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Research article

Simulator selection based on complex probabilistic hesitant fuzzy soft structure using multi-parameters group decision-making

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Abstract: Simulation software replicates the behavior of real electrical equipment using mathematical models. This is efficient not only in regard to time savings but also in terms of investment. It, at large scale for instance airplane pilots, chemical or nuclear plant operators, etc., provides valuable experiential learning without the risk of a catastrophic outcome. But the selection of a circuit simulator with effective simulation accuracy poses significant challenges for today's decision-makers because of uncertainty and ambiguity. Thus, better judgments with increased productivity and accuracy are crucial. For this, we developed a complex probabilistic hesitant fuzzy soft set (CPHFSS) to capture ambiguity and uncertain information with higher accuracy in application scenarios. In this manuscript, the novel concept of CPHFSS is explored and its fundamental laws are discussed. Additionally, we investigated several algebraic aspects of CPHFSS, including union, intersections, soft max-AND, and soft min-OR operators, and we provided numerical examples to illustrate these key qualities. The three decision-making strategies are also constructed using the investigated idea of CPHFSS. Furthermore, numerical examples related to bridges and circuit simulation are provided in order to assess the validity and efficacy of the proposed methodologies. The graphical expressions of the acquired results are also explored. Finally, we conclude the whole work.

Keywords: complex probabilistic hesitant fuzzy soft set; soft max-AND; soft min-OR operators; properties; algorithms; decision-making
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

Due to the rising complexity of current technology and engineering systems, simulation stability is of the greatest priority. Simulation is quite time-saving and cost-effective [1]. Simulation software is based on the technique of mathematically simulating real-world phenomena. Essentially, it is a tool that lets the user see a process via simulation without really conducting it [2]. As a result, engineers often turn to simulation software to ensure that their machines produce results as near to the intended specifications as feasible without resorting to costly alterations during production. When engineers get excellent and precise results from their simulation, they may help decrease the utilization of expensive hardware resources and only create hardware when strictly essential [3]. Real-time reaction simulation software is often used in engineering and when the penalty for improper operation is severe, a replica of the actual control panel is linked to a real-time simulation of the physical response, providing valuable training without the danger of a disastrous outcome.

A variety of simulators exist with diverse applications and features. The selection of a circuit simulator with effective simulation accuracy poses significant challenges for today's decision-makers. Decision-making procedures encompass crips data, but uncertainty performs a predominant role in any decisions pertaining to our daily life problems, and the data may not always be in crips form. To deal with the complications of decision-making and modeling of ambiguous or uncertain data, our researchers perform a significant role in various fields like process control, artificial intelligence, environmental sciences, computer vision, economics, medical science, engineering, image analysis, natural language processing, etc. Standard methods are always not fruitful for dealing with ambiguous or uncertain data. Throughout the preceding few decades, modeling of the natural world predicated on fuzzy logic and fuzzy sets has been manifested for explaining practical problems in such types of fields. Before providing the motivation for this paper, we describe the brief history that our researchers presented to deal with ambiguous or uncertain data. Lotfi Asker Zadeh in 1965 presented the concept of a fuzzy set to deal with such types of uncertainty [4]. The researcher is able to conduct a quantitative analysis of the uncertainty of an event by using this sort of mathematical model [5,6]. Researchers have defined numerous extensions of this impression [7–12].

One crucial problem shared by those theories is their inadequacy with parametrization tools, such as the level of the membership determined by the individual depending on the information received by the individual. For that reason, they are vulnerable to subjective components. Furthermore, distinct attributes in a single problem require them to be thought about in an integrated way. Molodtsov in 1999 presented the soft set theory to address these limitations [13]. The soft set theory is distinct from classical methods to deal with vagueness or uncertainties. The benefit of soft set theory is that it can solve problems that have multiple parameters and perform a very significant role in applications of numerous fields [14, 15]. This is especially the case in decision-making under an imprecise environment [16] and in data analysis [17]. Numerous extensions of soft set theory have been defined by researchers. For instance, Maji et al. [18] presented a fuzzy soft set by combining a fuzzy set and a soft set. Maji and Roy presented the impression of a fuzzy soft set and gave its operations, properties, and applications in decision-making [19–21].

Sometimes randomness and fuzziness related uncertainties occur in the system at the same time. Meghdadi in 2001 presented probabilistic fuzzy logic [22] to handle both sorts of uncertainties concurrently in a single framework. Probabilistic modeling is a very significant tool to deal with random uncertainties, such as power systems [23], robotic control systems [24], signal processing [25], medical applications [26], and control applications [27]. When there is only limited information about randomness, probabilistic models can be used by making appropriate assumptions about randomness statistics [28].

Ramot has examined the presented theories that have evaluated the decision-making problems under the fuzzy set and its generalizations, which are only capable of dealing with the vagueness and uncertainty that exist in data. These are not able to describe the fluctuations of data at a given point in time. Ramot in 2002 presented a complex fuzzy set to address these limitations. In contrast to a traditional fuzzy membership function, this range is not limited to [0, 1], but extended to the unit circle in the complex plane [29]. It is unique because it has a complex membership value that incorporates amplitude and phase terms. The phase term denotes the declaration of a complex fuzzy set that needs the second dimension of membership, while the amplitude term retains the concept of fuzziness. The complex fuzzy set offers the ability to resolve temporal issues and combine data from various aspects into a single, understandable collection. Two different components of the data are explained, each in its own way. As a result, it is more inclusive than the conventional fuzzy set. Dick presented a systematic analysis of complex fuzzy sets and logic, and also talked about their applications [30]. Alkouri presented complex intuitionistic fuzzy sets [31], their composition, relation, projection [32] and some of its operation [33]. Researchers have also defined numerous extensions of this impression [34–37].

Torra in 2010 presented a hesitant fuzzy set to deal with hesitant situations [38]. In this case, he regarded the membership function as a set of possible membership values, namely h(x), where h(x) is a finite subset of [0, 1]. When decision-makers are unable to determine which of many possible assessment values best reflects their point of view, hesitant fuzzy set may be used to address the situation by assigning different probabilities to each of the possible assessment values. Often, in probabilistic fuzzy settings, the probabilistic information is lost. This is why Pang [39] created a new class of probabilistic fuzzy linguistic objects to address these concerns in the setting of fuzzy language. An innovative concept known as probabilistic hesitant fuzzy sets was found thanks to the research of Xu and Zhou [40]. Hesitant fuzzy sets have attracted the concentration of many researchers who presented multiple extensions and their operators, which also perform a significant role in decision-making [41], such as in [42] where authors introduced the dual hesitant fuzzy soft set to group forecasting, in which they presented the example of a pharmaceutical company's board of directors' need to decide the further investment priorities for the subsidiaries based on net income. In [43] they developed the methods of E-VIKOR and TOPSIS and in [44] the EDAS method to solve multi-criteria decision-making problems with the hesitant fuzzy set information.

Humans have always struggled to attain a better perception of the developing world in which they live. In this developing world, we face vagueness and uncertainty created by fuzziness, hesitancy, and randomness when trying to make optimal decisions. A number of scholars investigated and developed distinct approaches for dealing with barriers to decision-making. Practical decision-making involves a lot of uncertainties, imprecise information and vagueness, whose representations and management are constantly the main concerns. In some real-world applications of decision theory, decision-makers deal with situations involving specific attributes in which the values of their membership degree need to deal with probability, phase of time, and hesitancy collectively. By using these terms, the evaluation is more precise and accurate as well as beyond one's preferences, which is the key point during making decisions. For instance, when treating mental diseases, doctors face many difficulties. A patient with a mental illness is frequently treated by several doctors, which causes uncertainty, reluctance, and randomness and makes diagnosis challenging. In such a situation, existing theories are unable to produce any results that are satisfactory. Thus, we created the concept of a complex probabilistic hesitant fuzzy soft set to address this issue, which plays a very significant role in making decisions in a short interval of time and as accurately as possible.

In complex probabilistic hesitant fuzzy soft set, membership grade is probabilistic hesitant and complex-valued which is illustrated in polar coordinate. The amplitude term of the membership grade expresses the belongingness of CPHFSS to some extent, and the phase term related to the membership grade gives additional information relevant to the phase of time and periodicity. This structure serves a crucial function in achieving more accurate and precise measurements in mechanical and especially electrical equipment. We consider multiple attributes to more accurately reflect our perceptions and it performs a very significant role in decision making to make optimal decisions. In the last part, we also present a real scenario based on numerical examples on decision making problems with the complex probabilistic hesitant fuzzy information to show the advantages of this methodology and efficacy.

The organization of the rest of the paper is as follows: Section 2 comprises of preliminaries, in which we recall the concept of a few basic definitions in a precise manner, enabling us to communicate easily in the rest of the sections. In Section 3, we introduce the model of a complex probabilistic hesitant fuzzy soft set. We discuss some basic operations (such as extended and restricted union, extended and restricted intersection, complement) and soft max-AND soft min-OR operators. Also, we verify their fundamental laws. Further, the numerical examples are solved to manifest the integrity and supremacy of the explored work. In Section 4, we establish three different algorithms for decision making with complex probabilistic hesitant fuzzy information under the environment of multiple attributes. In Section 5, we illustrate applications and case studies to make optimal decisions. In Section 6, we give the conclusion of this paper.

2. Preliminaries

In this section, a brief review of a few basic definitions which we use to establish the methods proposed in this paper is presented.

Definition 1. *Let $\mathcal{M}$ be a universe of discourse, then a fuzzy set $\widehat{A}$ defined on $\mathcal{M}$ as*

$$\widehat{A} = \left\{\langle m_i, \mu_{\widehat{A}}(m_i)\rangle \middle| m_i \in \mathcal{M}\right\},$$

where $\mu_{\widehat{A}}(m_i)$ is a membership grade of m_i in $\widehat{A}$ and $\mu_{\widehat{A}} : \mathcal{M} \to [0,1]$. If $\mu_{\widehat{A}}(m_i) = 1$ then it is full membership of m_i, if $\mu_{\widehat{A}}(m_i) = 0$ then it is non-membership of m_i, and if $\mu_{\widehat{A}}(m_i)$ has a value between 0 and 1 then it is partial membership of m_i [4].

Definition 2. *Let $\mathcal{M}$ be a universe of discourse, then a hesitant fuzzy set $\widehat{B}$ is in terms of a function h that when applied to $\mathcal{M}$ return a subset of [0,1] which is represented as*

$$\widehat{B} = \left\{\left\langle m_i, h_{\widehat{B}}(m_i)\right\rangle \middle| m_i \in \mathcal{M}\right\},$$

where $h_{\widehat{B}}(m_i)$ is a set of different finite elements in [0,1], representing the possible membership grades of the element $m_i \in \mathcal{M}$ to the set $\widehat{B}$ [38].

Definition 3. *Let $\mathcal{M}$ be a universe of discourse, then a probabilistic hesitant fuzzy set $\widehat{C}$ is defined as*

$$\widehat{C} = \left\{\left\langle m_i, h_{\widehat{C}}\left(\mu_x(m_i)\middle|P_{\widehat{C}_x}\right)\right\rangle \middle| m_i \in \mathcal{M}\right\},$$

where $h_{\widehat{C}}(\mu_x(m_i)|P_{\widehat{C}_x})$ is the set of different finite elements $(\mu_x(m_i)\big|P_{\widehat{C}_x})$ representing the hesitant fuzzy information along probabilities to the set $\widehat{C}, x = 1, 2, 3, ..., n$ where n is the number of possible elements in $h_{\widehat{C}}(\mu_x(m_i)|P_{\widehat{C}_x}), P_{\widehat{C}_x} \in [0, 1]$ is the hesitant probability of m_i and $\sum_x P_{\widehat{C}_x} = 1$ [45].

Definition 4. *Let $\mathcal{M}$ be a universe of discourse, then a complex fuzzy set $\widehat{D}$ is defined as*

$$\widehat{D} = \left\{\left\langle m_i, r_{\widehat{D}}(m_i)e^{2\pi i\omega_{\widehat{D}}(m_i)}\right\rangle \middle| m_i \in \mathcal{M}\right\},$$

where $r_{\widehat{D}}(m_i)e^{2\pi i\omega_{\widehat{D}}(m_i)}$ is a complex valued membership grade of m_i in $\widehat{D}$ and it may receive all values lie within the unit circle in the complex plane and $r_{\widehat{D}}(m_i) \in [0, 1]$, $\omega_{\widehat{D}}(m_i) \in (0, 1]$ and $i = \sqrt{-1}$ [29].

Definition 5. *Let $\mathcal{M}$ be a universe of discourse, then a complex probabilistic hesitant fuzzy set (CPHFS) $\widehat{E}$ is defined as*

$$\widehat{E} = \left\{\left\langle m_i, h_{\widehat{E}}\left(r_{\widehat{E}_x}(m_i)e^{2\pi i\omega_{\widehat{E}_x}(m_i)}\middle|P_{\widehat{E}_x}\right)\right\rangle \middle| m_i \in \mathcal{M}\right\},$$

where $h_{\widehat{E}}\left(r_{\widehat{E}_x}(m_i)e^{2\pi i\omega_{\widehat{E}_x}(m_i)}\middle|P_{\widehat{E}_x}\right)$ is the set of a few complex elements representing the complex hesitant fuzzy information along probabilities to the set $\widehat{E}, r_{\widehat{E}_x}(m_i) \in [0, 1]$ and $\omega_{\widehat{E}_x}(m_i) \in [0, 1], x = 1, 2, ..., n$ where n is the number of possible elements in $h_{\widehat{E}}\left(r_{\widehat{E}_x}(m_i)e^{2\pi i\omega_{\widehat{E}_x}(m_i)}\middle|P_{\widehat{E}_x}\right)$ and $P_{\widehat{E}_x} \in [0, 1]$ is the complex hesitant probability of $r_{\widehat{E}_x}(m_i)e^{2\pi i\omega_{\widehat{E}_x}(m_i)}$ and $\sum_x P_{\widehat{E}_x} = 1$.

Definition 6. *Let $\mathcal{M}$ be a universe of discourse and G be the set of attributes, for any non-empty set $\mathcal{E} \subseteq G$. A pair $(F, \mathcal{E})$ is called the soft set over $\mathcal{M}$ if there is a mapping $F : \mathcal{E} \to \mathcal{P}(\mathcal{M})$ where $\mathcal{P}(\mathcal{M})$ denotes the power set of $\mathcal{M}$.*
Thus, the soft set is a parametric family of the subsets of the universe of discourse. For each $e_j \in G$, we can interpret $F(e_j)$ as a subset of the universe of discourse $\mathcal{M}$. We can also consider $F(e_j)$ as a mapping $F(e_j) : \mathcal{M} \to \{0, 1\}$ and then $F(e_j)(m_i) = 1$ is equivalent to $m_i \in F(e_j)$, otherwise $F(e_j)(m_i) = 0$ [46]. Molodtsov considered many examples in [13] to illustrate the soft set.

Definition 7. *Let $\mathcal{M}$ be a universe of discourse and G be the set of attributes, for any non-empty set $\mathcal{E} \subseteq G$. A pair $(\widehat{H}, \mathcal{E})$ is called the fuzzy soft set over $\mathcal{M}$ if there is a mapping $\widehat{H} : \mathcal{E} \to \mathcal{F}(\mathcal{M})$ where $\mathcal{F}(\mathcal{M})$ denotes the fuzzy power set of $\mathcal{M}$ (all possible fuzzy subsets of $\mathcal{M}$) [47].*

3. Complex probabilistic hesitant fuzzy soft set

In this section, we develope the model of complex probabilistic hesitant fuzzy soft set and numerical example to illustrate this model. We discuss some basic operations and soft max-AND soft min-OR operators. Also, we verify their fundamental laws. Further, the numerical examples are solved to manifest the integrity and supremacy of the explored work.

Definition 8. *[48] Let $\mathcal{M}$ be a universe of discourse and G be the set of attributes, for any non-empty set $\mathcal{E} \subseteq G$. A pair $(\widehat{K}, \mathcal{E})$ is called the complex probabilistic hesitant fuzzy soft set over $\mathcal{M}$ if there is a mapping $\widehat{K} : \mathcal{E} \to \mathbb{F}(\mathcal{M})$ where $\mathbb{F}(\mathcal{M})$ denotes the all possible complex probabilistic hesitant fuzzy subset of $\mathcal{M}$. It is represented as*

$$\widehat{K}(e_j) = \left\{\left\langle m_i, h_{\widehat{K}}\left(r_{\widehat{K}_x}(m_i)e^{2\pi i\omega_{\widehat{K}_x}(m_i)}\Big|P_{\widehat{K}_x}\right)\right\rangle \Big| m_i \in \mathcal{M}\right\}, \ \forall e_j \in \mathcal{E} \subseteq G,$$

where $h_{\widehat{K}}\left(r_{\widehat{K}_x}(m_i)e^{2\pi i\omega_{\widehat{K}_x}(m_i)}\Big|P_{\widehat{K}_x}\right)$ is the set of a few complex elements representing the complex hesitant fuzzy information along probabilities to the set $\widehat{K}$, $r_{\widehat{K}_x}(m_i) \in [0,1]$ and $\omega_{\widehat{K}_x}(m_i) \in [0,1], x = 1,2,...,n$ where n is the number of possible elements in $h_{\widehat{K}}\left(r_{\widehat{K}_x}(m_i)e^{2\pi i\omega_{\widehat{K}_x}(m_i)}\Big|P_{\widehat{K}_x}\right)$ and $P_{\widehat{K}_x} \in [0,1]$ is the complex hesitant probability of $r_{\widehat{K}_x}(m_i)e^{2\pi i\omega_{\widehat{K}_x}(m_i)}$ and $\sum_x P_{\widehat{K}_x} = 1$.

