Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 17702–17718.
DOI: 10.3934/math.2023904
Received: 16 January 2023
Revised: 06 April 2023
Accepted: 02 May 2023
Published: 24 May 2023

Research article

Some sufficient conditions for a tree to have its weak Roman domination number be equal to its domination number plus 1

Jian Yang[1,*], Yuefen Chen[2] and Zhiqiang Li[3]

[1] Department of Public Basic Teaching, Henan College of Transportation, Zhengzhou, China
[2] Department of Public Basic Courses, Nanjing Vocational University of Industry Technology, Nanjing, China
[3] School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou, China

* **Correspondence:** Email: yangjian4088@163.com; Tel: +8613526525840.

Abstract: Let $G = (V, E)$ be a simple graph with vertex set V and edge set E, and let f be a function $f : V \mapsto \{0, 1, 2\}$. A vertex u with $f(u) = 0$ is said to be undefended with respect to f if it is not adjacent to a vertex with positive weight. The function f is a weak Roman dominating function (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f_u : V \mapsto \{0, 1, 2\}$, defined by $f_u(u) = 1$, $f_u(v) = f(v) - 1$ and $f_u(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight of f is $w(f) = \sum_{v \in V} f(v)$. The weak Roman domination number, denoted $\gamma_r(G)$, is the minimum weight of a WRDF in G. The domination number, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. In this paper, we give some sufficient conditions for a tree to have its weak Roman domination number be equal to its domination number plus 1 ($\gamma_r(T) = \gamma(T) + 1$) by recursion and construction.

Keywords: weak Roman domination number; domination number; tree; star
Mathematics Subject Classification: 05C50

1. Introduction

For a graph $G = (V, E)$, let $f : V \mapsto \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{ v \in V \mid f(v) = i \}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a $1 - 1$ correspondence between the functions $f : V \mapsto \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$.

Motivated by an article in *Scientific American* by I. Stewart [1] entitled "Defend the Roman Empire!", E. J. Cockayne et al. [2] defined a Roman dominating function (RDF) on a graph

$G = (V, E)$ to be a function $f : V \mapsto \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real-valued function $f : V \mapsto R$, the weight of f is $w(f) = \sum_{v \in V} f(v) = |V_1| + 2|V_2|$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. The Roman domination number, denoted $\gamma_R(G)$, is the minimum weight of a RDF in G; that is, $\gamma_R(G) = \min\{w(f) \mid f \text{ is a RDF in } G\}$. For example, the Roman domination number of the 2×5 grid graph $G_{2,5}$ is 6 (Figure 1(a)). A RDF of weight $\gamma_R(G)$ we call a $\gamma_R(G)$-function. Roman domination in graphs has been studied, for example, in [2,3].

(a) Roman domination in $G_{2,5}$ (b) Weak Roman domination in $G_{2,5}$

Figure 1. The constructions for grid graph $G_{2,5}$. Filled-in circles denote vertices in V_2, and empty circles denote vertices in V_1.

This definition of a Roman dominating function is motivated as follows. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered unsecured if no legions are stationed there (i.e., $f(v) = 0$) and secured otherwise (i.e., if $f(v) \in \{1, 2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from an adjacent location (an adjacent vertex u). However, Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured (i.e., if $f(v) = 1$). Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations.

M. A. Henning and S. T. Hedetniemi [4] explored a new strategy of defending the Roman Empire that has the potential of saving the Emperor Constantine the Great substantial costs of maintaining legions, while still defending the Roman Empire (from a single attack) . Let $G = (V, E)$ be a graph and f be a function $f = (V_0, V_1, V_2)$. A vertex $u \in V_0$ is undefended with respect to f, or simply undefended if the function f is clear from the context, if it is not adjacent to a vertex in V_1 or V_2. The function f is a weak Roman dominating function (WRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f_u : V \mapsto \{0, 1, 2\}$, defined by $f_u(u) = 1$, $f_u(v) = f(v) - 1$ and $f_u(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weak Roman domination number, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G, that is, $\gamma_r(G) = \min\{w(f) \mid f \text{ is a WRDF in } G\}$. For example, the weak Roman domination number of the 2×5 grid graph $G_{2,5}$ is 4 (Figure 1(b)). A WRDF of weight $\gamma_r(G)$ we call a $\gamma_r(G)$-function. Weak Roman domination in graphs has been studied, for example, in [4–7].

