AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 17827–17859.
DOI: 10.3934/math.2023909
Received: 04 January 2023
Revised: 27 April 2023
Accepted: 03 May 2023
Published: 24 May 2023

Research article

Blow-up and global existence of solutions for time-space fractional pseudo-parabolic equation

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Abstract: In this article, we consider the Cauchy problem for the following time-space fractional pseudo-parabolic equations

$$\begin{cases} {}_0^C D_t^\alpha (I - m\Delta)u + (-\Delta)^{\frac{\beta}{2}} u = |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $0 < \alpha < 1,\ 0 < \beta < 2,\ p > 1,\ m > 0,\ u_0 \in L^q\left(\mathbb{R}^N\right)$. An estimating $L^p - L^q$ for solution operator of time-space fractional pseudo-parabolic equations is obtained. The critical exponents of this problem are determined when $u_0 \in L^q(\mathbb{R}^N)$. Moreover, we also obtain global existence of the mild solution when $u_0 \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ small enough.

Keywords: fractional pseudo-parabolic equation; blow-up; global existence; Cauchy problems
Mathematics Subject Classification: 26A33, 35B33, 35K55, 74G25, 74G40, 74H35

1. Introduction

In this paper, we study blow-up and global existence of solutions for the following time-space fractional pseudo-parabolic equation in $\mathbb{R}^N$

$$\begin{cases} {}_0^C D_t^\alpha (I - m\Delta)u + (-\Delta)^{\frac{\beta}{2}} u = |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1,\ 0 < \beta \leq 2,\ p > 1,\ m > 0,\ u_0 \in L^q\left(\mathbb{R}^N\right)$ and

$${}_0^C D_t^\alpha u = \frac{\partial}{\partial t} {}_0 I_t^{1-\alpha} \left(u(t,x) - u_0(x)\right),$$

${}_0I_t^{1-\alpha}$ denotes left Riemann-Liouville fractional integrals of order 1–α and is defined by

$$ {}_0I_t^{1-\alpha}u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s)ds, $$

where Γ is the Gamma function. $(-\Delta)^{\beta/2}$ is the fractional Laplace operator, which may be defined as

$$ (-\Delta)^{\beta/2} v(x,t) = \mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x,t), $$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ represents the inverse Fourier transform in $L^2(\mathbb{R}^{\mathbb{N}})$.

As we all know, it has been seen that qualitative theory and applications of fractional derivatives are employed to describe the nonlocal effects in time and space and it appears to have better effects than the classical ones, like Levy flights in physics [20, 22], the physical model considering memory effects [11] and some corresponding engineering problems [1, 5, 16, 24, 26], especially the power-law memory (non-local effects) in time and space [5, 9]. Time fractional parabolic equations have been studied by many papers [6–8, 18, 21], many important physical models and practical problems require us to consider pseudo-parabolic model with fractional derivative rather than classical ones. The pseudo-parabolic equation with fractional derivatives, such as Eq (1.1), can be considered as a model for viscoelastic fluids. More and more work has been devoted to the investigation of fractional pseudo-parabolic equation, see [17, 27–29]. Zennir et al. [28] discussed the finite time blow-up that arise under an appropriate conditions and the nonsolvability of boundary value problem for damped pseudo-parabolic differential equations with variable exponents. In [29], they also investigated the global non-existence for a class of pseudo-parabolic equation with weak-viscoelasticity under suitable conditions on the variable exponents with negative initial energy.

If $\alpha = 1, m > 0, \beta = 2$, problem (1.1) becomes classical pseudo-parabolic equation. Cao et al. [4] considered the following semilinear pseudo-parabolic equation

$$ u_t - m\triangle u_t - \triangle u = u^p. $$

They investigated the existence, uniqueness for mild solutions and they also studied the large time behavior of solutions. They obtained that if $0 < p \leqslant 1$, then for any $0 \leqslant u_0 \in C^{2+\alpha}(\mathbb{R}^n)$, the classical solution of the Cauchy problem exists globally, and if $1 < p \leqslant 1+2/n$, then for any $0 \leqslant u_0 \in C^{2+\alpha}(\mathbb{R}^n)$, the classical solution of the Cauchy problem blow-up in a finite time. Jin et al. [12] considered the Cauchy problem of the following space-fractional pseudo-parabolic equation

$$ u_t - m\triangle u_t + (-\triangle)^{\sigma} u = u^p. $$

They discussed global existence, time-decay rates and large time behavior of solutions.

If $m = 0$, problem (1.1) is called space-time fractional diffusion equations, which are useful to model anomalous diffusion and Hamiltonian chaos, etc. [25]. In [14], Kirane et al. studied the following space-time fractional evolution problem

$$ \begin{cases} {}_0^CD_t^{\alpha}u + (-\Delta)^{\frac{\beta}{2}}u = h(x,t)|u|^{1+\tilde{p}}, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2} $$

They obtained the sufficient conditions such that problem (1.2) admits no global weak nonnegative solution other than the trivial one.

In [30], Zhang et al. discussed the following time-fractional diffusion equation

$$\begin{cases} {}_0^C D_t^\alpha u - \triangle u = |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0\left(\mathbb{R}^N\right)$. They proved that if $1 < p < 1 + \frac{2}{N}$, every nontrivial nonnegative solution blow-up in a finite time, and if $p \geq 1 + \frac{2}{N}$ and $\|u_0\|_{L^{\frac{N(p-1)}{2}}(\mathbb{R}^N)}$ is sufficiently small, then the problem has global solutions.

Tuan et al. [27] studied the following fractional pseudo-parabolic equations in a bounded domain

$$\begin{cases} \mathbb{D}_t^\alpha (u - m\Delta u)(x,t) + (-\Delta)^\sigma u(x,t) = \mathcal{N}(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\mathcal{N}(u)$ satisfies the Lipschitz condition. They established the results of existence, uniqueness and regularity of the local solution for problem (1.4). They also investigated global existence of the following time fractional pseudo-parabolic equations in $\mathbb{R}^N$ which is a special case of (1.1) with $\beta = 2$

$$\begin{cases} \mathbb{D}_t^\alpha (u - m\Delta u)(x,t) - \Delta u(x,t) = \mathcal{N}(u), & x \in \mathbb{R}^N, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

So far as is known to the authors, the study of Fujita exponent for problem (1.1) has not been carried out. Motivated by above results, in the present paper, our purpose is to determine the Fujita exponent of problem (1.1). By applying test function method we prove that if $1 < p < 1 + \frac{\beta}{N}$, the solutions blow-up in a finite time, and if $p \geq 1 + \frac{\beta}{N}$, the solutions exist globally when $u_0 \in L^q(\mathbb{R}^N)$. Moreover, we also obtain global existence of solutions when $u_0 \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ by using contraction mapping principle.

Note that if $m = 0$, the estimating $L^p - L^q$ is available for the corresponding solution operator. However, some estimates of the form $L^p - L^q$ is not available on the domain $\mathbb{R}^N$ for the time-space fractional pseudo-parabolic equations. So our main challenge is to prove $L^p - L^q$ estimates of the solution operator. Our conclusions extend the corresponding results in [27, 30] .

This paper is organized as follows: In Section 2, some preliminaries are presented; In Section 3, we establish the local existence and uniqueness of mild solution for problem (1.1); In Section 4, we show the blow-up and global existence of solutions to problem (1.1).

We will denote $L^p(\mathbb{R}^n)$-norm by $|\cdot|_p$,

$$\mathscr{F}u(\xi) = \hat{u}(\xi) \equiv \int \mathrm{e}^{-\mathrm{i}x\xi} u(x) \mathrm{d}x$$

denotes the Fourier transform of u and $\mathscr{F}^{-1}u(x) = \check{u}(x) \equiv (2\pi)^{-n} \int \mathrm{e}^{\mathrm{i}x\xi} u(\xi) \mathrm{d}\xi$–the inverse Fourier transform of u. The letters α, β, γ will denote multi-indices, i.e., $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i is a non-negative integer, and the number $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ is the order of the multi-index α. C will denote generic positive constants.

2. Preliminaries

In this section, we present some preliminaries that will be used in next sections.

The Mittag-Leffler function is defined for complex $z \in \mathbb{C}$ in [10, 15, 23]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,$$

and its Riemann-Liouville fractional integral satisfies

$${}_0I_t^{1-\alpha}\left(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})\right) = E_{\alpha,1}(\lambda t^{\alpha}) \quad \text{for } \lambda \in \mathbb{C},\ 0 < \alpha < 1.$$

We also need the following Wright type function which was considered by Mainardi [19]

$$\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} = \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!}, \tag{2.1}$$

for $0 < \alpha < 1$. ϕ_α is an entire function and has the following properties,

$(i)\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$.

$(ii) \int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$.

$(iii) \int_0^\infty \phi_\alpha(\theta)e^{-z\theta}d\theta = E_{\alpha,1}(-z), z \in \mathbb{C}$.

$(iv)\alpha \int_0^\infty \theta\phi_\alpha(\theta)e^{-z\theta}d\theta = E_{\alpha,\alpha}(-z), z \in \mathbb{C}$.

If ${}_0^C D_t^\alpha f \in L^1(0,T),\ g \in C^1([0,T])$ and $g(T) = 0$, then we have the following formula of integration by parts

$$\int_0^T g\,{}_0^C D_t^\alpha f dt = \int_0^T (f(t) - f(0))\,{}_t^C D_T^\alpha g dt, \tag{2.2}$$

where

$${}_t^C D_T^\alpha g = -\frac{d}{dt}\,{}_tI_T^{1-\alpha} g,$$

$${}_tI_T^{1-\alpha} g = \frac{1}{\Gamma(1-\alpha)}\int_t^T (s-t)^{-\alpha} g(s)ds.$$

We need know the Caputo fractional derivative of the following function, which will be used in next sections. If $\alpha \in (0,1)$, $w(t) = (1 - t/T)_+^\sigma,\ t \geq 0,\ T > 0,\ \sigma \gg 1$, then

$${}_0^C D_{t|T}^\alpha w(t) = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\alpha}\left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha}, \tag{2.3}$$

so

$$\left({}_0^C D_{t|T}^{\alpha} w\right)(T) = 0, \quad \left({}_0^C D_{t|T}^{\alpha} w\right)(0) = CT^{-\alpha},$$

where

$$C = (1-\alpha+\sigma)\Gamma(\sigma+1)/\Gamma(2-\alpha+\sigma).$$

In order to prove main results of this paper, we need consider the following linear problem.

$$\begin{cases} u_t - m\Delta u_t + (-\Delta)^{\frac{\beta}{2}} u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.4}$$

We know that as $(-\Delta)^{\beta/2}$ is a self-adjoint operator with $D(\mathcal{A}) = H^{\beta}\left(\mathbb{R}^N\right)$, thus,

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\beta/2} v(x) dx = \int_{\mathbb{R}^N} v(x)(-\Delta)^{\beta/2} u(x) dx,$$

for all $u, v \in H^{\beta}\left(\mathbb{R}^N\right)$, where $H^{\beta}\left(\mathbb{R}^N\right)$ is the homogeneous Sobolev space of order β defined by

$$H^{\beta}\left(\mathbb{R}^N\right) = \left\{u \in \mathcal{S}' | \mathcal{A}u \in L^2\left(\mathbb{R}^N\right)\right\}, \qquad \text{if } \beta \notin \mathbb{N},$$
$$H^{\beta}\left(\mathbb{R}^N\right) = \left\{u \in L^2\left(\mathbb{R}^N\right) | \mathcal{A}u \in L^2\left(\mathbb{R}^N\right)\right\}, \quad \text{if } \beta \in \mathbb{N},$$

where $\mathcal{S}'$ is the space of Schwartz distributions.

Let $T(t)$ denote the semigroup operator generated by $\mathcal{A} = (-\Delta)^{\frac{\beta}{2}}(m\Delta - I)^{-1}$. Taking the Fourier transform to (2.4) with respect to x, we can write the solution of (2.4) as

$$u(x,t) = T(t)u_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(t\Phi(\xi) + \mathrm{i}x\xi)\hat{u}_0(\xi) \mathrm{d}\xi, \tag{2.5}$$

where

$$\Phi(\xi) = -|\xi|^{\beta}\left(1+m|\xi|^2\right)^{-1}.$$

In order to obtain the behaviour of L^p-norms of (2.5), we need the following two Lemmas.

Lemma 2.1. *Suppose that $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ satisfies $\varphi(\xi) = 0$ for $|\xi| \geqslant \max\{1, \frac{1}{\sqrt{m}}\}$, $|\varphi(\xi)| \leqslant 1$, and denote*

$$\mathscr{I}_{\alpha}(x,t) = \int \xi^{\alpha} \varphi(\xi) \exp(t\Phi(\xi) + \mathrm{i}x\xi) \mathrm{d}\xi.$$

Then for every multi-index α and $1 \leqslant p \leqslant \infty$, there exists a constant C independent of t such that

$$|\mathscr{I}_{\alpha}(\cdot,t)|_p \leqslant C(1+t)^{(n(1/p-1)-|\alpha|)/\beta}.$$

Proof. First by using the Plancherel theorem we can estimate the L^2 -norm of $\mathscr{I}_{\alpha}$:

$$|\mathscr{I}_{\alpha}(\cdot,t)|_2^2 = (2\pi)^n \left|\hat{\mathscr{I}}_{\alpha}(\cdot,t)\right|_2^2 = (2\pi)^n \int |\xi^{\alpha}|^2 |\varphi(\xi)|^2 \exp\left(\frac{-2t|\xi|^{\beta}}{1+m|\xi|^2}\right) d\xi$$
$$\leqslant C \int_{|\xi| \leqslant k} |\xi|^{2|\alpha|} \exp\left(\frac{-2t|\xi|^{\beta}}{1+m|\xi|^2}\right) d\xi \leqslant C \int_{|\xi| \leqslant k} |\xi|^{2|\alpha|} \exp\left(-t|\xi|^{\beta}\right) d\xi$$

$$\leqslant Ct^{-n/\beta-2|\alpha|/\beta}\int_{\mathbb{R}^n}|w|^{2|\alpha|}\exp\left(-|w|^{\beta}\right)dw\leqslant Ct^{-n/\beta-2|\alpha|/\beta},\tag{2.6}$$

where we use

$$2|\xi|^{\beta}\left(1+m|\xi|^2\right)^{-1}\geqslant|\xi|^{\beta}$$

for $|\xi|\leqslant k$ where $k\leqslant\frac{1}{\sqrt{m}}$ and $k\leqslant 1$ and the change of variables $t^{1/\beta}\xi=w$. Hence,

$$|\mathscr{I}_{\alpha}(\cdot,t)|_2\leqslant C(1+t)^{-n/2\beta-|\alpha|/\beta}.\tag{2.7}$$