Example 1. *Let $\mathcal{M} = \{m_1, m_2, m_3\}$ be the set of energy projects, $\mathcal{E} \subseteq G$ be the set of attributes, such that $\mathcal{E} = \{e_1 = environmental, e_2 = economical, e_3 = technological\}$. Then, $(\widehat{K}, \mathcal{E})$ is the complex probabilistic hesitant fuzzy soft set over $\mathcal{M}$ as follows:*

$$\widehat{K}(e_1) = \left\{\left\langle m_1, \left\{0.6e^{2\pi i0.6}\Big|0.2, 0.7e^{2\pi i0.5}\Big|0.3, 0.8e^{2\pi i0.6}\Big|0.5\right\}\right\rangle, \left\langle m_2, \left\{0.9e^{2\pi i0.8}\Big|1\right\}\right\rangle, \left\langle m_3, \left\{0.5e^{2\pi i0.3}\Big|1\right\}\right\rangle\right\},$$

$$\widehat{K}(e_2) = \left\{\left\langle m_1, \left\{0.2e^{2\pi i0.3}\Big|1\right\}\right\rangle, \left\langle m_2, \left\{0.4e^{2\pi i0.5}\Big|0.5, 0.5e^{2\pi i0.4}\Big|0.5\right\}\right\rangle, \left\langle m_3, \left\{0.3e^{2\pi i0.8}\Big|0.1, 0.9e^{2\pi i0.1}\Big|0.9\right\}\right\rangle\right\},$$

$$\widehat{K}(e_3) = \left\{\left\langle m_1, \left\{0.5e^{2\pi i0.8}\Big|1\right\}\right\rangle, \left\langle m_2, \left\{0.1e^{2\pi i0.2}\Big|0.8, 0.2e^{2\pi i0.3}\Big|0.2\right\}\right\rangle, \left\langle m_3, \left\{0.7e^{2\pi i0.7}\Big|1\right\}\right\rangle\right\}.$$

We can write complex probabilistic hesitant fuzzy soft set in a tabular form as represented in Table 1.

Table 1. Tabular representation of CPHFSS.

$(\widehat{K}, \mathcal{E})$	e_1	e_2	e_3
m_1	$\left\{ 0.6e^{2\pi i0.6}\|0.2, 0.7e^{2\pi i0.5}\|0.3, 0.8e^{2\pi i0.6}\|0.5 \right\}$	$\left\{ 0.2e^{2\pi i0.3}\|1 \right\}$	$\left\{ 0.5e^{2\pi i0.8}\|1 \right\}$
m_2	$\left\{ 0.9e^{2\pi i0.8}\|1 \right\}$	$\left\{ 0.4e^{2\pi i0.5}\|0.5, 0.5e^{2\pi i0.4}\|0.5 \right\}$	$\left\{ 0.1e^{2\pi i0.2}\|0.8, 0.2e^{2\pi i0.3}\|0.2 \right\}$
m_3	$\left\{ 0.5e^{2\pi i0.3}\|1 \right\}$	$\left\{ 0.3e^{2\pi i0.8}\|0.1, 0.9e^{2\pi i0.1}\|0.9 \right\}$	$\left\{ 0.7e^{2\pi i0.7}\|1 \right\}$

Definition 9. *Let $\mathcal{M}$ be a universe of discourse and G be the set of attributes, for any non-empty set $\mathcal{E} \subseteq G$. A pair $(\widehat{K}, \mathcal{E})$ is called the empty complex probabilistic hesitant fuzzy soft set over $\mathcal{M}$. If $\widehat{K}(e_j) = \phi$ for all $e_i \in \mathcal{E}$.*

Definition 10. *Let $\mathcal{M}$ be a universe of discourse and G be the set of attributes, for any non-empty set $\mathcal{E} \subseteq G$. A pair $(\widehat{K}, \mathcal{E})$ is called the full complex probabilistic hesitant fuzzy soft set over $\mathcal{M}$. If $\widehat{K}(e_j) = 1$ for all $e_i \in \mathcal{E}$.*

3.1. Operations of CPHFSS

Definition 11. *Let $\mathcal{M}$ be a universe of discourse. If $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ are two CPHFSS over $\mathcal{M}$ then their restricted union is defined as*

$$(\widehat{Q}, O) = (\widehat{K_1}, \mathcal{E}_1) \cup_{\xi} (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{Q} = \widehat{K_1} \cup_{\xi} \widehat{K_2}$, $O = \mathcal{E}_1 \cap \mathcal{E}_2 \neq \phi$; $\forall o_j \in O$ *and* $m_i \in \mathcal{M}$, $\left\langle m_i, h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i) e^{2\pi i \omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x} \right) \right\rangle \in \widehat{Q}(o_j) \Longleftrightarrow r_{\widehat{Q}_x}(m_i) e^{2\pi i \omega_{\widehat{Q}_x}(m_i)} \Big| P_{\widehat{Q}_x}$

$$= \begin{cases} r_{\widehat{K_{1x}}}(m_i) e^{2\pi i \omega_{\widehat{K_{1x}}}(m_i)} \Big| P_{\widehat{K_{1x}}}, & \text{if } x \in h_{\widehat{K_{1x}}} - h_{\widehat{K_{2x}}}, \\ r_{\widehat{K_{2x}}}(m_i) e^{2\pi i \omega_{\widehat{K_{2x}}}(m_i)} \Big| P_{\widehat{K_{2x}}}, & \text{if } x \in h_{\widehat{K_{2x}}} - h_{\widehat{K_{1x}}}, \\ max\left\{ r_{\widehat{K_{1x}}}(m_i), r_{\widehat{K_{2x}}}(m_i) \right\} e^{2\pi i max\left\{ \omega_{\widehat{K_{1x}}}(m_i), \omega_{\widehat{K_{2x}}}(m_i) \right\}} \Big| P_{\widehat{K_{1x}}} \cdot P_{\widehat{K_{2x}}}, & \text{if } x \in h_{\widehat{K_{1x}}} \cap h_{\widehat{K_{2x}}}. \end{cases}$$

Example 2. *Let $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ be the two complex probabilistic hesitant fuzzy softs represented in Tables 2 and 3 respectively.*

Table 2. CPHFSS $(\widehat{K}_1, \mathcal{E}_1)$.

$(\widehat{K}_1, \mathcal{E}_1)$	e_1	e_2	e_3
m_1	$\{0.6e^{2\pi i 0.5}\|1\}$	$\{0.3e^{2\pi i 0.6}\|0.1, 0.2e^{2\pi i 0.5}\|0.2, 0.1e^{2\pi i 0.3}\|0.7\}$	$\{0.6e^{2\pi i 0.6}\|0.2, 0.9e^{2\pi i 0.5}\|0.3, 0.5e^{2\pi i 0.9}\|0.5\}$
m_2	$\{0.8e^{2\pi i 0.5}\|0.2, 0.3e^{2\pi i 0.3}\|0.8\}$	$\{0.7e^{2\pi i 0.6}\|1\}$	$\{0.8e^{2\pi i 0.7}\|0.2, 0.3e^{2\pi i 0.3}\|0.8\}$
m_3	$\{0.5e^{2\pi i 0.4}\|0.2, 0.3e^{2\pi i 0.8}\|0.4, 0.8e^{2\pi i 0.2}\|0.4\}$	$\{0.7e^{2\pi i 0.3}\|0.1, 0.4e^{2\pi i 0.7}\|0.9\}$	$\{0.4e^{2\pi i 0.6}\|0.3, 0.4e^{2\pi i 0.7}\|0.7\}$

Table 3. CPHFSS $(\widehat{K}_2, \mathcal{E}_2)$.

$(\widehat{K}_2, \mathcal{E}_2)$	e_1	e_2	e_4
m_1	$\{0.3e^{2\pi i 0.7}\|0.1, 0.6e^{2\pi i 0.4}\|0.3, 0.4e^{2\pi i 0.3}\|0.6\}$	$\{0.8e^{2\pi i 0.4}\|1\}$	$\{0.7e^{2\pi i 0.2}\|0.4, 0.7e^{2\pi i 0.8}\|0.6\}$
m_2	$\{0.6e^{2\pi i 0.9}\|1\}$	$\{0.5e^{2\pi i 0.2}\|0.1, 0.2e^{2\pi i 0.6}\|0.4, 0.8e^{2\pi i 0.9}\|0.5\}$	$\{0.3e^{2\pi i 0.3}\|0.3, 0.4e^{2\pi i 0.7}\|0.7\}$
m_3	$\{0.6e^{2\pi i 0.4}\|1\}$	$\{0.3e^{2\pi i 0.3}\|0.1, 0.3e^{2\pi i 0.4}\|0.9\}$	$\{0.5e^{2\pi i 0.3}\|1\}$

Then their restricted union is represented in Table 4.

Table 4. Tabular representation of restricted union.

$(\widehat{Q}, O)$	e_1	e_2
m_1	$\left\{\begin{matrix} 0.6e^{2\pi i0.7}\|0.1 \\ 0.6e^{2\pi i0.4}\|0.3 \\ 0.4e^{2\pi i0.3}\|0.6 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.8e^{2\pi i0.6}\|0.1, \\ 0.2e^{2\pi i0.5}\|0.2, \\ 0.1e^{2\pi i0.3}\|0.7 \end{matrix}\right\}$
m_2	$\left\{\begin{matrix} 0.8e^{2\pi i0.9}\|0.2, \\ 0.3e^{2\pi i0.3}\|0.8 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.7e^{2\pi i0.6}\|0.1 \\ 0.2e^{2\pi i0.6}\|0.4 \\ 0.8e^{2\pi i0.9}\|0.5 \end{matrix}\right\}$
m_3	$\left\{\begin{matrix} 0.6e^{2\pi i0.4}\|0.2, \\ 0.3e^{2\pi i0.8}\|0.4, \\ 0.8e^{2\pi i0.2}\|0.4 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.7e^{2\pi i0.3}\|0.01, \\ 0.4e^{2\pi i0.7}\|0.81 \end{matrix}\right\}$

Definition 12. *Let $\mathcal{M}$ be a universe of discourse. If $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ are two CPHFSS over $\mathcal{M}$, then their restricted intersection is defined as*

$$(\widehat{R}, O) = (\widehat{K_1}, \mathcal{E}_1) \cap_{\xi} (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{R} = \widehat{K_1} \cap_{\xi} \widehat{K_2}$, $O = \mathcal{E}_1 \cap \mathcal{E}_2 \neq \phi$; $\forall o_j \in O$ and $m_i \in \mathcal{M}$,

$$\left\langle m_i, h_{\widehat{R}}\left(r_{\widehat{R}_x}(m_i) e^{2\pi i \omega_{\widehat{R}_x}(m_i)} \middle| P_{\widehat{R}_x} \right) \right\rangle \in \widehat{R}(o_j) \Longleftrightarrow r_{\widehat{R}_x}(m_i) e^{2\pi i \omega_{\widehat{R}_x}(m_i)} \Big| P_{\widehat{R}_x}$$

$$= \begin{cases} r_{\widehat{K_1}_x}(m_i) e^{2\pi i \omega_{\widehat{K_1}_x}(m_i)} \Big| P_{\widehat{K_1}_x}, & \text{if } x \in h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ r_{\widehat{K_2}_x}(m_i) e^{2\pi i \omega_{\widehat{K_2}_x}(m_i)} \Big| P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_2}_x} - h_{\widehat{K_1}_x}, \\ min\left\{ r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i) \right\} e^{2\pi i min\left\{ \omega_{\widehat{K_1}_x}(m_i), \omega_{\widehat{K_2}_x}(m_i) \right\}} \Big| P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x}. \end{cases}$$

Example 3. *Consider $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ as described in Example 2. Then their restricted intersection is represented in Table 5.*

Table 5. Tabular representation of restricted intersection.

$(\widehat{R}, O)$	e_1	e_2
m_1	$\left\{\begin{matrix} 0.3e^{2\pi i0.5}\|0.1 \\ 0.6e^{2\pi i0.4}\|0.3 \\ 0.4e^{2\pi i0.3}\|0.6 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.3e^{2\pi i0.4}\|0.1, \\ 0.2e^{2\pi i0.5}\|0.2, \\ 0.1e^{2\pi i0.3}\|0.7 \end{matrix}\right\}$
m_2	$\left\{\begin{matrix} 0.6e^{2\pi i0.5}\|0.2, \\ 0.3e^{2\pi i0.3}\|0.8 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.5e^{2\pi i0.2}\|0.1 \\ 0.2e^{2\pi i0.6}\|0.4 \\ 0.8e^{2\pi i0.9}\|0.5 \end{matrix}\right\}$
m_3	$\left\{\begin{matrix} 0.5e^{2\pi i0.4}\|0.2, \\ 0.3e^{2\pi i0.8}\|0.4, \\ 0.8e^{2\pi i0.2}\|0.4 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.3e^{2\pi i0.3}\|0.01, \\ 0.3e^{2\pi i0.4}\|0.81 \end{matrix}\right\}$

Definition 13. *Let $\mathcal{M}$ be a universe of discourse. If $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ are two CPHFSS over $\mathcal{M}$,*

then their extended union is defined as

$$(\widehat{S}, P) = (\widehat{K_1}, \mathcal{E}_1) \cup_{\zeta} (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{S} = \widehat{K_1} \cup_{\zeta} \widehat{K_2}$, $P = \mathcal{E}_1 \cup \mathcal{E}_2$; $\forall p_j \in P$ *and* $m_i \in \mathcal{M}$, *with* $p_j^1 \in \mathcal{E}_1$ *and* $p_j^2 \in \mathcal{E}_2$.

$$\widehat{S}(p_j) = \begin{cases} \widehat{K_1}(p_j^1), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}(p_j^1) \cup_{\xi} \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

Example 4. *Consider* $(\widehat{K_1}, \mathcal{E}_1)$ *and* $(\widehat{K_2}, \mathcal{E}_2)$ *as described in Example 2. Then their extended union is represented in Table 6.*

Table 6. Tabular representation of extended union.

$(\widehat{S}, P)$	e_1	e_2	e_3	e_4
m_1	$\left\{\begin{array}{l} 0.6e^{2\pi i0.7}\|0.1 \\ 0.6e^{2\pi i0.4}\|0.3 \\ 0.4e^{2\pi i0.3}\|0.6 \end{array}\right\}$	$\left\{\begin{array}{l} 0.8e^{2\pi i0.6}\|0.1, \\ 0.2e^{2\pi i0.5}\|0.2, \\ 0.1e^{2\pi i0.3}\|0.7 \end{array}\right\}$	$\left\{\begin{array}{l} 0.6e^{2\pi i0.6}\|0.2, \\ 0.9e^{2\pi i0.5}\|0.3, \\ 0.5e^{2\pi i0.9}\|0.5 \end{array}\right\}$	$\left\{\begin{array}{l} 0.7e^{2\pi i0.2}\|0.4, \\ 0.7e^{2\pi i0.8}\|0.6 \end{array}\right\}$
m_2	$\left\{\begin{array}{l} 0.8e^{2\pi i0.9}\|0.2, \\ 0.3e^{2\pi i0.3}\|0.8 \end{array}\right\}$	$\left\{\begin{array}{l} 0.7e^{2\pi i0.6}\|0.1 \\ 0.2e^{2\pi i0.6}\|0.4 \\ 0.8e^{2\pi i0.9}\|0.5 \end{array}\right\}$	$\left\{\begin{array}{l} 0.8e^{2\pi i0.7}\|0.2, \\ 0.3e^{2\pi i0.3}\|0.8 \end{array}\right\}$	$\left\{\begin{array}{l} 0.3e^{2\pi i0.3}\|0.3, \\ 0.4e^{2\pi i0.7}\|0.7 \end{array}\right\}$
m_3	$\left\{\begin{array}{l} 0.6e^{2\pi i0.4}\|0.2, \\ 0.3e^{2\pi i0.8}\|0.4, \\ 0.8e^{2\pi i0.2}\|0.4 \end{array}\right\}$	$\left\{\begin{array}{l} 0.7e^{2\pi i0.3}\|0.01, \\ 0.4e^{2\pi i0.7}\|0.81 \end{array}\right\}$	$\left\{\begin{array}{l} 0.4e^{2\pi i0.6}\|0.3, \\ 0.4e^{2\pi i0.7}\|0.7 \end{array}\right\}$	$\left\{ 0.5e^{2\pi i0.3}\|1 \right\}$

Definition 14. *Let* $\mathcal{M}$ *be a universe of discourse. If* $(\widehat{K_1}, \mathcal{E}_1)$ *and* $(\widehat{K_2}, \mathcal{E}_2)$ *are two CPHFSS over* $\mathcal{M}$, *then their extended intersection is defined as*

$$(\widehat{T}, P) = (\widehat{K_1}, \mathcal{E}_1) \cap_{\zeta} (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{T} = \widehat{K_1} \cap_{\zeta} \widehat{K_2}$, $P = \mathcal{E}_1 \cup \mathcal{E}_2$; $\forall p_j \in P$ *and* $m_i \in \mathcal{M}$, *with* $p_j^1 \in \mathcal{E}_1$ *and* $p_j^2 \in \mathcal{E}_2$.

$$\widehat{T}(p_j) = \begin{cases} \widehat{K_1}(p_j^1), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}(p_j^1) \cap_{\xi} \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

Example 5. *Consider* $(\widehat{K_1}, \mathcal{E}_1)$ *and* $(\widehat{K_2}, \mathcal{E}_2)$ *as described in Example 2. Then their extended union is represented in Table 7.*

Table 7. Tabular representation of extended intersection.