This definition of a WRDF is motivated as follows. Using notation introduced earlier, we define a location to be undefended if the location and every location adjacent to it are unsecured (i.e., have no legion stationed there). Since an undefended location is vulnerable to an attack, we require that every unsecure location be adjacent to a secure location in such a way that the movement of a legion from the secure location to the unsecure location does not create an undefended location. Hence, every unsecure location can be defended without creating an undefended location. In this way Emperor Constantine the Great can still defend the Roman Empire. Such a placement of legions corresponds to a WRDF, and a minimum such placement of legions corresponds to a minimum WRDF.

Roman domination is a typical control problem with rich historical background and mathematical background [2,4,8–15]. E. J. Cockayne et al. [2] studied the Roman domination of graphs, including determining the Roman domination numbers of the paths, cycles, complete n-partite graphs and $2 \times n$ grid graphs; giving some upper and lower bounds of the Roman domination number; and characterizing the graphs with $\gamma_R(G) = \gamma(G)$, $\gamma_R(G) = \gamma(G) + 1$, $\gamma_R(G) = \gamma(G) + 2$ and the trees with $\gamma_R(T) = \gamma(T) + 1$, $\gamma_R(T) = \gamma(T) + 2$, etc. M. A. Henning and S. T. Hedetniemi [4] studied the weak Roman domination of graphs, including determining the weak Roman domination numbers of paths and cycles and characterizing the graphs with $\gamma_r(G) = \gamma(G)$, etc. Authors of this paper have also studied the weak Roman domination of graphs, including characterizing the trees with $\gamma_r(T) = \gamma(T)$ in [6] and determining that the path P_3, stars $K_{1,t}(t \geq 2)$ and trees T which consist of the center vertices of stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 3, i = 1, 2, \cdots, n)$ to form a path are weak Roman graphs in [7]. In this paper, we give some sufficient conditions for a tree to have its weak Roman domination number be equal to its domination number plus 1 ($\gamma_r(T) = \gamma(T) + 1$) by recursion and construction. The graphs considered in this paper are finite non-trivial simple graphs.

2. Notation and known conclusions

Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The degree of the vertex v is denoted as $d(v)$. A graph is called complete if every pair of different vertices is connected by exactly one edge, denoted by K_n. (where, K_1 represents a complete graph with vertex set V of order 1 and K_2 represents a complete graph with vertex set V of order 2.) A nonempty sequence of alternating vertices and edges $W = v_0e_1v_1e_2v_2 \cdots e_kv_k$ (where $v_0, v_1, \cdots, v_k$ are different) in a graph G is called a path, denoted by P_n. The open neighborhood of v is $N(v) = \{ u \in V | uv \in E \}$, and the closed neighborhood of v is $N[v] = \{ v \} \cup N(v)$. For a set $S \subseteq V$, its open neighborhood $N(S) = \cup_{v \in S} N(v) - S$, and its closed neighborhood $N[S] = N(S) \cup S$.

Let $G = (V, E)$ be a graph, and let $S \subseteq V$. A set S dominates a set U, denoted $S \succ U$, if every vertex in U is adjacent to a vertex of S. If $S \succ V - S$, then S is called a dominating set of G [16]. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G, namely, $\gamma(G) = \min\{ |S| |S \succ V - S \}$. A dominating set of cardinality $\gamma(G)$ we call a $\gamma(G)$-set.

Let $T = (V, E)$ be a tree with vertex set V of order n and edge set E. A leaf of T is a vertex of degree 1, while a support vertex of T is a vertex adjacent to a leaf. A strong support vertex is adjacent to at least two leaves. In this paper, we denote the set of all support vertices of T by $S(T)$, the set of all strong support vertices by $SS(T)$ and the set of leaves by $L(T)$. If $S \subseteq V$, and for all vertices in S, as long as there is an edge in the tree T, this edge appears in the subtree $T[S]$, then $T[S]$ is said to be the subtree of tree T induced by S, denoted as $T[S]$. For a positive integer t, the complete bipartite graph $K_{1,t}$ is called a star, the vertex whose degree is t ($t \geq 2$) is the center vertex, and a vertex whose degree is 1 is an outer vertex.

Previously known results on domination number and weak Roman domination number are the following.

