{cases}$$

where $C > 0$ is independent of ε.

Theorem 1.3. *Let $1 < p \leq p_G(3) = 2$. Assume that $(f, g) \in H^1(\Omega^c) \times L^2(\Omega^c)$ are non-negative functions and g does not vanish identically. If the solution u to problem* (1.1) *with $f(u, u_t) = |u_t|^p$ satisfies supp $(u, u_t) \subset \{(x,t) \in \Omega^c \times [0,T) \big| |x| \leq t + R\}$, then u blows up in finite time. Moreover, the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p-1}{2-p}}, & 1 < p < p_G(3), \\ \exp(C\varepsilon^{-1}), & p = p_G(3), \end{cases}$$

where $C > 0$ is independent of ε.

Theorem 1.4. *Let $p > 1$ and $1 < q < \min\{1 + \frac{2}{p-1}, 6\}$. Assume that f and g satisfy the conditions in Theorem 1.3. If a solution u to problem* (1.1) *with $f(u, u_t) = |u_t|^p + |u|^q$ satisfies supp $(u, u_t) \subset \{(x,t) \in \Omega^c \times [0,T) \big| |x| \leq t + R\}$, then u blows up in finite time. Moreover, the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C\varepsilon^{-\frac{p(q-1)}{q+1-p(q-1)}},$$

where $C > 0$ is independent of ε.

Theorem 1.5. *Let $p > 3$ and $1 < q < 2$. Assume that f and g satisfy the conditions in Theorem 1.3. If the solution u to problem* (1.1) *with $f(u, u_t) = |u_t|^p + |u|^q$ satisfies supp $(u, u_t) \subset \{(x,t) \in \Omega^c \times [0,T) \big| |x| \leq t + R\}$, then u blows up in finite time. Moreover, the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C\varepsilon^{-\frac{q-1}{2(2-q)}},$$

where $C > 0$ is independent of ε.

Theorem 1.6. *Let $1 < p < p_c(3)$. Let f and g satisfy the conditions in Theorem 1.1. Suppose that $b_1(t) \in C^1([0,\infty))$ and $r_2(t) \in L^1([0,\infty))$ satisfy*

$$\begin{cases} r_2'(t) + b_1(t)r_2(t) - r_2^2(t) = -b_2(t), \\ r_2(t)|_{t=0} = r_2(0). \end{cases}$$

$\rho'(0)$ is the initial value of $\rho'(t)$, where $\rho(t)$ is the solution to problem (5.1)*. It holds that*

$$\begin{cases} g(x) + r_2(0)f(x) \geq 0, \\ g(x) + (b_1(0) - \rho'(0))f(x) \geq 0. \end{cases}$$

There is no sign requirement for $b_1(t)$ and $b_2(t)$. If a solution u to problem (1.1) *with $f(u, u_t) = |u|^p$ satisfies supp $(u, u_t) \subset \{(x,t) \in \Omega^c \times [0,T) \big| |x| \leq t + R\}$, then u blows up in finite time. Moreover, the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C\varepsilon^{\frac{-2p(p-1)}{r(p,3)}},$$

where $C > 0$ is independent of ε.

Theorem 1.7. *Let $p > p_G(3+\mu)$, $q > q_S(3+\mu)$, μ, $\nu^2 \geq 0$ and $\delta = (\mu-1)^2 - 4\nu^2 \geq 0$. Assume that $\lambda(p, q, 3+\mu) < 4$, where $\lambda(p, q, n) = (q-1)((n-1)p-2) < 4$. The initial values $(f, g) \in H^1(\Omega^c) \times L^2(\Omega^c)$ are non-negative functions which do not vanish identically and satisfy*

$$\frac{\mu - 1 - \sqrt{\delta}}{2} f(x) + g(x) > 0. \tag{1.8}$$

If a solution u to problem (1.2) *with $f(u, u_t) = |u_t|^p + |u|^q$ satisfies supp $(u, u_t) \subset \{(x,t) \in \Omega^c \times [0,T) \big| \, |x| \leq t + R\}$, then u blows up in finite time. Moreover, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, R, p, q, \mu, \nu) > 0$ such that the lifespan estimate $T(\varepsilon)$ satisfies*

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(q-1)}{4-\lambda(p,q,3+\mu)}}, \tag{1.9}$$

where $C > 0$ is independent of ε.

Remark 1.1. Utilizing the Sobolev embedding theorem yields $H^1(\Omega^c) \hookrightarrow L^q(\Omega^c)$ when $n = 3$, $q < 6$ in Theorems 1.4 and 1.5. Consequently, the nonlinear term $|u|^q$ in problem (1.1) is integrable in the domain $\Omega^c \subset \mathbb{R}^3$.

Remark 1.2. Taking advantage of the Poincare's inequality, we conclude

$$\int_{\Omega^c} |\nabla u|^p \psi dx \geq \frac{1}{(t+R)^p} \int_{\Omega^c} |u|^p \psi dx \geq C \int_{\Omega^c} |u|^p \psi dx.$$

Similar to the proof of Theorem 1.1, we obtain the same result in (1.7) when nonlinear term is $f(u, u_t) = |\nabla u|^p$.

Remark 1.3 We call that u is a global solution of problems (1.1) and (1.2) if the maximal existence time of solution $T_{max} = \infty$. While in the case $T_{max} < \infty$, we call that u blows up in finite time.

2. Proofs of Theorems 1.1 and 1.2

2.1. Several related lemmas

Lemma 2.1. *[35] Let $b_1(t) \in C([0,\infty)) \cap L^1([0,\infty))$ be a non-negative function, which satisfies*

$$m_1(t) = \exp(-\int_t^\infty b_1(\tau) d\tau),$$

$m_1(0) \leq m_1(t) \leq 1$, $\frac{m_1'(t)}{m_1(t)} = b_1(t)$ *for $t \geq 0$.*

Lemma 2.2. *Let $\phi_1(x) = \phi_1(r) = \frac{1}{r}e^r$, where $x = (x_1, x_2, x_3)$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It holds that*

$$\Delta\phi_1 = (\partial_{rr} + \frac{2}{r}\partial_r)\phi_1 = \phi_1$$

and $\frac{\partial\phi_1}{\partial r}|_{r=1} = 0$. Setting $\psi = e^{-t}\phi_1(x)$, it satisfies

$$\int_{\Omega^c \cap \{|x| \leq t+R\}} \psi^{\frac{p}{p-1}} dx \leq C(R+t)^{2-\frac{p}{p-1}}, \qquad \Delta\psi = \psi,$$

where C is a positive constant.

Proof of Lemma 2.2. Direct calculation shows

$$\frac{\partial \phi_1}{\partial x_i} = \frac{\partial \phi_1}{\partial r}\frac{x_i}{r},$$
$$\frac{\partial^2 \phi_1}{\partial x_i^2} = \frac{\partial^2 \phi_1}{\partial r^2}\frac{x_i^2}{r^2} + \frac{r^2 - x_i^2}{r^3}\frac{\partial \phi_1}{\partial r},$$

where $i = 1,\ 2,\ 3$. Thus, we obtain

$$\begin{aligned}\Delta\phi_1 &= \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_1}{\partial x_2^2} + \frac{\partial^2 \phi_1}{\partial x_3^2}\\ &= \frac{\partial^2 \phi_1}{\partial r^2}(\frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} + \frac{x_3^2}{r^2}) + \frac{\partial \phi_1}{\partial r}(\frac{r^2 - x_1^2}{r^3} + \frac{r^2 - x_2^2}{r^3} + \frac{r^2 - x_3^2}{r^3})\\ &= (\partial_{rr} + \frac{2}{r}\partial_r)\phi_1 = \phi_1.\end{aligned}$$

Employing $\psi = e^{-t}\frac{1}{r}e^r$ gives rise to

$$\begin{aligned}\int_{\Omega^c\cap\{|x|\le t+R\}} \psi^{\frac{p}{p-1}} dx &= \int_{\mathbb{S}^2} dw \int_1^{t+R} [e^{-t}\frac{1}{r}e^r]^{\frac{p}{p-1}} r^2 dr\\ &\le C\int_0^{t+R} [e^{-(t-r)}]^{\frac{p}{p-1}} (R+r)^{2-\frac{p}{p-1}} dr \le C(R+t)^{2-\frac{p}{p-1}}.\end{aligned}$$

We complete the proof of Lemma 2.2.

2.2. *Proof of Theorem 1.1*

Let us set three functions

$$\begin{cases} F_0(t) = \int_{\Omega^c} u(x,t)dx,\\ F_1(t) = \int_{\Omega^c} u(x,t)\psi(x,t)dx,\\ F_2(t) = \int_{\Omega^c} u_t(x,t)\psi(x,t)dx, \end{cases}$$

where $\psi(x,t) = e^{-t}\phi_1(x) = e^{-t}\frac{1}{r}e^r$. It holds that

$$\Box\psi = 0,\quad \Delta\psi = \psi,\quad (\psi)_t = -\psi,\quad (\psi)_{tt} = \psi. \tag{2.1}$$

By straightforward computation, we achieve

$$\int_{\Omega^c} \Delta u dx = \int_{\partial\Omega^c} 1\frac{\partial u}{\partial n} dS - \int_{\Omega^c} \nabla 1 \cdot \nabla u dx = 0. \tag{2.2}$$

Choosing the test function $\phi(x,s) \equiv 1$ on $(x,s) \in \{\Omega^c \times [0,t] \big| |x| \le s+R\}$ in (1.4) with $f(u,\ u_t) = |u|^p$ and utilizing (2.2) yield

$$F_0''(t) + b_1(t)F_0'(t) = b_2(t)F_0(t) + \int_{\Omega^c} |u(x,t)|^p dx. \tag{2.3}$$

Multiplying (2.3) with $m_1(t)$ and integrating on $[0, t]$, we deduce

$$F_0'(t) \geq m_1(0) \int_0^t \int_{\Omega^c} |u(x, s)|^p dxds, \tag{2.4}$$

where we have used the fact $F_0'(0) \geq 0$ and $F_0(t) > 0$.