$(\widehat{T}, P)$	e_1	e_2	e_3	e_4
m_1	$\left\{\begin{matrix} 0.3e^{2\pi i0.5}\vert 0.1 \\ 0.6e^{2\pi i0.4}\vert 0.3 \\ 0.4e^{2\pi i0.3}\vert 0.6 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.3e^{2\pi i0.4}\vert 0.1, \\ 0.2e^{2\pi i0.5}\vert 0.2, \\ 0.1e^{2\pi i0.3}\vert 0.7 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.6e^{2\pi i0.6}\vert 0.2, \\ 0.9e^{2\pi i0.5}\vert 0.3, \\ 0.5e^{2\pi i0.9}\vert 0.5 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.7e^{2\pi i0.2}\vert 0.4, \\ 0.7e^{2\pi i0.8}\vert 0.6 \end{matrix}\right\}$
m_2	$\left\{\begin{matrix} 0.6e^{2\pi i0.5}\vert 0.2, \\ 0.3e^{2\pi i0.3}\vert 0.8 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.5e^{2\pi i0.2}\vert 0.1 \\ 0.2e^{2\pi i0.6}\vert 0.4 \\ 0.8e^{2\pi i0.9}\vert 0.5 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.8e^{2\pi i0.7}\vert 0.2, \\ 0.3e^{2\pi i0.3}\vert 0.8 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.3e^{2\pi i0.3}\vert 0.3, \\ 0.4e^{2\pi i0.7}\vert 0.7 \end{matrix}\right\}$
m_3	$\left\{\begin{matrix} 0.5e^{2\pi i0.4}\vert 0.2, \\ 0.3e^{2\pi i0.8}\vert 0.4, \\ 0.8e^{2\pi i0.2}\vert 0.4 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.3e^{2\pi i0.3}\vert 0.01, \\ 0.3e^{2\pi i0.4}\vert 0.81 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.4e^{2\pi i0.6}\vert 0.3, \\ 0.4e^{2\pi i0.7}\vert 0.7 \end{matrix}\right\}$	$\left\{ 0.5e^{2\pi i0.3}\vert 1 \right\}$

Definition 15. *Let $\mathcal{M}$ be a universe of discourse. If $(\widehat{K}, \mathcal{E})$ is the CPHFSS over $\mathcal{M}$, then its complement is defined as*

$$\widehat{K}^c(e_j) = \left\{ \left\langle m_i, h_{\widehat{K}^c}\left((1 - r_{\widehat{K}_x}(m_i)) e^{2\pi i\left(1-\omega_{\widehat{K}_x}(m_i)\right)} \middle| 1 - P_{\widehat{K}_x} \right) \right\rangle \middle| m_i \in \mathcal{M} \right\}, \ \forall e_j \in \mathcal{E} \subseteq G.$$

Example 6. *Consider $(\widehat{K}, \mathcal{E})$ as described in Example 1. Then its complement is represented in Table 8.*

Table 8. Tabular representation of CPHFSS.

$(\widehat{K}, \mathcal{E})$	e_1	e_2	e_3
m_1	$\left\{\begin{matrix} 0.4e^{2\pi i0.4}\vert 0.2, \\ 0.3e^{2\pi i0.5}\vert 0.3, \\ 0.2e^{2\pi i0.4}\vert 0.5 \end{matrix}\right\}$	$\left\{ 0.8e^{2\pi i0.7}\vert 1 \right\}$	$\left\{ 0.5e^{2\pi i0.2}\vert 1 \right\}$
m_2	$\left\{ 0.1e^{2\pi i0.2}\vert 1 \right\}$	$\left\{\begin{matrix} 0.6e^{2\pi i0.5}\vert 0.5, \\ 0.5e^{2\pi i0.6}\vert 0.5 \end{matrix}\right\}$	$\left\{\begin{matrix} 0.9e^{2\pi i0.8}\vert 0.8, \\ 0.8e^{2\pi i0.7}\vert 0.2 \end{matrix}\right\}$
m_3	$\left\{ 0.5e^{2\pi i0.7}\vert 1 \right\}$	$\left\{\begin{matrix} 0.7e^{2\pi i0.2}\vert 0.1, \\ 0.1e^{2\pi i0.9}\vert 0.9 \end{matrix}\right\}$	$\left\{ 0.3e^{2\pi i0.3}\vert 1 \right\}$

Definition 16. *Soft max-AND operation of two CPHFSS $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) defined as*

$$(\widehat{K_1}, \mathcal{E}_1) \wedge (\widehat{K_2}, \mathcal{E}_2) = (\widehat{Q}, L),$$

where $\widehat{Q} : L \to \mathbb{F}(\mathcal{M})$ such that $\widehat{Q} = \widehat{K_1} \cup \widehat{K_2}$ and $L = \mathcal{E}_1 \times \mathcal{E}_2$; $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$. $\left\langle m_i, h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i) e^{2\pi i\omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x} \right) \right\rangle \in \widehat{Q}(l_i, l_j) \Longleftrightarrow r_{\widehat{Q}_x}(m_i) e^{2\pi i\omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x}(m_i) e^{2\pi i\omega_{\widehat{K_1}_x}(m_i)} \Big| P_{\widehat{K_1}_x}, & \text{if } x \in h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ r_{\widehat{K_2}_x}(m_i) e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)} \Big| P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_2}_x} - h_{\widehat{K_1}_x}, \\ max\left\{ r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i) \right\} e^{2\pi i max\left\{ \omega_{\widehat{K_1}_x}(m_i), \omega_{\widehat{K_2}_x}(m_i) \right\}} \Big| P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x}. \end{cases}$$

Definition 17. *Soft min-OR operation of two CPHFSS* $(\widehat{K_1}, \mathcal{E}_1)$ *and* $(\widehat{K_2}, \mathcal{E}_2)$ $($*where* $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ *and* $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M}))$ *defined as*

$$(\widehat{K_1}, \mathcal{E}_1) \vee (\widehat{K_2}, \mathcal{E}_2) = (\widehat{R}, L),$$

where $\widehat{R} : L \to \mathbb{F}(\mathcal{M})$ *such that* $\widehat{R} = \widehat{K_1} \cap \widehat{K_2}$ *and* $L = \mathcal{E}_1 \times \mathcal{E}_2$*;* $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$.
$\left\langle m_i, h_{\widehat{R}}\left(r_{\widehat{R}_x}(m_i)e^{2\pi i \omega_{\widehat{R}_x}(m_i)}\middle| P_{\widehat{R}_x}\right)\right\rangle \in \widehat{R}(l_i, l_j) \Longleftrightarrow r_{\widehat{R}_x}(m_i)e^{2\pi i \omega_{\widehat{R}_x}(m_i)}\Big| P_{\widehat{R}_x}$

$$= \begin{cases} r_{\widehat{K}_{1x}}(m_i)e^{2\pi i \omega_{\widehat{K}_{1x}}(m_i)}\Big| P_{\widehat{K}_{1x}}, & \text{if } x \in h_{\widehat{K}_{1x}} - h_{\widehat{K}_{2x}}, \\ r_{\widehat{K}_{2x}}(m_i)e^{2\pi i \omega_{\widehat{K}_{2x}}(m_i)}\Big| P_{\widehat{K}_{2x}}, & \text{if } x \in h_{\widehat{K}_{2x}} - h_{\widehat{K}_{1x}}, \\ min\left\{r_{\widehat{K}_{1x}}(m_i), r_{\widehat{K}_{2x}}(m_i)\right\} e^{2\pi i min\left\{\omega_{\widehat{K}_{1x}}(m_i), \omega_{\widehat{K}_{2x}}(m_i)\right\}}\Big| P_{\widehat{K}_{1x}} \cdot P_{\widehat{K}_{2x}}, & \text{if } x \in h_{\widehat{K}_{1x}} \cap h_{\widehat{K}_{2x}}. \end{cases}$$

3.2. Fundamental laws

Proposition 1. *Given that* $(\widehat{K_1}, \mathcal{E}_1)$*,* $(\widehat{K_2}, \mathcal{E}_2)$ *and* $(\widehat{K_3}, \mathcal{E}_3)$ *are any three CPHFSS on* $\mathcal{M}$*, then the following holds:*

Idempotent laws.

(i). $(\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_1}, \mathcal{E}_1) = (\widehat{K_1}, \mathcal{E}_1)$,
(ii). $(\widehat{K_1}, \mathcal{E}_1) \cap_\zeta (\widehat{K_1}, \mathcal{E}_1) = (\widehat{K_1}, \mathcal{E}_1)$,
(iii). $(\widehat{K_1}, \mathcal{E}_1) \cup_\xi (\widehat{K_1}, \mathcal{E}_1) = (\widehat{K_1}, \mathcal{E}_1)$,
(iv). $(\widehat{K_1}, \mathcal{E}_1) \cap_\xi (\widehat{K_1}, \mathcal{E}_1) = (\widehat{K_1}, \mathcal{E}_1)$,

Commutative laws.

(v). $(\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_2}, \mathcal{E}_2) = (\widehat{K_2}, \mathcal{E}_2) \cup_\zeta (\widehat{K_1}, \mathcal{E}_1)$,
(vi). $(\widehat{K_1}, \mathcal{E}_1) \cap_\zeta (\widehat{K_2}, \mathcal{E}_2) = (\widehat{K_2}, \mathcal{E}_2) \cap_\zeta (\widehat{K_1}, \mathcal{E}_1)$,
(vii). $(\widehat{K_1}, \mathcal{E}_1) \cup_\xi (\widehat{K_2}, \mathcal{E}_2) = (\widehat{K_2}, \mathcal{E}_2) \cup_\xi (\widehat{K_1}, \mathcal{E}_1)$,
(viii). $(\widehat{K_1}, \mathcal{E}_1) \cap_\xi (\widehat{K_2}, \mathcal{E}_2) = (\widehat{K_2}, \mathcal{E}_2) \cap_\xi (\widehat{K_1}, \mathcal{E}_1)$,

Associative laws.

(ix). $(\widehat{K_1}, \mathcal{E}_1) \cup_\zeta \left((\widehat{K_2}, \mathcal{E}_2) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3)\right) = \left((\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_2}, \mathcal{E}_2)\right) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3)$,
(x). $(\widehat{K_1}, \mathcal{E}_1) \cap_\zeta \left((\widehat{K_2}, \mathcal{E}_2) \cap_\zeta (\widehat{K_3}, \mathcal{E}_3)\right) = \left((\widehat{K_1}, \mathcal{E}_1) \cap_\zeta (\widehat{K_2}, \mathcal{E}_2)\right) \cap_\zeta (\widehat{K_3}, \mathcal{E}_3)$,
(xi). $(\widehat{K_1}, \mathcal{E}_1) \cup_\xi \left((\widehat{K_2}, \mathcal{E}_2) \cup_\xi (\widehat{K_3}, \mathcal{E}_3)\right) = \left((\widehat{K_1}, \mathcal{E}_1) \cup_\xi (\widehat{K_2}, \mathcal{E}_2)\right) \cup_\xi (\widehat{K_3}, \mathcal{E}_3)$,
(xii). $(\widehat{K_1}, \mathcal{E}_1) \cap_\xi \left((\widehat{K_2}, \mathcal{E}_2) \cap_\xi (\widehat{K_3}, \mathcal{E}_3)\right) = \left((\widehat{K_1}, \mathcal{E}_1) \cap_\xi (\widehat{K_2}, \mathcal{E}_2)\right) \cap_\xi (\widehat{K_3}, \mathcal{E}_3)$.

Proof. (i)–(viii) laws are straight-forward and follow from their definition.

(ix) **L.H.S:** Let $(\widehat{K_2}, \mathcal{E}_2)$ and $(\widehat{K_3}, \mathcal{E}_3)$ (where $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_3} : \mathcal{E}_3 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS on $\mathcal{M}$. By definition of extended union we have $(\widehat{W}, M)$ (where $\widehat{W} : M \to \mathbb{F}(\mathcal{M})$) such that

$$(\widehat{W}, M) = (\widehat{K_2}, \mathcal{E}_2) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3),$$

where $\widehat{W} = \widehat{K_2} \cup_\zeta \widehat{K_3}$, $M = \mathcal{E}_2 \cup \mathcal{E}_3$; $\forall m_j \in M$ and $m_i \in \mathcal{M}$, with $m_j^1 \in \mathcal{E}_2$ and $m_j^2 \in \mathcal{E}_3$.

$$\widehat{W}(m_j) = \begin{cases} \widehat{K_2}(m_j^1), & \text{if } m_j \in \mathcal{E}_2 - \mathcal{E}_3, \\ \widehat{K_3}(m_j^2), & \text{if } m_j \in \mathcal{E}_3 - \mathcal{E}_2, \\ \widehat{K_2}(m_j^1) \cup_\xi \widehat{K_3}(m_j^2), & \text{if } m_j \in \mathcal{E}_2 \cap \mathcal{E}_3. \end{cases}$$

As, $(\widehat{K_1}, \mathcal{E}_1) \cup_\zeta \left((\widehat{K_2}, \mathcal{E}_2) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3)\right) = (\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{W}, M)$. Suppose that $(\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{W}, M) = (\widehat{X}, N)$ such that $\widehat{X} : N \to \mathbb{F}(\mathcal{M})$, where $\widehat{X} = (\widehat{K_1} \cup_\zeta \widehat{W})$, $N = \mathcal{E}_1 \cup M = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$; $\forall n_j \in N$ with $n_j^1 \in \mathcal{E}_1$, $n_j^2 \in \mathcal{E}_2$ and $n_j^3 \in \mathcal{E}_3$,

$$\widehat{X}(n_j) = \begin{cases} \widehat{K_1}(n_j^1), & \text{if } n_j \in \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\ \widehat{K_2}(n_j^2), & \text{if } n_j \in \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_1, \\ \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_3 - \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_1}(n_j^1) \cap_\xi \widehat{K_2}(n_j^2), & \text{if } n_j \in \mathcal{E}_1 \cap \mathcal{E}_2 - \mathcal{E}_3, \\ \widehat{K_2}(n_j^2) \cap_\xi \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_2 \cap \mathcal{E}_3 - \mathcal{E}_1, \\ \widehat{K_3}(n_j^3) \cap_\xi \widehat{K_1}(n_j^1), & \text{if } n_j \in \mathcal{E}_3 \cap \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_1}(n_j^1) \cap_\xi \widehat{K_2}(n_j^2) \cap_\xi \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3. \end{cases}$$

R.H.S: Let $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS over $\mathcal{M}$. By definition of extended union we have $(\widehat{S}, P)$ (where $\widehat{S} : P \to \mathbb{F}(\mathcal{M})$) such that

$$(\widehat{S}, P) = (\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{S} = \widehat{K_1} \cup_\zeta \widehat{K_2}$, $P = \mathcal{E}_1 \cup \mathcal{E}_2$; $\forall p_j \in P$ and $m_i \in \mathcal{M}$, with $p_j^1 \in \mathcal{E}_1$ and $p_j^2 \in \mathcal{E}_2$.

$$\widehat{S}(p_j) = \begin{cases} \widehat{K_1}(p_j^1), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}(p_j^1) \cup_\xi \widehat{K_2}(p_j^2), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

As, $\left((\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_2}, \mathcal{E}_2)\right) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3) = (\widehat{S}, P) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3)$. Suppose that $(\widehat{S}, P) \cup_\zeta (\widehat{K_3}, \mathcal{E}_3) = (\widehat{Y}, N)$ such that $\widehat{Y} : N \to \mathbb{F}(\mathcal{M})$, where $\widehat{Y} = (\widehat{S} \cup_\zeta \widehat{K_3})$, $N = P \cup \mathcal{E}_3 = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$; $\forall n_j \in N$ with $n_j^1 \in \mathcal{E}_1$, $n_j^2 \in \mathcal{E}_2$ and $n_j^3 \in \mathcal{E}_3$,

$$\widehat{Y}(n_j) = \begin{cases} \widehat{K_1}(n_j^1), & \text{if } n_j \in \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \\ \widehat{K_2}(n_j^2), & \text{if } n_j \in \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_1, \\ \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_3 - \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_1}(n_j^1) \cap_\xi \widehat{K_2}(n_j^2), & \text{if } n_j \in \mathcal{E}_1 \cap \mathcal{E}_2 - \mathcal{E}_3, \\ \widehat{K_2}(n_j^2) \cap_\xi \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_2 \cap \mathcal{E}_3 - \mathcal{E}_1, \\ \widehat{K_3}(n_j^3) \cap_\xi \widehat{K_1}(n_j^1), & \text{if } n_j \in \mathcal{E}_3 \cap \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_1}(n_j^1) \cap_\xi \widehat{K_2}(n_j^2) \cap_\xi \widehat{K_3}(n_j^3), & \text{if } n_j \in \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3. \end{cases}$$

Then $\widehat{X}(n_j) = \widehat{Y}(n_j)$, $\forall n_j \in N$.
Thus L.H.S = R.H.S.
Hence, (ix) is hold. □

Proof. (xi) **L.H.S:** Let $(\widehat{K_2}, \mathcal{E}_2)$ and $(\widehat{K_3}, \mathcal{E}_3)$ (where $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_3} : \mathcal{E}_3 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS on $\mathcal{M}$. By definition of restricted union we have $(\widehat{U}, M')$ (where $\widehat{U} : M' \to \mathbb{F}(\mathcal{M})$) such that

$$(\widehat{U}, M') = (\widehat{K_2}, \mathcal{E}_2) \cup_\xi (\widehat{K_3}, \mathcal{E}_3),$$

where $\widehat{U}=\widehat{K_2}\cup_{\xi}\widehat{K_3}$, $M'=\mathcal{E}_2\cap\mathcal{E}_3$; $\forall m'_j\in M'$ and $m_i\in\mathcal{M}$,
$\left\langle m_i, h_{\widehat{U}}\left(r_{\widehat{U}_x}(m_i)e^{2\pi i\omega_{\widehat{U}_x}(m_i)}\middle|P_{\widehat{U}_x}\right)\right\rangle\in\widehat{U}(m'_j)\Longleftrightarrow r_{\widehat{U}_x}(m_i)e^{2\pi i\omega_{\widehat{U}_x}(m_i)}\Big|P_{\widehat{U}_x}$

$$=\begin{cases} r_{\widehat{K_2}_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)}\Big|P_{\widehat{K_2}_x}, & \text{if } x\in h_{\widehat{K_2}_x}-h_{\widehat{K_3}_x},\\ r_{\widehat{K_3}_x}(m_i)e^{2\pi i\omega_{\widehat{K_3}_x}(m_i)}\Big|P_{\widehat{K_3}_x}, & \text{if } x\in h_{\widehat{K_3}_x}-h_{\widehat{K_2}_x},\\ max\Big\{r_{\widehat{K_2}_x}(m_i), r_{\widehat{K_3}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_3}_x}(m_i)\Big\}}\Big|P_{\widehat{K_2}_x}\cdot P_{\widehat{K_3}_x}, & \text{if } x\in h_{\widehat{K_2}_x}\cap h_{\widehat{K_3}_x},\end{cases}$$