Lemma 1. *[4]* For any graph G, $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Lemma 2. *[4]* For any $n \geq 1$, $\gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$. For any $n \geq 4$, $\gamma_r(C_n) = \lceil \frac{3n}{7} \rceil$.
Lemma 3. *[6]* For any tree T with $\gamma_r(T) = \gamma(T)$, tree T does not contain any strong support vertex, that is, $\forall v \in V, v \notin SS(T)$.
Lemma 4. *[17]* For any $n \geq 1$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.
Lemma 5. *[6]* Let $T = (V, E)$ be a tree with vertex set V of order n and edge set E. Then, $\gamma_r(T) = \gamma(T)$ if and only if one of the following is true:

Case 1. m (m is a positive integer) is even, the tree T contains $\dfrac{m}{2}$ K_2 (let $u_i, v_i \in V$, $u_i v_i \in E, i = 1, 2, \cdots, \dfrac{m}{2}$), and the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $v_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_{\frac{m}{2}}\}$, and $L(T) = \{v_1, v_2, \cdots, v_{\frac{m}{2}}\}$.

(b) $\forall u \in S(T)$, except for its leaf, u is adjacent to other vertices in $S(T)$.

Case 2. The tree T does not contain any strong support vertex and contains m (m is a positive integer) K_2 (let $u_i, v_i \in V$, $u_i v_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}$ ($t_i \geq 2, i = 1, 2, \cdots, n$) (the set formed by their center vertices is C, and the set formed by their outer vertices is W), and the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $v_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{v_1, v_2, \cdots, v_m\}$.

(b) $\forall w \in W$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$.

(c) $\forall u \in S(T)$, except for its leaf, either u is adjacent to an outer vertex $w \in W$, or u is adjacent to other vertices in $S(T)$.

3. Some necessary and some sufficient conditions for trees T to have $\gamma_r(T) = \gamma(T) + 1$

3.1. Results on dominating sets and weak Roman dominating sets in trees

Theorem 1. For any tree T, let the set S be any $\gamma(T)$-set of the tree T. Then, $SS(T) \subseteq S$.
Proof. Suppose the contrary, that is, there is $v \in SS(T) \setminus S$, and let $u_1, u_2, \cdots, u_l$ ($l \geq 2$) be the leaves of v. Because $v \notin S$, $u_1, u_2, \cdots, u_l \in S$ ($l \geq 2$). Let $S_1 = S \cup \{v\} \setminus \{u_1, u_2, \cdots, u_l \ (l \geq 2)\}$. Then, obviously, S_1 is a dominating set of the tree T, and $|S_1| = |S \cup \{v\} \setminus \{u_1, u_2, \cdots, u_l \ (l \geq 2)\}| \leq |S| - 1$. This contradicts S being the $\gamma(T)$-set of the tree T. Therefore, $SS(T) \subseteq S$.
Theorem 2. For any tree T, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$.
Proof. From Theorem 1 we know $SS(T) \subseteq S$. Let S_0 be a $\gamma(T)$-set of the tree $T = (V, E)$. Then, $SS(T) \subseteq S_0$. If $S(T) \setminus SS(T) \subseteq S_0$, then the conclusion is true.

Otherwise, let $v \in S(T) \setminus S_0$, and $v \notin SS(T)$. Then, let u be the only leaf of v, that is, $uv \in E$, and $d(u) = 1$. Thus, $N(u) = \{v\}$, but $v \notin S_0$, so $u \in S_0$. Let $S_1 = S_0 \cup \{v\} \setminus \{u\}$. Then, obviously, S_1 is a dominating set of the tree T. Also, $|S_1| = |S_0| = \gamma(T)$, so S_1 is also a $\gamma(T)$-set of the tree T, and the number of support vertices in S_1 is 1 more than that in S_0.

If $S(T) \setminus SS(T) \subseteq S_1$, then the conclusion is true. Otherwise, $\exists v_1 \in S(T) \setminus S_1$, and $v_1 \notin SS(T)$. Similar to the above, find S_2 such that S_2 is also a $\gamma(T)$-set of the tree T, and the number of support vertices in S_2 is 1 more than that in S_1.

$\cdots\cdots$

Repeat the above operation. Since the tree T is finite, the number of support vertices must also be finite. Each repetition will increase the number of support vertices by 1, so a $\gamma(T)$-set S_k must exist after a finite number of repetitions such that $S(T) \setminus SS(T) \subseteq S_k$. Hence, the conclusion follows.

Theorem 3. For any tree $T = (V, E)$, there is a $\gamma_r(T)$-function $f = (V_0, V_1, V_2)$, such that $SS(T) \subseteq V_2$.