Note that for all smooth rapidly decreasing functions $w=w(x)$ defined in $\mathbb{R}^n$ and for every integer $N>n/2$,

$$|\hat{w}|_1\leqslant C|w|_2^{1-n/(2N)}\sum_{|\kappa|=N}\left(\left|\partial_x^{\kappa}w\right|_2\right)^{n/(2N)}\tag{2.8}$$

and

$$|w|_{\infty}\leqslant C|w|_2^{1-n/(2N)}\sum_{|\kappa|=N}\left(\left|\partial_x^{\kappa}w\right|_2\right)^{n/(2N)}.\tag{2.9}$$

Equation (2.8) was proofed in ([2], Example 2). It is obvious that (2.9) is an immediate consequence of (2.8) if we use the fact $|w|_{\infty}\leqslant(2\pi)^{-n}|\hat{w}|_1$. Hence, using (2.9), noting that $\partial_x^{\kappa}\mathscr{I}_{\alpha}=\mathrm{i}^{|\kappa|}\mathscr{I}_{\kappa+\alpha}$, and applying (2.7) we can deduce that

$$\begin{aligned}|\mathscr{I}_{\alpha}(\cdot,t)|_{\infty}&\leqslant C\,|\mathscr{I}_{\alpha}(\cdot,t)|_2^{1-n/(2N)}\sum_{|\kappa|=N}\left(\left|\partial_x^{\kappa}\mathscr{I}_{\alpha}(\cdot,t)\right|_2\right)^{n/(2N)}\\&\leqslant C(1+t)^{-(n/2\beta+|\alpha|/\beta)(1-n/(2N))}(1+t)^{-\frac{1}{\beta}(n/2+|\alpha|+N))n/(2N)}\\&=C(1+t)^{-(n+|\alpha|)/\beta}.\end{aligned}$$

Now we estimate the L^1 -norm of $\mathscr{I}_{\alpha}$

$$\begin{aligned}|\mathscr{I}_{\alpha}(\cdot,t)|_1\leqslant C&\left(\int|\xi^{\alpha}|^2\,|\varphi(\xi)|^2\exp\left(\frac{-2t|\xi|^{\beta}}{1+m|\xi|^2}\right)d\xi\right)^{(1-n/(2N))/2}\\&\times\sum_{|\kappa|=N}\left(\int\left|\partial_{\xi}^{\kappa}\left(\xi^{\alpha}\varphi(\xi)\exp\left(\frac{-|\xi|^{\beta}}{1+m|\xi|^2}t\right)\right)\right|^2d\xi\right)^{n/(4N)}.\end{aligned}\tag{2.10}$$

The first term on the right-hand side of (2.10) decays like $(1+t)^{-(n/2\beta+|\alpha|/\beta)(1-n/(2N))}$ by an argument similar to that in (2.6). We use the Leibnitz formula to the integrand in the second term in (2.10). It suffices to show that for fixed $|\zeta|\leqslant N$ the L^2-norm of expressions

$$\varphi(\xi)\left(\partial_{\xi}^{\zeta-\gamma}\xi^{\alpha}\right)\partial_{\xi}^{\gamma}\exp(t\tilde{\Phi}(\xi)),\tag{2.11}$$

then it can be bounded by

$$(1+t)^{(-n/2\beta-(|\alpha|-N)/\beta)(n/(2N)}\tag{2.12}$$

for each multi-index $|\gamma| \leqslant |\zeta|$, all $t \geqslant 1$, and a positive constant C.

Observe that we can always assume that $(\zeta-\gamma)_i \leqslant \alpha_i$ for every $i \in \{1, \ldots, n\}$ (if it does not hold, then (2.11) equals 0). Note that considering $\tilde{\Phi}$ instead Φ allows us to postulate that $\partial_{\xi_i}\tilde{\Phi}(\xi) = \xi_i\Theta_i(\xi)$ for some $\Theta_i(\xi) \in C^\infty(\mathbb{R}^n)$.

If $|\gamma| = 0$, as in (2.6), the L^2 -norm of (2.11) is bounded by

$$Ct^{-n/2\beta-|\alpha-\zeta|/\beta} = Ct^{-n/2\beta-(|\alpha-\zeta|)/\beta} \leqslant Ct^{-n/2\beta-(|\alpha|-N)/\beta}.$$

In the proof we may therefore assume that $|\gamma| > 0$, and that the decay rate (2.12) is already proved for all multi-indices γ' such that $|\gamma'| < |\gamma|$.

Now let us replace γ by $\gamma + e_i$ in (2.11), where $e_i = (0, \ldots, 1, \ldots, 0)$. Then (2.11) takes the form

$$\varphi(\xi)\left(\partial_\xi^{\zeta-\gamma-e_i}\xi^\alpha\right)\partial_\xi^\gamma\left(t\partial_{\xi_i}\tilde{\Phi}(\xi)\exp(t\tilde{\Phi}(\xi))\right) = \varphi(\xi)\left(\partial_\xi^{\zeta-\gamma-e_i}\xi^\alpha\right)\sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1\gamma_2}\partial_\xi^{\gamma_1}\left(t\partial_{\xi_i}\tilde{\Phi}(\xi)\right)\partial_\xi^{\gamma_2}\exp(t\tilde{\Phi}(\xi)).$$

By (2.6) with φ replaced by $\varphi(\xi)\partial_\xi^{\gamma_1+e_i}\tilde{\Phi}(\xi)$, and the recurrence hypothesis (here $|\gamma_2| < |\gamma|$)), we can estimate the L^2-norm of each term in the sum above by

$$Ct \cdot t^{-n/2\beta-|\alpha-\zeta+\gamma+e_i|/\beta} \leqslant Ct^{-n/2\beta-(|\alpha|-N)/\beta},$$

because $|\gamma| \geqslant 1$. Since

$$-\left(\frac{n}{2\beta}+\frac{|\alpha|}{\beta}\right)\left(1-\frac{n}{2N}\right)-\left(\frac{n}{2\beta}+\frac{|\alpha|-N}{\beta}\right)\frac{n}{2N} = -\frac{|\alpha|}{\beta},$$

$$|\mathscr{F}_\alpha(\cdot,t)|_1 \leqslant C(1+t)^{-|\alpha|/\beta}.$$

Finally, using the following inequality

$$|w|_p \leqslant |w|_\infty^{1-1/p}|w|_1^{1/p},$$

which completes the proof. □

Lemma 2.2. *(see [13], Lemma 4.2) Let $k > 0$ and $a(t,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfy*

$$\left|\partial_\xi^\alpha a(t,\xi)\right| \leqslant C_\alpha e^{-\varepsilon t}(1+|\xi|)^{-k-|\alpha|} \tag{2.13}$$

for each α and some positive constants C_α, ε independent of t and ξ. Then the operators

$$\mathscr{T}(t)v(x) = \int a(t,\xi)e^{ix}\hat{v}(\xi)d\xi$$

are bounded on $L^p(\mathbb{R}^n)$ for every $p \in [1,\infty]$. Moreover, there exists a positive constant C_p independent of v and t such that

$$|\mathscr{T}(t)u|_p \leqslant C_p e^{-\varepsilon t}|u|_p$$

for every $v \in L^p(\mathbb{R}^n)$ and $t \geqslant 0$.

The following Lemma describes the behaviour of the L^p-norms of (2.5).

Lemma 2.3. *Let* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $n \geqslant 1$. *There exist positive constants* C, ε *(independent of* t *and* u_0*) such that for every* $u_0 \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

$$|T(t)u_0|_p \leqslant C(1+t)^{n(1/p-1/q)/\beta} |u_0|_q + Ce^{-\varepsilon t} |u_0|_p \tag{2.14}$$

for all $t \geqslant 0$.

Proof. We decompose $T(t)u_0$ into two integrals using the cut-off function φ from Lemma 2.1 with the additional assumption $\varphi(\xi) \equiv 1$ for $|\xi| \leqslant \delta$ where $\delta \leqslant \frac{1}{2}$ and $\delta \leqslant \frac{1}{\sqrt{m}}$

$$\begin{aligned}(2\pi)^n T(t)u_0(x) = &\int \varphi(\xi)\exp(t\Phi(\xi) + \mathrm{i}x\xi)\hat{u}_0(\xi)d\xi \\ &+ \int (1-\varphi(\xi))\exp(t\Phi(\xi) + \mathrm{i}x\xi)\hat{u}_0(\xi)d\xi \\ \equiv &I_1(x,t) + I_2(x,t).\end{aligned}$$

Observe that $I_1(x,t) = (\mathscr{I}_0(\cdot,t) * u_0)(x)$, where $\mathscr{I}_0$ is defined in Lemma 2.1 with $\alpha = (0,\ldots,0)$. Hence, by the Young inequality and Lemma 2.1,

$$|I_1(\cdot,t)|_p \leqslant C(1+t)^{(n/\beta)(1/p-1/q)} |u_0|_q, \ 1 \leqslant q \leqslant p \leqslant \infty$$

and this is the second term on right-hand side of (2.14). The estimates of L^p-norms of $I_2(\cdot,t)$ are based on Lemma 2.2. We see that

$$\begin{aligned}I_2(x,t) &= \int (1-\varphi(\xi))\exp\left(\frac{-t|\xi|^\beta}{1+m|\xi|^2} + \mathrm{i}x\xi\right)\hat{u}_0(\xi)d\xi \\ &= \int (1-\varphi(\xi))e^{-\frac{1}{m}t}\exp\left(t\frac{\frac{1}{m} - |\xi|^\beta + |\xi|^2}{1+m|\xi|^2} + \mathrm{i}x\xi\right)\hat{u}_0(\xi)d\xi.\end{aligned}$$

Note that

$$\Phi(\xi) = -\frac{1}{m} + \frac{\frac{1}{m} - |\xi|^\beta + |\xi|^2}{1+m|\xi|^2},$$

let us denote by $\mathscr{T}_0(t)$ and $\mathscr{T}_A(t)$ the multiplier operators defined by the symbols $(1-\varphi(\xi))e^{-\frac{1}{m}t}$ and

$$A(t,\xi) = (1-\varphi(\xi))\left(\exp(t\Phi(\xi)) - e^{-\frac{1}{m}t}\right),$$

respectively. Since we have

$$\begin{aligned}\mathscr{T}_0(t)u_0(x) = &e^{-\frac{1}{m}t}\int (1-\varphi(\xi))e^{\mathrm{i}x\xi}\hat{u}_0(\xi)d\xi = (2\pi)^n e^{-\frac{1}{m}t}u_0(x) + e^{-\frac{1}{m}t}\int \varphi(\xi)e^{\mathrm{i}x\xi}\hat{u}_0(\xi)d\xi \\ = &(2\pi)^n e^{-\frac{1}{m}t}u_0(x) + (2\pi)^n e^{-\frac{1}{m}t}\check{\varphi} * u_0(x),\end{aligned}$$

by the Fourier inversion formula, we have

$$|\mathscr{T}_0(t)u_0|_p \leqslant (2\pi)^n e^{-\frac{1}{m}t}(1 + |\check{\varphi}|_1)|u_0|_p = Ce^{-\frac{1}{m}t}|u_0|_p \tag{2.15}$$

for all $p \in [1,\infty]$.

In order to estimate $\mathscr{T}_A(t)$, we observe that

$$|A(t,\xi)| = \left| e^{-\frac{1}{m}t}(1-\varphi(\xi))\left(\exp\left(t\frac{\frac{1}{m}-|\xi|^\beta+|\xi|^2}{1+m|\xi|^2}\right)-1\right)\right|$$
$$\leq \left| e^{-\frac{1}{m}t}(1-\varphi(\xi))\left(\exp\left(t\frac{\frac{1}{m}}{1+m|\xi|^2}\right)-1\right)\right|,$$
$$\sup_{\xi\in\mathbb{R}^n}\left|\partial_\xi^\alpha A(t,\xi)\right| \leqslant C_\alpha(1+|\xi|)^{-1-|\alpha|}t^{|\alpha|}e^{(-\nu+\delta)t},$$

where

$$\delta = \sup_{|\xi|\geqslant\delta}\nu\left(1+m|\xi|^2\right)^{-1} = \nu\left(1+m|\delta|^2\right)^{-1}.$$

This argument shows that $A(t,\xi)$ satisfies (2.14) with $k=1$ and each $\varepsilon\in(0,\nu-\nu\left(1+m|\delta|^2\right)^{-1})$. Now by Lemma 2.2 we obtain

$$|\mathscr{T}_A(t)v_0|_p \leqslant Ce^{-\varepsilon t}|v_0|_p \tag{2.16}$$

for each $1\leqslant p\leqslant\infty$ and all $v_0\in L^p(\mathbb{R}^n)$. Since

$$I_2(x,t)\leqslant \mathscr{T}_0(t)v_0(x)+\mathscr{T}_A(t)v_0(x),$$

then summing up (2.15) and (2.16) we obtain the second term on the right-hand side of (2.14). Hence, the conclusion holds. □

Define the operators $P_\alpha(t)$ and $S_\alpha(t)$ as

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0 d\theta,\ t\geq 0, \tag{2.17}$$
$$S_\alpha(t)u_0 = \alpha\int_0^\infty \theta\phi_\alpha(\theta)T(t^\alpha\theta)u_0 d\theta,\ t\geq 0, \tag{2.18}$$

where $T(t)$ is given by (2.5). Later on, we will use the following results.