We are in the position to establish the lower bound of $F_1(t)$. Elementary computation leads to

$$\frac{\partial u}{\partial n}|_{\partial\Omega^c} = \frac{\partial u}{\partial n}|_{r=1} = 0, \quad \frac{\partial \psi}{\partial n}|_{\partial\Omega^c} = \frac{\partial \psi}{\partial n}|_{r=1} = 0. \tag{2.5}$$

Applying (2.5) and the Green formula yields

$$\int_{\Omega^c} (\Delta u\psi - u\Delta\psi)dx = \int_{\partial\Omega^c} (\frac{\partial u}{\partial n}\psi - u\frac{\partial \psi}{\partial n})dS = 0.$$

Thus, we have

$$\int_{\Omega^c} \Delta u\psi dx = \int_{\Omega^c} u\Delta\psi dx = \int_{\Omega^c} u\psi dx. \tag{2.6}$$

Utilizing (2.1), (2.6) and replacing $\phi(x, s)$ in (1.4) with $f(u, u_t) = |u|^p$ by $\psi(x, s)$, we obtain

$$\begin{aligned}
& m_1(t) \int_{\Omega^c} u_t(x, t)\psi(x, t)dx - m_1(0)\varepsilon \int_{\Omega^c} g(x)\psi(x, 0)dx \\
& -m_1(t) \int_{\Omega^c} u(x, t)\psi_t(x, t)dx + m_1(0)\varepsilon \int_{\Omega^c} f(x)\psi_t(x, 0)dx \\
& + \int_0^t \int_{\Omega^c} m_1(s)b_1(s)u(x, s)\psi_t(x, s)dxds \\
& = \int_0^t \int_{\Omega^c} m_1(s)b_2(s)u(x, s)\psi(x, s)dxds \\
& + \int_0^t \int_{\Omega^c} m_1(s)|u(x, s)|^p\psi(x, s)dxds.
\end{aligned}$$

That is

$$\begin{aligned}
m_1(t)\{F_1'(t) + 2F_1(t)\} &= m_1(0)\varepsilon \int_{\Omega^c} \{f(x) + g(x)\}\phi_1(x)dx \\
&+ \int_0^t m_1(s)\{b_1(s) + b_2(s)\}F_1(s)ds + \int_0^t \int_{\Omega^c} m_1(s)|u(x, s)|^p\psi(x, s)dxds,
\end{aligned}$$

which leads to

$$F_1'(t) + 2F_1(t) \geq m_1(0)C_{f,g}\varepsilon + \int_0^t m_1(s)\{b_1(s) + b_2(s)\}F_1(s)ds,$$

where $C_{f,g} = \int_{\Omega^c} \{f(x) + g(x)\}\phi_1(x)dx > 0$.

Thanks to the positivity of $F_1(t)$ and $F_1(0)$, we deduce

$$F_1(t) > \frac{1 - e^{-2t}}{2} m_1(0)C_{f,g}\varepsilon, \tag{2.7}$$

where $t > 2$. Employing (2.4) and the Holder inequality yields

$$F_0(t) > C_1 m_1(0) \int_0^t ds \int_0^s (r+R)^{-3(p-1)} F_0^p(r) dr. \tag{2.8}$$

Making use of the Holder inequality and Lemma 2.2 gives rise to

$$\int_{\Omega^c} |u(x,t)|^p dx \geq \left(\int_{\Omega^c \cap \{|x| \leq t+R\}} (\psi(x,t))^{\frac{p}{p-1}} dx \right)^{-(p-1)} |F_1(t)|^p \geq C(t+R)^{2-p} |F_1(t)|^p. \tag{2.9}$$

Taking advantage of (2.4), (2.7) and (2.9), we acquire

$$F_0(t) > \frac{C_2 \varepsilon^p}{12} (R+t)^{-p} t^4.$$

We denote

$$F_0(t) > D_j (R+t)^{-a_j} t^{b_j}, \tag{2.10}$$

where

$$D_1 = \frac{C_2 \varepsilon^p}{12}, \quad a_1 = p, \quad b_1 = 4. \tag{2.11}$$

Combining (2.8) with (2.10), we derive

$$F_0(t) > \frac{C_1 m_1(0) D_j^p}{(pb_j + 2)^2} (R+t)^{-3(p-1)-pa_j} t^{pb_j + 2}.$$

Thus, we define the sequences $\{D_j\}_{j \in \mathbb{N}}$, $\{a_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}}$ by

$$D_{j+1} \geq \frac{C_1 m_1(0) D_j^p}{(pb_j + 2)^2}, \quad a_{j+1} = pa_j + 3(p-1), \quad b_{j+1} = pb_j + 2. \tag{2.12}$$

Exploiting (2.11), (2.12) and iterative argument gives rise to

$$a_j = p^{j-1}(p+3) - 3, \quad b_j = p^{j-1}\left(4 + \frac{2}{p-1}\right) - \frac{2}{p-1},$$
$$D_j \geq C_3 \frac{D_{j-1}^p}{p^{2(j-1)}} \geq \exp\{p^{j-1}(\log D_1 - S_p(\infty))\},$$

where $S_p(\infty)$ is obtained by using the d'Alembert's criterion. Moreover, $S_p(j) = \sum_{k=1}^{j-1} \frac{2k \log p - \log C_3}{p^k}$ converges to $S_p(\infty)$ as $j \to \infty$. As a consequence, making use of (2.10) yields

$$F_0(t) \geq (t+R)^3 t^{-\frac{2}{p-1}} \exp(p^{j-1} J(t)) \tag{2.13}$$

and

$$J(t) = -(p+3)\log(t+R) + \left(4 + \frac{2}{p-1}\right)\log t + \log D_1 - S_p(\infty) \geq \log\left(D_1 t^{\frac{r(p,3)}{2(p-1)}}\right) - C_4,$$

where $C_4 = (p+3)\log 2 + S_p(\infty) > 0$ and $t \geq R > 2$. Utilizing the condition $p < p_c(3)$, we arrive at $J(t) > 1$ when $t \geq C_5 \varepsilon^{-\frac{2p(p-1)}{r(p,3)}}$. Sending $j \to \infty$ in (2.13) yields $F_0(t) \to \infty$. Therefore, we derive the lifespan estimate

$$T(\varepsilon) \leq C_5 \varepsilon^{-\frac{2p(p-1)}{r(p,3)}}.$$

The proof of Theorem 1.1 is finished.

2.3. Proof of Theorem 1.2

Integrating (2.3) on $[0, t]$, we acquire

$$\begin{aligned} F_0(t) = F_0(0) + m_1(0)F_0'(0) \int_0^t \frac{1}{m_1(s)} ds \\ + \int_0^t \frac{1}{m_1(s)} ds \int_0^s m_1(r) b_2(r) F_0(r) dr \\ + \int_0^t \frac{1}{m_1(s)} ds \int_0^s m_1(r) dr \int_{\Omega^c} |u(x,s)|^p dx. \end{aligned} \tag{2.14}$$

Let us define two functions

$$\begin{aligned} \widetilde{F}_0(t) = \frac{1}{2} F_0(0) + \frac{m_1(0)}{2} F_0'(0) t + m_1(0) \int_0^t ds \int_0^s b_2(r) \widetilde{F}_0(r) dr \\ + m_1(0) \int_0^t ds \int_0^s dr \int_{\Omega^c} |u(x,r)|^p dx \end{aligned} \tag{2.15}$$

and

$$G(t) = (1+t)^{k+\lambda} F_0(t).$$

Thanks to $m_1(0) < m_1(t) < 1$ and $\nu_2 > 0$, we achieve

$$F_0(t) - \widetilde{F}_0(t) \geq \frac{1}{2} F_0(0) + \frac{m_1(0)}{2} F_0'(0) t + m_1(0) \int_0^t ds \int_0^s b_2(r) [F_0(r) - \widetilde{F}_0(r)] dr.$$

Applying comparison argument, we conclude $F_0(t) \geq \widetilde{F_0}(t)$. Employing (2.15) and the formula (4.2) with $\mu_1 = 0$, $\mu_2 = -m_1(0)\nu_2$ in [23] gives rise to

$$\widetilde{F}_0''(t) - b_2(t) m_1(0) \widetilde{F}_0(t) = m_1(0) \int_{\Omega^c} |u(x,t)|^p dx. \tag{2.16}$$

Similar to the derivation in the proof of Theorem 5 in [23], we derive

$$\begin{aligned} \widetilde{F}_0(t) = \widetilde{F}_0(0)(1+t)^{-k} + [k\widetilde{F}_0(0) + \widetilde{F}_0'(0)](1+t)^{-k} \int_0^t (1+s)^{-\lambda} ds \\ + (1+t)^{-k} \int_0^t (1+s)^{-\lambda} ds \int_0^s (1+r)^{k+\lambda} dr \times \int_{\Omega^c} |u(x,r)|^p dx, \end{aligned} \tag{2.17}$$

$$G(t) \gtrsim \int_{T_0}^t ds \int_{T_0}^s r^{-(3+k+\lambda)(p-1)} G(r)^p dr \tag{2.18}$$

and

$$G(t) \gtrsim \varepsilon t^{\lambda}. \tag{2.19}$$

Here, $A \gtrsim B$ means that there exists a positive constant C such that $A \geq CB$. Taking into account (2.7) and the Holder inequality, we obtain

$$\int_{\Omega^c} |u(x,t)|^p \gtrsim \varepsilon^p (t+R)^{2-p},$$

which together with (2.17) results in

$$\widetilde{F}_0(t) \gtrsim \varepsilon^p (1+t)^{-k} \int_{T_1}^t (1+s)^{-\lambda} ds \int_{T_1}^s (1+r)^{q+\sqrt{\delta}-1} dr,$$

where $t \geq T_1 > 0$, $q = -\frac{1+\sqrt{\delta}}{2} - p + 4$. Therefore, we arrive at

$$G(t) \gtrsim \varepsilon^p \begin{cases} t^{\lambda+q}, \quad q > 0, \\ t^{\lambda} \ln(1+t), \quad q = 0, \\ t^{\lambda}, \quad q < 0. \end{cases} \tag{2.20}$$