As, $(\widehat{K_1},\mathcal{E}_1)\cup_{\xi}\left((\widehat{K_2},\mathcal{E}_2)\cup_{\xi}(\widehat{K_3},\mathcal{E}_3)\right)=(\widehat{K_1},\mathcal{E}_1)\cup_{\xi}(\widehat{U},M')$. Suppose that $(\widehat{K_1},\mathcal{E}_1)\cup_{\xi}(\widehat{U},M')=(\widehat{V},N')$ such that $\widehat{V}:N'\to\mathbb{F}(\mathcal{M})$, where $\widehat{V}=(\widehat{K_1}\cup_{\xi}\widehat{U})$, $N'=\mathcal{E}_1\cap M'=\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$; $\forall n'_j\in N'$,
$\left\langle m_i, h_{\widehat{V}}\left(r_{\widehat{V}_x}(m_i)e^{2\pi i\omega_{\widehat{V}_x}(m_i)}\middle|P_{\widehat{V}_x}\right)\right\rangle\in\widehat{V}(n'_j)\Longleftrightarrow r_{\widehat{V}_x}(m_i)e^{2\pi i\omega_{\widehat{V}_x}(m_i)}\Big|P_{\widehat{V}_x}$

$$=\begin{cases} r_{\widehat{K_1}_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}_x}(m_i)}\Big|P_{\widehat{K_1}_x}, & \text{if } x\in h_{\widehat{K_1}_x}-h_{\widehat{K_2}_x}-h_{\widehat{K_3}_x},\\ r_{\widehat{K_2}_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)}\Big|P_{\widehat{K_2}_x}, & \text{if } x\in h_{\widehat{K_2}_x}-h_{\widehat{K_3}_x}-h_{\widehat{K_1}_x},\\ r_{\widehat{K_3}_x}(m_i)e^{2\pi i\omega_{\widehat{K_3}_x}(m_i)}\Big|P_{\widehat{K_3}_x}, & \text{if } x\in h_{\widehat{K_3}_x}-h_{\widehat{K_1}_x}-h_{\widehat{K_2}_x},\\ max\Big\{r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_1}_x}(m_i),\omega_{\widehat{K_2}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\in}, & \text{if } x\in h_{\widehat{K_1}_x}\cap h_{\widehat{K_2}_x}-h_{\widehat{K_3}_x},\\ max\Big\{r_{\widehat{K_2}_x}(m_i), r_{\widehat{K_3}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_3}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\ni}, & \text{if } x\in h_{\widehat{K_2}_x}\cap h_{\widehat{K_3}_x}-h_{\widehat{K_1}_x},\\ max\Big\{r_{\widehat{K_3}_x}(m_i), r_{\widehat{K_1}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_3}_x}(m_i),\omega_{\widehat{K_1}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\triangle}, & \text{if } x\in h_{\widehat{K_3}_x}\cap h_{\widehat{K_1}_x}-h_{\widehat{K_2}_x},\\ max\Big\{r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i), r_{\widehat{K_3}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_3}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\triangledown}, & \text{if } x\in h_{\widehat{K_1}_x}\cap h_{\widehat{K_2}_x}\cap h_{\widehat{K_3}_x},\end{cases}$$

where $\mathcal{P}_{\in}=P_{\widehat{K_1}_x}\cdot P_{\widehat{K_2}_x}, \mathcal{P}_{\ni}=P_{\widehat{K_2}_x}\cdot P_{\widehat{K_3}_x}, \mathcal{P}_{\triangle}=P_{\widehat{K_3}_x}\cdot P_{\widehat{K_1}_x}$ and $\mathcal{P}_{\triangledown}=P_{\widehat{K_1}_x}\cdot P_{\widehat{K_2}_x}\cdot P_{\widehat{K_3}_x}$.
R.H.S: Let $(\widehat{K_1},\mathcal{E}_1)$ and $(\widehat{K_2},\mathcal{E}_2)$ (where $\widehat{K_1}:\mathcal{E}_1\to\mathbb{F}(\mathcal{M})$ and $\widehat{K_2}:\mathcal{E}_2\to\mathbb{F}(\mathcal{M})$) be two CPHFSS over $\mathcal{M}$. By definition of restricted union we have $(\widehat{Q},O)$ (where $\widehat{Q}:O\to\mathbb{F}(\mathcal{M})$) such that

$$(\widehat{Q},O)=(\widehat{K_1},\mathcal{E}_1)\cup_{\xi}(\widehat{K_2},\mathcal{E}_2),$$

where $\widehat{S}=\widehat{K_1}\cup_{\xi}\widehat{K_2}$, $O=\mathcal{E}_1\cap\mathcal{E}_2$; $\forall o_j\in O$ and $m_i\in\mathcal{M}$,
$\left\langle m_i, h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i)e^{2\pi i\omega_{\widehat{Q}_x}(m_i)}\middle|P_{\widehat{Q}_x}\right)\right\rangle\in\widehat{Q}(o_j)\Longleftrightarrow r_{\widehat{Q}_x}(m_i)e^{2\pi i\omega_{\widehat{Q}_x}(m_i)}\Big|P_{\widehat{Q}_x}$

$$=\begin{cases} r_{\widehat{K_1}_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}_x}(m_i)}\Big|P_{\widehat{K_1}_x}, & \text{if } x\in h_{\widehat{K_1}_x}-h_{\widehat{K_2}_x},\\ r_{\widehat{K_2}_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)}\Big|P_{\widehat{K_2}_x}, & \text{if } x\in h_{\widehat{K_2}_x}-h_{\widehat{K_1}_x},\\ max\Big\{r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i)\Big\}e^{2\pi i max\Big\{\omega_{\widehat{K_1}_x}(m_i),\omega_{\widehat{K_2}_x}(m_i)\Big\}}\Big|P_{\widehat{K_1}_x}\cdot P_{\widehat{K_2}_x}, & \text{if } x\in h_{\widehat{K_1}_x}\cap h_{\widehat{K_2}_x},\end{cases}$$

As, $\left((\widehat{K_1},\mathcal{E}_1)\cup_{\xi}(\widehat{K_2},\mathcal{E}_2)\right)\cup_{\xi}(\widehat{K_3},\mathcal{E}_3)=(\widehat{S},P)\cup_{\xi}(\widehat{K_3},\mathcal{E}_3)$. Suppose that $(\widehat{Q},O)\cup_{\xi}(\widehat{K_3},\mathcal{E}_3)=(\widehat{Z},N')$ such that $\widehat{Z}:N'\to\mathbb{F}(\mathcal{M})$, where $\widehat{Z}=(\widehat{Q}\cup_{\xi}\widehat{K_3})$, $N'=O\cap\mathcal{E}_3=\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$; $\forall n'_j\in N'$,
$\left\langle m_i, h_{\widehat{Z}}\left(r_{\widehat{Z}_x}(m_i)e^{2\pi i\omega_{\widehat{Z}_x}(m_i)}\middle|P_{\widehat{Z}_x}\right)\right\rangle\in\widehat{Z}(n'_j)\Longleftrightarrow r_{\widehat{Z}_x}(m_i)e^{2\pi i\omega_{\widehat{Z}_x}(m_i)}\Big|P_{\widehat{Z}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}_x}(m_i)}\Big|P_{\widehat{K_1}_x}, & \text{if } x \in h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x} - h_{\widehat{K_3}_x}, \\ r_{\widehat{K_2}_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)}\Big|P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_2}_x} - h_{\widehat{K_3}_x} - h_{\widehat{K_1}_x}, \\ r_{\widehat{K_3}_x}(m_i)e^{2\pi i\omega_{\widehat{K_3}_x}(m_i)}\Big|P_{\widehat{K_3}_x}, & \text{if } x \in h_{\widehat{K_3}_x} - h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ max\Big\{r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i)\Big\}e^{2\pi imax\Big\{\omega_{\widehat{K_1}_x}(m_i),\omega_{\widehat{K_2}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\in}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x} - h_{\widehat{K_3}_x}, \\ max\Big\{r_{\widehat{K_2}_x}(m_i), r_{\widehat{K_3}_x}(m_i)\Big\}e^{2\pi imax\Big\{\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_3}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\ni}, & \text{if } x \in h_{\widehat{K_2}_x} \cap h_{\widehat{K_3}_x} - h_{\widehat{K_1}_x}, \\ max\Big\{r_{\widehat{K_3}_x}(m_i), r_{\widehat{K_1}_x}(m_i)\Big\}e^{2\pi imax\Big\{\omega_{\widehat{K_3}_x}(m_i),\omega_{\widehat{K_1}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\triangle}, & \text{if } x \in h_{\widehat{K_3}_x} \cap h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ max\Big\{r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i), r_{\widehat{K_3}_x}(m_i)\Big\}e^{2\pi imax\Big\{\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_2}_x}(m_i),\omega_{\widehat{K_3}_x}(m_i)\Big\}}\Big|\mathcal{P}_{\triangledown}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x} \cap h_{\widehat{K_3}_x}, \end{cases}$$

where $\mathcal{P}_{\in} = P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x}, \mathcal{P}_{\ni} = P_{\widehat{K_2}_x} \cdot P_{\widehat{K_3}_x}, \mathcal{P}_{\triangle} = P_{\widehat{K_3}_x} \cdot P_{\widehat{K_1}_x}$ and $\mathcal{P}_{\triangledown} = P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x} \cdot P_{\widehat{K_3}_x}$.
Then $\widehat{V}(n'_j) = \widehat{Z}(n'_j), \forall n'_j \in N'$.
Thus L.H.S = R.H.S.
Hence, (xi) is hold. □

The proof of (x) is similar to (ix) and (xii) is similar to (xi).

Proposition 2. *Given that* $(\widehat{K_1}, \mathcal{E}_1)$ *and* $(\widehat{K_2}, \mathcal{E}_2)$ *are any two CPHFSS on* $\mathcal{M}$*, then the following holds:*
Involution law.

(i). $\left(\left(\widehat{K_1}^c\right)^c, \mathcal{E}_1\right) = (\widehat{K_1}, \mathcal{E}_1)$,

De-Morgan's law.

(ii). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cap_{\xi} \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \cup_{\xi} \widehat{K_2})^c, (\mathcal{E}_1 \cap \mathcal{E}_2)\right)$,
(iii). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cup_{\xi} \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \cap_{\xi} \widehat{K_2})^c, (\mathcal{E}_1 \cap \mathcal{E}_2)\right)$,
(iv). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cap_{\zeta} \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \cup_{\zeta} \widehat{K_2})^c, (\mathcal{E}_1 \cup \mathcal{E}_2)\right)$,
(v). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cup_{\zeta} \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \cap_{\zeta} \widehat{K_2})^c, (\mathcal{E}_1 \cup \mathcal{E}_2)\right)$.
(vi). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \wedge \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \vee \widehat{K_2})^c, (\mathcal{E}_1 \times \mathcal{E}_2)\right)$,
(vii). $\left(\widehat{K_1}^c, \mathcal{E}_1\right) \vee \left(\widehat{K_2}^c, \mathcal{E}_2\right) = \left((\widehat{K_1} \wedge \widehat{K_2})^c, (\mathcal{E}_1 \times \mathcal{E}_2)\right)$.

Proof. (i) is straight-forward.

(ii) **L.H.S:** Let $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS on $\mathcal{M}$. Then by definition of complement and restricted intersection we have

$$\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cap_{\xi} \left(\widehat{K_2}^c, \mathcal{E}_2\right) = (\widehat{I}, O),$$

where $\widehat{I} = \left(\widehat{K_1}^c \cap_{\xi} \widehat{K_2}^c\right)$ and $O = \mathcal{E}_1 \cap \mathcal{E}_2; \forall o_j \in O$,
$\Big\langle m_i, h_{\widehat{I}}\Big(r_{\widehat{I}_x}(m_i)e^{2\pi i\omega_{\widehat{I}_x}(m_i)}\Big|P_{\widehat{I}_x}\Big)\Big\rangle \in \widehat{I}(o_j) \Longleftrightarrow r_{\widehat{I}_x}(m_i)e^{2\pi i\omega_{\widehat{I}_x}(m_i)}\Big|P_{\widehat{I}_x}$

$$= \begin{cases} r_{\widehat{K_1}^c_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}^c_x}(m_i)}\Big|P_{\widehat{K_1}^c_x}, & \text{if } x \in h_{\widehat{K_1}^c_x} - h_{\widehat{K_2}^c_x}, \\ r_{\widehat{K_2}^c_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}^c_x}(m_i)}\Big|P_{\widehat{K_2}^c_x}, & \text{if } x \in h_{\widehat{K_2}^c_x} - h_{\widehat{K_1}^c_x}, \\ min\Big\{r_{\widehat{K_1}^c_x}(m_i), r_{\widehat{K_2}^c_x}(m_i)\Big\}e^{2\pi imin\Big\{\omega_{\widehat{K_1}^c_x}(m_i),\omega_{\widehat{K_2}^c_x}(m_i)\Big\}}\Big|P_{\widehat{K_1}^c_x} \cdot P^c_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}^c_x} \cap h_{\widehat{K_2}^c_x}. \end{cases}$$

R.H.S: By the definition of restricted union we have $(\widehat{Q}, O) = (\widehat{K_1}, \mathcal{E}_1) \cup_\xi (\widehat{K_2}, \mathcal{E}_2)$ where $\widehat{Q} = \widehat{K_1} \cup_\xi \widehat{K_2}$, $O = \mathcal{E}_1 \cap \mathcal{E}_2 \neq \phi$; $\forall o_j \in O$ and $m_i \in \mathcal{M}$,
$\left\langle m_i, h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i)e^{2\pi i\omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x}\right)\right\rangle \in \widehat{Q}(o_j) \Longleftrightarrow r_{\widehat{Q}_x}(m_i)e^{2\pi i\omega_{\widehat{Q}_x}(m_i)}\Big| P_{\widehat{Q}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}_x}(m_i)}\Big| P_{\widehat{K_1}_x}, & \text{if } x \in h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ r_{\widehat{K_2}_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}_x}(m_i)}\Big| P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_2}_x} - h_{\widehat{K_1}_x}, \\ max\left\{ r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i)\right\} e^{2\pi i max\left\{\omega_{\widehat{K_1}_x}(m_i), \omega_{\widehat{K_2}_x}(m_i)\right\}}\Big| P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x}. \end{cases}$$

Now, by the definition of complement $r_{\widehat{Q}^c_x}(m_i)e^{2\pi i\omega_{\widehat{Q}^c_x}(m_i)}\Big| P_{\widehat{Q}^c_x}$

$$= \begin{cases} r_{\widehat{K_1}^c_x}(m_i)e^{2\pi i\omega_{\widehat{K_1}^c_x}(m_i)}\Big| P_{\widehat{K_1}^c_x}, & \text{if } x \in h_{\widehat{K_1}^c_x} - h_{\widehat{K_2}^c_x}, \\ r_{\widehat{K_2}^c_x}(m_i)e^{2\pi i\omega_{\widehat{K_2}^c_x}(m_i)}\Big| P_{\widehat{K_2}^c_x}, & \text{if } x \in h_{\widehat{K_2}^c_x} - h_{\widehat{K_1}^c_x}, \\ min\left\{ r_{\widehat{K_1}^c_x}(m_i), r_{\widehat{K_2}^c_x}(m_i)\right\} e^{2\pi i min\left\{\omega_{\widehat{K_1}^c_x}(m_i), \omega_{\widehat{K_2}^c_x}(m_i)\right\}}\Big| P_{\widehat{K_1}^c_x} \cdot P^c_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}^c_x} \cap h_{\widehat{K_2}^c_x}. \end{cases}$$

Then, $\widehat{I}(o_j) = \widehat{Q}^c(o_j)$; $\forall o_j \in O$.
Thus L.H.S = R.H.S.
Hence, (ii) is hold. □

Proof. (iv) **L.H.S:** Let $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS on $\mathcal{M}$. Then by definition of complement and extended intersection we have

$$\left(\widehat{K_1}^c, \mathcal{E}_1\right) \cap_\zeta \left(\widehat{K_2}^c, \mathcal{E}_2\right) = (\widehat{J}, P),$$

where $\widehat{J} = \left(\widehat{K_1}^c \cap_\zeta \widehat{K_2}^c\right)$ and $P = \mathcal{E}_1 \cup \mathcal{E}_2$; $\forall p_j \in P$ and $m_i \in \mathcal{M}$, with $p^1_j \in \mathcal{E}_1$ and $p^2_j \in \mathcal{E}_2$.

$$\widehat{J}(p_j) = \begin{cases} \widehat{K_1}^c(p^1_j), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}^c(p^2_j), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}^c(p^1_j) \cap_\xi \widehat{K_2}^c(p^2_j), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

R.H.S: By the definition of extended union we have $(\widehat{S}, P) = (\widehat{K_1}, \mathcal{E}_1) \cup_\zeta (\widehat{K_2}, \mathcal{E}_2)$ where $\widehat{S} = \widehat{K_1} \cup_\zeta \widehat{K_2}$, $P = \mathcal{E}_1 \cup \mathcal{E}_2$; $\forall p_j \in P$ and $m_i \in \mathcal{M}$, with $p^1_j \in \mathcal{E}_1$ and $p^2_j \in \mathcal{E}_2$.

$$\widehat{S}(p_j) = \begin{cases} \widehat{K_1}(p^1_j), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}(p^2_j), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}(p^1_j) \cup_\xi \widehat{K_2}(p^2_j), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

Now, by the definition of complement we have

$$\widehat{S}^c(p_j) = \begin{cases} \widehat{K_1}^c(p^1_j), & \text{if } p_j \in \mathcal{E}_1 - \mathcal{E}_2, \\ \widehat{K_2}^c(p^2_j), & \text{if } p_j \in \mathcal{E}_2 - \mathcal{E}_1, \\ \widehat{K_1}^c(p^1_j) \cap_\xi \widehat{K_2}^c(p^2_j), & \text{if } p_j \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

Then, $\widehat{J}(p_j) = \widehat{S}^c(p_j); \forall p_j \in P$.
Thus L.H.S = R.H.S.
Hence, (iv) is hold. □