Lemma 2.4. *(See [4]) Let $\mathscr{G} = -(m\Delta - I)^{-1}$, if $1\leq q$ and $f\in L^q\left(\mathbb{R}^N\right)$ then there exists $M>0$ such that*

$$\|\mathscr{G}f\|_{L^q(\mathbb{R}^N)}\leq M\|f\|_{L^q(\mathbb{R}^N)}.$$

Lemma 2.5. *For $u_0\in L^p\left(\mathbb{R}^N\right)$, we have*

$${}_0^C D_t^\alpha P_\alpha(t)u_0 = \mathcal{A}P_\alpha(t)u_0,\ t>0.$$

Proof. Let $X = L^p\left(\mathbb{R}^N\right)$. First, we prove if $u_0\in X$, then $P_\alpha(t)u_0\in D(\mathcal{A})$. In fact, for $u_0\in X$,

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0 d\theta$$
$$= \int_0^1(\phi_\alpha(\theta)-\phi_\alpha(0))T(t^\alpha\theta)u_0 d\theta + \phi_\alpha(0)\int_0^1 T(t^\alpha\theta)u_0 d\theta + \int_1^\infty \phi_\alpha(\theta)T(t^\alpha\theta)u_0 d\theta.$$

Clearly, $\int_0^1 T\left(t^\alpha\theta\right)u_0 d\theta \in D(\mathcal{A})$. Note that there exists positive constant C such that

$$\|\mathcal{A}T\left(t^\alpha\theta\right)u_0\|_X \leq C\frac{\|u_0\|_X}{t^\alpha\theta},\quad t>0,\ \theta>0,$$

we get that

$$\int_1^\infty \phi_\alpha(\theta)T\left(t^\alpha\theta\right)u_0 d\theta \in D(\mathcal{A}).$$

Next, we show that

$$\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)T\left(t^\alpha\theta\right)u_0 d\theta \in D(\mathcal{A}).$$

In fact, for every $h>0$,

$$\begin{aligned}&\frac{1}{h}\left[T(h)\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)T\left(t^\alpha\theta\right)u_0 d\theta-\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)T\left(t^\alpha\theta\right)u_0 d\theta\right]\\&=\frac{1}{h}\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)\left(T\left(t^\alpha\theta+h\right)-T\left(t^\alpha\theta\right)\right)u_0 d\theta.\end{aligned}$$

Since

$$\left\|\frac{\left(T\left(t^\alpha\theta+h\right)-T\left(t^\alpha\theta\right)\right)u_0}{h}\right\|_X \leq \frac{C}{t^\alpha\theta}\|u_0\|_X\,,\qquad \left|\frac{\phi_\alpha(\theta)-\phi_\alpha(0)}{\theta}\right|\leq C,$$

for some constant $C>0$ independent of θ and h, so, by Lebesuge dominated convergence theorem, we know

$$\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)T\left(t^\alpha\theta\right)u_0 d\theta \in D(\mathcal{A}).$$

Note that

$$\begin{aligned}\mathcal{A}P_\alpha(t)u_0 =&\mathcal{A}\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)T\left(t^\alpha\theta\right)u_0 d\theta+\phi_\alpha(0)\mathcal{A}\int_0^1 T\left(t^\alpha\theta\right)u_0 d\theta+\mathcal{A}\int_1^\infty \phi_\alpha(\theta)T\left(t^\alpha\theta\right)u_0 d\theta\\=&\int_0^1 \left(\phi_\alpha(\theta)-\phi_\alpha(0)\right)\mathcal{A}T\left(t^\alpha\theta\right)u_0 d\theta+\frac{\phi_\alpha(0)\left(T\left(t^\alpha\right)u_0-u_0\right)}{t^\alpha}+\int_1^\infty \phi_\alpha(\theta)\mathcal{A}T\left(t^\alpha\theta\right)u_0 d\theta.\end{aligned}$$

Therefore

$$\|\mathcal{A}P_\alpha(t)u_0\|_X \leq \frac{C}{t^\alpha}\|u_0\|_X\,. \tag{2.19}$$

Via dominated convergence theorem, we obtain that for $u_0 \in L^p\left(\mathbb{R}^N\right)$,

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}\mathcal{A}S_\alpha(t)u_0,\ t>0.$$

Furthermore, if $u_0 \in D(\mathcal{A})$, then

$$\frac{d}{dt}P_\alpha(t)u_0 = t^{\alpha-1}S_\alpha(t)\mathcal{A}u_0,\ t>0.$$

Since

$$
{}_0I_t^{1-\alpha}\left(t^{\alpha-1}S_\alpha(t)\mathcal{A}u_0\right)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{s^{\alpha-1}}{(t-s)^\alpha}\int_0^\infty\alpha\theta\phi_\alpha(\theta)T\left(s^\alpha\theta\right)\mathcal{A}u_0d\theta ds,
$$

$$
\int_0^\infty\alpha\theta\phi_\alpha(\theta)T\left(s^\alpha\theta\right)\mathcal{A}u_0d\theta=\frac{1}{2\pi i}\int_\Gamma E_{\alpha,\alpha}\left(\lambda s^\alpha\right)(\lambda-\mathcal{A})^{-1}\mathcal{A}u_0d\lambda,
$$

where Γ is a path composed from two rays

$$
\Gamma_1=\{\rho e^{i\tau}|\rho\geq 1,\ \frac{\pi}{2}<\tau<\pi\},\ \Gamma_2=\{\rho e^{-i\tau}|\rho\geq 1,\frac{\pi}{2}<\tau<\pi\}
$$

and a curve $\Gamma_3=\{e^{i\beta}|-\tau\leq\beta\leq\tau\}$.

$$
{}_0I_t^{1-\alpha}\left(t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right)\right)=E_{\alpha,1}\left(\lambda t^\alpha\right),
$$

so,

$$
{}_0I_t^{1-\alpha}\left(t^{\alpha-1}S_\alpha(t)\mathcal{A}u_0\right)=P_\alpha(t)\mathcal{A}u_0=\mathcal{A}P_\alpha(t)u_0. \tag{2.20}
$$

Therefore, we get ${}_0^CD_t^\alpha P_\alpha(t)u_0=\mathcal{A}P_\alpha(t)u_0,\ t>0$. Next, we prove that the conclusion also holds if $u_0\in X$.

In fact, if $u_0\in X$, then we can find $\{u_{0,n}\}\subset D(\mathcal{A})$ such that $u_{0,n}\to u_0$ in X. By (2.20), we know

$$
{}_0^CD_t^\alpha P_\alpha(t)u_{0,n}=\mathcal{A}P_\alpha(t)u_{0,n}\text{ and }\left\|P_\alpha(t)u_{0,n}\right\|_X\leq\left\|u_{0,n}\right\|_X.
$$

We denote $u_n=P_\alpha(t)u_{0,n}$. Then, there exists $u\in X$ such that for every $T>0,\ u_n\to u$ uniformly in X for $t\in[0,T]$ as $n\to\infty$. Since

$$
\left\|{}_0I_t^{1-\alpha}u_n\right\|_X\leq\frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\left\|u_n\right\|_{L^\infty((0,T),X)},\ t\in[0,T],
$$

we have ${}_0I_t^{1-\alpha}u_n\to{}_0I_t^{1-\alpha}u$ in X. By (2.19),

$$
\left\|{}_0^CD_t^\alpha u_n\right\|_X\leq\frac{C}{t^\alpha}\left\|u_{0,n}\right\|_X,\text{ for some constant }C>0,\ t>0.
$$

Hence, for every $\delta>0$, there exists $w\in C([\delta,\infty),X)$ such that ${}_0^CD_t^\alpha u_n\to w$ uniformly in X on $t\in[\delta,\infty)$.

Note that for $t\in[\delta,\infty)$,

$$
{}_0^CD_t^\alpha u_n=\frac{d}{dt}\left({}_0I_t^{1-\alpha}\left(P_\alpha(t)u_{0,n}-u_{0,n}\right)\right)=\mathcal{A}u_n,
$$

so

$$
w=\frac{d}{dt}{}_0I_t^{1-\alpha}\left(u-u_0\right)={}_0^CD_t^\alpha u,\ t\in[\delta,\infty).
$$

Since $\mathcal{A}$ is closed, we have $w=\mathcal{A}u$, that is ${}_0^CD_t^\alpha u=\mathcal{A}u=\mathcal{A}P_\alpha(t)u_0,\ t\in[\delta,\infty)$. By arbitrariness of δ, we cn conclude ${}_0^CD_t^\alpha u=\mathcal{A}P_\alpha(t)u_0,\ t>0$. □

Lemma 2.6. *If* $u_0 \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 \le p \le q \le +\infty$ *and* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} < \frac{\beta}{N}$, *then*

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le A_1 t^{-\frac{\alpha N}{\beta r}} \|u_0\|_{L^p(\mathbb{R}^N)} + A_2 t^{-\alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)}, \tag{2.21}$$

where

$$A_1 = \frac{C_1\Gamma\left(1-\frac{N}{\beta r}\right)}{\Gamma\left(1-\frac{\alpha N}{\beta r}\right)},\ A_2 = \frac{C_2\vartheta^\vartheta e^{-\vartheta}\Gamma(1-\vartheta)}{\Gamma(1-\alpha\vartheta)}.$$

Proof. By the properties of ϕ_α, Lemma 2.3 and the fact that $\mathrm{e}^{-\theta t^\alpha} \le \vartheta^\vartheta \mathrm{e}^{-\vartheta}(\theta t^\alpha)^{-\vartheta}$ for any $\vartheta \in (0,1)$, we find that

$$\begin{aligned}
&\left\|\int_0^{+\infty} \phi_\alpha(\theta) T(t^\alpha\theta) u_0 d\theta\right\|_{L^q(\mathbb{R}^N)}\\
\le& C\int_0^{+\infty} \phi_\alpha(\theta)(t^\alpha\theta)^{-\frac{N}{\beta r}} \|u_0\|_{L^p(\mathbb{R}^N)} d\theta + C\int_0^{+\infty} \phi_\alpha(\theta) e^{-\theta t^\alpha} \|u_0\|_{L^q(\mathbb{R}^N)} d\theta\\
\le& Ct^{-\frac{\alpha N}{\beta r}} \int_0^{+\infty} \phi_\alpha(\theta)\theta^{-\frac{N}{\beta r}} d\theta \|u_0\|_{L^p(\mathbb{R}^N)} + C\vartheta^\vartheta e^{-\vartheta} t^{-\alpha\vartheta} \int_0^{+\infty} \phi_\alpha(\theta)\theta^{-\vartheta} d\theta \|u_0\|_{L^q(\mathbb{R}^N)}\\
=& C\frac{\Gamma\left(1-\frac{N}{\beta r}\right)}{\Gamma\left(1-\frac{\alpha N}{\beta r}\right)} t^{-\frac{\alpha N}{\beta r}} \|u_0\|_{L^p(\mathbb{R}^N)} + C\frac{\vartheta^\vartheta e^{-\vartheta}\Gamma(1-\vartheta)}{\Gamma(1-\alpha\vartheta)} t^{-\alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)}\\
=& A_1 t^{-\frac{\alpha N}{\beta r}} \|u_0\|_{L^p(\mathbb{R}^N)} + A_2 t^{-\alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)}.
\end{aligned}$$

Hence, we get it. □

Lemma 2.7. *For* $u_0 \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *and* $1 \le p \le q \le +\infty$, *let* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, *if* $\frac{1}{r} < \frac{2\beta}{N}$, *then*

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le A_3 t^{-\frac{\alpha N}{\beta r}} \|u_0\|_{L^p(\mathbb{R}^N)} + A_4 t^{-\alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)},$$

where

$$A_3 = \frac{C_1\Gamma\left(2-\frac{N}{\beta r}\right)}{\Gamma\left(1+\alpha-\frac{\alpha N}{\beta r}\right)},\ A_4 = \frac{C_2\vartheta^\vartheta e^{-\vartheta}\Gamma(2-\vartheta)}{\Gamma(1+\alpha-\alpha\vartheta)}.$$

Proof. The proof is similar to that of Lemma 2.6. □

Lemma 2.8. *Assume* $f \in L^q\left((0,T), L^p\left(\mathbb{R}^N\right)\right)$, $q > 1$. *Let*

$$w(t) = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathcal{G} f(s) ds,$$

then

$${}_0I_t^{1-\alpha} w = \int_0^t P_\alpha(t-s)\mathcal{G} f(s) ds.$$

Furthermore, suppose $1 < p < +\infty, 1 < q \le +\infty$ *and* $r \ge p$ *satisfy*

$$\frac{1}{p} - \frac{1}{r} < \frac{2}{N}\left(1 - \frac{1}{\alpha q}\right) \text{ and } \vartheta < 1 - \frac{1}{\alpha q}.$$

If $f \in L^q\left((0,T), L^p\left(\mathbb{R}^N\right)\right)$, *then* $w \in C\left([0,T], L^r\left(\mathbb{R}^N\right)\right)$.

Proof. It follows from Fubini theorem and (2.20) that

$$\begin{aligned}
{}_0I_t^{1-\alpha}w &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\int_0^s (s-\tau)^{\alpha-1}S_\alpha(s-\tau)\mathscr{G}f(\tau)d\tau ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_\tau^t (t-s)^{-\alpha}(s-\tau)^{\alpha-1}S_\alpha(s-\tau)\mathscr{G}f(\tau)dsd\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^{t-\tau} (t-s-\tau)^{-\alpha}s^{\alpha-1}S_\alpha(s)\mathscr{G}f(\tau)dsd\tau \\
&= \int_0^t P_\alpha(t-\tau)\mathscr{G}f(\tau)d\tau.
\end{aligned}$$

Since $1 < q < +\infty$, we can assume that $f \in L^q\left((0,T), W^{2,p}\left(\mathbb{R}^N\right)\right)$ by using a regularizing sequence. Thus, one obtains $f \in L^q\left((0,T), L^r\left(\mathbb{R}^N\right)\right)$. And using dominated convergence theorem, we have $w \in C\left([0,T], L^r\left(\mathbb{R}^N\right)\right)$. By Lemma 2.7,

$$\frac{1}{p}-\frac{1}{r} < \frac{2}{N}\left(1-\frac{1}{\alpha q}\right) \text{ and } \vartheta < 1-\frac{1}{\alpha q},$$

we can get

$$\begin{aligned}
\|w\|_{L^r(\mathbb{R}^N)} \leq & C\int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|f(s)\|_{L^p(\mathbb{R}^N)}ds + C\int_0^t (t-s)^{\alpha-1-\alpha\vartheta}\|f(s)\|_{L^r(\mathbb{R}^N)}ds \\
\leq & C\left(\int_0^t (t-s)^{\left[\alpha-1-\frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)\right]\frac{q}{q-1}}ds\right)^{\frac{q-1}{q}} \|f(s)\|_{L^p\left((0,T),L^p(\mathbb{R}^N)\right)} \\
& + C\left(\int_0^t (t-s)^{[\alpha-1-\alpha\vartheta]\frac{q}{q-1}}ds\right)^{\frac{q-1}{q}} \|f(s)\|_{L^p\left((0,T),L^r(\mathbb{R}^N)\right)} \\
\leq & C(T)\|f\|_{L^q\left((0,T),L^p(\mathbb{R}^N)\right)}.
\end{aligned}$$

Thus, a approximate argument leads to $w \in C\left([0,T], L^r\left(\mathbb{R}^N\right)\right)$ if $f \in L^q((0,T), L^p\left(\mathbb{R}^N\right))$. □

3. Local existence

This section is dedicated to proving the local existence and uniqueness of mild solutions to problem (1.1). First, we give the definition of mild solution foe problem (1.1).