Utilizing (2.18)–(2.20) and the Kato lemma in Sub-section 4.3 in [23], we finishes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

Direct computation gives rise to

$$\begin{aligned} &\frac{d}{dt}[m_1(t) \int_{\Omega^c} \{u_t(x,t) + u(x,t)\}\psi(x,t)dx] \\ &= b_1(t)m_1(t) \int_{\Omega^c} \{u_t(x,t) + u(x,t)\}\psi(x,t)dx + m_1(t)\frac{d}{dt}\int_{\Omega^c} \{u_t(x,t) + u(x,t)\}\psi(x,t)dx. \end{aligned} \tag{3.1}$$

Making use of (1.4) and (2.6), we acquire

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega^c} \{u_t(x,t) + u(x,t)\}\psi(x,t)dx \\ &= \int_{\Omega^c} |u_t(x,t)|^p \psi(x,t)dx - b_1(t)\int_{\Omega^c} u_t(x,t)\psi(x,t)dx \\ &\quad + b_2(t)\int_{\Omega^c} u(x,t)\psi(x,t)dx. \end{aligned} \tag{3.2}$$

Plugging (3.2) into (3.1) yields

$$\begin{aligned} &\frac{d}{dt}[m_1(t) \int_{\Omega^c} \{u_t(x,t) + u(x,t)\}\psi(x,t)dx] \\ &= b_1(t)m_1(t) \int_{\Omega^c} u(x,t)\psi(x,t)dx + b_2(t)m_1(t) \int_{\Omega^c} u(x,t)\psi(x,t)dx \\ &\quad + m_1(t) \int_{\Omega^c} |u_t(x,t)|^p \psi(x,t)dx, \end{aligned} \tag{3.3}$$

which together with (2.7) results in

$$\begin{aligned}&m_1(t)\int_{\Omega^c}\{u_t(x,t)+u(x,t)\}\psi(x,t)dx\\&\geq m_1(0)\varepsilon\int_{\Omega^c}\{f(x)+g(x)\}\phi_1(x)dx\\&\quad+\int_0^t ds\int_{\Omega^c}m_1(s)|u_t(x,s)|^p\psi(x,s)dx.\end{aligned}\tag{3.4}$$

Combining (1.4), (2.1) and (2.6), we have

$$\begin{aligned}&\frac{d}{dt}[m_1(t)\int_{\Omega^c}u_t(x,t)\psi(x,t)dx]+m_1(t)\int_{\Omega^c}\{u_t(x,t)-u(x,t)\}\psi(x,t)dx\\&=m_1(t)\int_{\Omega^c}|u_t(x,t)|^p\psi(x,t)dx+m_1(t)b_2(t)\int_{\Omega^c}u(x,t)\psi(x,t)dx.\end{aligned}\tag{3.5}$$

An application of (3.4) and (3.5) gives rise to

$$\begin{aligned}&\frac{d}{dt}[m_1(t)\int_{\Omega^c}u_t(x,t)\psi(x,t)dx]+2m_1(t)\int_{\Omega^c}u_t(x,t)\psi(x,t)dx\\&\geq m_1(0)\varepsilon\int_{\Omega^c}\{f(x)+g(x)\}\phi_1(x)dx+m_1(t)\int_{\Omega^c}|u_t(x,t)|^p\psi(x,t)dx\\&\quad+\int_0^t ds\int_{\Omega^c}m_1(s)|u_t(x,s)|^p\psi(x,s)dx.\end{aligned}\tag{3.6}$$

We set

$$\begin{aligned}G(t)=&\,m_1(t)\int_{\Omega^c}u_t(x,t)\psi(x,t)dx-\frac{m_1(0)}{2}\varepsilon\int_{\Omega^c}g(x)\phi_1(x)dx\\&-\frac{1}{2}\int_0^t m_1(s)ds\int_{\Omega^c}|u_t(x,s)|^p\psi(x,s)dx,\end{aligned}\tag{3.7}$$

where $G(0)=\frac{m_1(0)\varepsilon}{2}\int_{\Omega^c}g(x)\phi_1(x)dx>0$. Taking into account (3.6), we acquire

$$G'(t)+2G(t)\geq\frac{m_1(t)}{2}\int_{\Omega^c}|u_t(x,t)|^p\psi(x,t)dx+m_1(0)\varepsilon\int_{\Omega^c}f(x)\phi_1(x)dx\geq 0.$$

It follows that $G(t)\geq e^{-2t}G(0)>0$ for $t\geq 0$. Thus, we conclude

$$\int_{\Omega^c}u_t(x,t)\psi(x,t)dx\geq\frac{m_1(0)\varepsilon}{2}\int_{\Omega^c}g(x)\phi_1(x)dx.\tag{3.8}$$

We define

$$H(t)=\frac{1}{2}\int_0^t m_1(s)ds\int_{\Omega^c}|u_t(x,s)|^p\psi(x,s)dx+\frac{m_1(0)}{2}\varepsilon\int_{\Omega^c}g(x)\phi_1(x)dx.$$

Applying the Holder inequality and (3.8) yields

$$H'(t)\geq\frac{C^{1-p}}{2(R+t)^{p-1}}H^p(t).$$

As a direct consequence, we have

$$-\frac{d}{dt}[H^{-p+1}(t)] \geq \frac{C^{1-p}}{2(R+t)^{p-1}}.$$

It is worth noticing that $H(0) = \frac{m_1(0)}{2}\varepsilon \int_{\Omega^c} g(x)\phi_1(x)dx$. Therefore, employing the assumption $1 < p \leq 2$, we derive the lifespan estimate in Theorem 1.3. The proof of Theorem 1.3 is finished.

4. Proofs of Theorems 1.4 and 1.5

4.1. Proof of Theorem 1.4

We are in the position to establish the estimate of $F_0(t)$. Choosing the test function $\phi(x,t) = 1$ in (1.4) yields

$$F_0''(t) + b_1(t)F_0'(t) = \int_{\Omega^c} \{|u_t(x,t)|^p + |u(x,t)|^q\}dx + b_2(t)F_0(t). \tag{4.1}$$

Multiplying (4.1) with $m_1(t)$ and integrating on $[0,t]$ yield

$$F_0'(t) \geq m_1(0)\int_0^t \int_{\Omega^c} \{|u_t(x,s)|^p + |u(x,s)|^q\}dxds, \tag{4.2}$$

where we have applied the fact $F_0'(0) \geq 0$ and $F_0(t) > 0$.

Similar to the estimates in (2.7) and (3.8), we obtain the estimates

$$F_1(t) \geq \frac{m_1(0)\varepsilon}{2}\int_{\Omega^c} f(x)\phi_1(x)dx \geq 0, \quad F_2(t) \geq \frac{m_1(0)\varepsilon}{2}\int_{\Omega^c} g(x)\phi_1(x)dx \geq 0$$

when nonlinear term is $f(u,\ u_t) = |u_t|^p + |u|^q$. Taking advantage of Lemma 2.2 and (3.8), we derive

$$\int_{\Omega^c} |u_t(x,t)|^p dx \geq \frac{|F_2(t)|^p}{\left(\int_{\Omega^c\cap\{|x|\leq t+R\}} (\psi(x,t))^{\frac{p}{p-1}}dx\right)^{p-1}} \geq \overline{C}_1\varepsilon^p(t+R)^{2-p}, \tag{4.3}$$

where $\overline{C}_1 = C^{1-p}(\frac{m_1(0)}{2}\int_{\Omega^c} g(x)\phi_1(x)dx)^p$. Plugging (4.3) into (4.2) leads to

$$F_0(t) \geq m_1(0)\overline{C}_1\varepsilon^p \int_0^t \int_0^s (r+R)^{2-p}drds \geq \overline{C}_2\varepsilon^p(t+R)^{-p}t^4. \tag{4.4}$$

Recalling (4.2), we acquire

$$F_0(t) \geq \overline{C}_3 m_1(0)\int_0^t \int_0^s (r+R)^{-3(q-1)}F_0^q(r)drds. \tag{4.5}$$

We set

$$F_0(t) \geq D_j(t+R)^{-a_j}t^{b_j}, \tag{4.6}$$

where

$$D_1 = \overline{C}_2\varepsilon^p, \quad a_1 = p, \quad b_1 = 4. \tag{4.7}$$

Inserting (4.6) into (4.5), we come to the estimate

$$F_0(t) \geq \frac{\overline{C}_3 m_1(0) D_j^q}{(qb_j+2)^2}(t+R)^{-3(q-1)-qa_j} t^{qb_j+2}.$$

Therefore, we denote the sequences $\{D_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$ by

$$D_{j+1} \geq \frac{\overline{C}_3 m_1(0) D_j^q}{(qb_j+2)^2}, \quad a_{j+1} = 3(q-1) + qa_j, \quad b_{j+1} = qb_j + 2. \tag{4.8}$$

Taking advantage of (4.7), (4.8) and iterative argument gives rise to

$$\begin{aligned} a_j &= q^{j-1}(p+3) - 3, \quad b_j = q^{j-1}(4 + \frac{2}{q-1}) - \frac{2}{q-1}, \\ D_j &\geq \overline{C}_4 \frac{D_{j-1}^q}{q^{2(j-1)}} \geq \exp\{q^{j-1}(\log D_1 - S(\infty))\}, \end{aligned}$$

where $S(j) = \sum\limits_{k=1}^{j-1} \frac{2k\log q - \log \overline{C}_4}{q^k}$ converges to $S(\infty)$ as $j \to \infty$.