Proof. (vi) **L.H.S:** Let $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) be two CPHFSS on $\mathcal{M}$. Then by definition of complement and soft max-AND operation we have

$$(\widehat{K_1}^c, \mathcal{E}_1) \wedge (\widehat{K_2}^c, \mathcal{E}_2) = (\widehat{J}, L),$$

where $\widehat{J} = \left(\widehat{K_1}^c \cup \widehat{K_2}^c\right)$ and $L = \mathcal{E}_1 \times \mathcal{E}_2$; $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$,
$\left\langle m_i, h_{\widehat{J}}\left(r_{\widehat{J}_x}(m_i) e^{2\pi i \omega_{\widehat{J}_x}(m_i)} \middle| P_{\widehat{J}_x} \right) \right\rangle \in \widehat{J}(l_i, l_j) \Longleftrightarrow r_{\widehat{J}_x}(m_i) e^{2\pi i \omega_{\widehat{J}_x}(m_i)} \Big| P_{\widehat{J}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x^c}(m_i) e^{2\pi i \omega_{\widehat{K_1}_x^c}(m_i)} \Big| P_{\widehat{K_1}_x^c}, & \text{if } x \in h_{\widehat{K_1}_x^c} - h_{\widehat{K_2}_x^c}, \\ r_{\widehat{K_2}_x^c}(m_i) e^{2\pi i \omega_{\widehat{K_2}_x^c}(m_i)} \Big| P_{\widehat{K_2}_x^c}, & \text{if } x \in h_{\widehat{K_2}_x^c} - h_{\widehat{K_1}_x^c}, \\ max\left\{ r_{\widehat{K_1}_x^c}(m_i), r_{\widehat{K_2}_x^c}(m_i) \right\} e^{2\pi i max\left\{ \omega_{\widehat{K_1}_x^c}(m_i), \omega_{\widehat{K_2}_x^c}(m_i) \right\}} \Big| P_{\widehat{K_1}_x^c} \cdot P_{\widehat{K_2}_x^c}, & \text{if } x \in h_{\widehat{K_1}_x^c} \cap h_{\widehat{K_2}_x^c}. \end{cases}$$

R.H.S: By the definition of soft min-OR we have

$$(\widehat{K_1}, \mathcal{E}_1) \vee (\widehat{K_2}, \mathcal{E}_2) = (\widehat{R}, L),$$

where $\widehat{R} : L \to \mathbb{F}(\mathcal{M})$ such that $\widehat{R} = \widehat{K_1} \cap \widehat{K_2}$ and $L = \mathcal{E}_1 \times \mathcal{E}_2$; $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$.
$\left\langle m_i, h_{\widehat{R}}\left(r_{\widehat{R}_x}(m_i) e^{2\pi i \omega_{\widehat{R}_x}(m_i)} \middle| P_{\widehat{R}_x} \right) \right\rangle \in \widehat{R}(l_i, l_j) \Longleftrightarrow r_{\widehat{R}_x}(m_i) e^{2\pi i \omega_{\widehat{R}_x}(m_i)} \Big| P_{\widehat{R}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x}(m_i) e^{2\pi i \omega_{\widehat{K_1}_x}(m_i)} \Big| P_{\widehat{K_1}_x}, & \text{if } x \in h_{\widehat{K_1}_x} - h_{\widehat{K_2}_x}, \\ r_{\widehat{K_2}_x}(m_i) e^{2\pi i \omega_{\widehat{K_2}_x}(m_i)} \Big| P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_2}_x} - h_{\widehat{K_1}_x}, \\ min\left\{ r_{\widehat{K_1}_x}(m_i), r_{\widehat{K_2}_x}(m_i) \right\} e^{2\pi i min\left\{ \omega_{\widehat{K_1}_x}(m_i), \omega_{\widehat{K_2}_x}(m_i) \right\}} \Big| P_{\widehat{K_1}_x} \cdot P_{\widehat{K_2}_x}, & \text{if } x \in h_{\widehat{K_1}_x} \cap h_{\widehat{K_2}_x}. \end{cases}$$

Now, by the definition of complement we have $r_{\widehat{R}_x^c}(m_i) e^{2\pi i \omega_{\widehat{R}_x^c}(m_i)} \Big| P^c_{\widehat{R}_x}$

$$= \begin{cases} r_{\widehat{K_1}_x^c}(m_i) e^{2\pi i \omega_{\widehat{K_1}_x^c}(m_i)} \Big| P_{\widehat{K_1}_x^c}, & \text{if } x \in h_{\widehat{K_1}_x^c} - h_{\widehat{K_2}_x^c}, \\ r_{\widehat{K_2}_x^c}(m_i) e^{2\pi i \omega_{\widehat{K_2}_x^c}(m_i)} \Big| P_{\widehat{K_2}_x^c}, & \text{if } x \in h_{\widehat{K_2}_x^c} - h_{\widehat{K_1}_x^c}, \\ max\left\{ r_{\widehat{K_1}_x^c}(m_i), r_{\widehat{K_2}_x^c}(m_i) \right\} e^{2\pi i max\left\{ \omega_{\widehat{K_1}_x^c}(m_i), \omega_{\widehat{K_2}_x^c}(m_i) \right\}} \Big| P_{\widehat{K_1}_x^c} \cdot P_{\widehat{K_2}_x^c}, & \text{if } x \in h_{\widehat{K_1}_x^c} \cap h_{\widehat{K_2}_x^c}. \end{cases}$$

Then, $\widehat{J}(l_i, l_j) = \widehat{R}^c(l_i, l_j); \forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2)$.
Thus L.H.S = R.H.S.
Hence, (vi) is hold. □

The proof of (iii) is similar to (ii), (v) is similar to (iv) and (vii) is similar to (vi).

4. Algorithms

In this section, we will present the algorithms by comparison method, soft max-AND operation and soft min-OR operation on one and two complex probabilistic hesitant fuzzy soft sets for decision making. If the evaluation is done by single expert then we use Algorithm 1. If the evaluation is done by two experts and the person wants the decision based on supreme quality of parameters then we use Algorithm 2 and if the evaluation is done by two experts and the person wants the decision based on lowest quality of parameters then we use Algorithm 3. If the experts are more than two then the Algorithms 2 and 3 can also be used with the same working policy except where we use two sets in Step 2 we will use more than two sets with the same selection.

Definition 18. *In the comparison algorithm the score function $\bar{\mathcal{S}}$ is defined as*

$$\bar{\mathcal{S}} = \left(\tilde{\mathbb{S}}(r_{\widehat{K}}) + \tilde{\mathbb{S}}(\omega_{\widehat{K}})\right) \times \tilde{\mathbb{S}}(P_{\widehat{K}}),$$

where $\tilde{\mathbb{S}}(r_{\widehat{K}})$ is obtained by subtracting the column sum from the row sum of amplitude terms in the comparison table, $\tilde{\mathbb{S}}(\omega_{\widehat{K}})$ is obtained by subtracting the column sum from the row sum of phase terms in the comparison table, and $\tilde{\mathbb{S}}(P_{\widehat{K}})$ is obtained by taking the product of probability choice values.

Algorithm 1: Proposed Algorithm-1

Step 1: Input universal set $\mathcal{M} = \{m_1, m_2, m_3, ..., m_l\}$ and set of attributes $\mathcal{E}$ where $\mathcal{E} \subseteq G$.
Step 2: Design CPHFSS $(\widehat{K}, \mathcal{E})$ where $\widehat{K} : \mathcal{E} \to \mathbb{F}(\mathcal{M})$ on $\mathcal{M}$.
Step 3: Design tables for complex valued membership $h_{\widehat{K}}\left(r_{\widehat{K}_x}(m_i)e^{2\pi i\omega_{\widehat{K}_x}(m_i)}\middle|P_{\widehat{K}_x}\right)$ of CPHFSS $(\widehat{K}, \mathcal{E})$ separately for $r_{\widehat{K}}$, $\omega_{\widehat{K}}$ and $P_{\widehat{K}}$.
Step 4: Compute the choice value $\check{C}$ in the tabular form for amplitude term $\check{C}(r_{\widehat{K}})$ and phase term $\check{C}(\omega_{\widehat{K}})$ by taking average of hesitant fuzzy set and for probability $\check{C}(P_{\widehat{K}})$ by taking their product.
Step 5: Compute the comparison tables for amplitude term $\tilde{\mathbb{C}}(r_{\widehat{K}})$ and phase term $\tilde{\mathbb{C}}(\omega_{\widehat{K}})$, where $\tilde{\mathbb{C}}(m_i, m_{\grave{i}}) = \sum_{e_j \in \mathcal{E}}$ (if m_i w.r.t $e_j \geq m_{\grave{i}}$ w.r.t e_j then 1 otherwise 0) where $i, \grave{i} = \{1, 2, 3, ...,$ no. of the universal elements$\}$.
Step 6: Calculate the $\tilde{\mathbb{S}}(r_{\widehat{K}})$, $\tilde{\mathbb{S}}(\omega_{\widehat{K}})$ and $\tilde{\mathbb{S}}(P_{\widehat{K}})$ by using Definition 18.
Step 7: Evaluate the final score $\bar{\mathcal{S}}$ by using Definition 18.
Step 8: Any of the alternative is selected as optimal decision with the highest score value.

The flow chart of Algorithm 1 is given in Figure 1.

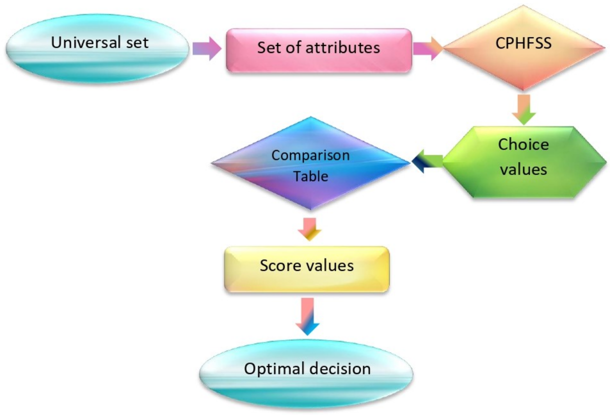

Figure 1. Flow chart of Algorithm 1.

Algorithm 2: Proposed Algorithm-2

Step 1: Input universal set $\mathcal{M} = \{m_1, m_2, m_3, ..., m_l\}$ and set of attributes $\mathcal{E}$ where $\mathcal{E} \subseteq G$.
Step 2: Design two CPHFSSs $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) on universal set $\mathcal{M}$.
Step 3: Evaluate

$$(\widehat{Q}, L) = (\widehat{K_1}, \mathcal{E}_1) \wedge (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{Q} : L \to \mathbb{F}(\mathcal{M})$; $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$.
Step 4: Figure out the choice value $\complement(l_i, l_j)(m_i);\ \forall\ m_i \in \mathcal{M}, (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2)$ defined as

$$\complement(l_i, l_j)(m_i) = \frac{\sum\limits_x \left(r_{\widehat{Q}_x}(m_i) \times \omega_{\widehat{Q}_x}(m_i)\right)}{\sum\limits_x \left(r_{\widehat{Q}_x}(m_i)\right)} \Bigg| \prod_x P_{\widehat{Q}_x},$$

where $x = 1, 2, 3, ..., n$ and n is the number of possible elements in $h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i) e^{2\pi i \omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x}\right)$.
Step 5: Figure out the score values $\mathbb{S}(l_i, l_j)(m_i)$ by product of the maximum of $\complement(l_i, l_j)$ against each m_i with corresponding probability.
Step 6: The weighted value for each $\mathbb{S}(l_i, l_j)(m_i)$ is:

$$\mathbb{S}(m_i) = \sum_{(l_i, l_j)) \in (\mathcal{E}_1 \times \mathcal{E}_2)} \mathbb{S}(l_i, l_j)(m_i).$$

Step 7: Any of the alternative is selected as optimal decision for which:

$$¥ = max\left\{\mathbb{S}(m_1), \mathbb{S}(m_2), ..., \mathbb{S}(m_l)\right\}.$$

The flow chart of Algorithm 2 is given in Figure 2.

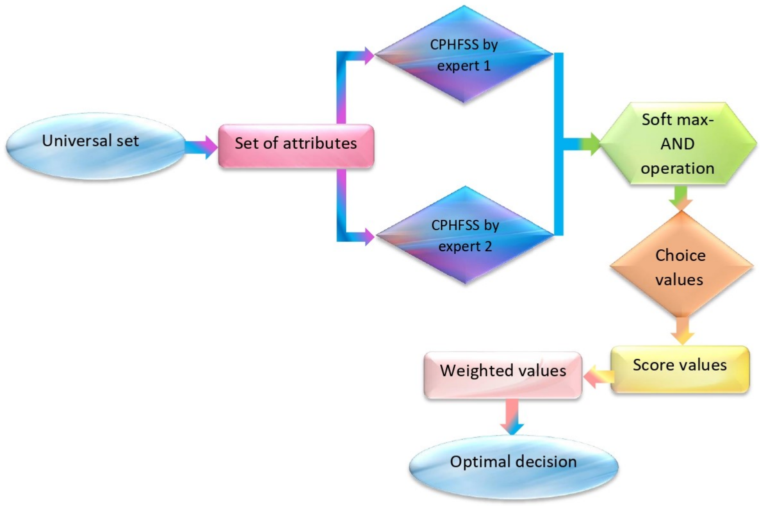

Figure 2. Flow chart of Algorithm 2.

Algorithm 3: Proposed Algorithm-3

Step 1: Input universal set $\mathcal{M} = \{m_1, m_2, m_3, ..., m_l\}$ and set of attributes $\mathcal{E}$ where $\mathcal{E} \subseteq G$.
Step 2: Design two CPHFSSs $(\widehat{K_1}, \mathcal{E}_1)$ and $(\widehat{K_2}, \mathcal{E}_2)$ (where $\widehat{K_1} : \mathcal{E}_1 \to \mathbb{F}(\mathcal{M})$ and $\widehat{K_2} : \mathcal{E}_2 \to \mathbb{F}(\mathcal{M})$) on universal set $\mathcal{M}$.
Step 3: Evaluate

$$(\widehat{R}, L) = (\widehat{K_1}, \mathcal{E}_1) \vee (\widehat{K_2}, \mathcal{E}_2),$$

where $\widehat{R} : L \to \mathbb{F}(\mathcal{M})$; $\forall (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2),\ l_i, l_j \in \Lambda$.
Step 4: Figure out the choice value $\complement(l_i, l_j)(m_i)$; $\forall\, m_i \in \mathcal{M}, (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2)$ defined as

$$\complement(l_i, l_j)(m_i) = \frac{\sum_x \left(r_{\widehat{Q}_x}(m_i) \times \omega_{\widehat{Q}_x}(m_i) \right)}{\sum_x \left(r_{\widehat{Q}_x}(m_i) \right)} \Bigg| \prod_x P_{\widehat{Q}_x},$$

where $x = 1, 2, 3, ..., n$ and n is the number of possible elements in $h_{\widehat{Q}}\left(r_{\widehat{Q}_x}(m_i) e^{2\pi i \omega_{\widehat{Q}_x}(m_i)} \middle| P_{\widehat{Q}_x} \right)$.
Step 5: Figure out the score values $\mathbb{S}(l_i, l_j)(m_i)$ by product of the maximum of $\complement(l_i, l_j)$ against each m_i with corresponding probability.
Step 6: The weighted value for each $\mathbb{S}(l_i, l_j)(m_i)$ is

$$\mathbb{S}(m_i) = \sum_{(l_i, l_j)) \in (\mathcal{E}_1 \times \mathcal{E}_2)} \mathbb{S}(l_i, l_j)(m_i).$$

Step 7: Any of the alternative is selected as optimal decision for which

$$¥ = max \left\{ \mathbb{S}(m_1), \mathbb{S}(m_2), ..., \mathbb{S}(m_l) \right\}.$$

The flow chart of Algorithm 3 is given in Figure 3.

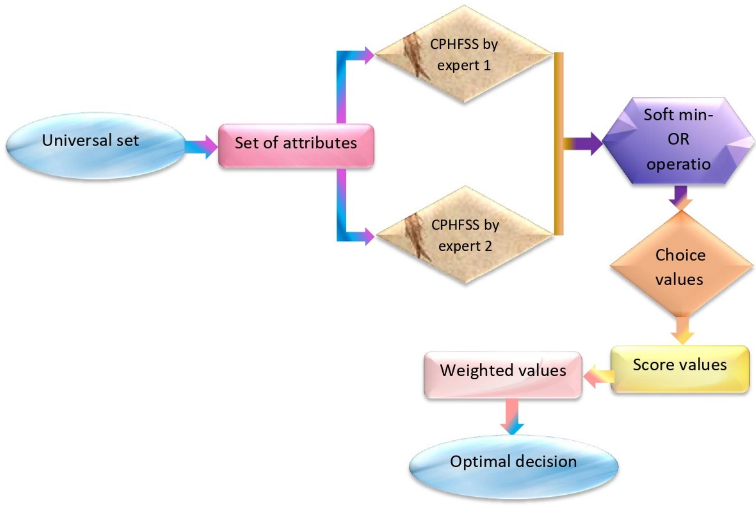

Figure 3. Flow chart of Algorithm 3.

5. Applications using multi-parameter group decision making

In this section, we illustrate applications and case studies of our proposed sets to manifest the integrity and supremacy of the explored work. In this paper, we deal with hesitancy, randomness, and fuzziness using parameterized families simultaneously, which is efficient and useful. Complex probabilistic hesitant fuzzy sets play a very significant role in decision making. It helps to save time while dealing with hesitancy, randomness, and fuzziness separately. In this paper, the membership function is in complex form. In the engineering field, complex forms or complex numbers play a very very significant role. For example, in the modelling of bridges, signal processing, computational intelligence, data analysis, image processing, radar systems, mobile communication systems, computer graphics, launching a satellite, electrical appliances, and many more.

One of the biggest failures due to lack of knowledge of complex numbers is, the Tacoma Bridge in the USA in Washington State near Puget Sound, which was opened on 1^{st} July 1940, but unfortunately on 7^{th} November 1940 it collapsed for certain reasons [49]. In the modeling of bridges, the imaginary part of complex numbers measures the frequency of the vibrations, and the real part gives you the measure of the amount of decay in the motion. Professor Ahmer Wadee of Imperial College London spoke about the strength of complex numbers and how they work to keep bridges from collapsing [50].