Definition 3.1. *Let* $u_0 \in L^q\left(\mathbb{R}^N\right)$*,* $T > 0$*, we call that* $u \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right)$ *is a mild solution of* (1.1)*, if u satisfies the following integral equation*

$$u = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathscr{G}|u|^{p-1}ud\tau. \tag{3.1}$$

Theorem 3.1. *Let* $0 < \alpha < 1$ *and* $q_c = \frac{N(p-1)}{\beta}$, $u_0 \in L^q\left(\mathbb{R}^N\right)$, $\alpha q_c < q < +\infty$. *Then there exists* $T > 0$ *such that problem* (1.1) *has a mild solution u in*

$$C\left([0,T], L^q\left(\mathbb{R}^N\right)\right) \cap C\left((0,T], L^r\left(\mathbb{R}^N\right)\right)$$

and

$$\sup_{t\in(0,T)} t^{b_r}\|u(t)\|_{L^r(\mathbb{R}^N)} < \infty,$$

where $b_r = \frac{\alpha N}{\beta}\left(\frac{1}{q}-\frac{1}{r}\right)$ *and* $r \in (q,+\infty]$ *satisfies* $\frac{1}{q}-\frac{1}{r} < \frac{\beta}{N}$. *This solution is unique in the class*

$$\left\{u \in L^\infty_{loc}\left((0,T), L^{pq}\left(\mathbb{R}^N\right)\right) \mid \sup_{t\in(0,T)} t^{\frac{N\alpha}{\beta}\left(\frac{1}{q}-\frac{1}{pq}\right)}\|u\|_{L^{pq}(\mathbb{R}^N)} < \infty\right\}. \tag{3.2}$$

Furthermore, if r *satisfies* $pq \le r \le +\infty$ *and* $\frac{1}{q}-\frac{1}{r} < \frac{\beta}{Np\alpha}$, *then* u *can be extended to a maximal interval* $[0,T^*)$ *such that*

$$u \in C\left([0,T^*), L^q\left(\mathbb{R}^N\right)\right) \cap C\left((0,T^*), L^r\left(\mathbb{R}^N\right)\right)$$

and either $T^* = +\infty$ *or* $T^* < +\infty$ *with* $\|u(t)\|_{L^r(\mathbb{R}^N)} \to +\infty$ *as* $t \to T^{*-}$.

Proof. For given $T > 0$, let

$$E_{pq,T} = \left\{u \in L^\infty_{loc}\left((0,T), L^{pq}\left(\mathbb{R}^N\right)\right) \mid \|u\|_{E_{pq,T}} < \infty\right\}, \quad \|u\|_{E_{pq,T}} = \sup_{t\in(0,T)} t^{b_{pq}}\|u(t)\|_{L^{pq}(\mathbb{R}^N)},$$

where

$$b_{pq} = \frac{\alpha N}{\beta}\left(\frac{1}{q}-\frac{1}{pq}\right)$$

and $b_{pq} - \alpha\vartheta > 0$. Then, $E_{pq,T}$ is a Banach space. Choose

$$M > A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2 T^{b_{pq}-\alpha\vartheta}\|u_0\|_{L^q(\mathbb{R}^N)},$$

where A_1 and A_2 are given by Lemma 2.6, let B_K denote the closed ball in $E_{pq,T}$ with center 0 and radius K. We define the operator G on $E_{pq,T}$ as

$$G(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathcal{G}|u|^{p-1}u ds.$$

It follows from Lemmas 2.6 and 2.7 that there exists a constant $C > 0$ such that for $u \in B_K$ and $t \in (0,T)$

$$\begin{aligned}
& t^{b_{pq}}\|G(u)(t)\|_{L^{pq}(\mathbb{R}^N)} \\
\le & C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2 t^{b_{pq}-\alpha\vartheta}\|u_0\|_{L^{pq}(\mathbb{R}^N)}\right) \\
& + CA_3 t^{b_{pq}}\int_0^t (t-s)^{\alpha-1}\|u(s)\|^p_{L^{pq}(\mathbb{R}^N)}ds + CA_4 t^{b_{pq}}\int_0^t (t-s)^{\alpha-1-\alpha\vartheta}\|u(s)\|^p_{L^{pq}(\mathbb{R}^N)}ds \\
\le & C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2 T^{b_{pq}-\alpha\vartheta}\|u_0\|_{L^q(\mathbb{R}^N)}\right) \\
& + CA_3 K^p t^{b_{pq}}\int_0^t (t-s)^{\alpha-1}s^{-pb_{pq}}ds + CA_4 K^p t^{b_{pq}}\int_0^t (t-s)^{\alpha-1-\alpha\vartheta}s^{-pb_{pq}}ds \\
\le & C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2 T^{b_{pq}-\alpha\vartheta}\|u_0\|_{L^q(\mathbb{R}^N)}\right) \\
& + CA_3 K^p T^{\alpha-pb_{pq}+b_{pq}}\int_0^1 (1-s)^{\alpha-1}s^{-pb_{pq}}ds + CA_4 K^p T^{\alpha-pb_{pq}+b_{pq}-\alpha\vartheta}\int_0^1 (1-s)^{\alpha-1-\alpha\vartheta}s^{-pb_{pq}}ds
\end{aligned}$$

$$\leq CM + CA_3K^pT^{\alpha-pb_{pq}+b_{pq}} + CA_4K^pT^{\alpha-pb_{pq}+b_{pq}-\alpha\vartheta}.$$

The fact that $q > \alpha q_c > q_c$ guarantees $\alpha - b_{pq} > 0, pb_{pq} < 1$ and $\alpha - pb_{pq} > 0$. So, all the integrals above are convergent. Choose $K > 0$ and $T > 0$ such that

$$CM + CA_3K^pT^{\alpha-pb_{pq}+b_{pq}} + CA_4K^pT^{\alpha-pb_{pq}+b_{pq}-\alpha\vartheta} \leq K. \tag{3.3}$$

Hence, G maps B_K into itself. Note that

$$\||u|^p - |v|^p\|_{L^q(\mathbb{R}^N)} \leq C\left(\|u\|^{p-1}_{L^{pq}(\mathbb{R}^N)} + \|v\|^{p-1}_{L^{pq}(\mathbb{R}^N)}\right)\|u-v\|_{L^{pq}(\mathbb{R}^N)}$$

for some constant $C > 0$ independent of u and v. Similar calculations show that G is a strict contraction on B_K if T is chosen small enough. Therefore, G possesses a unique fixed point u in B_K.

Note that $\sup_{t\in(0,T)} t^{pb_{pq}} \||u|^p\|_{L^q(\mathbb{R}^N)} < +\infty$. Then we deduce from Lemma 2.7 and $pb_{pq} < \alpha$ that

$$\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)|u|^p ds \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right).$$

Thus $u \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right)$.

Since $r > q$ satisfies $1/q - 1/r < \beta/N$, using Lemmas 2.6 and 2.7 the fact that $pb_{pq} < 1 < \alpha$, we have

$$\begin{aligned}
&t^{b_r}\|G(u)(t)\|_{L^r(\mathbb{R}^N)}\\
\leq &C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2t^{b_r-\alpha\vartheta}\|u_0\|_{L^r(\mathbb{R}^N)}\right)\\
+&CA_3t^{b_r}\int_0^t (t-s)^{\alpha-1}\|u(s)\|^p_{L^{pq}(\mathbb{R}^N)}ds + CA_4t^{b_r}\int_0^t (t-s)^{\alpha-1-\alpha\vartheta}\|u(s)\|^p_{L^{pq}(\mathbb{R}^N)}ds\\
\leq &C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2T^{b_r-\alpha\vartheta}\|u_0\|_{L^r(\mathbb{R}^N)}\right)\\
+&CA_3K^pt^{b_r}\int_0^t (t-s)^{\alpha-1}s^{-pb_{pq}}ds + CA_4K^pt^{b_r}\int_0^t (t-s)^{\alpha-1-\alpha\vartheta}s^{-pb_{pq}}ds\\
\leq &C\left(A_1\|u_0\|_{L^q(\mathbb{R}^N)} + A_2T^{b_{pq}-\alpha\vartheta}\|u_0\|_{L^q(\mathbb{R}^N)}\right)\\
+&CA_3K^pT^{\alpha-pb_{pq}+b_r}\int_0^1 (1-s)^{\alpha-1}s^{-pb_{pq}}ds + CA_4K^pT^{\alpha-pb_{pq}+b_r-\alpha\vartheta}\int_0^1 (1-s)^{\alpha-1-\alpha\vartheta}s^{-p\beta_{pq}}ds\\
\leq &+\infty.
\end{aligned}$$

In addition, observe that $u \in E_{pq,T}$ and by a simple calculation we find that $u \in C\left((0,T], L^r\left(\mathbb{R}^N\right)\right)$. Consequently, $u \in E_{r,T} \cap C\left((0,T], L^r\left(\mathbb{R}^N\right)\right)$.

Next we prove the uniqueness of the solution. Let $u, v \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right) \cap E_{pq,T}$ be two mild solutions of (1.1) for some $T > 0$. Suppose $u, v \in B_{K'}$. Then, we can take $T' < T$ small enough such that (3.3) holds with K replaced by K'. Thus, $u(t) = v(t)$ for $t \in [0, T']$. When $T' \leq t \leq T$, we have

$$\|u(t)-v(t)\|_{L^{pq}(\mathbb{R}^N)} \leq C\left(A_3\int_{T'}^t (t-s)^{\alpha-\frac{\alpha N(p-1)}{\beta pq}-1} + A_4\int_{T'}^t (t-s)^{\alpha-\alpha\vartheta-1}\right)\|u(s)-v(s)\|_{L^{pq}(\mathbb{R}^N)}ds$$

for some constant $C > 0$ independent of u and v. Hence, Gronwall's inequality yields $u(t) = v(t)$ for $t \in [T', T]$.

Finally, we prove that the existence of maximal time provided r satisfies

$$pq \leq r \leq +\infty \text{ and } \frac{1}{q} - \frac{1}{r} < \frac{\beta}{Np\alpha}.$$

Set

$$T^* = \sup\left\{T > 0 \mid u \in E_{r,T} \cap C\left((0,T], L^r\left(\mathbb{R}^N\right)\right) \text{ is a mild solution}\right\}.$$

Assume $T^* < +\infty$ and there exists $M_1 > 0$ such that $\sup_{t\in(0,T^*)} t^{b_r} \|u(t)\|_{L^r\left(\mathbb{R}^N\right)} \leq M_1$.

We claim that there exists $\tilde{M}_1 > 0$ such that

$$\sup_{t\in(0,T^*)} t^{b_{pq}} \|u(t)\|_{L^{pq}\left(\mathbb{R}^N\right)} < \tilde{M}_1 \text{ and } \sup_{t\in(0,T^*)} \|u(t)\|_{L^q\left(\mathbb{R}^N\right)} < +\infty. \tag{3.4}$$

If $r = pq$, we have

$$\begin{aligned}
\|u(t)\|_{L^q\left(\mathbb{R}^N\right)} \leq& C\left(\|u_0\|_{L^q\left(\mathbb{R}^N\right)} + T^{*b_r - \alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)}\right) \\
&+ C\int_0^t (t-s)^{\alpha-1} \|u(s)\|^p_{L^{pq}\left(\mathbb{R}^N\right)} ds + C\int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \|u(s)\|^p_{L^{pq}\left(\mathbb{R}^N\right)} ds \\
\leq& C\left(\|u_0\|_{L^q\left(\mathbb{R}^N\right)} + T^{*b_r - \alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)}\right) \\
&+ C\left(T^*\right)^{\alpha - \frac{\alpha N(p-1)}{\beta pq}} \int_0^1 (1-s)^{\alpha-1} s^{-pb_{pq}} ds + + C\left(T^*\right)^{\alpha - \alpha\vartheta - \frac{\alpha N(p-1)}{\beta pq}} \int_0^1 (1-s)^{\alpha-1} s^{-pb_{pq}} ds \\
<& +\infty.
\end{aligned}$$

For the case of $pq < r < +\infty$, since

$$\frac{p}{r} - \frac{1}{r} < \frac{\beta}{N}, \ \frac{1}{q} - \frac{1}{r} < \frac{\beta}{Np\alpha} \text{ and } \frac{1}{q} - \frac{1}{pq} < \frac{\beta}{N},$$

we can take $n \in \mathbb{N}$ large enough such that

$$\frac{r}{p} < r\left(\frac{pq}{r}\right)^{\frac{1}{n}} < r, \ \frac{\left(\frac{pq}{r}\right)^{\frac{1}{n}} p - 1}{pq} < \frac{\beta}{N} \text{ and } \frac{p}{r} - \frac{1}{r\left(\frac{pq}{r}\right)^{\frac{1}{n}}} < \frac{\beta}{N}.$$