From (4.6), we have

$$F_0(t) \geq (t+R)^3 t^{\frac{-2}{q-1}} \exp\{q^{j-1} J(t)\} \tag{4.9}$$

and

$$\begin{aligned} J(t) &\geq -(p+3)\log(2t) + (4 + \frac{2}{q-1})\log t + \log D_1 - S(\infty) \\ &= \log(t^{1+\frac{2}{q-1}-p} D_1) - \overline{C}_5, \end{aligned}$$

where $\overline{C}_5 = (p+3)\log 2 + S(\infty) > 0,\ t \geq R > 2$. Recalling the assumption $q < 1 + \frac{2}{p-1}$, we deduce that $J(t) > 1$ when $t > \overline{C}_6 \varepsilon^{\frac{-p(q-1)}{q+1-p(q-1)}}$. Sending $j \to \infty$ in (4.9) yields $F_0(t) \to \infty$. Therefore, we achieve the lifespan estimate

$$T \leq \overline{C}_7 \varepsilon^{\frac{-p(q-1)}{q+1-p(q-1)}}.$$

The proof of Theorem 1.4 is finished.

4.2. Proof of Theorem 1.5

We set $I[f] = \int_{\Omega^c} f(x)dx$. Utilizing (4.4) gives rise to

$$F_0(t) \geq C\varepsilon^p t^{4-p}$$

for sufficiently large t, where $C > 0$ is independent of ε. Thus, we deduce that (4.4) is weaker than the linear growth when $p > 3$. An application of (4.2) leads to

$$F_0'(t) \geq \frac{m_1(0)}{m_1(t)} F_0'(0) \geq m_1(0)\varepsilon \int_{\Omega^c} g(x)dx. \tag{4.10}$$

That is

$$F_0(t) \geq \overline{C}_8 \varepsilon t. \tag{4.11}$$

It is deduced from (4.5) and (4.11) that

$$F_0(t) \geq \overline{C}_9 \varepsilon^q \int_0^t \int_0^s (R+r)^{-3(q-1)} r^q dr ds \geq \overline{C}_{10} \varepsilon^q (R+t)^{-3(q-1)} t^{q+2}.$$

We assume

$$F_0(t) \geq \overline{D}_j (R+t)^{-\overline{a}_j} t^{\overline{b}_j}, \tag{4.12}$$

where

$$\overline{D}_1 = \overline{C}_{10} \varepsilon^q, \ \ \overline{a}_1 = 3(q-1), \ \ \overline{b}_1 = q+2. \tag{4.13}$$

Plugging (4.12) into (4.5), we derive

$$F_0(t) \geq \overline{D}_{j+1} (R+t)^{-q\overline{a}_j - 3(q-1)} t^{q\overline{b}_j + 2} \tag{4.14}$$

with

$$\overline{D}_{j+1} \geq \frac{\overline{C}_{11} m_1(0) \overline{D}_j^q}{(q\overline{b}_j + 2)^2}, \ \ \overline{a}_{j+1} = 3(q-1) + q\overline{a}_j, \ \ \overline{b}_{j+1} = q\overline{b}_j + 2. \tag{4.15}$$

Making use of (4.13) and (4.15), we conclude

$$\begin{aligned} &\overline{a}_j = 3q^j - 3, \ \ \overline{b}_j = q^{j-1}(q + 2 + \frac{2}{q-1}) - \frac{2}{q-1}, \\ &\overline{D}_j \geq \overline{C}_{12} \frac{\overline{D}_{j-1}^q}{q^{2(j-1)}} \geq \exp\{q^{j-1}(\log \overline{D}_1 - \overline{S}_q(\infty))\}. \end{aligned}$$

Applying (4.12) gives rise to

$$F_0(t) \geq (R+t)^3 t^{-\frac{2}{q-1}} \exp(q^{j-1} \overline{J}(t))$$

and

$$\overline{J}(t) = -3q \log(R+t) + (q + 2 + \frac{2}{q-1}) \log t + \log \overline{D}_1 - \overline{S}_q(\infty).$$

Bearing in mind $1 < q < 2$, we arrive at the lifespan estimate in Theorem 1.5. This completes the proof of Theorem 1.5.

5. Proof of Theorem 1.6

To outline the proof of Theorem 1.6, we recall the following Lemmas.

Lemma 5.1. *[17] Let $\rho(t)$ be a solution of the second order ODE*

$$\begin{cases}\rho''(t)-b_1(t)\rho'(t)+(-b_2(t)-1-b_1'(t))\rho(t)=0,\\ \rho(0)=1,\quad \rho(\infty)=0,\end{cases}\tag{5.1}$$

where $\rho(t)$ decays as e^{-t} for large t.

Lemma 5.2. *Let $\phi_1(x)=\phi_1(r)=\frac{1}{r}e^r$, where $x=(x_1,x_2,x_3)$ and $r=\sqrt{x_1^2+x_2^2+x_3^2}$. Setting $\psi_1(x,t)=\rho(t)\phi_1(x)$, it holds that*

$$\int_{\Omega^c\cap\{|x|\le t+R\}}(\psi_1(x,t))^{\frac{p}{p-1}}dx\le C(R+t)^{2-\frac{p}{p-1}},\qquad \Delta\psi_1=\psi_1,$$

where $\rho(t)\sim e^{-t}$, C is a positive constant.

Proof of Lemma 5.2. Taking into account $\psi_1=\rho(t)\frac{1}{r}e^r$, we obtain

$$\begin{aligned}\int_{\Omega^c\cap\{|x|\le t+R\}}(\psi_1)^{\frac{p}{p-1}}dx=&\int_{\mathbb{S}^2}dw\int_1^{t+R}[\rho(t)\frac{1}{r}e^r]^{\frac{p}{p-1}}r^2dr\\ \le& C\int_0^{t+R}[\rho(t-r)]^{\frac{p}{p-1}}(R+r)^{2-\frac{p}{p-1}}dr\le C(R+t)^{2-\frac{p}{p-1}}.\end{aligned}$$

We finish the proof of Lemma 5.2.

Proof of Theorem 1.6. Let us define the functions

$$\begin{cases}F_0(t)=\int_{\Omega^c}u(x,t)dx,\\ F_1(t)=\int_{\Omega^c}u(x,t)\psi_1(x,t)dx,\end{cases}$$

where $\psi_1(x,t)=\rho(t)\phi_1(x)$.

Choosing the test function $\phi(x,s)\equiv 1$ on $\{(x,s)\in\Omega^c\times[0,t]\big|\,|x|\le s+R\}$ in (1.4) with $f(u,\,u_t)=|u|^p$, we have

$$F_0''(t)+b_1(t)F_0'(t)-b_2(t)F_0(t)=\int_{\Omega^c}|u(x,t)|^pdx.\tag{5.2}$$

We rewrite (5.2) into the form

$$\begin{aligned}&F_0''(t)+b_1(t)F_0'(t)-b_2(t)F_0(t)\\ &=[F_0'(t)+r_2(t)F_0(t)]'+r_1(t)[F_0'(t)+r_2(t)F_0(t)],\end{aligned}\tag{5.3}$$

where $r_1(t)$ and $r_2(t)$ satisfy

$$\begin{cases}r_1(t)+r_2(t)=b_1(t),\\ r_2'(t)+r_1(t)r_2(t)=-b_2(t).\end{cases}$$

Multiplying both sides of (5.3) by $\exp\int_{s_2}^{s_1} r_1(\tau)d\tau$, integrating over $[0, s_2]$ and applying $g(x) + r_2(0)f(x) \geq 0$ yield

$$F_0'(s_2) + r_2(s_2)F_0(s_2) \geq \int_0^{s_2} e^{\int_{s_2}^{s_1} r_1(\tau)d\tau} \int_{\Omega^c} |u(x, s_1)|^p dxds_1. \tag{5.4}$$

Multiplying (5.4) by $\exp\int_t^{s_2} r_2(\tau)d\tau$ leads to

$$F_0(t) \geq \int_0^t e^{\int_t^{s_2} r_2(\tau)d\tau} \int_0^{s_2} e^{\int_{s_2}^{s_1} r_1(\tau)d\tau} \int_{\Omega^c} |u(x, s_1)|^p dxds_1ds_2. \tag{5.5}$$

Replacing $\phi(x, s)$ with $\psi_1(x, s)$ in (1.4) in the case $f(u, u_t) = |u|^p$ and employing (2.6), we derive

$$\begin{aligned}
&\int_0^t \int_{\Omega^c} u_{tt}(x, s)\psi_1(x, s)dxds - \int_0^t \int_{\Omega^c} u(x, s)\psi_1(x, s)dxds \\
&+ \int_0^t \int_{\Omega^c} \partial_s(b_1(s)\psi_1(x, s)u(x, s)) - \partial_s(b_1(s)\psi_1(x, s))u(x, s)dxds \\
&- \int_0^t \int_{\Omega^c} b_2(s)\psi_1(x, s)u(x, s)dxds \\
&= \int_0^t \int_{\Omega^c} |u(x, s)|^p\psi_1(x, s)dxds.
\end{aligned} \tag{5.6}$$

Employing Lemma 5.1 and (5.6), we deduce

$$F_1'(t) + (b_1(t) - 2\frac{\rho'(t)}{\rho(t)})F_1(t) \geq \varepsilon C_{f,g}, \tag{5.7}$$

where $C_{f,g} = \int_{\Omega^c}(g(x) + (b_1(0) - \rho'(0))f(x))\phi_1(x)dx > 0$.