Case study 1. *A suspension bridge can be characterized as a pedestrian bridge or footbridge, a very environment-friendly, light-weight shape that allows pedestrians to go through risky areas such as highways, rivers, and ravines. Suspension bridges can span somewhere between 2,000 to 13,000 feet, which is farther than different kinds of bridges. This is why it is frequently the layout of preference when connecting very far-off locations. Suspension bridges are an antique but smart technology.*

Suppose the X country wants to construct the world's best suspension bridge and $\mathcal{M} = \{m_1, m_2, m_3, m_4, m_5\}$ *be the set of the engineers who presented the bridge model. The government wants to* *take into account the following three specific features $\mathcal{E} = \{e_1, e_2, e_3\}$ where e_1 = "capability to bear forces (such as wind force, gravity force, restoring force)", e_2 ="safety during natural disasters" and e_3 ="durability". The government wants to select the best model among them. As a result, experts gave the CPHFSS $(\widehat{K}, \mathcal{E})$ which furnishes an approximate narration of complex probabilistic hesitant fuzzy information of five models of engineers and their features represented in the Table 9.*

Table 9. Tabular representation of CPHFSS $(\widehat{K}, \mathcal{E})$.

$(\widehat{K}, \mathcal{E})$	e_1	e_2	e_3
m_1	$\left\{\begin{array}{l} 0.4e^{2\pi i0.8}\vert 0.7, \\ 0.2e^{2\pi i0.6}\vert 0.2, \\ 0.3e^{2\pi i0.2}\vert 0.1 \end{array}\right\}$	$\left\{\begin{array}{l} 0.7e^{2\pi i0.7}\vert 0.4, \\ 0.1e^{2\pi i0.5}\vert 0.3, \\ 0.3e^{2\pi i0.9}\vert 0.3 \end{array}\right\}$	$\left\{\begin{array}{l} 0.9e^{2\pi i0.9}\vert 0.6, \\ 0.8e^{2\pi i0.1}\vert 0.4 \end{array}\right\}$
m_2	$\left\{\begin{array}{l} 0.1e^{2\pi i0.7}\vert 0.8, \\ 0.8e^{2\pi i0.3}\vert 0.2 \end{array}\right\}$	$\left\{\begin{array}{l} 0.4e^{2\pi i0.5}\vert 0.5, \\ 0.5e^{2\pi i0.4}\vert 0.5 \end{array}\right\}$	$\left\{ 0.2e^{2\pi i0.3}\vert 1 \right\}$
m_3	$\left\{ 0.2e^{2\pi i0.6}\vert 1 \right\}$	$\left\{ 0.8e^{2\pi i0.3}\vert 1 \right\}$	$\left\{\begin{array}{l} 0.7e^{2\pi i0.3}\vert 0.9, \\ 0.9e^{2\pi i0.7}\vert 0.1 \end{array}\right\}$
m_4	$\left\{\begin{array}{l} 0.5e^{2\pi i0.5}\vert 0.7, \\ 0.7e^{2\pi i0.8}\vert 0.3 \end{array}\right\}$	$\left\{\begin{array}{l} 0.8e^{2\pi i0.6}\vert 0.2, \\ 0.9e^{2\pi i0.5}\vert 0.3, \\ 0.8e^{2\pi i0.8}\vert 0.5 \end{array}\right\}$	$\left\{\begin{array}{l} 0.3e^{2\pi i0.4}\vert 0.8, \\ 0.7e^{2\pi i0.8}\vert 0.2 \end{array}\right\}$
m_5	$\left\{ 0.5e^{2\pi i0.3}\vert 1 \right\}$	$\left\{\begin{array}{l} 0.3e^{2\pi i0.8}\vert 0.9, \\ 0.9e^{2\pi i0.7}\vert 0.1 \end{array}\right\}$	$\left\{ 0.7e^{2\pi i0.7}\vert 1 \right\}$

For illustration, the above table is of CPHFSS $(\widehat{K}, \mathcal{E})$ established on specific parameters (forces, safety, and durability) of the bridge models presented by engineers. Where the top left cell contains the complex membership of the bridge model of m_1 with respect to e_1 = forces. Similarly, the bottom right cell contains the complex membership of the bridge model of m_5 with respect to e_3 = durability.

By using Step 3 of the Algorithm 1, Tables 10–12 represent the complex membership values for the amplitude term, phase term, and probability separately.

Table 10. Tabular representation of the membership of amplitude term.

$r_{\widehat{K}}$	e_1	e_2	e_3
m_1	{0.4, 0.2, 0.3}	{0.7, 0.1, 0.3}	{0.9, 0.8}
m_2	{0.1, 0.8}	{0.4, 0.5}	{0.2}
m_3	{0.2}	{0.8}	{0.7, 0.9}
m_4	{0.5, 0.7}	{0.8, 0.9, 0.8}	{0.3 0.7}
m_5	{0.5}	{0.3, 0.9}	{0.7}

Table 11. Tabular representation of the membership of phase term.

$\omega_{\widehat{K}}$	e_1	e_2	e_3
m_1	{0.8, 0.6, 0.2}	{0.7, 0.5, 0.9}	{0.9, 0.1}
m_2	{0.7, 0.3}	{0.5, 0.4}	{0.3}
m_3	{0.6}	{0.3}	{0.3, 0.7}
m_4	{0.5, 0.8}	{0.6, 0.5, 0.8}	{0.4, 0.8}
m_5	{0.3}	{0.8, 0.7}	{0.7}

Table 12. Tabular representation of the probability of membership.

$P_{\widehat{K}}$	e_1	e_2	e_3
m_1	{0.7, 0.2, 0.1}	{0.4, 0.3, 0.3}	{0.6, 0.4}
m_2	{0.8, 0.2}	{0.5, 0.5}	{1}
m_3	{1}	{1}	{0.9, 0.1}
m_4	{0.7, 0.3}	{0.2, 0.3, 0.5}	{0.8, 0.2}
m_5	{1}	{0.9, 0.1}	{1}

Computation by using Step 4 of the Algorithm 1, Tables 13–15 represent the choice values for amplitude term, phase term, and probability separately.

Table 13. Choice value of amplitude term.

$\check{C}(r_{\widehat{K}})$	e_1	e_2	e_3
m_1	0.3	0.36	0.85
m_2	0.45	0.45	0.2
m_3	0.2	0.8	0.8
m_4	0.6	0.83	0.5
m_5	0.5	0.6	0.7

Table 14. Choice value of phase term.

$\check{C}(\omega_{\widehat{K}})$	e_1	e_2	e_3
m_1	0.53	0.7	0.5
m_2	0.5	0.45	0.3
m_3	0.6	0.3	0.5
m_4	0.65	0.63	0.6
m_5	0.3	0.75	0.7

Table 15. Choice value of probability.

$\check{C}(P_{\widehat{K}})$	e_1	e_2	e_3
m_1	0.014	0.036	0.24
m_2	0.16	0.25	1
m_3	1	1	0.09
m_4	0.21	0.03	0.16
m_5	1	0.09	1

Computation by using Step 5 of the Algorithm 1, Tables 16 and 17 represent the comparison of amplitude term and phase term.

Table 16. Comparision table for amplitude term.

$\tilde{\mathbb{C}}(r_{\widehat{K}})$	m_1	m_2	m_3	m_4	m_5
m_1	3	1	2	1	1
m_2	2	3	1	0	0
m_3	1	2	3	1	2
m_4	2	3	2	3	2
m_5	2	3	1	1	3

Table 17. Comparison table for phase term.

$\tilde{\mathbb{C}}(\omega_{\widehat{K}})$	m_1	m_2	m_3	m_4	m_5
m_1	3	3	2	1	1
m_2	0	3	1	0	1
m_3	2	2	3	0	1
m_4	2	3	3	3	1
m_5	2	2	2	2	3

Evaluation by using Step 6 of the Algorithm 1, Table 18 represent the score values of $\tilde{\mathbb{S}}(r_{\widehat{K}})$, $\tilde{\mathbb{S}}(\omega_{\widehat{K}})$ *and* $\tilde{\mathbb{S}}(P_{\widehat{K}})$.

Table 18. Score values.

	row sum $\tilde{\mathbb{C}}(r_{\widehat{K}})$	col. sum $\tilde{\mathbb{C}}(r_{\widehat{K}})$	$\tilde{\mathbb{S}}(r_{\widehat{K}})$	row sum $\tilde{\mathbb{C}}(\omega_{\widehat{K}})$	col. sum $\tilde{\mathbb{C}}(\omega_{\widehat{K}})$	$\tilde{\mathbb{S}}(\omega_{\widehat{K}})$	$\tilde{\mathbb{S}}(P_{\widehat{K}})$
m_1	8	10	-2	10	9	1	0.00012096
m_2	6	12	-6	5	13	-8	0.04
m_3	9	9	0	8	11	-3	0.09
m_4	12	6	6	12	6	6	0.001008
m_5	10	8	2	11	7	4	0.09

Evaluation by using Step 7 of the Algorithm 1, the final score $\bar{S}$ *is*

$$\bar{S}(m_1) = -0.00012,\ \ \bar{S}(m_2) = -0.56,\ \ \bar{S}(m_3) = -0.27,\ \ \bar{S}(m_4) = 0.012096,\ \ \bar{S}(m_5) = 0.54.$$

By using Step 8 of the Algorithm 1, the best bridge model among five engineers is of m_5. So, the government will select the model of m_5 engineer to build the suspension bridge.

The graphical representation of ranking is given in Figure 4.

Figure 4. Ranking results.

Case study 2. *Circuit simulation is an essential phase of developing digital products. With the assistance of simulators, we are able to compute the competencies of circuits as well as analyze their overall performance without making the circuit. It is cost effiient as well as time-saving. With the assistance of software, we can design complicated circuits in no time.*

There is a wide range of circuit simulator software and equipment in the market. For instance, Proteus, Droid Tesla, NI's Multisim, Ngspice, Synopsys, Cadence Spectre, PSIM and many more. These assist agencies by means of saving their time and money. However, it is difficult to pick out exceptional circuit simulation software by giving a significant number of options.

Suppose the manager of the multinational company Y wants to buy complicated multi-purpose circuit simulator software. For that, the manager hires the two experts and provides them with five circuit simulator software and $\mathcal{M} = \{m_1, m_2, m_3, m_4, m_5\}$ be the set of the simulator softwares. The two experts evaluate with CPHF information based on the given set of parameters G. Let the first expert $\widehat{K_1}$ evaluate with the set of parameters $\mathcal{E}_1 = \{e_1, e_2, e_3, e_4\} \subseteq G$ and the second expert $\widehat{K_2}$ evaluate with the set of parameters $\mathcal{E}_2 = \{e_5, e_6, e_7, e_8\} \subseteq G$ where e_1 = library parts, e_2 = 3D viewing features, e_3 = graphical representation, e_4 = analog and digital components, e_5 = real looking, e_6 = draw circuit quickly, e_7 = readymade circuit and e_8 = expensive. The manager wants to buy the circuit simulator software, which consists of supreme quality parameters. Then the right decision for selecting the software is done by using the max-AND operator. The evaluation given by the experts is presented in Tables 19 and 20.

Table 19. Tabular representation of CPHFSS $(\widehat{K_1}, \mathcal{E}_1)$.

$(\widehat{K_1}, \mathcal{E}_1)$	e_1	e_2	e_3	e_4
m_1	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.7, \\ 0.7e^{2\pi i0.7}\|0.3 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.8}\|0.6, \\ 0.9e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ 0.1e^{2\pi i0.4}\|1 \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.1}\|0.1, \\ 0.7e^{2\pi i0.5}\|0.1, \\ 0.3e^{2\pi i0.3}\|0.8 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.4}\|0.8, \\ 0.7e^{2\pi i0.8}\|0.2 \end{array} \right\}$	$\left\{ 0.5e^{2\pi i0.8}\|1 \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.2}\|0.9, \\ 0.2e^{2\pi i0.3}\|0.1 \end{array} \right\}$	$\left\{ 0.1e^{2\pi i0.3}\|1 \right\}$
m_3	$\left\{ 0.4e^{2\pi i0.4}\|1 \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.2, \\ 0.8e^{2\pi i0.4}\|0.4, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.1}\|0.8, \\ 0.7e^{2\pi i0.3}\|0.2 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.9}\|0.6, \\ 0.3e^{2\pi i0.5}\|0.4 \end{array} \right\}$
m_4	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.6}\|0.1, \\ 0.9e^{2\pi i0.2}\|0.2, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ 0.4e^{2\pi i0.1}\|1 \right\}$	$\left\{ \begin{array}{l} 0.7e^{2\pi i0.2}\|0.7, \\ 0.2e^{2\pi i0.4}\|0.3 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.6, \\ 0.4e^{2\pi i0.7}\|0.4 \end{array} \right\}$
m_5	$\left\{ 0.3e^{2\pi i0.3}\|1 \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.9, \\ 0.8e^{2\pi i0.8}\|0.1 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.5e^{2\pi i0.1}\|0.2, \\ 0.9e^{2\pi i0.4}\|0.3, \\ 0.6e^{2\pi i0.6}\|0.5 \end{array} \right\}$	$\left\{ 0.2e^{2\pi i0.8}\|1 \right\}$

Table 20. Tabular representation of CPHFSS $(\widehat{K_2}, \mathcal{E}_2)$.

$(\widehat{K_2}, \mathcal{E}_2)$	e_5	e_6	e_7	e_8
m_1	$\left\{ 0.6e^{2\pi i0.5}\|1 \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.7}\|0.3, \\ 0.1e^{2\pi i0.6}\|0.3, \\ 0.3e^{2\pi i0.3}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.9}\|0.3, \\ 0.7e^{2\pi i0.2}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.7}\|0.2, \\ 0.7e^{2\pi i0.8}\|0.8 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.3}\|0.2, \\ 0.9e^{2\pi i0.8}\|0.2, \\ 0.1e^{2\pi i0.7}\|0.6 \end{array} \right\}$	$\left\{ 0.1e^{2\pi i0.1}\|1 \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.4}\|0.3, \\ 0.9e^{2\pi i0.2}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.7e^{2\pi i0.9}\|0.6, \\ 0.9e^{2\pi i0.7}\|0.4 \end{array} \right\}$
m_3	$\left\{ 0.7e^{2\pi i0.9}\|1 \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.3}\|0.5, \\ 0.2e^{2\pi i0.4}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.3}\|0.2, \\ 0.6e^{2\pi i0.7}\|0.4, \\ 0.8e^{2\pi i0.9}\|0.4 \end{array} \right\}$	$\left\{ 0.2e^{2\pi i0.6}\|1 \right\}$
m_4	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.6}\|0.2, \\ 0.8e^{2\pi i0.2}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.5}\|0.1, \\ 0.3e^{2\pi i0.3}\|0.9 \end{array} \right\}$	$\left\{ 0.5e^{2\pi i0.7}\|1 \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.8}\|0.3, \\ 0.5e^{2\pi i0.4}\|0.3, \\ 0.8e^{2\pi i0.1}\|0.4 \end{array} \right\}$
m_5	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.2, \\ 0.5e^{2\pi i0.3}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.2}\|0.2, \\ 0.3e^{2\pi i0.4}\|0.3, \\ 0.6e^{2\pi i0.3}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.3}\|0.3, \\ 0.8e^{2\pi i0.6}\|0.7 \end{array} \right\}$	$\left\{ 0.9e^{2\pi i0.3}\|1 \right\}$

Evaluate the soft max-AND operation $(\widehat{Q}, L) = (\widehat{K_1}, \mathcal{E}_1) \wedge (\widehat{K_2}, \mathcal{E}_2)$ *by using Step 3 of Algorithm 2, represented in Table 21.*

Table 21. Evaluation by using soft max-AND operation.

$(\widehat{Q}, L)$	$(e_1 \times e_5)$	$(e_1 \times e_6)$	$(e_1 \times e_7)$	$(e_1 \times e_8)$
m_1	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.5}\|0.7, \\ 0.7e^{2\pi i0.7}\|0.3 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.7}\|0.21, \\ 0.7e^{2\pi i0.7}\|0.09, \\ 0.3e^{2\pi i0.3}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.9}\|0.21 \\ 0.7e^{2\pi i0.7}\|0.21 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.7}\|0.14, \\ 0.7e^{2\pi i0.8}\|0.24 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.4}\|0.16, \\ 0.9e^{2\pi i0.8}\|0.04, \\ 0.1e^{2\pi i0.7}\|0.6 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.4}\|0.8, \\ 0.7e^{2\pi i0.8}\|0.2 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.4}\|0.24, \\ 0.9e^{2\pi i0.8}\|0.14 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.9}\|0.48, \\ 0.9e^{2\pi i0.8}\|0.08 \end{array} \right\}$
m_3	$\left\{ 0.7e^{2\pi i0.9}\|1 \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.4}\|0.5, \\ 0.2e^{2\pi i0.4}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.4}\|0.2, \\ 0.6e^{2\pi i0.7}\|0.4, \\ 0.8e^{2\pi i0.9}\|0.4 \end{array} \right\}$	$\left\{ 0.4e^{2\pi i0.6}\|1 \right\}$
m_4	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.6}\|0.02, \\ 0.9e^{2\pi i0.2}\|0.16, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.6}\|0.01, \\ 0.9e^{2\pi i0.3}\|0.18, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.5e^{2\pi i0.7}\|0.1, \\ 0.9e^{2\pi i0.2}\|0.2, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.8}\|0.03, \\ 0.9e^{2\pi i0.4}\|0.06, \\ 0.8e^{2\pi i0.8}\|0.28 \end{array} \right\}$
m_5	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.2, \\ 0.5e^{2\pi i0.3}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.2, \\ 0.3e^{2\pi i0.4}\|0.3, \\ 0.6e^{2\pi i0.3}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.3}\|0.3, \\ 0.8e^{2\pi i0.6}\|0.7 \end{array} \right\}$	$\left\{ 0.9e^{2\pi i0.3}\|1 \right\}$
$(\widehat{Q}, L)$	$(e_2 \times e_5)$	$(e_2 \times e_6)$	$(e_2 \times e_7)$	$(e_2 \times e_8)$
m_1	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.8}\|0.6, \\ 0.9e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.8}\|0.18, \\ 0.9e^{2\pi i0.6}\|0.12 \\ 0.3e^{2\pi i0.3}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.9}\|0.18, \\ 0.9e^{2\pi i0.2}\|0.28 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.8}\|0.12, \\ 0.9e^{2\pi i0.8}\|0.32 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.5e^{2\pi i0.8}\|0.2, \\ 0.9e^{2\pi i0.8}\|0.2, \\ 0.1e^{2\pi i0.7}\|0.6 \end{array} \right\}$	$\left\{ 0.5e^{2\pi i0.8}\|1 \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.8}\|0.3, \\ 0.9e^{2\pi i0.2}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.7e^{2\pi i0.9}\|0.6, \\ 0.9e^{2\pi i0.7}\|0.4 \end{array} \right\}$
m_3	$\left\{ \begin{array}{l} 0.7e^{2\pi i0.9}\|0.2, \\ 0.8e^{2\pi i0.4}\|0.4, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.1, \\ 0.8e^{2\pi i0.4}\|0.2, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.04, \\ 0.8e^{2\pi i0.7}\|0.16 \\ 0.8e^{2\pi i0.9}\|0.16 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.2, \\ 0.8e^{2\pi i0.4}\|0.4, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$
m_4	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.6}\|0.2, \\ 0.8e^{2\pi i0.2}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.5}\|0.1, \\ 0.3e^{2\pi i0.3}\|0.9 \end{array} \right\}$	$\left\{ 0.5e^{2\pi i0.7}\|1 \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.8}\|0.3, \\ 0.5e^{2\pi i0.4}\|0.3, \\ 0.8e^{2\pi i0.1}\|0.4 \end{array} \right\}$
m_5	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.18, \\ 0.8e^{2\pi i0.8}\|0.08 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.18, \\ 0.8e^{2\pi i0.8}\|0.03, \\ 0.6e^{2\pi i0.3}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.27, \\ 0.8e^{2\pi i0.8}\|0.07 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.9, \\ 0.8e^{2\pi i0.8}\|0.1 \end{array} \right\}$

Table 21. Evaluation by using soft max-AND operation (Continued).