Set $\chi = \left(\frac{pq}{r}\right)^{\frac{1}{n}}$ and $q_1 = r, \ q_k = q_{k-1}\chi = q_1\chi^{k-1}, \ k = 2, 3, \ldots, n+1$. Observing that $\chi < 1$ and

$$\begin{gathered}
0 < \frac{p}{q_k} - \frac{1}{q_{k+1}} = \frac{1}{\chi^{k-1}}\left(\frac{p}{r} - \frac{1}{r\chi}\right) \leq \frac{1}{\chi^{n-1}}\left(\frac{p}{r} - \frac{1}{r\chi}\right) = \frac{\chi p}{pq} - \frac{1}{pq} < \frac{\beta}{N}, \ k = 1, 2, \ldots, n, \\
\frac{1}{q} - \frac{1}{q_k} \leq \frac{1}{q} - \frac{1}{q_1} = \frac{1}{q} - \frac{1}{r} < \frac{\beta}{Np\alpha}, \ k = 1, 2, \ldots, n+1,
\end{gathered}$$

we know that if

$$\sup_{t\in(0,T^*)} t^{b_{q_k}} \|u(t)\|_{L^{q_k}\left(\mathbb{R}^N\right)} < +\infty,$$

there exists a constant $C > 0$ such that

$$
\begin{aligned}
&t^{b_{q_{k+1}}} \|u(t)\|_{L^{q_{k+1}}(\mathbb{R}^N)} \\
\leq & C \|u_0\|_{L^q(\mathbb{R}^N)} + C T^{*b_r - \alpha\vartheta} \|u_0\|_{L^{q_{k+1}}(\mathbb{R}^N)} \\
+ & C t^{b_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \frac{\alpha N}{\beta}\left(\frac{p}{q_k} - \frac{1}{q_{k+1}}\right) - 1} \|u(s)\|^p_{L^{q_k}(\mathbb{R}^N)} ds + C t^{b_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \alpha\vartheta - 1} \|u(s)\|^p_{L^{q_k}(\mathbb{R}^N)} ds \\
\leq & C \|u_0\|_{L^q(\mathbb{R}^N)} + C T^{*b_r - \alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)} \\
+ & C t^{b_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \frac{\alpha N}{\beta}\left(\frac{p}{q q_k} - \frac{1}{q_{k+1}}\right) - 1} s^{-p b_{q_k}} ds + C t^{b_{q_{k+1}}} \int_0^t (t-s)^{\alpha - \alpha\vartheta - 1} s^{-p b_{q_k}} ds \\
\leq & C \left(\|u_0\|_{L^q(\mathbb{R}^N)} + T^{*b_r - \alpha\vartheta} \|u_0\|_{L^q(\mathbb{R}^N)} \right) \quad + C t^{\alpha - \frac{\alpha N (p-1)}{\beta q}} \int_0^1 (1-s)^{\alpha - \frac{\alpha N}{\beta}\left(\frac{p}{q_k} - \frac{1}{q_{k+1}}\right) - 1} s^{-p b_{q_k}} ds \\
+ & C t^{\alpha - \frac{\alpha N}{\beta q}\left(\frac{1-p}{q} - \frac{1}{q_{k+1}} \frac{1}{q_k}\right) - \alpha\vartheta} \int_0^1 (1-s)^{\alpha - \alpha\vartheta - 1} s^{-p b_{q_k}} ds \\
< & +\infty.
\end{aligned}
$$

Thus, the assumption that $\sup_{t\in(0,T^*)} t^{b_r} \|u(t)\|_{L^r(\mathbb{R}^N)} \leq M_1$ implies

$$\sup_{t\in(0,T^*)} t^{b_{pq}} \|u(t)\|_{L^{pq}(\mathbb{R}^N)} < +\infty,$$

and then $\sup_{t\in(0,T^*)} \|u(t)\|_{L^q(\mathbb{R}^N)} < +\infty$. Therefore, the claims are proved.

Next we verify that $\lim_{t\to T^{*-}} u(t)$ exists in $L^r\left(\mathbb{R}^N\right) \cap L^{pq}\left(\mathbb{R}^N\right)$. Indeed, for $\frac{T^*}{2} < t < \tau < T^*$, by a similar proof of Lemma 2.8 and using Lemmas 2.6 and 2.7, there exists a $\tilde{m} > 0$ such that

$$
\begin{aligned}
\|u(t) - u(\tau)\|_{L^r(\mathbb{R}^N)} &\leq C(\tau - t)\left(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_0\|_{L^r(\mathbb{R}^N)} \right) + C M_1^p (\tau - t)^{\tilde{m}}, \\
\|u(t) - u(\tau)\|_{L^{pq}(\mathbb{R}^N)} &\leq C(\tau - t)\left(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_1\|_{L^{pq}(\mathbb{R}^N)} \right) + C \tilde{M}_1^p (\tau - t)^{\tilde{m}}.
\end{aligned} \tag{3.5}
$$

Therefore, $\lim_{t\to T^{*-}} u(t)$ exists in $L^r\left(\mathbb{R}^N\right) \cap L^{pq}\left(\mathbb{R}^N\right)$. Denote $u_{T^*} = \lim_{t\to T^{*-}} u(t)$ and define $u\left(T^*\right) = u_{T^*}$. For $h > 0$ and $\delta > 0$, let

$$\tilde{E}_{h,\delta} = \left\{ u \in C\left([T^*, T^* + h], L^r\left(\mathbb{R}^N\right) \cap L^{pq}\left(\mathbb{R}^N\right) \right) \mid u\left(T^*\right) = u_{T^*}, d\left(u, u_{T^*}\right) \leq \delta \right\},$$

where

$$d\left(u, u_{T^*}\right) = \max_{t\in[T^*, T^*+h]} \|u(t) - u_{T^*}\|_{L^r(\mathbb{R}^N)} + \max_{t\in[T^*, T^*+h]} \|u(t) - u_{T^*}\|_{L^{pq}(\mathbb{R}^N)}.$$

It follows from (3.4) and Lemma 2.8 that $u \in C\left((0, T^*], L^{pq}\left(\mathbb{R}^N\right) \cap L^r\left(\mathbb{R}^N\right) \right)$. Then we can define the operator K on $\tilde{E}_{h,\delta}$ as

$$
\begin{aligned}
K(v)(t) = & P_\alpha(t) u_0 + \int_0^{T^*} (t-\tau)^{\alpha-1} S_\alpha(t-\tau) \mathcal{G} |u(\tau)|^p d\tau \\
& + \int_{T^*}^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) \mathcal{G} |v(\tau)|^p d\tau, \quad v \in \tilde{E}_{h,\delta}.
\end{aligned}
$$

We can easily see that $K(v) \in C\left([T^*, T^*+\ h],\ L^r\left(\mathbb{R}^N\right) \cap L^{pq}\left(\mathbb{R}^N\right)\right)$ and $K(v)(T^*) = u_{T^*}$ by using (3.5), Lemmas 2.6 and 2.7. For $v \in \tilde{E}_{h,\delta}$ and $t \in [T^*, T^*+h]$, it follows from the same arguments as above that

$$\begin{aligned} \|K(v)(t) - u_{T^*}\|_{L^r(\mathbb{R}^N)} \leq & C\,(t-T^*)\left(\|u_0\|_{L^q(\mathbb{R}^N)} + \|u_0\|_{L^r(\mathbb{R}^N)}\right) + CM_1^p\,(t-T^*)^{\tilde{m}} \\ & + C\left(\|u_{T^*}\|_{L^r(\mathbb{R}^N)} + \delta\right)^p \left[(t-T^*)^{\alpha - \frac{\alpha N(p-1)}{\beta r}} + (t-T^*)^{\alpha-\alpha\vartheta}\right] \end{aligned} \tag{3.6}$$

for some positive constant C. Moreover, (3.6) also holds if r is replaced by pq. So we can choose h small enough such that $d(u, u_{T^*}) \leq \delta$.

On the other hand, for every $w, v \in \tilde{E}_{h,\delta}$, there exists a positive constant C such that

$$\begin{aligned} & \|Kw - Kv\|_{L^r(\mathbb{R}^N)} \\ \leq & C\left(\int_{T^*}^t (t-\tau)^{\alpha - \frac{\alpha N(p-1)}{\beta r} - 1} + \int_{T^*}^t (t-\tau)^{\alpha-\alpha\vartheta-1}\right)\left(\|w\|_{L^r(\mathbb{R}^N)}^{p-1} + \|v\|_{L^r(\mathbb{R}^N)}^{p-1}\right)\|w-v\|_{L^r(\mathbb{R}^N)} d\tau \\ \leq & C\left(\|u_{T^*}\|_{L^r(\mathbb{R}^N)} + \delta\right)^{p-1}\left(h^{\alpha - \frac{\alpha N(p-1)}{\beta r}} + h^{\alpha-\alpha\vartheta}\right) \max_{t\in[T^*, T^*+h]} \|w-v\|_{L^r(\mathbb{R}^N)}, \end{aligned}$$

and

$$\|Kw - Kv\|_{L^{pq}(\mathbb{R}^N)} \leq C\left(\|u_{T^*}\|_{L^{pq}(\mathbb{R}^N)} + \delta\right)^{p-1}\left(h^{\alpha - \frac{\alpha N(p-1)}{\beta r}} + h^{\alpha-\alpha\vartheta}\right) \max_{t\in[T^*, T^*+h]} \|w-v\|_{L^{pq}(\mathbb{R}^N)}.$$

Thus, choosing h small enough such that

$$C\left[\left(\|u_{T^*}\|_{L^r(\mathbb{R}^N)} + \delta\right)^{p-1} + \left(\|u_{T^*}\|_{L^{pq}(\mathbb{R}^N)} + \delta\right)^{p-1}\right]\left(h^{\alpha - \frac{\alpha N(p-1)}{\beta r}} + h^{\alpha-\alpha\vartheta}\right) \leq \frac{1}{2},$$

we know G is a strict contraction on $\tilde{E}_{h,\delta}$. So the contraction mapping principle implies G has a fixed point $v \in \tilde{E}_{h,\delta}$. Define

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [0, T^*], \\ v(t), & t \in [T^*, T^*+h]. \end{cases}$$

Since

$$v(T^*) = G(v(T^*)) = u(T^*),$$

we can easily verify that

$$\tilde{u} \in E_{r,T^*+h} \cap C\left((0, T^*+h], L^r\left(\mathbb{R}^N\right)\right)$$

and

$$\tilde{u}(t) = P_\alpha(t)u_0 s + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}|\tilde{u}(\tau)|^p ds.$$

Obverse that $u \in E_{pq,T^*+h}$, we have

$$u \in C\left([0, T^*+h], L^q\left(\mathbb{R}^N\right)\right).$$

Thus, $\tilde{u}(t)$ is a mild solution of (1.1), which contradicts the definition of T^*. □

4. Blow-up and global existence

In this section, we prove the results of blow-up and global existence of solutions for problem (1.1). First, we give the definition of weak solution for problem (1.1).

Definition 4.1. *For* $u_0 \in L^q\left(\mathbb{R}^N\right)$ *and* $T > 0$*,we call* $u \in L^p\left((0,T), L^q\left(\mathbb{R}^N\right)\right)$*is a weak solution of problem* (1.1) *if*

$$\int_0^T \int_{\mathbb{R}^N} (u-u_0)_t^C D_T^\alpha \varphi dxdt = -\int_0^T \int_{\mathbb{R}^N} \mathscr{G} u\,(-\Delta)^{\frac{\beta}{2}} \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \mathscr{G}\left(|u|^{p-1}u\right)\varphi dxdt$$

for every $\varphi \in C^1\left([0,T], H^\beta\left(\mathbb{R}^N\right)\right)$ *with* $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ *and* $\varphi(\cdot,T)=0$.

Lemma 4.1. *Assume* $u_0 \in L^q\left(\mathbb{R}^N\right)$*, let* $u \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right)$ *be a mild solution of* (1.1)*, then u is also a weak solution of* (1.1).

Proof. Assume that $u \in C\left([0,T], L^q\left(\mathbb{R}^N\right)\right)$ is a mild solution of (1.1), we have

$$u-u_0 = P_\alpha(t)u_0 - u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}|u|^{p-1}uds,$$

where $\mathscr{G} = -(m\Delta - I)^{-1}$. Note that by Lemma 2.8,

$${}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)|u|^{p-1}\mathscr{G}u(s)ds\right) = \int_0^t P_\alpha(t-s)|u|^{p-1}\mathscr{G}u(s)ds,$$

so

$${}_0I_t^{1-\alpha}(u-u_0) = {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0) + \int_0^t P_\alpha(t-s)\mathscr{G}|u|^{p-1}u(s)ds.$$

Then, for every $\varphi \in C^1\left([0,T], H^\beta\left(\mathbb{R}^N\right)\right)$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(\cdot,T)=0$, we have

$$\int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(u-u_0)\varphi dx = I_1(t) + I_2(t), \tag{4.1}$$

where

$$I_1(t) = \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\varphi dx,$$

$$I_2(t) = \int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)\mathscr{G}|u|^{p-1}uds\varphi dx.$$

By Lemma 2.5,

$$\frac{dI_1}{dt} = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta}{2}} \mathscr{G}(P_\alpha(t)u_0)\varphi dx + \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}(P_\alpha(t)u_0 - u_0)\varphi_t dx. \tag{4.2}$$

For every $h > 0, t \in [0, T)$ and $t + h \leq T$, we have

$$\begin{aligned}\frac{1}{h}\left(I_2(t+h) - I_2(t)\right) =& \frac{1}{h}\int_0^{t+h}\int_{\mathbb{R}^N} P_\alpha(t+h-s)\mathscr{G}|u|^{p-1}uds\varphi(t+h,x)dx\\ &-\frac{1}{h}\int_0^t\int_{\mathbb{R}^N} P_\alpha(t-s)\mathscr{G}|u|^{p-1}uds\varphi(t,x)dx\\ =& I_3 + I_4 + I_5,\end{aligned}$$

where

$$\begin{aligned}I_3 &= \frac{1}{h}\int_{\mathbb{R}^N}\int_t^{t+h}\int_0^\infty \phi_\alpha(\theta)T\left((t+h-s)^\alpha\theta\right)\mathscr{G}|u|^{p-1}u(s)d\theta ds\varphi(t+h,x)dx,\\ I_4 &= \frac{1}{h}\int_{\mathbb{R}^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)\left(T\left((t+h-s)^\alpha\theta\right) - T\left((t-s)^\alpha\theta\right)\right)\mathscr{G}|u|^{p-1}u(s)d\theta ds\varphi(t,x)dx,\\ I_5 &= \frac{1}{h}\int_{\mathbb{R}^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)T\left((t+h-s)^\alpha\theta\right)\mathscr{G}|u|^{p-1}u(s)d\theta ds(\varphi(t+h,x) - \varphi(t,x))dx.\end{aligned}$$

By dominated convergence theorem, we deduce that

$$\begin{aligned}I_3 &\to \int_{\mathbb{R}^N}\mathscr{G}\left(|u|^{p-1}u\right)\varphi dx \text{ as } h\to 0,\\ I_5 &\to \int_{\mathbb{R}^N}\int_0^t\int_0^\infty \phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\mathscr{G}|u|^{p-1}u(s)d\theta ds\varphi_t dx\\ &= \int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)\mathscr{G}|u|^{p-1}u(s)ds\varphi_t dx \text{ as } h\to 0.\end{aligned}$$