Multiplying (5.7) with $\frac{1}{\rho^2(t)}e^{\int_0^t b_1(\tau)d\tau}$ yields

$$F_1(t) \geq \varepsilon C_{f,g,b_1(t)} \int_0^t \frac{\rho^2(t)}{\rho^2(s)}ds. \tag{5.8}$$

Utilizing Lemma 5.2 gives rise to

$$\int_{\Omega^c} |u(x, t)|^p dx \geq \frac{|F_1(t)|^p}{(\int_{\Omega^c\cap\{|x|\leq t+R\}}(\psi_1(x, t))^{\frac{p}{p-1}}dx)^{p-1}} \geq C\varepsilon^p\langle t\rangle^{2-p}, \tag{5.9}$$

where $\langle t\rangle = 3 + |t|$. Taking advantage of (5.5) and Lemma 2.1 in [17] leads to

$$F_0(t) \geq C_{r_1,r_2} \int_0^t \int_0^{s_2} F_0^p(s_1)(s_1 + R)^{3(1-p)}ds_1ds_2. \tag{5.10}$$

Similar to the iteration argument in Theorem 1.1, we derive the lifespan estimate in Theorem 1.6. The proof of Theorem 1.6 is finished.

6. Proof of Theorem 1.7

6.1. Several lemmas

Lemma 6.1. *[11] Assume that $\rho_1(t)$ is solution of*

$$\frac{d^2\rho_1(t)}{dt^2}-\rho_1(t)-\frac{d}{dt}(\frac{\mu}{1+t}\rho_1(t))+\frac{\nu^2}{(1+t)^2}\rho_1(t)=0. \tag{6.1}$$

The expression of $\rho_1(t)$ is

$$\rho_1(t)=(1+t)^{\frac{\mu+1}{2}}K_{\frac{\sqrt{\delta}}{2}}(1+t),$$

where $K_\xi(t)=\sqrt{\frac{\pi}{2t}}e^{-t}(1+O(t^{-1}))$ as $t\to\infty$ and $K'_\xi(t)=-K_{\xi+1}(t)+\frac{\xi}{t}K_\xi(t)$. It holds that

$$\frac{\rho_1'(t)}{\rho_1(t)}=-1+O(t^{-1}),\quad t\to\infty. \tag{6.2}$$

Let $\phi_1(x)=\phi_1(r)=\frac{1}{r}e^r$, where $x=(x_1,x_2,x_3)$ and $r=\sqrt{x_1^2+x_2^2+x_3^2}$. Setting $\psi_2(x,t)=\rho_1(t)\phi_1(x)$, it holds that

$$\partial_t^2\psi_2(x,t)-\Delta\psi_2(x,t)-\frac{\partial}{\partial t}(\frac{\mu}{1+t}\psi_2(x,t))+\frac{\nu^2}{(1+t)^2}\psi_2(x,t)=0 \tag{6.3}$$

and

$$\int_{\Omega^c\cap\{|x|\le t+R\}}(\psi_2(x,t))^{\frac{p}{p-1}}dx\le C\rho_1^{\frac{p}{p-1}}e^{\frac{pt}{p-1}}(t+R)^{2-\frac{p}{p-1}} \tag{6.4}$$

for some positive constant C.

Proof of Lemma 6.1. Applying $\psi_2=\rho_1(t)\frac{1}{r}e^r$ gives rise to

$$\begin{aligned}\int_{\Omega^c\cap\{|x|\le t+R\}}(\psi_2)^{\frac{p}{p-1}}dx=&\int_{\mathbb{S}^2}dw\int_1^{t+R}[\rho_1(t)\frac{1}{r}e^r]^{\frac{p}{p-1}}r^2dr\\ \le&C\int_0^{t+R}[\rho_1(t)e^r]^{\frac{p}{p-1}}r^{2-\frac{p}{p-1}}dr\\ \le&C\rho_1^{\frac{p}{p-1}}e^{\frac{pt}{p-1}}(t+R)^{2-\frac{p}{p-1}}.\end{aligned}$$

We complete the proof of Lemma 6.1.

We denote two functions

$$\begin{cases}G_1(t)=\int_{\Omega^c}u(x,t)\psi_2(x,t)dx,\\ G_2(t)=\int_{\Omega^c}u_t(x,t)\psi_2(x,t)dx.\end{cases}$$

Lemma 6.2. *Let u be a weak solution of problem* (1.2)*. If $(p,\ q)$ and $(f(x),\ g(x))$ satisfy the conditions in Theorem 1.7, then there exists $T_0=T_0(\mu,\nu)>1$ such that*

$$G_1(t)\ge C_{G_1}\varepsilon, \tag{6.5}$$

where $t\ge T_0$, C_{G_1} is a positive constant which depends on $f,\ g,\ \mu,\ \nu$.

Proof of Lemma 6.2. Replacing $\phi(x,t)$ in (1.5) by $\psi_2(x,t) = \rho_1(t)\phi_1(x)$ and employing (6.3), we derive

$$\begin{aligned}&\int_{\Omega^c} (u_t(x,t)\psi_2(x,t) - u(x,t)\partial_t\psi_2(x,t) + \frac{\mu}{1+t}u(x,t)\psi_2(x,t))dx \\ &= \int_0^t \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)\psi_2(x,s)dxds + \varepsilon C(f,g),\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}C(f,g) = K_{\frac{\sqrt{\delta}}{2}}(1)\int_{\Omega^c} ((\frac{\mu-1-\sqrt{\delta}}{2}f(x) + g(x))\phi_1(x)dx \\ +K_{\frac{\sqrt{\delta}}{2}+1}(1)\int_{\Omega^c} g(x)\phi_1(x)dx > 0.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&G_1'(t) + (\frac{\mu}{1+t} - 2\frac{\rho_1'(t)}{\rho_1(t)})G_1(t) \\ &= \int_0^t \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)\psi_2(x,s)dxds + \varepsilon C(f,g).\end{aligned} \tag{6.7}$$

Multiplying (6.7) by $\frac{1}{\rho_1^2(t)}(1+t)^\mu$, integrating over $(0,t)$ and exploiting Lemma 6.1 yield

$$\begin{aligned}G_1(t) &\geq G_1(0)\frac{\rho_1^2(t)}{(1+t)^\mu} + \varepsilon C(f,g)\frac{\rho_1^2(t)}{(1+t)^\mu}\int_0^t \frac{(1+s)^\mu}{\rho_1^2(s)}ds \\ &\geq \varepsilon C(f,g)(1+t)K^2_{\frac{\sqrt{\delta}}{2}}(1+t)\int_{\frac{t}{2}}^t \frac{1}{K^2_{\frac{\sqrt{\delta}}{2}}(1+s)}ds \\ &\geq \frac{\varepsilon}{4}C(f,g)e^{-2t}\int_{\frac{t}{2}}^t e^{2s}ds \\ &\geq \frac{\varepsilon}{16}C(f,g)\end{aligned} \tag{6.8}$$

for $t > T_0(\mu,\nu) > 1$, where $G_1(0) = \varepsilon K_{\frac{\sqrt{\delta}}{2}}(1)\int_{\Omega^c} f(x)\phi_1(x)dx > 0$. This finishes the proof of Lemma 6.2.

Lemma 6.3. *Let u be a weak solution of problem* (1.2)*. If* (p, q) *and* $(f(x), g(x))$ *satisfy the conditions in Theorem 1.7, it holds that*

$$G_2(t) + C\nu^2(1 + \nu^{\frac{2}{p-1}}e^{\frac{p}{p-1}t}(1+t)) \geq 0, \tag{6.9}$$

where C is a positive constant which depends on p, f, g, R, ε_0, μ but not on ε, ν.

Proof of Lemma 6.3. We define two functions

$$\begin{cases}F_1(t) = \int_{\Omega^c} u(x,t)\psi(x,t)dx, \\ F_2(t) = \int_{\Omega^c} u_t(x,t)\psi(x,t)dx.\end{cases}$$

Replacing $\phi(x,s)$ in (1.6) by $\psi(x,t)$ and using the fact $F_1'(t)+F_1(t)=F_2(t)$ lead to

$$
\begin{aligned}
& m(t)(F_1(t)+F_2(t))-\varepsilon C(f,g)+\int_0^t \frac{\nu^2 m(s)}{(1+s)^2}F_1(s)ds \\
&= \int_0^t m(s)\int_{\Omega^c}(|u_t(x,s)|^p+|u(x,s)|^q)\psi(x,s)dxds+\int_0^t m'(s)F_1(s)ds,
\end{aligned}
\tag{6.10}
$$

where $C(f,g)=\int_{\Omega^c}(f(x)+g(x))\phi_1(x)dx$.

Therefore, we arrive at

$$
\begin{aligned}
& \frac{d}{dt}(F_2(t)m(t))+2m(t)F_2(t) \\
&= m(t)(F_1(t)+F_2(t))-\frac{\nu^2 m(t)}{(1+t)^2}F_1(t) \\
&\quad +m(t)\int_{\Omega^c}(|u_t(x,t)|^p+|u(x,t)|^q)\psi(x,t)dx.
\end{aligned}
\tag{6.11}
$$

Combining (6.8), (6.10) and (6.11), we deduce

$$
\begin{aligned}
& \frac{d}{dt}(F_2(t)m(t))+2m(t)F_2(t) \\
&= \varepsilon C(f,g)+\int_0^t m(s)\int_{\Omega^c}(|u_t(x,s)|^p+|u(x,s)|^q)\psi(x,s)dxds \\
&\quad +m(t)\int_{\Omega^c}(|u_t(x,t)|^p+|u(x,t)|^q)\psi(x,t)dx \\
&\quad +\int_0^t m'(s)F_1(s)ds-\nu^2\int_0^t \frac{m(t)}{(1+s)^2}F_1(s)ds \\
&\quad -\nu^2\frac{m(t)}{(1+t)^2}F_1(t) \\
&\geq \int_0^t\int_{\Omega^c}|u_t(x,s)|^p\psi(x,s)dxds-C\varepsilon\nu^2-C\nu^2\int_0^t e^s|F_2(s)|ds,
\end{aligned}
\tag{6.12}
$$

where $C(f,g)=\int_{\Omega^c}(f(x)+g(x))\phi_1(x)dx$, we have applied the facts $G_1(t)=e^t\rho_1(t)F_1(t)$, $F_1'(t)+F_1(t)=F_2(t)$ and $m(t)\geq 1$.