$(\widehat{Q}, L)$	$(e_3 \times e_5)$	$(e_3 \times e_6)$	$(e_3 \times e_7)$	$(e_3 \times e_8)$
m_1	$\left\{ 0.6e^{2\pi i0.5}\mid 1 \right\}$	$\left\{ 0.2e^{2\pi i0.7}\mid 0.3, 0.1e^{2\pi i0.6}\mid 0.3, 0.3e^{2\pi i0.3}\mid 0.4 \right\}$	$\left\{ 0.4e^{2\pi i0.9}\mid 0.3, 0.7e^{2\pi i0.2}\mid 0.7 \right\}$	$\left\{ 0.6e^{2\pi i0.7}\mid 0.2, 0.7e^{2\pi i0.8}\mid 0.8 \right\}$
m_2	$\left\{ 0.4e^{2\pi i0.3}\mid 0.18, 0.9e^{2\pi i0.8}\mid 0.02, 0.1e^{2\pi i0.7}\mid 0.6 \right\}$	$\left\{ 0.1e^{2\pi i0.2}\mid 0.9, 0.2e^{2\pi i0.3}\mid 0.1 \right\}$	$\left\{ 0.8e^{2\pi i0.4}\mid 0.27, 0.9e^{2\pi i0.3}\mid 0.07 \right\}$	$\left\{ 0.7e^{2\pi i0.9}\mid 0.54, 0.9e^{2\pi i0.7}\mid 0.04 \right\}$
m_3	$\left\{ 0.7e^{2\pi i0.9}\mid 0.8, 0.7e^{2\pi i0.3}\mid 0.2 \right\}$	$\left\{ 0.1e^{2\pi i0.3}\mid 0.4, 0.7e^{2\pi i0.4}\mid 0.1 \right\}$	$\left\{ 0.6e^{2\pi i0.3}\mid 0.16, 0.7e^{2\pi i0.7}\mid 0.08, 0.8e^{2\pi i0.9}\mid 0.4 \right\}$	$\left\{ 0.2e^{2\pi i0.6}\mid 0.8, 0.7e^{2\pi i0.3}\mid 0.2 \right\}$
m_4	$\left\{ 0.7e^{2\pi i0.6}\mid 0.14, 0.8e^{2\pi i0.4}\mid 0.24 \right\}$	$\left\{ 0.7e^{2\pi i0.5}\mid 0.07, 0.3e^{2\pi i0.4}\mid 0.27 \right\}$	$\left\{ 0.7e^{2\pi i0.7}\mid 0.7, 0.2e^{2\pi i0.4}\mid 0.3 \right\}$	$\left\{ 0.7e^{2\pi i0.8}\mid 0.21, 0.5e^{2\pi i0.4}\mid 0.09, 0.8e^{2\pi i0.1}\mid 0.4 \right\}$
m_5	$\left\{ 0.5e^{2\pi i0.3}\mid 0.04, 0.9e^{2\pi i0.4}\mid 0.24, 0.6e^{2\pi i0.6}\mid 0.5 \right\}$	$\left\{ 0.5e^{2\pi i0.2}\mid 0.04, 0.9e^{2\pi i0.4}\mid 0.09, 0.6e^{2\pi i0.6}\mid 0.25 \right\}$	$\left\{ 0.8e^{2\pi i0.3}\mid 0.06, 0.9e^{2\pi i0.6}\mid 0.21, 0.6e^{2\pi i0.6}\mid 0.5 \right\}$	$\left\{ 0.9e^{2\pi i0.3}\mid 0.2, 0.9e^{2\pi i0.4}\mid 0.3, 0.6e^{2\pi i0.6}\mid 0.5 \right\}$
$(\widehat{Q}, L)$	$(e_4 \times e_5)$	$(e_4 \times e_6)$	$(e_4 \times e_7)$	$(e_4 \times e_8)$
m_1	$\left\{ 0.6e^{2\pi i0.5}\mid 0.1, 0.7e^{2\pi i0.5}\mid 0.1, 0.3e^{2\pi i0.3}\mid 0.8 \right\}$	$\left\{ 0.6e^{2\pi i0.7}\mid 0.03, 0.7e^{2\pi i0.6}\mid 0.03, 0.3e^{2\pi i0.3}\mid 0.32 \right\}$	$\left\{ 0.6e^{2\pi i0.9}\mid 0.03, 0.7e^{2\pi i0.5}\mid 0.07, 0.3e^{2\pi i0.3}\mid 0.8 \right\}$	$\left\{ 0.6e^{2\pi i0.7}\mid 0.02, 0.7e^{2\pi i0.8}\mid 0.08, 0.3e^{2\pi i0.3}\mid 0.8 \right\}$
m_2	$\left\{ 0.4e^{2\pi i0.3}\mid 0.2, 0.9e^{2\pi i0.8}\mid 0.2, 0.1e^{2\pi i0.7}\mid 0.6 \right\}$	$\left\{ 0.1e^{2\pi i0.3}\mid 1 \right\}$	$\left\{ 0.8e^{2\pi i0.4}\mid 0.3, 0.9e^{2\pi i0.2}\mid 0.7 \right\}$	$\left\{ 0.7e^{2\pi i0.9}\mid 0.6, 0.9e^{2\pi i0.7}\mid 0.4 \right\}$
m_3	$\left\{ 0.7e^{2\pi i0.9}\mid 0.6, 0.3e^{2\pi i0.5}\mid 0.4 \right\}$	$\left\{ 0.2e^{2\pi i0.9}\mid 0.3, 0.3e^{2\pi i0.5}\mid 0.2 \right\}$	$\left\{ 0.6e^{2\pi i0.9}\mid 0.12, 0.6e^{2\pi i0.7}\mid 0.16, 0.8e^{2\pi i0.9}\mid 0.4 \right\}$	$\left\{ 0.2e^{2\pi i0.9}\mid 0.6, 0.3e^{2\pi i0.5}\mid 0.4 \right\}$
m_4	$\left\{ 0.6e^{2\pi i0.9}\mid 0.12, 0.8e^{2\pi i0.7}\mid 0.32 \right\}$	$\left\{ 0.6e^{2\pi i0.9}\mid 0.06, 0.4e^{2\pi i0.7}\mid 0.36 \right\}$	$\left\{ 0.6e^{2\pi i0.9}\mid 0.6, 0.4e^{2\pi i0.7}\mid 0.4 \right\}$	$\left\{ 0.6e^{2\pi i0.9}\mid 0.18, 0.5e^{2\pi i0.7}\mid 0.12, 0.8e^{2\pi i0.1}\mid 0.4 \right\}$
m_5	$\left\{ 0.3e^{2\pi i0.8}\mid 0.2, 0.5e^{2\pi i0.3}\mid 0.8 \right\}$	$\left\{ 0.2e^{2\pi i0.8}\mid 0.2, 0.3e^{2\pi i0.4}\mid 0.3, 0.6e^{2\pi i0.3}\mid 0.5 \right\}$	$\left\{ 0.8e^{2\pi i0.8}\mid 0.3, 0.8e^{2\pi i0.6}\mid 0.7 \right\}$	$\left\{ 0.9e^{2\pi i0.8}\mid 1 \right\}$

Computation of the choice value $\complement(l_i, l_j)(m_i)$*;* $\forall\ m_i \in \mathcal{M}, (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2)$ *by using Step 4 of the Algorithm 2, represented in Table 22.*

Table 22. Tabular representation of choice value.

C	$(e_1 \times e_1)$	$(e_1 \times e_2)$	$(e_1 \times e_3)$	$(e_1 \times e_4)$
m_1	$0.60769 \mid 0.21$	$0.6 \mid 0.00756$	$0.77272 \mid 0.0441$	$0.75384 \mid 0.0336$
m_2	$0.60526 \mid 0.00384$	$0.575 \mid 0.16$	$0.6 \mid 0.0336$	$0.85 \mid 0.0384$
m_3	$0.9 \mid 1$	$0.4 \mid 0.25$	$0.69 \mid 0.032$	$0.6 \mid 1$
m_4	$0.44705 \mid 0.00224$	$0.515 \mid 0.00126$	$0.48947 \mid 0.014$	$0.62 \mid 0.000504$
m_5	$0.3 \mid 0.16$	$0.325 \mid 0.03$	$0.45 \mid 0.21$	$0.3 \mid 1$
C	$(e_2 \times e_1)$	$(e_2 \times e_2)$	$(e_2 \times e_3)$	$(e_2 \times e_4)$
m_1	$0.38 \mid 0.24$	$0.58 \mid 0.00864$	$0.41538 \mid 0.0504$	$0.8 \mid 0.0384$
m_2	$0.79333 \mid 0.024$	$0.8 \mid 1$	$0.48235 \mid 0.21$	$0.7875 \mid 0.24$
m_3	$0.46363 \mid 0.032$	$0.44285 \mid 0.008$	$0.82727 \mid 0.001024$	$0.44285 \mid 0.032$
m_4	$0.33333 \mid 0.16$	$0.43333 \mid 0.09$	$0.7 \mid 1$	$0.35294 \mid 0.036$
m_5	$0.53529 \mid 0.0144$	$0.47391 \mid 0.0027$	$0.53529 \mid 0.0189$	$0.53529 \mid 0.09$
C	$(e_3 \times e_1)$	$(e_3 \times e_2)$	$(e_3 \times e_3)$	$(e_3 \times e_4)$
m_1	$0.5 \mid 1$	$0.48333 \mid 0.036$	$0.45454 \mid 0.21$	$0.75384 \mid 0.16$
m_2	$0.65 \mid 0.00216$	$0.26666 \mid 0.09$	$0.34705 \mid 0.0189$	$0.7875 \mid 0.0216$
m_3	$0.6 \mid 0.16$	$0.3875 \mid 0.04$	$0.66190 \mid 0.00512$	$0.36666 \mid 0.16$
m_4	$0.49333 \mid 0.0336$	$0.47 \mid 0.0189$	$0.63333 \mid 0.21$	$0.42 \mid 0.00756$
m_5	$0.435 \mid 0.0048$	$0.41 \mid 0.0009$	$0.49565 \mid 0.0063$	$0.4125 \mid 0.03$
C	$(e_4 \times e_1)$	$(e_4 \times e_2)$	$(e_4 \times e_3)$	$(e_4 \times e_4)$
m_1	$0.4625 \mid 0.008$	$0.58125 \mid 0.000288$	$0.6125 \mid 0.00168$	$0.66875 \mid 0.00128$
m_2	$0.65 \mid 0.024$	$0.3 \mid 1$	$0.29411 \mid 0.21$	$0.7875 \mid 0.24$
m_3	$0.78 \mid 0.24$	$0.66 \mid 0.06$	$0.84 \mid 0.00768$	$0.66 \mid 0.24$
m_4	$0.78571 \mid 0.0384$	$0.82 \mid 0.0216$	$0.82 \mid 0.24$	$0.51052 \mid 0.00864$
m_5	$0.4875 \mid 0.16$	$0.41818 \mid 0.03$	$0.7 \mid 0.21$	$0.8 \mid 1$

Computation of the the score values $\mathbb{S}(l_i, l_j)(m_i)$ by using Step 5 of Algorithm 2, represented in Table 23.

Table 23. Tabular representation of score value.

$\mathbb{S}(l_i, l_j)(m_i)$	The Score	$\mathbb{S}(l_i, l_j)(m_i)$	The Score
$\mathbb{S}(e_1 \times e_1)(m_3)$	0.9	$\mathbb{S}(e_3 \times e_1)(m_2)$	0.00140
$\mathbb{S}(e_1 \times e_2)(m_1)$	0.00453	$\mathbb{S}(e_3 \times e_2)(m_1)$	0.0174
$\mathbb{S}(e_1 \times e_3)(m_1)$	0.03407	$\mathbb{S}(e_3 \times e_3)(m_3)$	0.00338
$\mathbb{S}(e_1 \times e_4)(m_2)$	0.03264	$\mathbb{S}(e_3 \times e_4)(m_2)$	0.01701
$\mathbb{S}(e_2 \times e_1)(m_2)$	0.01904	$\mathbb{S}(e_4 \times e_1)(m_4)$	0.03017
$\mathbb{S}(e_2 \times e_2)(m_2)$	0.8	$\mathbb{S}(e_4 \times e_2)(m_4)$	0.01771
$\mathbb{S}(e_2 \times e_3)(m_3)$	0.00084	$\mathbb{S}(e_4 \times e_3)(m_3)$	0.00645
$\mathbb{S}(e_2 \times e_4)(m_1)$	0.03072	$\mathbb{S}(e_4 \times e_4)(m_5)$	0.8

Evaluation of weighted values for each $\mathbb{S}(l_i, l_j)(m_i)$ by using Step 6 of Algorithm 2 is

$$\mathbb{S}(m_1) = 0.08673,\ \mathbb{S}(m_2) = 0.87009,\ \mathbb{S}(m_3) = 0.91068,\ \mathbb{S}(m_4) = 0.04788,\ \mathbb{S}(m_5) = 0.8.$$

By using Step 7 of Algorithm 2 the best circuit simulator software is m_3. So, the manager will buy the m_3 software.

The graphical representation of ranking is given in Figure 5.

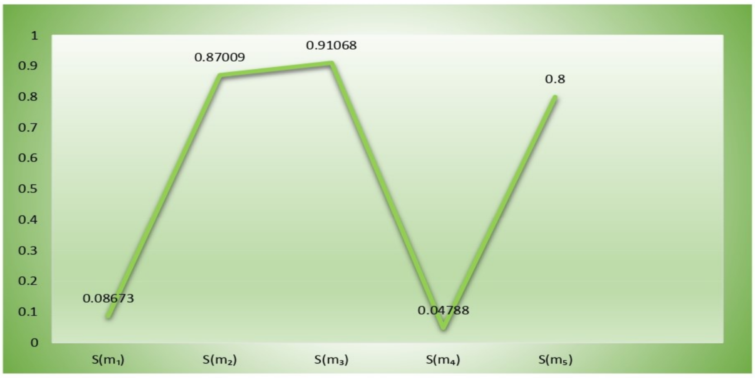

Figure 5. Ranking results.

Case study 3. *Consider the Case Study 2. Manager wants to buy the circuit simulator software which consists of the lowest quality of parameters. Then, the right decision for selecting the software is done by using the min-OR operator. Evaluation given by the experts is represented in Tables 19 and 20.*

First of all evaluate the soft min-OR operation $(\widehat{R}, L) = (\widehat{K_1}, \mathcal{E}_1) \vee (\widehat{K_2}, \mathcal{E}_2)$ by using Step 3 of Algorithm 3, represented in Table 24.

Table 24. Evaluation by using soft min-OR operation.