Since

$$\begin{aligned}I_4 &= -\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}(-\Delta)^{\frac{\beta}{2}}\mathscr{G}\left(T\left((t+\tau h-s)^\alpha\theta\right)\right)\mathscr{G}|u|^{p-1}u(s)d\tau d\theta ds\varphi dx\\ &= -\int_{\mathbb{R}^N}(-\Delta)^{\frac{\beta}{2}}\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}\mathscr{G}T\left((t+\tau h-s)^\alpha\theta\right)\mathscr{G}|u|^{p-1}u(s)d\tau d\theta ds\varphi dx\\ &= -\int_{\mathbb{R}^N}\int_0^t\int_0^\infty\int_0^1 \alpha\theta\phi_\alpha(\theta)(t+\tau h-s)^{\alpha-1}\mathscr{G}T\left((t+\tau h-s)^\alpha\theta\right)\mathscr{G}|u|^{p-1}u(s)d\tau d\theta ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dx,\end{aligned}$$

using dominated convergence theorem again, we know

$$I_4 \to -\int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}\mathscr{G}S_\alpha(t-s)\mathscr{G}|u|^{p-1}u(s)ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dx \text{ as } h\to 0.$$

Hence, the right derivative of I_2 on [0, T) is

$$\begin{aligned}\int_{\mathbb{R}^N}\mathscr{G}|u|^{p-1}u\varphi dx - \int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}\mathscr{G}S_\alpha(t-s)\mathscr{G}|u|^{p-1}u(s)ds\Delta^{\frac{\beta}{2}}\varphi dx\\ + \int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)\mathscr{G}|u|^{p-1}u(s)ds\varphi_t dx\end{aligned}$$

and it is continuous in $[0,T)$. Therefore,

$$
\begin{aligned}
\frac{dI_2}{dt} &= \int_{\mathbb{R}^N} \mathcal{G}\left(|u|^{p-1}u\right)\varphi dx - \int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}\mathcal{G}S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dx \\
&+ \int_{\mathbb{R}^N}\int_0^t P_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\varphi_t dx \\
&= \int_{\mathbb{R}^N} \mathcal{G}\left(|u|^{p-1}u\right)\varphi dx - \int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}\mathcal{G}S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dx \\
&+ \int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\right)\varphi_t dx,\ t\in[0,T).
\end{aligned}
\tag{4.3}
$$

Combining (4.1)–(4.3), we conclude that

$$
\begin{aligned}
0 &= \int_0^T \frac{d}{dt}\int_{\mathbb{R}^N} I_t^{1-\alpha}(u-u_0)\varphi dxdt = \int_0^T \frac{dI_5}{dt} + \frac{dI_6}{dt}dt \\
&= \int_0^T\int_{\mathbb{R}^N} \mathcal{G}\left(P_\alpha(t)u_0\right)(-\Delta)^{\frac{\beta}{2}}\varphi dxdt + \int_0^T\int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}\left(P_\alpha(t)u_0-u_0\right)\varphi_t dxdt \\
&+ \int_0^T\int_{\mathbb{R}^N} \mathcal{G}\left(|u|^{p-1}u\right)\varphi dxdt - \int_0^T\int_{\mathbb{R}^N}\int_0^t (t-s)^{\alpha-1}\mathcal{G}S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt \\
&+ \int_0^T\int_{\mathbb{R}^N} {}_0I_t^{1-\alpha}\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds\right)\varphi_t dxdt \\
&= -\int_0^T\int_{\mathbb{R}^N} \mathcal{G}u\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt - \int_0^T\int_{\mathbb{R}^N}(u-u_0)\,{}_t^C D_T^\alpha\varphi dxdt + \int_0^T\int_{\mathbb{R}^N}\mathcal{G}\left(|u|^{p-1}u\right)\varphi dxdt,
\end{aligned}
$$

so, we can get the following equation

$$
0 = -\int_0^T\int_{\mathbb{R}^N} \mathcal{G}u\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt - \int_0^T\int_{\mathbb{R}^N}(u-u_0)\,{}_t^C D_T^\alpha\varphi dxdt + \int_0^T\int_{\mathbb{R}^N}\mathcal{G}\left(|u|^{p-1}u\right)\varphi dxdt.
$$

Hence, this completes the proof. □

We say the solution u of problem (1.1) blow-up in a finite time T if

$$
\lim_{t\to T}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty.
$$

Now, we give a blow-up result for problem (1.1).

Theorem 4.1. *Let $u_0\in L^q\left(\mathbb{R}^N\right)$, $u_0\geq 0$ and $u_0\not\equiv 0$, then*
(i) If $p<1+\frac{\beta}{N}$, then the mild solution of (1.1) blow-up in a finite time.
(ii) If $p\geq 1+\frac{\beta}{N}$ and $\|u_0\|_{L^{q_c}}$ is sufficiently small, where $q_c=\frac{N(p-1)}{\beta}$, then the mild solution of (1.1) exists globally.

Proof. (i) The proof is by contradiction. Suppose that u is a global mild solution of (1.1), then u is a solution of (1.1) and $u\in C\left([0,T],L^q\left(\mathbb{R}^N\right)\right)$. Then, Lemma 4.1 tells us

$$
\int_0^T\int_{\mathbb{R}^N}(u-u_0)\,{}_t^C D_T^\alpha\varphi dxdt = -\int_0^T\int_{\mathbb{R}^N} \mathcal{G}u\,(-\Delta)^{\frac{\beta}{2}}\varphi dxdt + \int_0^T\int_{\mathbb{R}^N}\mathcal{G}\left(|u|^{p-1}u\right)\varphi dxdt,
$$

for all $\varphi \in C^1\left([0,T], H^\beta\left(\mathbb{R}^N\right)\right)$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(\cdot, T) = 0$. Now we take

$$\varphi(x,t) = (\varphi_1(x))^\ell \varphi_2(t)$$

with

$$\varphi_1(x) = \Phi\left(|x|/T^{1/\beta}\right),\ \varphi_2(t) = (1 - t/T)_+^\eta,$$

where

$$\ell \geq p/(p-1),\ \eta \geq \max\{(\alpha p + 1)/(p-1), \alpha + 1\}$$

and Φ is a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq 1, \\ 0, & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$. We have,

$$\int_0^T \int_{\mathbb{R}^N} u_0 {}_t^C D_T^\alpha \varphi dxdt = \int_{\Omega_T} u {}_t^C D_T^\alpha \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \mathscr{G} u\, (-\Delta)^{\frac{\beta}{2}} \varphi dxdt - \int_0^T \int_{\mathbb{R}^N} \mathscr{G}\left(|u|^{p-1}u\right) \varphi dxdt,$$

where

$$\Omega_T = [0,T] \times \Omega \ \text{ for } \ \Omega = \left\{x \in \mathbb{R}^N \middle| |x| \leq 2T^{1/\beta}\right\}.$$

So, from Ju's inequality $(-\Delta)^{\beta/2}\left(\varphi_1^\ell\right) \leq \ell \varphi_1^{\ell-1} (-\Delta)^{\beta/2} (\varphi_1)$, we can obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} u_0 {}_t^C D_T^\alpha \varphi dxdt &= \int_{\Omega_T} u {}_t^C D_T^\alpha \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} \mathscr{G} u\, (-\Delta)^{\frac{\beta}{2}} \varphi dxdt - \int_0^T \int_{\mathbb{R}^N} \mathscr{G}\left(|u|^{p-1}u\right) \varphi dxdt \\
&\leq \int_{\Omega_T} u {}_t^C D_T^\alpha \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} |\mathscr{G} u| \left|(-\Delta)^{\frac{\beta}{2}} \varphi\right| dxdt + \int_0^T \int_{\mathbb{R}^N} |\mathscr{G}(|u|^p)| |\varphi| \, dxdt \\
&\leq \int_{\Omega_T} u {}_t^C D_T^\alpha \varphi dxdt + \int_0^T \int_{\mathbb{R}^N} |\mathscr{G} u| \left|\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2} \varphi_1(x) \varphi_2(t)\right| dxdt \\
&+ \int_0^T \int_{\mathbb{R}^N} |\mathscr{G}(|u|^p)| |\varphi| \, dxdt.
\end{aligned}$$

Therefore, using *Hölder* inequality, Lemma 2.4 and $u_0 \geq 0$,

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} u_0 {}_t^C D_T^\alpha \varphi dxdt &\leq \left(\int_{\Omega_T} u^p dxdt\right)^{\frac{1}{p}} \left(\int_{\Omega_T} \left|{}_t^C D_T^\alpha \varphi\right|^{\tilde{p}} dxdt\right)^{\frac{1}{\tilde{p}}} + \int_{\Omega_T} |\mathscr{G}(|u|^p)| \, dxdt \\
&+ \left(\int_{\Omega_T} |\mathscr{G}(|u|^p)|^p \, dxdt\right)^{\frac{1}{p}} \left(\int_{\Omega_T} \left|\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2} \varphi_1(x) \varphi_2(t)\right|^{\tilde{p}} dxdt\right)^{\frac{1}{\tilde{p}}} \\
&\leq \left(\int_{\Omega_T} u^p dxdt\right)^{\frac{1}{p}} \left(\int_{\Omega_T} \left|{}_t^C D_T^\alpha \varphi\right|^{\tilde{p}} dxdt\right)^{\frac{1}{\tilde{p}}} + M_1 \int_{\Omega_T} |u|^p dxdt \\
&+ M_2 \left(\int_{\Omega_T} |u|^p \, dxdt\right)^{\frac{1}{p}} \left(\int_{\Omega_T} \left|\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2} \varphi_1(x) \varphi_2(t)\right|^{\tilde{p}} dxdt\right)^{\frac{1}{\tilde{p}}}, \qquad (4.4)
\end{aligned}$$

where $p\tilde{p} = p + \tilde{p}, p > 1, \tilde{p} > 1$.

By changing the variables: $\tau = T^{-1}t,\ \xi = T^{-1/\beta}x$ and using formulas (2.3) in the right hand-side of (4.4), we get

$$T^{1-\alpha}\int_{\Omega} u_0\varphi_1^{\ell} \le CT^{-\alpha+\left(1+\frac{N}{\beta}\right)\frac{1}{\tilde{p}}}.$$

Hence,

$$\int_{\Omega} u_0\varphi_1^{\ell} \le CT^{-\delta}, \tag{4.5}$$

where

$$\delta = 1 - (1 + N/\beta)\widetilde{p},\ C = C\left(|\Omega_1|, |\Omega_2|\right),$$

($|\Omega_i|$ stands for the measure of Ω_i, for $i = 1, 2$), with

$$\Omega_1 = \left\{\xi \in \mathbb{R}^N \Big| |\xi| \le 2\right\}, \quad \Omega_2 = \{\tau \ge 0 \big| \tau \le 1\}.$$

Since

$$p \le p^* = 1 + \frac{\beta}{N},$$

passing to the limit in (4.5) as $T \to \infty$, we get

$$\lim_{T\to\infty}\int_0^T \int_{|x|\le 2T^{1/\beta}} u_0\varphi_1^{\ell} dxdt = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of u, we infer that

$$\int_0^{\infty}\int_{\mathbb{R}^N} u_0\varphi_1^{\ell} dxdt = 0 \quad \Longrightarrow \quad u_0 \equiv 0.$$

Contradiction.

(ii) We construct the global solution of (1.1) by the contraction mapping principle. As $q > p \ge 1 + \frac{\beta}{2}$, then we have the possibility to take a positive constant $q > 0$ such that

$$\frac{\alpha}{p-1} - \frac{1}{p} < \frac{\alpha N}{\beta q} < \frac{\alpha}{p-1} \tag{4.6}$$

and

$$\frac{\alpha}{p-1} - \alpha < \frac{\alpha N}{\beta q}. \tag{4.7}$$

Let

$$b = \frac{\alpha N}{\beta}\left(\frac{1}{q_c} - \frac{1}{q}\right) = \frac{\alpha}{p-1} - \frac{\alpha N}{\beta q}. \tag{4.8}$$

We verify that

$$0 < pb < 1, \quad \alpha = \frac{\alpha N(p-1)}{\beta q} + (p-1)b. \tag{4.9}$$

Assume that u_0 satisfies

$$\sup_{t>0} t^b \left\|P_\alpha(t)u_0\right\|_{L^q\left(\mathbb{R}^N\right)} = \eta < +\infty. \tag{4.10}$$

If $u_0 \in L^{q_c}\left(\mathbb{R}^N\right)$, Lemma 2.6 implies (4.10) holds. Note that we can take $u_0(x) \le C|x|^{-\frac{2}{p-1}}$ instead of $u_0 \in L^{q_c}\left(\mathbb{R}^N\right)$ for some constant $C > 0$.

Let

$$Y = \left\{u \in L^\infty\left((0,\infty), L^q\left(\mathbb{R}^N\right)\right) \mid \|u\|_Y < \infty\right\},$$

where

$$\|u\|_Y = \sup_{t>0} t^b \|u(t)\|_{L^q\left(\mathbb{R}^N\right)}.$$

For $u \in Y$, we define

$$\Phi(u)(t) = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathcal{G}|u|^{p-1}u(s)ds.$$

Denote

$$B_M = \{u \in Y \mid \|u\|_Y \le M\}.$$

For any $u, v \in B_M, t \ge 0$,

$$t^b\|\Phi(u)(t) - \Phi(v)(t)\|_{L^q\left(\mathbb{R}^N\right)} \le t^b \int_0^t (t-s)^{\alpha-1} \|S_\alpha(t-s)\mathcal{G}\left(u^p(s) - v^p(s)\right)\|_{L^q\left(\mathbb{R}^N\right)} ds. \tag{4.11}$$