Taking advantage of the Holder inequality and Lemma 2.2 yields

$$
C\nu^2\int_0^t e^s|F_2(s)|ds \leq \int_0^t\int_{\Omega^c}|u_t(x,s)|^p\psi(x,s)dxds+C\nu^{\frac{2p}{p-1}}e^{\frac{p}{p-1}t}(1+t). \tag{6.13}
$$

Making use of (6.12) and (6.13), we have

$$
\frac{d}{dt}(e^{2t}F_2(t)m(t))+C\nu^2e^{2t}+C\nu^{\frac{2p}{p-1}}e^{\frac{3p-2}{p-1}t}(1+t)\geq 0. \tag{6.14}
$$

As a consequent, it holds that

$$
G_2(t)+C\nu^2e^t\rho_1(t)(1+t)^{-\mu}+C\nu^{\frac{2p}{p-1}}e^t\rho_1(t)e^{\frac{p}{p-1}t}(1+t)^{1-\mu}\geq 0, \tag{6.15}
$$

where we have used $G_2(t) = e^t\rho_1(t)F_2(t)$.

An application of (6.15) and the fact $\rho_1(t)e^t \le C(1+t)^{\frac{\mu}{2}}$ gives rise to

$$G_2(t) + C\nu^2(1 + \nu^{\frac{2}{p-1}} e^{\frac{p}{p-1}t}(1+t)) \ge 0. \tag{6.16}$$

This ends the proof of Lemma 6.3.

Lemma 6.4. *Let u be a weak solution of problem* (1.2)*. If* (p, q) *and* $(f(x), g(x))$ *satisfy the conditions in Theorem 1.7, then there exists* $T_1 > 0$ *such that*

$$G_2(t) \ge C_{G_2}\varepsilon, \quad t \ge T_1 = -\ln(\varepsilon), \tag{6.17}$$

where C_{G_2} *is a positive constant which depends on* $p, f, g, R, \varepsilon_0, \nu, \mu$.

Proof of Lemma 6.4. Applying (6.7) and the fact $G_1'(t) - \frac{\rho_1'(t)}{\rho_1(t)}G_1(t) = G_2(t)$ leads to

$$\begin{aligned} &G_2(t) + (\frac{\mu}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)})G_1(t) \\ &= \int_0^t \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)\psi_2(x,s)dxds + \varepsilon C(f,g). \end{aligned} \tag{6.18}$$

Taking into account (6.1), (6.2), (6.18) and Lemma 6.2, we derive

$$\begin{aligned} &G_2'(t) + \frac{3}{4}(\frac{\mu}{1+t} - 2\frac{\rho_1'(t)}{\rho_1(t)})G_2(t) \\ &\ge I_4(t) + I_5(t) + \int_{\Omega^c} (|u_t(x,t)|^p + |u(x,t)|^q)\psi_2(x,t)dx \\ &\ge C\varepsilon, \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} I_4(t) &= (-\frac{\rho_1'(t)}{2\rho_1(t)} - \frac{\mu}{4(1+t)})(G_2(t) + (\frac{\mu}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)})G_1(t)) \\ &\ge C\varepsilon + \frac{1}{4}\int_0^t \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)\psi_2(x,s)dxds \end{aligned}$$

for $t > \widetilde{T}_1(\mu,\nu) \ge T_0$,

$$I_5(t) = (1 - \frac{\nu^2}{(1+t)^2} + (\frac{\rho_1'(t)}{2\rho_1(t)} + \frac{\mu}{4(1+t)})(\frac{\mu}{1+t} - \frac{\rho_1'(t)}{\rho_1(t)}))G_1(t) \ge 0$$

for $t > \widetilde{T}_2(\mu,\nu) \ge \widetilde{T}_1(\mu,\nu)$.

Utilizing (6.19) and Lemma 6.3, we conclude

$$G_2(t) \ge C_{G_2}\varepsilon \tag{6.20}$$

for $t \ge T_1 = -\ln\varepsilon$. This completes the proof of Lemma 6.4.

6.2. Proof of Theorem 1.7

We define the function

$$F(t) = \int_{\Omega^c} u(x,t)dx. \tag{6.21}$$

Choosing the test function $\phi(x,t) \equiv 1$ in (1.5) yields

$$F''(t) + \frac{\mu}{1+t}F'(t) + \frac{\nu^2}{(1+t)^2}F(t) = \int_{\Omega^c} (|u_t(x,t)|^p + |u(x,t)|^q)dx. \tag{6.22}$$

Therefore, we obtain

$$(F'(t) + \frac{r_1}{1+t}F(t))' + \frac{r_2+1}{1+t}(F'(t) + \frac{r_1}{1+t}F(t)) = \int_{\Omega^c} (|u_t(x,t)|^p + |u(x,t)|^q)dx, \tag{6.23}$$

where $r_1 = \frac{\mu-1-\sqrt{\delta}}{2}$ and $r_2 = \frac{\mu-1+\sqrt{\delta}}{2}$ are real roots of the quadratic equation $r^2 - (\mu-1)r + \nu^2 = 0$.

It is deduced from (1.8) and (6.23) that

$$F(t) \geq \int_0^t (\frac{1+\tau}{1+t})^{r_1} d\tau \int_0^\tau (\frac{1+s}{1+\tau})^{r_2+1} ds \int_{\Omega^c} (|u_t(x,s)|^p + |u(x,s)|^q)dx. \tag{6.24}$$

Making use of the Holder inequality and (6.24), we acquire

$$F(t) \geq C\int_0^t (\frac{1+\tau}{1+t})^{r_1} d\tau \int_0^\tau (\frac{1+s}{1+\tau})^{r_2+1} (1+s)^{-3(q-1)}|F(s)|^q ds, \tag{6.25}$$

where $C = (means(B_1))^{1-q}R^{-3(q-1)} > 0$.

Employing Lemma 6.4, (6.4) and the fact $\rho_1(t)e^t \leq C(1+t)^{\frac{\mu}{2}}$ gives rise to

$$\begin{aligned}\int_{\Omega^c} |u_t(x,t)|^p dx &\geq G_2^p(t)(\int_{\Omega^c \cap \{|x|\leq t+R\}} (\psi_2(x,t))^{\frac{p}{p-1}} dx)^{-(p-1)} \\ &\geq \widetilde{C}_1 \varepsilon^p (t+R)^{-\frac{\mu p+2(p-2)}{2}}.\end{aligned} \tag{6.26}$$

Plugging (6.26) into (6.24), we deduce

$$\begin{aligned}F(t) &\geq \widetilde{C}_1\varepsilon^p \int_0^t (\frac{1+\tau}{1+t})^{r_1} d\tau \int_0^\tau (\frac{1+s}{1+\tau})^{r_2+1} (s+R)^{-\frac{\mu p+2(p-2)}{2}} ds \\ &\geq \widetilde{C}_1\varepsilon^p (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1-(2+\mu)\frac{p}{2}} d\tau \int_{T_0}^\tau (1+s)^{3+r_2} ds \\ &\geq \widetilde{C}_1\varepsilon^p (1+t)^{-r_2-1-(2+\mu)\frac{p}{2}} \int_{T_0}^t d\tau \int_{T_0}^\tau (s-T_0)^{3+r_2} ds \\ &\geq \frac{\widetilde{C}_1}{(4+r_2)(5+r_2)} \varepsilon^p (t+R)^{-r_2-1-(2+\mu)\frac{p}{2}} (t-T_0)^{5+r_2}\end{aligned} \tag{6.27}$$

for $t > T_0$.

We set

$$F(t) \geq D_j (t+R)^{-a_j} (t-T_0)^{b_j}, \tag{6.28}$$

where

$$D_1 = \frac{\widetilde{C}_1}{(4+r_2)(5+r_2)}, \ \ a_1 = r_2 + 1 + (2+\mu)\frac{p}{2}, \ \ b_1 = 5 + r_2. \tag{6.29}$$

Utilizing (6.25) and (6.28), we have

$$\begin{aligned} F(t) &\geq C D_j^q (1+t)^{-r_2-1-3(q-1)-qa_j} \int_{T_0}^t \int_{T_0}^\tau (s-T_0)^{r_2+1+qb_j} ds d\tau \\ &\geq \frac{C D_j^q}{(r_2+qb_j+2)(r_2+qb_j+3)} (t+R)^{-r_2-1-3(q-1)-qa_j} \\ &\quad \times (t-T_0)^{r_2+qb_j+3}. \end{aligned} \tag{6.30}$$

We denote the sequences $\{D_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$ by

$$D_{j+1} \geq \frac{C D_j^q}{(r_2+qb_j+2)(r_2+qb_j+3)}, \tag{6.31}$$

$$a_{j+1} = r_2 + 1 + 3(q-1) + qa_j, \quad b_{j+1} = r_2 + qb_j + 3. \tag{6.32}$$

Taking advantage of (6.29), (6.31) and (6.32) leads to

$$a_j = q^{j-1}(a_1 + 3 + \frac{r_2+1}{q-1}) - (3 + \frac{r_2+1}{q-1}), \tag{6.33}$$

$$b_j = q^{j-1}(b_1 + \frac{r_2+3}{q-1}) - \frac{r_2+3}{q-1}, \tag{6.34}$$

$$D_j \geq \exp\{q^{j-1}(\log D_1 - S_q(\infty))\}, \tag{6.35}$$

where $S_q(j) = \frac{2q\log q}{(q-1)^2} - \frac{q\log C}{q-1}$ converges to $S_q(\infty)$ as $j \to \infty$.