$(\widehat{R}, L)$	$(e_1 \times e_5)$	$(e_1 \times e_6)$	$(e_1 \times e_7)$	$(e_1 \times e_8)$
m_1	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.7, \\ 0.7e^{2\pi i0.7}\|0.3 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.21, \\ 0.1e^{2\pi i0.6}\|0.09, \\ 0.3e^{2\pi i0.3}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.21 \\ 0.7e^{2\pi i0.2}\|0.21 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.14, \\ 0.7e^{2\pi i0.7}\|0.24 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.3}\|0.16, \\ 0.7e^{2\pi i0.8}\|0.04, \\ 0.1e^{2\pi i0.7}\|0.6 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.1}\|0.8, \\ 0.7e^{2\pi i0.8}\|0.2 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.4}\|0.24, \\ 0.7e^{2\pi i0.2}\|0.14 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.7e^{2\pi i0.4}\|0.48, \\ 0.7e^{2\pi i0.7}\|0.08 \end{array} \right\}$
m_3	$\left\{ 0.4e^{2\pi i0.4}\|1 \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.3}\|0.5, \\ 0.2e^{2\pi i0.4}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.3}\|0.2, \\ 0.6e^{2\pi i0.7}\|0.4, \\ 0.8e^{2\pi i0.9}\|0.4 \end{array} \right\}$	$\left\{ 0.2e^{2\pi i0.4}\|1 \right\}$
m_4	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.6}\|0.02, \\ 0.8e^{2\pi i0.2}\|0.16, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.5}\|0.01, \\ 0.3e^{2\pi i0.2}\|0.18, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.6}\|0.1, \\ 0.9e^{2\pi i0.2}\|0.2, \\ 0.5e^{2\pi i0.8}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.6}\|0.03, \\ 0.5e^{2\pi i0.2}\|0.06, \\ 0.5e^{2\pi i0.1}\|0.28 \end{array} \right\}$
m_5	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.2, \\ 0.5e^{2\pi i0.3}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.2}\|0.2, \\ 0.3e^{2\pi i0.4}\|0.3, \\ 0.6e^{2\pi i0.3}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.3, \\ 0.8e^{2\pi i0.6}\|0.7 \end{array} \right\}$	$\left\{ 0.3e^{2\pi i0.3}\|1 \right\}$
$(\widehat{R}, L)$	$(e_2 \times e_5)$	$(e_2 \times e_6)$	$(e_2 \times e_7)$	$(e_2 \times e_8)$
m_1	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.5}\|0.6, \\ 0.9e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.7}\|0.18, \\ 0.1e^{2\pi i0.1}\|0.12 \\ 0.3e^{2\pi i0.3}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.8}\|0.18, \\ 0.7e^{2\pi i0.1}\|0.28 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.7}\|0.12, \\ 0.7e^{2\pi i0.1}\|0.32 \end{array} \right\}$
m_2	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.3}\|0.2, \\ 0.9e^{2\pi i0.8}\|0.2, \\ 0.1e^{2\pi i0.7}\|0.6 \end{array} \right\}$	$\left\{ 0.1e^{2\pi i0.1}\|1 \right\}$	$\left\{ \begin{array}{l} 0.5e^{2\pi i0.4}\|0.3, \\ 0.9e^{2\pi i0.2}\|0.7 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.5e^{2\pi i0.8}\|0.6, \\ 0.9e^{2\pi i0.7}\|0.4 \end{array} \right\}$
m_3	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.9}\|0.2, \\ 0.8e^{2\pi i0.4}\|0.4, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.3}\|0.1, \\ 0.2e^{2\pi i0.4}\|0.2, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.6e^{2\pi i0.3}\|0.04, \\ 0.6e^{2\pi i0.4}\|0.16 \\ 0.7e^{2\pi i0.1}\|0.16 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.6}\|0.2, \\ 0.8e^{2\pi i0.4}\|0.4, \\ 0.7e^{2\pi i0.1}\|0.4 \end{array} \right\}$
m_4	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.2, \\ 0.8e^{2\pi i0.2}\|0.8 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.4e^{2\pi i0.1}\|0.1, \\ 0.3e^{2\pi i0.3}\|0.9 \end{array} \right\}$	$\left\{ 0.4e^{2\pi i0.1}\|1 \right\}$	$\left\{ \begin{array}{l} 0.2e^{2\pi i0.1}\|0.3, \\ 0.5e^{2\pi i0.4}\|0.3, \\ 0.8e^{2\pi i0.1}\|0.4 \end{array} \right\}$
m_5	$\left\{ \begin{array}{l} 0.3e^{2\pi i0.3}\|0.18, \\ 0.5e^{2\pi i0.3}\|0.08 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.1e^{2\pi i0.2}\|0.18, \\ 0.3e^{2\pi i0.4}\|0.03, \\ 0.6e^{2\pi i0.3}\|0.5 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.8e^{2\pi i0.3}\|0.27, \\ 0.8e^{2\pi i0.6}\|0.07 \end{array} \right\}$	$\left\{ \begin{array}{l} 0.9e^{2\pi i0.3}\|0.9, \\ 0.8e^{2\pi i0.8}\|0.1 \end{array} \right\}$

Table 24. Evaluation by using soft min-OR operation (Continued).

$(\widehat{R}, L)$	$(e_3 \times e_5)$	$(e_3 \times e_6)$	$(e_3 \times e_7)$	$(e_3 \times e_8)$
m_1	$\{0.1e^{2\pi i0.4}\|1\}$	$\{0.1e^{2\pi i0.4}\|0.3, 0.1e^{2\pi i0.6}\|0.3, 0.3e^{2\pi i0.3}\|0.4\}$	$\{0.1e^{2\pi i0.4}\|0.3, 0.7e^{2\pi i0.2}\|0.7\}$	$\{0.1e^{2\pi i0.4}\|0.2, 0.7e^{2\pi i0.8}\|0.8\}$
m_2	$\{0.1e^{2\pi i0.2}\|0.18, 0.2e^{2\pi i0.3}\|0.02, 0.1e^{2\pi i0.7}\|0.6\}$	$\{0.1e^{2\pi i0.1}\|0.9, 0.2e^{2\pi i0.3}\|0.1\}$	$\{0.1e^{2\pi i0.2}\|0.27, 0.1e^{2\pi i0.2}\|0.07\}$	$\{0.1e^{2\pi i0.2}\|0.54, 0.2e^{2\pi i0.3}\|0.04\}$
m_3	$\{0.1e^{2\pi i0.1}\|0.8, 0.7e^{2\pi i0.3}\|0.2\}$	$\{0.1e^{2\pi i0.1}\|0.4, 0.2e^{2\pi i0.3}\|0.1\}$	$\{0.1e^{2\pi i0.1}\|0.16, 0.6e^{2\pi i0.3}\|0.08, 0.8e^{2\pi i0.9}\|0.4\}$	$\{0.1e^{2\pi i0.1}\|0.8, 0.7e^{2\pi i0.3}\|0.2\}$
m_4	$\{0.2e^{2\pi i0.2}\|0.14, 0.2e^{2\pi i0.2}\|0.24\}$	$\{0.6e^{2\pi i0.2}\|0.07, 0.2e^{2\pi i0.3}\|0.27\}$	$\{0.5e^{2\pi i0.2}\|0.7, 0.2e^{2\pi i0.4}\|0.3\}$	$\{0.2e^{2\pi i0.2}\|0.21, 0.2e^{2\pi i0.4}\|0.09, 0.8e^{2\pi i0.1}\|0.4\}$
m_5	$\{0.3e^{2\pi i0.1}\|0.04, 0.5e^{2\pi i0.3}\|0.24, 0.6e^{2\pi i0.6}\|0.5\}$	$\{0.1e^{2\pi i0.1}\|0.04, 0.3e^{2\pi i0.4}\|0.09, 0.6e^{2\pi i0.3}\|0.25\}$	$\{0.5e^{2\pi i0.1}\|0.06, 0.8e^{2\pi i0.4}\|0.21, 0.6e^{2\pi i0.6}\|0.5\}$	$\{0.5e^{2\pi i0.1}\|0.2, 0.9e^{2\pi i0.4}\|0.3, 0.6e^{2\pi i0.6}\|0.5\}$
$(\widehat{R}, L)$	$(e_4 \times e_5)$	$(e_4 \times e_6)$	$(e_4 \times e_7)$	$(e_4 \times e_8)$
m_1	$\{0.6e^{2\pi i0.1}\|0.1, 0.7e^{2\pi i0.5}\|0.1, 0.3e^{2\pi i0.3}\|0.8\}$	$\{0.2e^{2\pi i0.1}\|0.03, 0.1e^{2\pi i0.5}\|0.03, 0.3e^{2\pi i0.3}\|0.32\}$	$\{0.4e^{2\pi i0.1}\|0.03, 0.7e^{2\pi i0.2}\|0.07, 0.3e^{2\pi i0.3}\|0.8\}$	$\{0.6e^{2\pi i0.1}\|0.02, 0.7e^{2\pi i0.5}\|0.08, 0.3e^{2\pi i0.3}\|0.8\}$
m_2	$\{0.1e^{2\pi i0.3}\|0.2, 0.9e^{2\pi i0.8}\|0.2, 0.1e^{2\pi i0.7}\|0.6\}$	$\{0.1e^{2\pi i0.1}\|1\}$	$\{0.1e^{2\pi i0.3}\|0.3, 0.9e^{2\pi i0.2}\|0.7\}$	$\{0.1e^{2\pi i0.3}\|0.6, 0.9e^{2\pi i0.7}\|0.4\}$
m_3	$\{0.2e^{2\pi i0.9}\|0.6, 0.3e^{2\pi i0.5}\|0.4\}$	$\{0.1e^{2\pi i0.3}\|0.3, 0.2e^{2\pi i0.4}\|0.2\}$	$\{0.2e^{2\pi i0.3}\|0.12, 0.3e^{2\pi i0.5}\|0.16, 0.8e^{2\pi i0.9}\|0.4\}$	$\{0.2e^{2\pi i0.6}\|0.6, 0.3e^{2\pi i0.5}\|0.4\}$
m_4	$\{0.2e^{2\pi i0.6}\|0.12, 0.4e^{2\pi i0.2}\|0.32\}$	$\{0.6e^{2\pi i0.5}\|0.06, 0.3e^{2\pi i0.3}\|0.36\}$	$\{0.5e^{2\pi i0.7}\|0.6, 0.4e^{2\pi i0.7}\|0.4\}$	$\{0.2e^{2\pi i0.8}\|0.18, 0.4e^{2\pi i0.4}\|0.12, 0.8e^{2\pi i0.1}\|0.4\}$
m_5	$\{0.2e^{2\pi i0.3}\|0.2, 0.5e^{2\pi i0.3}\|0.8\}$	$\{0.1e^{2\pi i0.2}\|0.2, 0.3e^{2\pi i0.4}\|0.3, 0.6e^{2\pi i0.3}\|0.5\}$	$\{0.2e^{2\pi i0.3}\|0.3, 0.8e^{2\pi i0.6}\|0.7\}$	$\{0.2e^{2\pi i0.3}\|1\}$

Computation of the choice value $\complement(l_i, l_j)(m_i)$; $\forall\ m_i \in \mathcal{M}, (l_i, l_j) \in (\mathcal{E}_1 \times \mathcal{E}_2)$ by using Step 4 of Algorithm 3, represented in Table 25.

Table 25. Tabular representation of choice value.

$\complement$	$(e_1 \times e_1)$	$(e_1 \times e_2)$	$(e_1 \times e_3)$	$(e_1 \times e_4)$
m_1	0.56666|0.21	0.28333|0.00756	0.17777|0.0441	0.56666|0.0336
m_2	0.625|0.00384	0.7125|0.16	0.30666|0.0336	0.55|0.0384
m_3	0.4|1	0.36666|0.25	0.7|0.032	0.4|1
m_4	0.45333|0.00224	0.55454|0.00126	0.44705|0.014	0.225|0.000504
m_5	0.3|0.16	0.32|0.03	0.51818|0.21	0.3|1
$\complement$	$(e_2 \times e_1)$	$(e_2 \times e_2)$	$(e_2 \times e_3)$	$(e_2 \times e_4)$
m_1	0.2|0.24	0.4|0.00864	0.31|0.0504	0.28|0.0384
m_2	0.65|0.024	0.1|1	0.27142|0.21	0.73571|0.24
m_3	0.44285|0.032	0.18|0.008	0.25789|0.001024	0.3|0.032
m_4	0.18|0.16	0.18571|0.09	0.1|1	0.2|0.036
m_5	0.3|0.0144	0.32|0.0027	0.45|0.0189	0.53529|0.09
$\complement$	$(e_3 \times e_1)$	$(e_3 \times e_2)$	$(e_3 \times e_3)$	$(e_3 \times e_4)$
m_1	0.4|1	0.38|0.036	0.225|0.21	0.75|0.16
m_2	0.375|0.00216	0.23333|0.09	0.2|0.0189	0.26666|0.0216
m_3	0.275|0.16	0.23333|0.04	0.60666|0.00512	0.275|0.16
m_4	0.2|0.0336	0.225|0.0189	0.25714|0.21	0.16666|0.00756
m_5	0.38571|0.0048	0.31|0.0009	0.38421|0.0063	0.385|0.03
$\complement$	$(e_4 \times e_1)$	$(e_4 \times e_2)$	$(e_4 \times e_3)$	$(e_4 \times e_4)$
m_1	0.3125|0.008	0.26666|0.000288	0.19285|0.00168	0.3125|0.00128
m_2	0.74545|0.024	0.1|1	0.21|0.21	0.66|0.24
m_3	0.66|0.24	0.36666|0.06	0.71538|0.00768	0.54|0.24
m_4	0.33333|0.0384	0.43333|0.0216	0.7|0.24	0.28571|0.00864
m_5	0.3|0.16	0.32|0.03	0.54|0.21	0.3|1

Computation of the the score values $\mathbb{S}(l_i, l_j)(m_i)$ by using Step 5 of Algorithm 3, represented in Table 26.

Table 26. Tabular representation of score value.

$\mathbb{S}(l_i, l_j)(m_i)$	The Score	$\mathbb{S}(l_i, l_j)(m_i)$	The Score
$\mathbb{S}(e_1 \times e_1)(m_2)$	0.0024	$\mathbb{S}(e_3 \times e_1)(m_1)$	0.4
$\mathbb{S}(e_1 \times e_2)(m_2)$	0.114	$\mathbb{S}(e_3 \times e_2)(m_1)$	0.01368
$\mathbb{S}(e_1 \times e_3)(m_3)$	0.0224	$\mathbb{S}(e_3 \times e_3)(m_3)$	0.00310
$\mathbb{S}(e_1 \times e_4)(m_1)$	0.01904	$\mathbb{S}(e_3 \times e_4)(m_1)$	0.12
$\mathbb{S}(e_2 \times e_1)(m_2)$	0.0156	$\mathbb{S}(e_4 \times e_1)(m_2)$	0.01789
$\mathbb{S}(e_2 \times e_2)(m_1)$	0.00345	$\mathbb{S}(e_4 \times e_2)(m_4)$	0.00936
$\mathbb{S}(e_2 \times e_3)(m_5)$	0.00850	$\mathbb{S}(e_4 \times e_3)(m_3)$	0.00549
$\mathbb{S}(e_2 \times e_4)(m_2)$	0.17657	$\mathbb{S}(e_4 \times e_4)(m_2)$	0.1584

Evaluation of weighted values for each $\mathbb{S}(l_i, l_j)(m_i)$ by using Step 6 of Algorithm 3 is

$$\mathbb{S}(m_1) = 0.55617,\ \mathbb{S}(m_2) = 0.48486,\ \mathbb{S}(m_3) = 0.031,\ \mathbb{S}(m_4) = 0.00936,\ \mathbb{S}(m_5) = 0.00850,$$

optimal decision

$$¥ = max\Big\{\mathbb{S}(m_1), \mathbb{S}(m_2), \mathbb{S}(m_3), \mathbb{S}(m_4), \mathbb{S}(m_5)\Big\}.$$

By using Step 7 of Algorithm 3 the best circuit simulator software is m_1. *So, the manager will buy the* m_1 *software.*

The graphical representation of ranking is given in Figure 6.

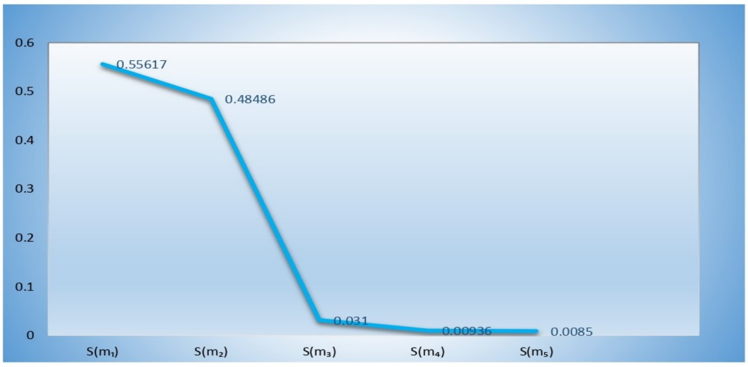

Figure 6. Ranking results.

5.1. Comparative analysis

We devised an approach by combining the theory of constraints with multiattribute and multiobjective decision-making techniques in order to find a solution to a challenging problem which still exists in the real-world. In the context of the CPHFSS, we examined three different algorithms. All three approaches provide a ranked list of all viable options regardless of whether there is a single best answer. However, none of the existing work is up to the task of handling Soft max-AND or min-OR kind of information supplied by CPHFSS to a decision-maker. While the suggested method is capable of handling data from pre-existing methods as well, it has the potential to significantly improve upon them. Several structures supplied by various researchers were compared to show how well the proposed method performed in contrast to the existing methods for multi-parameter group decision-making. The comparison is given in Table 27 and concludes that our proposed study is superior and more reliable than the those currently in use.

Table 27. The characteristic comparison with existing approaches.

Methods	Capable of making decisions using probability	Capable of handling two dimensional information	Flexible to adapt decision-makers' choices	Capable of integrate information	capable of handling multi-parameter information
Zadeh [4]	no	no	no	no	no
Roy and Maji [16]	no	no	no	no	yes
Torra [38]	no	no	yes	no	no
Babitha and John [51]	no	no	yes	no	yes
Garg et al. [52]	no	yes	yes	no	no
Proposed approach	yes	yes	yes	yes	yes

6. Conclusions

Many struggles have been made in the literature about the enhancement of data measures. However, there is nonetheless a lot of room to enhance these measures in a higher way and use them to locate new functions and novel directions. As computational intelligence has grown to be more popular these days due to big data, superior algorithms, and elevated computing electricity and storage, these structures are becoming an embedded factor of digital systems and, more specifically, are having a profound impact on human selection making. As a result, there is a growing demand for information systems researchers to look at and recognize the implications of computational intelligence for selection making and to make a contribution to theoretical development, especially in engineering applications. In this paper, we developed the complex probabilistic hesitant fuzzy set and its operations such as restricted union, restricted intersection, extended union, extended intersection, complement, soft max-AND operator and soft min-OR operator by integrating randomness, hesitancy, and uncertainty with parameterized families. Several examples are introduced to exhibit the suitability and validity of the proposed methods. Based on computational results, it is viewed that the proposed techniques are realistic and well-suited to the surroundings of CPHFSS. We prove their fundamental laws, such as commutativity, associativity, idempotency, involution, and de-Morgan laws. We also proposed three different algorithms and case studies based on bridges and simulation for decision-making to exhibit the validity and effectiveness of the new approach. Furthermore, the limitation of this structure is that we can not take the membership values beyond the complex plane unit circle. In further research, the average or Einstein aggregation operators of complex probabilistic hesitant fuzzy soft set is an important and interesting issue to be addressed, and TODIM, EDAS, TAOV and AHP methodologies can also be investigated. The proposed metrics will be applied to various applications like risk analysis, investment strategy, feature extractions, bipartite consensus control models, and unmanned aerial vehicles [53–55].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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