Hence, *Hölder* inequality and Lemma 2.7 imply that there exists a constant $C > 0$ such that

$$\begin{aligned}
& t^b\|\Phi(u)(t) - \Phi(v)(t)\|_{L^q\left(\mathbb{R}^N\right)} \\
&\le Ct^b \int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{\beta}\left(\frac{p}{q}-\frac{1}{q}\right)} \|\mathcal{G}\,|u^p - v^p|\|_{L^{\frac{q}{p}}\left(\mathbb{R}^N\right)} ds + Ct^b \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \|\mathcal{G}\,|u^p - v^p|\|_{L^q\left(\mathbb{R}^N\right)} ds \\
&\le Ct^b \int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{\beta}\left(\frac{p}{q}-\frac{1}{q}\right)} \left(\|u\|^{p-1}_{L^q\left(\mathbb{R}^N\right)} + \|v\|^{p-1}_{L^q\left(\mathbb{R}^N\right)}\right) \|u-v\|_{L^q\left(\mathbb{R}^N\right)} ds \\
&+ Ct^b M^{p-1} \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} s^{-pb} ds \|u-v\|_Y \\
&\le Ct^b M^{p-1} \int_0^t (t-s)^{\alpha-1-\frac{\alpha N(p-1)}{\beta q}} s^{-pb} ds \|u-v\|_Y + Ct^b M^{p-1} \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} s^{-pb} ds \|u-v\|_Y \\
&= CM^{p-1} t^{b-pb-\frac{\alpha N(p-1)}{\beta q}+\alpha} \int_0^1 (1-\tau)^{-\frac{\alpha N(p-1)}{\beta q}+\alpha-1} \tau^{-pb} d\tau \|u-v\|_Y \\
&+ CM^{p-1} t^{b-pb-\alpha\vartheta+\alpha} \int_0^1 (1-\tau)^{-\alpha\vartheta+\alpha-1} \tau^{-pb} d\tau \|u-v\|_Y \\
&\le CM^{p-1} \int_0^1 (1-\tau)^{-\frac{\alpha N(p-1)}{\beta q}+\alpha-1} \tau^{-pb} d\tau \|u-v\|_Y + CM^{p-1} \int_0^1 (1-\tau)^{-\alpha\vartheta+\alpha-1} \tau^{-pb} d\tau \|u-v\|_Y \\
&\le CM^{p-1} \frac{\Gamma((p-1)b)\Gamma(1-pb)}{\Gamma(1-b)} \|u-v\|_Y + CM^{p-1} \frac{\Gamma(-\alpha\vartheta+\alpha)\Gamma(1-pb)}{\Gamma(1-b)} \|u-v\|_Y,
\end{aligned}$$

where

$$\vartheta = \frac{N(p-1)}{\beta q}.$$

If we choose M small enough such that

$$CM^{p-1}\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)}+CM^{p-1}\frac{\Gamma(-\alpha\vartheta+\alpha)\Gamma(1-pb)}{\Gamma(1-b)}<\frac{1}{2},$$

then

$$\|\Phi(u)-\Phi(v)\|_Y\le\frac{1}{2}\|u-v\|_Y.$$

Since

$$\begin{aligned}t^{\beta}\|\Phi(u)(t)\|_{L^q(\mathbb{R}^N)}\le&\eta+CM^pt^b\int_0^t(t-s)^{-\frac{\alpha N}{\beta}\left(\frac{p}{q}-\frac{1}{q}\right)-1+\alpha}s^{-pb}ds+CM^pt^b\int_0^t(t-s)^{-\alpha\vartheta-1+\alpha}s^{-pb}ds\\ \le&\eta+CM^p\frac{\Gamma((p-1)\beta)\Gamma(1-p\beta)}{\Gamma(1-\beta)}+CM^{p-1}\frac{\Gamma(-\alpha\vartheta+\alpha)\Gamma(1-pb)}{\Gamma(1-b)},\ t\in[0,+\infty).\end{aligned}$$

Therefore, by contraction mapping principle we know Φ has a fixed point $u\in B_M$.

Next, we will prove $u\in C\left([0,T],L^q\left(\mathbb{R}^N\right)\right)$.

First, we claim that for $T>0$ small enough, $u\in C\left([0,T],L^q\left(\mathbb{R}^N\right)\right)$. In fact, the above proof shows that u is the unique solution in

$$B_{M,T}=\left\{u\in L^{\infty}_{loc}\left((0,T),L^r\left(\mathbb{R}^N\right)\right)\mid\sup_{0<t<T}t^{\beta}\|u(t)\|_{L^r(\mathbb{R}^N)}\le M\right\}.$$

Since $u_0\in L^q\left(\mathbb{R}^N\right)$ and $r>q_c$, we know $u_0\in L^{\tilde{q}}\left(\mathbb{R}^N\right)$ for every $\tilde{q}\in(q_c,q)$ and $\tilde{q}<n$. Observe that the assumption

$$p>1+\frac{\beta\alpha}{\alpha N+\beta-\beta\alpha}$$

implies

$$p>1+\frac{\beta}{\alpha N}\ \text{ and } q_c>\frac{\alpha Np}{\alpha N+\beta}.$$

Then, using Theorem 3.1, we know that (1.1) has a unique solution

$$\tilde{u}\in C\left([0,T],L^q\left(\mathbb{R}^N\right)\cap L^{\tilde{q}}\left(\mathbb{R}^N\right)\right)\cap C\left((0,T],L^{\infty}\left(\mathbb{R}^N\right)\right)$$

if T is small enough, and

$$\sup_{0<t<T}t^{\frac{\alpha N}{\beta q}}\|\tilde{u}(t)\|_{L^{\infty}(\mathbb{R}^N)}<+\infty.$$

Note that $\tilde{q}>q_c$ and there exists a constant $C>0$ such that

$$t^{\beta}\|\tilde{u}(t)\|_{L^r(\mathbb{R}^N)}\le t^{\beta}\|\tilde{u}(t)\|^{1-\frac{\tilde{q}}{r}}_{L^{\infty}(\mathbb{R}^N)}\|\tilde{u}(t)\|^{\frac{\tilde{q}}{r}}_{L^{\tilde{q}}(\mathbb{R}^N)}\le Ct^{\frac{\alpha N}{2}\left(\frac{1}{qc}-\frac{1}{q}\right)}\|\tilde{u}(t)\|^{\frac{\tilde{q}}{r}}_{L^{\tilde{q}}(\mathbb{R}^N)},\ t\in(0,T).$$

It follows that we can take T small enough such that

$$\sup_{0<t<T}t^{\beta}\|\tilde{u}(t)\|_{L^r(\mathbb{R}^N)}\le M.$$

Thus, by uniqueness, $u\equiv\tilde{u}$ for $t\in[0,T]$. Consequently, we get that

$$u\in C\left((0,T],L^{\infty}\left(\mathbb{R}^N\right)\right)\cap C\left([0,T],L^q\left(\mathbb{R}^N\right)\cap L^{\tilde{q}}\left(\mathbb{R}^N\right)\right).$$

Finally, we prove $u \in C\left([T,\infty), L^q\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)\right)$. Indeed, for $t > T$, we have

$$\begin{aligned} u - P_\alpha(t)u_0 &= \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}|u|^p ds \\ &= \int_0^T (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}|u|^p \, \mathrm{d}s + \int_T^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}|u|^p ds \\ &= I_5 + I_6. \end{aligned}$$

Since

$$u \in C\left([0,T], L^{\tilde{q}}\left(\mathbb{R}^N\right)\right) \cap C\left((0,T], L^\infty\left(\mathbb{R}^N\right)\right)$$

and

$$\sup_{0<t<T} t^{\frac{\alpha N}{\beta q}} \|u(t)\|_{L^\infty\left(\mathbb{R}^N\right)} < \infty,$$

we obtain

$$I_5 \in C\left([T,\infty), L^\infty\left(\mathbb{R}^N\right)\right) \cap C\left([T,\infty), L^{\tilde{q}}\left(\mathbb{R}^N\right)\right)$$

by an argument similar to the proof of Lemma 2.8.

For given

$$T_1 > T, |u|^p \in L^\infty\left((T,T_1), L^{\frac{r}{p}}\left(\mathbb{R}^N\right)\right).$$

Since $r > \frac{N(p-1)}{\beta}$, we can choose $\tilde{m} > r$ such that

$$\frac{N}{\beta}\left(\frac{p}{r} - \frac{1}{\tilde{m}}\right) < 1.$$

Observing

$$0 < \frac{p}{r} - \frac{1}{\tilde{q}} < \frac{p}{r} - \frac{1}{\tilde{m}} < \frac{\beta}{N},$$

an argument similar to that used in Lemma 2.8 shows that

$$I_6 \in C\left([T,T_1], L^{\tilde{m}}\left(\mathbb{R}^N\right) \cap L^{\tilde{q}}\left(\mathbb{R}^N\right)\right).$$

By the arbitrariness of T_1, we know

$$I_6 \in C\left([T,\infty), L^{\tilde{m}}\left(\mathbb{R}^N\right) \cap L^{\tilde{q}}\left(\mathbb{R}^N\right)\right).$$

Note that the term

$$P_\alpha(\cdot)u_0 \in C\left([T,\infty), L^{\tilde{m}}\left(\mathbb{R}^N\right) \cap L^{\tilde{q}}\left(\mathbb{R}^N\right)\right).$$

Consequently,

$$u \in C\left([T,\infty), L^{\tilde{m}}\left(\mathbb{R}^N\right)\right) \cap C\left([0,\infty), L^{\tilde{q}}\left(\mathbb{R}^N\right)\right).$$

Let

$$\chi = \frac{\tilde{m}}{r}.$$

Observe that $\chi > 1$ and

$$\frac{N}{\beta}\left(\frac{p}{r\chi^{i-1}} - \frac{1}{r\chi^i}\right) < 1, i = 1, 2, \cdots.$$

Repeating the above procedures, we deduce that if

$$u \in C\left([T,\infty), L^{r\chi^{i-1}}\left(\mathbb{R}^N\right)\right),$$

then

$$u \in C\left([T,\infty), L^{r\chi^{i}}\left(\mathbb{R}^N\right)\right).$$

After finite steps, we get

$$\frac{p}{r\chi^i} < \frac{\beta}{N}.$$

Then

$$u \in C\left((0,\infty), L^{\infty}\left(\mathbb{R}^N\right)\right).$$

Therefore,

$$u \in C\left([0,+\infty), L^{q}\left(\mathbb{R}^N\right)\right) \cap C((0,\infty), L^{\infty}\left(\mathbb{R}^N\right)).$$

This completes the proof. □

Lemma 4.2. *(See [3]) Let $\omega_1 > -1, \omega_2 > -1$ such that $\omega_1 + \omega_2 \geq -1, h > 0$ and $t \in [0,T]$. Then the following limit holds for $\mu > 0$,*

$$\lim_{\mu\to\infty}\left(\sup_{t\in[0,T]} t^h \int_0^1 s^{\omega_1}(1-s)^{\omega_2} e^{-\mu t(1-s)} \mathrm{d}s\right) = 0.$$

Theorem 4.2. *Let $\frac{1}{2} < \alpha < 1, 0 < \vartheta < 1 - \frac{1}{2\alpha}$, b satisfies*

$$1 + \alpha\vartheta - \alpha < b < \alpha - \alpha\vartheta, \tag{4.12}$$

$$p > \max\left(1, \frac{1}{1-\frac{\beta(\alpha-b)}{\alpha N}}, \frac{1}{1-\frac{\beta(\alpha+b-1)}{\alpha N}}\right) \tag{4.13}$$

and

$$q > \max\left(\frac{1}{\frac{1}{b}+\frac{\beta(\alpha-b)}{\alpha N}}, \frac{1}{\frac{1}{b}+\frac{\beta(\alpha+b-1)}{\alpha N}}\right). \tag{4.14}$$

Suppose that

$$u_0 \in L^p\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right),$$

then if $\|u_0\|_{L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)}$ is sufficiently small, then the mild solution of (1.1) *exists globally.*

Proof. We construct the global solution of (1.1) by contraction mapping principle. First, we consider the following function

$$\Psi v(t) := P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathcal{G}|u|^{p-1}u ds. \tag{4.15}$$

Let B be a Banach space. Define the Banach space $\mathbb{Z}^{b,\mu}((0,T];\mathrm{B})$ of all Bochner integrable functions $u:[0,\infty)\to\mathrm{B}$ such that $t^b u$ are bounded continuous functions, endowed with the norm

$$\|v\|_{\mathbb{Z}^{b,\mu}((0,T];\mathrm{B})} := \sup_{t\in(0,T]} t^b e^{-\mu t}\|v(\cdot,t)\|_{\mathrm{B}} < \infty, b \geq 0, \mu \geq 0.$$

Since $u_0 \in L^p\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)$, we know that

$$\|P_\alpha(t)u_0\|_{L^q\left(\mathbb{R}^N\right)} \leq A_1^{-\frac{\alpha N}{\beta r}} \|u_0\|_{L^p\left(\mathbb{R}^N\right)} + A_2 t^{-\alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)}, \tag{4.16}$$

and

$$\|P_\alpha(t)u_0\|_{L^q\left(\mathbb{R}^N\right)} \leq A_1 \|u_0\|_{L^p\left(\mathbb{R}^N\right)} + A_2 t^{-\alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)}. \tag{4.17}$$

From (4.16) and (4.17) yield that

$$\begin{aligned} t^b e^{-\mu t} \|P_\alpha(t)u_0\|_{L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)} \leq & e^{-\mu t}\left(A_1 t^{b-\frac{\alpha N}{\beta r}} + A_1 t^b + A_2 t^{b-\alpha\vartheta}\right)\|u_0\|_{L^p\left(\mathbb{R}^N\right)} + e^{-\mu t} A_2 t^{b-\alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)} \\ \leq & \left(A_1 T^{b-\frac{\alpha N}{\beta r}} + A_1 T^b + A_2 T^{b-\alpha\vartheta}\right)\|u_0\|_{L^p\left(\mathbb{R}^N\right)} + A_2 T^{b-\alpha\vartheta} \|u_0\|_{L^q\left(\mathbb{R}^N\right)}. \end{aligned} \tag{4.18}$$

It follows from (4.18) that,

$$P_\alpha(t)u_0 \in \mathbb{Z}^{b,\mu}\left((0,T]; L^p\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)\right)$$

for any $\mu > 0$ and b satisfies (4.12).