Employing (6.28), (6.29) and (6.33)–(6.35), we achieve

$$F(t) \geq \exp(q^{j-1}J(t))(t+R)^{3+\frac{r_2+1}{q-1}}(t-T_0)^{\frac{r_2+3}{q-1}} \tag{6.36}$$

and

$$\begin{aligned} J(t) &= \log D_1 - S_q(\infty) - (a_1 + 3 + \frac{r_2+1}{q-1}) \log(t+R) \\ &\quad + (b_1 + \frac{r_2+3}{q-1}) \log(t-T_0) \\ &\geq \log\left(D_1 (t-T_0)^{\frac{4-((2+\mu)p-2)(q-1)}{2(q-1)}}\right) - S_q(\infty) \\ &\quad - (a_1 + 3 + \frac{r_2+1}{q-1}) \log 2 \end{aligned} \tag{6.37}$$

for $t > 2T_0 + 1$. Recalling $p > p_G(3+\mu)$, $q > q_S(3+\mu)$ and $\lambda(p, q, 3+\mu) < 4$, we conclude lifespan estimate (1.9) in Theorem 1.7. This completes the proof of Theorem 1.7.

7. Numerical simulation

We are in the position to show variation of wave for the Cauchy problem of semilinear wave equation in two dimensions. All codes are written and run with Matlab2014a on Windows 10 (64bite), RAM:8G and CPU 3.60 GHz. That is,

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = (\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2})u + |u|^3, \end{cases} \tag{7.1}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = (\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2})u - u_t + u + |u|^3. \end{cases} \tag{7.2}$$

Suppose that the initial values satisfy

$$u|_{t=0} = e^{-20[(x-0.4)^2+(y+0.4)^2]} + e^{-20[(x+0.4)^2+(y-0.4)^2]}, \quad \frac{\partial u}{\partial t}|_{t=0} = 0.$$

The following two group figures indicate the propagation of wave in two dimensions.

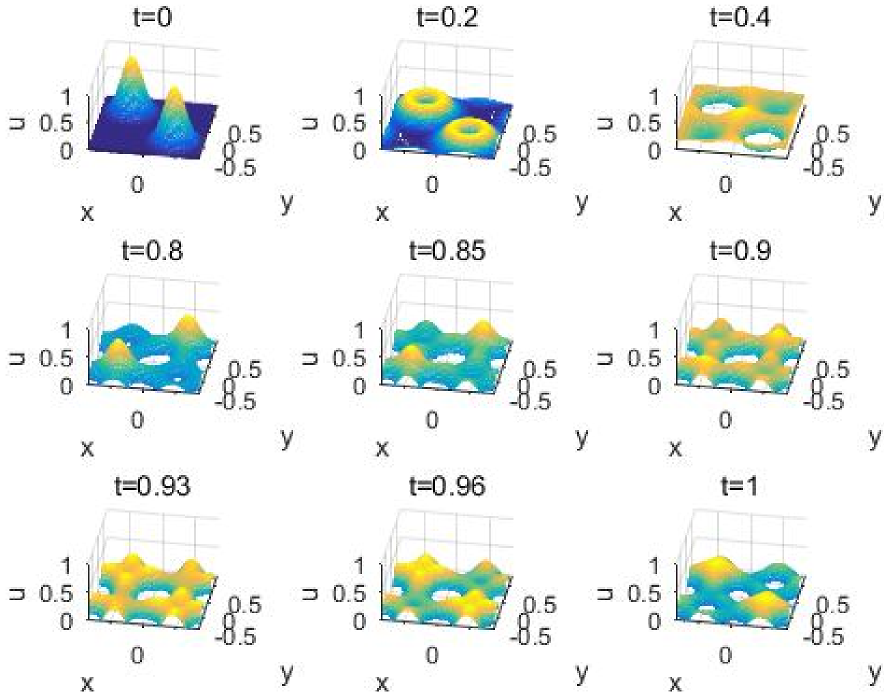

Figure 1. Wave variation of semilinear wave equation.

Figure 1 represents the trend of wave from $t = 0$ s to $t = 1$ s when nonlinear term is $|u|^3$ in problem (7.1). It indicates that there are two peaks of wave when $t = 0$ s. With the increase of time, two peaks of wave move downward until they disappear. Then, two new wave peaks appear at different positions when $t = 0.8$ s. From $t = 0.8$ s to $t = 0.9$ s, the new wave amplitudes decrease gradually. However, when $t = 0.9$ s $\sim$ 1 s, the old wave amplitudes increase constantly. When $t = 1$ s, the wave peaks appear again and the position of wave peaks is same as the position of $t = 0$ s.

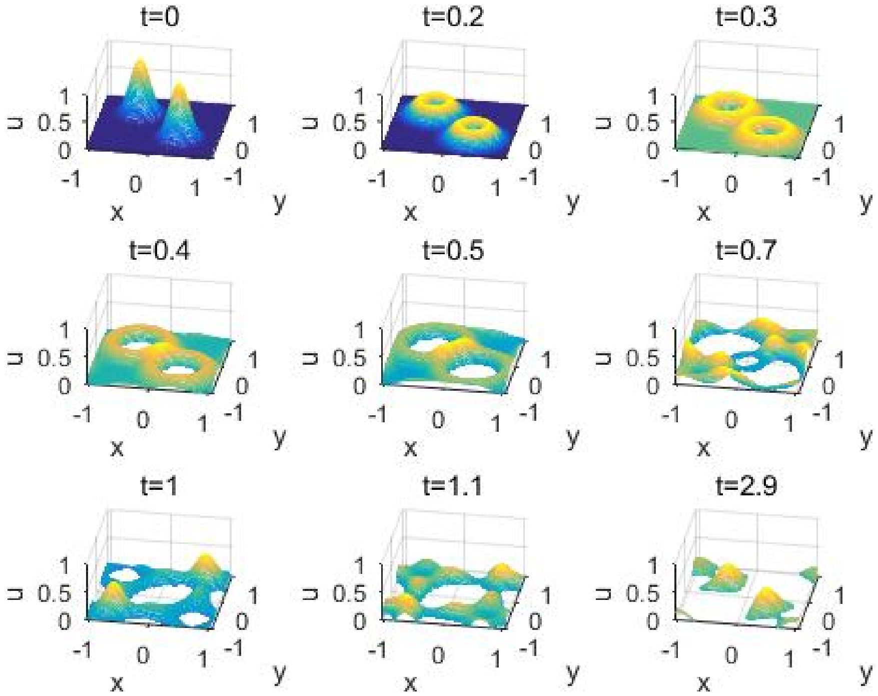

Figure 2. Wave variation of semilinear wave equation with frictional damping term and negative mass term.

Figure 2 stands for the trend of wave from $t = 0$ s to $t = 2.9$ s when nonlinear term is $|u|^3$ in problem (7.2). When $t = 0$ s, it shows the initial state of wave with two peaks. From $t = 0$ s to $t = 0.5$ s, the wave peaks continue to drop and begin to stack when they meet. When $t = 0.5$ s $\sim$ 1 s, two new waves appear at different positions and the amplitude increases continuously to form two new wave peaks. When $t = 1$ s $\sim$ 1.1 s, the amplitudes of wave decreases gradually. When $t = 2.9$ s, two wave peaks appears again and the position is same as $t = 0$ s.

From our observation of the above two groups of figures, we obtain that the frictional damping and negative mass terms have an effect on the wave propagation and wave amplitude.

8. Conclusions

This article is dedicated to investigating blow-up results and lifespan estimates of solutions to the initial boundary value problems of semilinear wave equations with damping term and mass term as well as Neumann boundary conditions on exterior domain in three dimensions. Our main new contribution is that upper bound lifespan estimates of solutions are related to the Strauss exponent and Glassey exponent. We extend the Cauchy problem investigated in the related papers to problems (1.1) and (1.2) with damping term, mass term and Neumann boundary condition on exterior domain in three dimensions. Applying test function technique ($\psi_2(x,t) = \rho_1(t)\frac{1}{r}e^r$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$) and iterative approach, upper bound lifespan estimates of solutions to problems (1.1) and (1.2) are deduced (see Theorems 1.1–1.7). In addition, we characterize the variation of wave by employing numerical simulation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author Sen Ming would like to express his sincere thanks to Professor Yi Zhou for his guidance and encouragement during the postdoctoral study in Fudan University. The author Sen Ming also would like to express his sincere thanks to Professors Han Yang and Ning-An Lai for their helpful suggestions and discussions. The project is supported by the Fundamental Research Program of Shanxi Province (No. 20210302123021, No. 20210302123045, No. 20210302124657, No. 20210302123182), the Program for the Innovative Talents of Higher Education Institutions of Shanxi Province, the Innovative Research Team of North University of China (No. TD201901), National Natural Science Foundation of China (No. 11601446).

Conflict of interest

This work has no conflict of interest.