Now, We show that for any

$$v_1, v_2 \in \mathbb{Z}^{b,\mu_0}\left((0,T]; L^p\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right)\right)$$

and the constant ϱ which is independent of t, there exists a μ_0 such that

$$\|\Psi v_1 - \Psi v_2\|_{\mathbb{Z}^{b,\mu_0}\left((0,T];L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)\right)} \leq \varrho \|v_1 - v_2\|_{\mathbb{Z}^{b,\mu_0}\left((0,T];L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)\right)}. \tag{4.19}$$

Indeed, we have

$$\begin{aligned} t^b e^{-\mu t} \|\Psi v_1 - \Psi v_2\|_{L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)} &= t^b e^{-\mu t}\left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}\left|v_1^p - v_2^p\right| ds\right\|_{L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)} \\ &= t^b e^{-\mu t}\left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}\left|v_1^p - v_2^p\right| ds\right\|_{L^p\left(\mathbb{R}^N\right)} \\ &+ t^b e^{-\mu t}\left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\mathscr{G}\left|v_1^p - v_2^p\right| ds\right\|_{L^q\left(\mathbb{R}^N\right)} \\ &= (I) + (II). \end{aligned} \tag{4.20}$$

Hence, we have

$$
\begin{aligned}
(I) &\le t^b e^{-\mu t} A_3 \int_0^t (t-s)^{\alpha-1} \left\| \mathscr{G} \left| v_1^p - v_2^p \right| \right\|_{L^p(\mathbb{R}^N)} ds \\
&+ t^b e^{-\mu t} A_4 \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \left\| \mathscr{G} \left| v_1^p - v_2^p \right| \right\|_{L^p(\mathbb{R}^N)} ds \\
&\le CMA_3 t^b e^{-\mu t} \int_0^t (t-s)^{\alpha-1} \| v_1(s) - v_2(s) \|_{L^p(\mathbb{R}^N)} ds \\
&+ CMA_4 t^b e^{-\mu t} \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \| v_1(s) - v_2(s) \|_{L^p(\mathbb{R}^N)} ds \\
&= (III) + (IV),
\end{aligned} \tag{4.21}
$$

and

$$
\begin{aligned}
(II) &\le t^b e^{-\mu t} A_3 \int_0^t (t-s)^{\alpha-1} (t-s)^{-\frac{\alpha N}{\beta r}} \left\| \mathscr{G} \left| v_1^p - v_2^p \right| \right\|_{L^p(\mathbb{R}^N)} ds \\
&+ t^b e^{-\mu t} A_4 \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \left\| \mathscr{G} \left| v_1^p - v_2^p \right| \right\|_{L^q(\mathbb{R}^N)} ds \\
&\le CMA_3 t^b e^{-\mu t} \int_0^t (t-s)^{\alpha-1-\frac{\alpha N}{\beta r}} \| v_1(s) - v_2(s) \|_{L^p(\mathbb{R}^N)} ds \\
&+ CMA_4 t^b e^{-\mu t} \int_0^t (t-s)^{\alpha-1-\alpha\vartheta} \| v_1(s) - v_2(s) \|_{L^q(\mathbb{R}^N)} ds \\
&= (V) + (VI),
\end{aligned} \tag{4.22}
$$

thanks to the following inequality

$$
\left| |u|^{p-1} u - |v|^{p-1} v \right| \le C|u-v| \left(|u|^{p-1} + |v|^{p-1} \right).
$$

We treat the term (III) as follows

$$
\begin{aligned}
(III) &= CMA_3 t^b \int_0^t e^{-\mu(t-s)} s^{-b} (t-s)^{\alpha-1} s^b e^{-\mu s} \| v_1(s) - v_2(s) \|_{L^p(\mathbb{R}^N)} ds \\
&\le CMA_3 t^b \left(\int_0^t e^{-\mu(t-s)} s^{-b} (t-s)^{\alpha-1} ds \right) \sup_{0 \le s \le T} s^b e^{-\mu s} \| v_1(s) - v_2(s) \|_{L^p(\mathbb{R}^N)} \\
&= CMA_3 t^b L_{1,\mu}(t,\alpha,b) \| v_1 - v_2 \|_{\mathbb{Z}^{b,\mu}((0,T];L^p(\mathbb{R}^N))} .
\end{aligned} \tag{4.23}
$$

By a similar argument, we can also obtain some following estimates

$$
\begin{aligned}
(IV) &\le CMA_4 t^b L_{2,\mu}(t,\alpha,b) \| v_1 - v_2 \|_{\mathbb{Z}^{b,\mu}((0,T];L^p(\mathbb{R}^N))} , \\
(V) &\le CMA_3 t^b L_{3,\mu}(t,\alpha,b) \| v_1 - v_2 \|_{\mathbb{Z}^{b,\mu}((0,T];L^p(\mathbb{R}^N))} , \\
(VI) &\le CMA_3 t^b L_{2,\mu}(t,\alpha,b) \| v_1 - v_2 \|_{\mathbb{Z}^{b,\mu}((0,T];L^q(\mathbb{R}^N))} ,
\end{aligned} \tag{4.24}
$$

where

$$
L_{2,\mu}(t,\alpha,b) = \int_0^t e^{-\mu(t-s)} s^{-b} (t-s)^{\alpha-1-\alpha\vartheta} ds,
$$

$$L_{3,\mu}(t,\alpha,b) = \int_0^t e^{-\mu(t-s)} s^{-b}(t-s)^{\alpha-1-\frac{\alpha N}{\beta r}} ds.$$

Combining some above observations, we obtain

$$\begin{aligned}
&\|\Psi v_1 - \Psi v_2\|_{\mathbb{Z}^{b,\mu}\left((0,T];L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)\right)} \\
&\leq \sup_{0\leq t\leq T} t^b e^{-\mu t} \| \int_0^t (t-s)^{\alpha-1} S_\alpha(t) \mathscr{G}\left(v_1(s) - v_2(s)\right) ds \|_{L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)} \\
&\leq 2CM\left(A_3 + A_4\right) \sup_{0\leq t\leq T} \left(t^b L_{1,\mu}(t,\alpha,b) + t^b L_{2,\mu}(t,\alpha,b) + t^b L_{3,\mu}(t,\alpha,b)\right) \qquad (4.25)\\
&\times \|v_1 - v_2\|_{\mathbb{Z}^{b,\mu}\left((0,T];L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)\right)} \\
&\leq \varrho \|v_1 - v_2\|_{\mathbb{Z}^{b,\mu}\left((0,T];L^p\left(\mathbb{R}^N\right)\cap L^q\left(\mathbb{R}^N\right)\right)}.
\end{aligned}$$

We first consider the term $L_{2,\mu}(t,\alpha,b)$. By changing variable $s = ts'$, we get

$$t^b L_{2,\mu}(t,\alpha,b) = t^{b+\alpha-1-\alpha\vartheta} \int_0^1 e^{-\mu t(1-s)} s^{-b}(1-s)^{\alpha-1-\alpha\vartheta} ds. \qquad (4.26)$$

Using (4.12), we can easily verify that the following conditions hold

$$b+\alpha-1-\alpha\vartheta > 0,\ -b > -1,\ \alpha-1-\alpha\vartheta > -1,\ \alpha-1-\alpha\vartheta-b > -1.$$

By Lemma 4.2, we have

$$\lim_{\mu\to\infty} t^b L_{2,\mu}(t,\alpha,b) = \lim_{\mu\to\infty}\left(\sup_{t\in[0,T]} t^{b+\alpha-1-\alpha\vartheta} \int_0^1 e^{-\mu t(1-s)} s^{-b}(1-s)^{\alpha-1-\alpha\vartheta} ds\right) = 0. \qquad (4.27)$$

Noting that $L_{1,\mu}(t,\alpha,b) \leq T^{\alpha\vartheta} L_{2,\mu}(t,\alpha,b)$, we deduce that

$$\lim_{\mu\to\infty} t^b L_{2,\mu}(t,\alpha,b) = 0. \qquad (4.28)$$

It follows from (4.12)–(4.14) that the following conditions hold

$$b+\alpha-1-\tfrac{\alpha N}{\beta r} > 0,\ -b > -1,\ \alpha-1-\tfrac{\alpha N}{\beta r} > -1,\ \alpha-1-\tfrac{\alpha N}{\beta r}-b > -1.$$

By Lemma 4.2, we obtain

$$\lim_{\mu\to\infty} t^b L_{3,\mu}(t,\alpha,b) = \lim_{\mu\to\infty}\left(\sup_{t\in[0,T]} t^{b+\alpha-1-\frac{\alpha N}{\beta r}} \int_0^1 e^{-\mu t(1-s)} s^{-b}(1-s)^{\alpha-1-\frac{\alpha N}{\beta r}} ds\right) = 0. \qquad (4.29)$$

From (4.27)–(4.29), we know that there exists a μ_0 small enough such that

$$\varrho := 2CM\left(A_3 + A_4\right) \sup_{0\leq t\leq T}\left(t^b L_{1,\mu_0}(t,\alpha,\beta) + t^\beta L_{2,\mu_0}(t,\alpha,b) + t^b L_{3,\mu_0}(t,\alpha,b)\right) < 1. \qquad (4.30)$$

We find that Ψ is a contraction in the space

$$\mathbb{Z}^{b,\mu_0}((0,T]; L^p\left(\mathbb{R}^N\right) \cap L^q\left(\mathbb{R}^N\right))$$

by combining (4.25) and (4.30), it means that there exists a unique global solution u satisfies Lemma 4.2. Hence, the proof is finished. □

5. Conclusions

In this work, we considered the blow-up and global existence of a class of space-time fractional pseudo-parabolic equations. A family of solution operators is defined based on a kind of density function and semigroup, and the $L^p - L^q$ estimate for solutions of the corresponding linear problem is investigated and this is our main contribution. On this basis, the local existence of solutions to a class of space-time fractional pseudo-parabolic equations is studied by using the fixed-point theorem. The definition of weak solutions is given and it is proved that mild solutions are also weak solutions. The global existence of solutions is proved by using the contraction mapping principle, while the blow-up of solutions is proved by using the test function method. In this direction, we can study the global existence and blow-up of solutions for space-time fractional pseudo-parabolic equations when $\alpha \in (1, 2)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper was supported by NSF of China (No. 11801276).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. B. de Andrade, A. Viana, Abstract Volterra integro-differential equations with applications to parabolic models with memory, *Math. Ann.*, **369** (2017), 1131–1175. https://doi.org/10.1007/s00208-016-1469-z
2. P. Biler, J. Dziubanski, W. Hebisch, Scattering of small solutions to generalized Benjamin-Bona-Mahony equation in several space dimensions, *Commun. Partial Differ. Equations*, **17** (1992), 1737–1758. https://doi.org/10.1080/03605309208820902
3. Y. Chen, H. Gao, M. J. Garrido-Atienza, B. Schmalfuss, Pathwise solutions of SPDEs driven by Holder-continuous integrators with exponent larger than $\frac{1}{2}$ and random dynamical systems, *ArXiv*, 2014. https://doi.org/10.48550/arXiv.1305.6903
4. Y. Cao, J. Yin, C. Wang, Cauchy problems of semilinear pseudo-parabolic equations, *J. Differ. Equations*, **246** (2009), 4568–4590. https://doi.org/10.1016/j.jde.2009.03.021
5. H. Dong, D. Kim, L_p-estimates for time fractional parabolic equations with coefficients measurable in time, *Adv. Math.*, **345** (2019), 289–345. https://doi.org/10.1016/j.aim.2019.01.016
6. M. Fardi, M. Ghasemi, A numerical solution strategy based on error analysis for time-fractional mobile/immobile transport model, *Soft Comput.*, **25** (2021), 11307–11331. https://doi.org/10.1007/s00500-021-05914-y

7. M. Fardi, Y. Khan, A fast difference scheme on a graded mesh for time-fractional and space distributed-order diffusion equation with nonsmooth data, *Int. J. Mod. Phys. B*, **36** (2022), 15. https://doi.org/10.1142/S021797922250076X

8. M. Fardi, S. K. Q. Al-Omari, S. Araci, A pseudo-spectral method based on reproducing kernel for solving the time-fractional diffusion-wave equation, *Adv. Contin. Discret Model.*, **2022** (2022), 54. https://doi.org/10.1186/s13662-022-03726-4

9. Y. Giga, T. Namba, Well-posedness of Hamilton-Jacobi equations with Caputo's time fractional derivative, *Commun. Partial Differ. Equation*, **42** (2017), 1088–1120. https://doi.org/10.1080/03605302.2017.1324880

10. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer, 2014.

11. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, 2000.

12. L. Jin, L. Li, S. Fang, The global existence and time-decay for the solutions of the fractional pseudo-parabolic equation, *Comput. Math. Appl.*, **73** (2017), 2221–2232. https://doi.org/10.1016/j.camwa.2017.03.005

13. G. Karch, Asymptotic behaviour of solutions to some pesudoparabolic equations, *Math. Methods Appl. Sci.*, **20** (1997), 271–289.

14. M. Kirane, Y. Laskri, N. E. Tatar, Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives, *J. Math. Anal. Appl.*, **312** (2005), 488–501. https://doi.org/10.1016/j.jmaa.2005.03.054

15. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science, 2006.

16. L. Li, J. G. Liu, L. Wang, Cauchy problems for Keller-Segel type time-space fractional diffusion equation, *J. Differ. Equations*, **265** (2018), 1044–1096. https://doi.org/10.1016/j.jde.2018.03.025

17. Y. Li, Y. Yang, The critical exponents for a semilinear fractional pseudo-parabolic equation with nonlinear memory in a bounded domain, *Electron. Res. Arch.*, **31** (2023), 2555–2567. https://doi.org/10.3934/era.2023129

18. Y. Li, Q. Zhang, Blow-up and global existence of solutions for a time fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **21** (2018), 1619–1640. https://doi.org/10.1515/fca-2018-00859

19. F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, *Waves Stab. Contin. Media*, **1994** (1994), 246–251.

20. B. B. Mandelbrot, J. W. V. Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.*, **10** (1968), 422–437. https://doi.org/10.1137/1010093

21. R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A*, **37** (2004), 161–208. https://doi.org/10.1088/0305-4470/37/31/R01

22. E. Orsingher, L. Beghin, Fractional diffusion equations and processes with randomly varying time, *Ann. Probab.*, **37** (2009), 206–249. https://doi.org/10.1214/08-AOP401

23. I. Podlubny, *Fractional differential euations*, Elsevier Science, 1999.

24. M. Ralf, K. Joseph, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

25. W. R. Schneider, W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.*, **30** (1989), 134–144. https://doi.org/10.1063/1.528578

26. Y. F. Sun, Z. Zeng, J. Song, Quasilinear iterative method for the boundary value problem of nonlinear fractional differential equation, *Numer. Algebra Control Optim.*, **10** (2020), 157–164. https://doi.org/10.3934/naco.2019045

27. N. H. Tuan, V. V. Au, R. Xu, Semilinear Caputo time-fractional pseudo-parabolic equations, *Commun. Pure Appl. Anal.*, **20** (2020), 583–621. https://doi.org/10.3934/cpaa.2020282

28. K. Zennir, H. Dridi, S. Alodhaibi, S. Alkhalaf, Nonexistence of global solutions for coupled system of pseudoparabolic equations with variable exponents and weak memories, *J. Funct. Space*, **2021** (2021), 5573959. https://doi.org/10.1155/2021/5573959

29. K. Zennir, T. Miyasita, Lifespan of solutions for a class of pseudo-parabolic equation with weak-memory, *Alexandria Eng. J.*, **59** (2020), 957–964. https://doi.org/10.1016/j.aej.2020.03.016

30. Q. Zhang, H. Sun, The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation, *Topol. Methods Nonlinear Anal.*, **46** (2015), 69–92. https://doi.org/10.12775/TMNA.2015.038