References

1. W. H. Chen, T. A. Dao, On the Cauchy problem for semilinear regularity-loss-type σ-evolution models with memory term, *Nonlinear Anal. Real World Appl.*, **59** (2021), 103265. https://doi.org/10.1016/j.nonrwa.2020.103265
2. W. H. Chen, T. A. Dao, Sharp lifespan estimates for the weakly coupled system of semilinear damped wave equations in the critical case, *Math. Ann.*, **385** (2023), 101–130. https://doi.org/10.1007/s00208-021-02335-y
3. Y. X. Chen, R. Z. Xu, Global well-posedness of solutions for fourth order dispersive wave equation with nonlinear weak damping, linear strong damping and logarithmic nonlinearity, *Nonlinear Anal.*, **192** (2020), 111664. https://doi.org/10.1016/j.na.2019.111664
4. F. A. Chiarello, G. Girardi, S. Lucente, Fujita modified exponent for scale invariant damped semilinear wave equation, *arXiv*, 2020. https://doi.org/10.48550/arXiv.2002.03418
5. W. Dai, H. Kubo, M. Sobajima, Blow-up for Strauss type wave equation with damping and potential, *Nonlinear Anal.: Real World Appl.*, **57** (2021), 103195. https://doi.org/10.1016/j.nonrwa.2020.103195
6. T. A. Dao, A result for non-existence of global solutions to semilinear structural damped wave model, *arXiv*, 2019. https://doi.org/10.48550/arXiv.1912.07066

7. T. A. Dao, A. Z. Fino, Critical exponent for semilinear structurally damped wave equation of derivative type, *Math. Methods Appl. Sci.*, **43** (2020), 9766–9775. https://doi.org/10.1002/mma.6649 .
8. F. Q. Du, J. H. Hao, Energy decay for wave equation of variable coefficients with dynamic boundary conditions and time varying delay, *J. Geom. Anal.*, **33** (2023), 119. https://doi.org/10.1007/s12220-022-01161-1
9. V. Georgiev, A. Palmieri, Critical exponent of Fujita type for the semilinear damped wave equation on the Heisenberg group with power nonlinearity, *J. Differ. Equations*, **269** (2020), 420–448. https://doi.org/10.1016/j.jde.2019.12.009
10. V. Georgiev, A. Palmieri, Lifespan estimates for local in time solutions to the semilinear heat equation on the Heisenberg group, *Ann. Mat. Pura Appl.*, **200** (2021), 999–1032. https://doi.org/10.1007/s10231-020-01023-z
11. M. Hamouda, M. A. Hamza, A blow-up result for the wave equation with localized initial data: the scale invariant damping and mass term with combined nonlinearities, *arXiv*, 2020. https://doi.org/10.48550/arXiv.2010.05455
12. M. Hamouda, M. A. Hamaz, Blow-up for wave equation with the scale invariant damping and combined nonlinearities, *Math. Methods Appl. Sci.*, **44** (2021), 1127–1136. https://doi.org/10.1002/mma.6817
13. W. Han, Y. Zhou, Blow-up for some semilinear wave equations in multi-space dimensions, *Commun. Partial Differ. Equ.*, **39** (2014), 651–665. https://doi.org/10.1080/03605302.2013.863916
14. K. Hidano, C. B. Wang, K. Yokoyama, Combined effects of two nonlinearities in lifespan of small solutions to semilinear wave equations, *Math. Ann.*, **366** (2016), 667–694. https://doi.org/10.1007/s00208-015-1346-1
15. M. Ikeda, M. Sobajima, K. Wakasa, Blow-up phenomena of semilinear wave equations and their weakly coupled systems, *J. Differ. Equations*, **267** (2019), 5165–5201. https://doi.org/10.1016/j.jde.2019.05.029
16. M. Ikeda, T. Tanaka, K. Wakasa, Critical exponent for the wave equation with a time dependent scale invariant damping and a cubic convolution, *J. Differ. Equations*, **270** (2021), 916–946. https://doi.org/10.1016/j.jde.2020.08.047
17. M. Ikeda, Z. H. Tu, K. Wakasa, Small data blow-up of semilinear wave equation with scattering dissipation and time dependent mass, *Evol. Equ. Control The.*, **11** (2021), 515–536. https://doi.org/10.3934/eect.2021011
18. T. Imai, M. Kato, H. Takamura, K. Wakasa, The lifespan of solutions of semilinear wave equations with the scale invariant damping in two space dimensions, *J. Differ. Equations*, **269** (2020), 8387–8424. https://doi.org/10.1016/j.jde.2020.06.019
19. F. John, Blow-up for quasilinear wave equations in three space dimensions, *Commun. Pure Appl. Math.*, **34** (1981), 29–51. https://doi.org/10.1002/cpa.3160340103
20. S. Kitamura, K. Morisawa, H. Takamura, The lifespan of classical solutions of semilinear wave equations with spatial weights and compactly supported data in one space dimension, *J. Differ. Equations*, **307** (2022), 486–516. https://doi.org/10.1016/j.jde.2021.10.062

21. N. A. Lai, M. Y. Liu, Z. H. Tu, C. B. Wang, Lifespan estimates for semilinear wave equations with space dependent damping and potential, *Calc. Var. Partial Differ. Equ.*, **62** (2023), 44. https://doi.org/10.1007/s00526-022-02388-0
22. N. A. Lai, N. M. Schiavone, H. Takamura, Wave-like blow-up for semilinear wave equations with scattering damping and negative mass term, In: M. D'Abbicco, M. Ebert, V. Georgiev, T. Ozawa, *New tools for nonlinear PDEs and application*, Trends in Mathematics, Birkhäuser, Cham, 2019, 217–240. https://doi.org/10.1007/978-3-030-10937-0_8
23. N. A. Lai, N. M. Schiavone, H. Takamura, Heat like and wave like lifespan estimates for solutions of semilinear damped wave equations via a Kato's type lemma, *J. Differ. Equations*, **269** (2020), 11575–11620. https://doi.org/10.1016/j.jde.2020.08.020
24. N. A. Lai, H. Takamura, Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case, *Nonlinear Anal.*, **168** (2018), 222–237. https://doi.org/10.1016/J.NA.2017.12.008
25. N. A. Lai, H. Takamura, Non-existence of global solutions of nonlinear wave equations with weak time dependent damping related to Glassey's conjecture, *Differ. Integr. Equ.*, **32** (2019), 37–48. https://doi.org/10.57262/die/1544497285
26. N. A. Lai, H. Takamura, Non-existence of global solutions of wave equations with weak time dependent damping and combined nonlinearity, *Nonlinear Anal.: Real World Appl.*, **45** (2019), 83–96. https://doi.org/10.1016/j.nonrwa.2018.06.008
27. N. A. Lai, H. Takamura, K. Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super Fujita exponent, *J. Differ. Equations*, **263** (2017), 5377–5394. https://doi.org/10.1016/j.jde.2017.06.017
28. N. A. Lai, Z. H. Tu, Strauss exponent for semilinear wave equations with scattering space dependent damping, *J. Math. Anal. Appl.*, **489** (2020), 124189. https://doi.org/10.1016/j.jmaa.2020.124189
29. N. A. Lai, Y. Zhou, Blow-up and lifespan estimate to a nonlinear wave equation in Schwarzschild spacetime, *J. Math. Pures Appl.*, **173** (2023), 172–194. https://doi.org/10.1016/j.matpur.2023.02.009
30. Q. Lei, H. Yang, Global existence and blow-up for semilinear wave equations with variable coefficients, *Chin. Ann. Math. Ser. B*, **39** (2018), 643–664. https://doi.org/10.1007/s11401-018-0087-3
31. Y. H. Lin, N. A. Lai, S. Ming, Lifespan estimate for semilinear wave equation in Schwarzschild spacetime, *Appl. Math. Lett.*, **99** (2020), 105997. https://doi.org/10.1016/j.aml.2019.105997
32. M. Y. Liu, C. B. Wang, Blow-up for small amplitude semilinear wave equations with mixed nonlinearities on asymptotically Euclidean manifolds, *J. Differ. Equations*, **269** (2020), 8573–8596. https://doi.org/10.1016/j.jde.2020.06.032
33. S. Ming, S. Y. Lai, X. M. Fan, Lifespan estimates of solutions to quasilinear wave equations with scattering damping, *J. Math. Anal. Appl.,* **492** (2020), 124441. https://doi.org/10.1016/j.jmaa.2020.124441
34. S. Ming, S. Y. Lai, X. M. Fan, Blow-up for a coupled system of semilinear wave equations with scattering dampings and combined nonlinearities, *Appl. Anal.*, **101** (2022), 2996–3016. https://doi.org/10.1080/00036811.2020.1834086

35. A. Palmieri, H. Takamura, Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities, *Nonlinear Anal.*, **187** (2019), 467–492. https://doi.org/10.1016/j.na.2019.06.016
36. A. Palmieri, Z. H. Tu, Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity, *J. Math. Anal. Appl.*, **470** (2019), 447–469. https://doi.org/10.1016/j.jmaa.2018.10.015
37. A. Palmieri, Z. H. Tu, A blow-up result for a semilinear wave equation with scale invariant damping and mass and nonlinearity of derivative type, *Calc. Var. Partial Differ. Equ.*, **60** (2021), 72. https://doi.org/10.1007/s00526-021-01948-0
38. K. Wakasa, B. Yordanov, On the non-existence of global solutions for critical semilinear wave equations with damping in the scattering case, *Nonlinear Anal.*, **180** (2019), 67–74. https://doi.org/10.1016/j.na.2018.09.012
39. K. Wakasa, B. Yordanov, Blow-up of solutions to critical semilinear wave equations with variable coefficients, *J. Differ. Equations*, **266** (2019), 5360–5376. https://doi.org/10.1016/j.jde.2018.10.028
40. Y. Zhou, Blow up of solutions to semilinear wave equations with critical exponent in high dimensions, *Chin. Ann. Math. Ser. B*, **28** (2007), 205–212. https://doi.org/10.1007/s11401-005-0205-x
41. Y. Zhou, W. Han, Blow-up of solutions to semilinear wave equations with variable coefficients and boundary, *J. Math. Anal. Appl.*, **374** (2011), 585–601. https://doi.org/10.1016/j.jmaa.2010.08.052
42. Y. Zhou, W. Han, Lifespan of solutions to critical semilinear wave equations, *Commun. Partial Differ. Equ.*, **39** (2014), 439–451. https://doi.org/10.1080/03605302.2013.863914