Proof. Suppose there is $v \in SS(T) \setminus V_2$, and then $v \in V_0 \cup V_1$. Because $v \in SS(T)$, v has at least two leaves, denoted $u_1, u_2, \cdots, u_l$ $(l \geq 2)$. If $v \in V_0$, since $u_1, u_2, \cdots, u_l$ $(l \geq 2)$ are only adjacent to v, $u_1, u_2, \cdots, u_l \in V_1$. Let $f_v = (V_0^*, V_1^*, V_2^*) = (V_0 \cup \{u_1, u_2, \cdots, u_l\} \setminus \{v\}, V_1 \setminus \{u_1, u_2, \cdots, u_l\}, V_2 \cup \{v\})$. Then, f_v has no undefended vertex, and $w(f_v) = |V_1 \setminus \{u_1, u_2, \cdots, u_l\}| + 2|V_2 \cup \{v\}| = |V_1| - l + 2|V_2| + 2 = |V_1| + 2|V_2| + 2 - l \leq |V_1| + 2|V_2| = w(f)$. So, f_v is also a $\gamma_r(T)$-function, and $v \in V_2^*$.

If $v \in V_1$, at most one of $u_1, u_2, \cdots, u_l$ $(l \geq 2)$ belongs to V_0 (otherwise, if $u_1, u_2 \in V_0$, send a legion from v to u_1 for security defense, such that u_2 is an undefended vertex, a contradiction), denoted by $u_1 \in V_0$. Then, $u_2, u_3, \cdots, u_l \in V_1$. Let $f_v = (V_0^*, V_1^*, V_2^*) = (V_0 \cup \{u_2, u_3, \cdots, u_l\}, V_1 \setminus \{v, u_2, u_3, \cdots, u_l\}, V_2 \cup \{v\})$. Then, f_v has no undefended vertex, and $w(f_v) = |V_1 \setminus \{v, u_2, u_3, \cdots, u_l\}| + 2|V_2 \cup \{v\}| = |V_1| - l + 2|V_2| + 2 = |V_1| + 2|V_2| + 2 - l \leq |V_1| + 2|V_2| = w(f)$. So, f_v is also a $\gamma_r(T)$-function, and $v \in V_2^*$. Hence, there is a $\gamma_r(T)$-function $f = (V_0, V_1, V_2)$, such that $SS(T) \subseteq V_2$.

3.2. A necessary condition for trees T to have $\gamma_r(T) = \gamma(T) + 1$

Theorem 4. For any tree T with $\gamma_r(T) = \gamma(T) + 1$, tree T contains at most 1 strong support vertex.

Proof. Let $T = (V, E)$ be a tree with $\gamma_r(T) = \gamma(T) + 1$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_r(T)$-function of the tree T. $V_1 \cup V_2 \succ V_0$, so $\gamma(T) \leq |V_1| + |V_2|$, and then $|V_2| \leq 1$ (otherwise, if $|V_2| \geq 2$, then $\gamma_r(T) = |V_1| + 2|V_2| = |V_1| + |V_2| + |V_2| \geq |V_1| + |V_2| + 2 \geq \gamma(T) + 2 > \gamma(T) + 1$, which contradicts the assumptions).

Suppose the contrary, that is, T has at least two strong support vertices, say, $v_1, v_2 \in SS(T)$. By Theorem 3, there is a $\gamma_r(T)$-function $f = (V_0, V_1, V_2)$ such that $SS(T) \subseteq V_2$, and then $v_1, v_2 \in V_2$. Therefore, $|V_2| \geq 2$, a contradiction. So, tree T contains at most 1 strong support vertex.

3.3. Sufficient conditions for trees T to have $\gamma_r(T) = \gamma(T) + 1$

Theorem 5. If the tree $T = (V, E)$ has one and only one strong support vertex, and there is a $\gamma(T)$-set S such that every vertex in S is a support vertex, and $G[S]$ is a tree, as shown in Figure 2, then, $\gamma_r(T) = \gamma(T) + 1$.

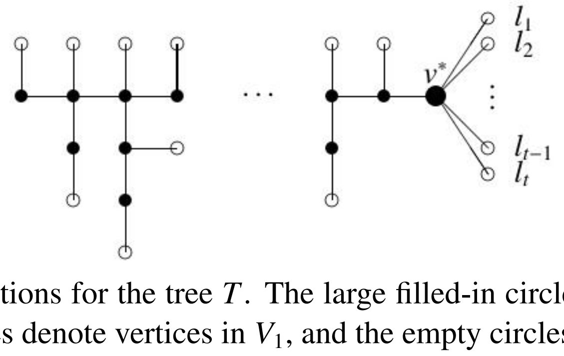

Figure 2. The constructions for the tree T. The large filled-in circles denote vertices in V_2, the small filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Proof. By Theorem 2, there is a $\gamma(T)$-set S_1 such that $S(T) \subseteq S_1$. There is also a $\gamma(T)$-set S such that every vertex in S is a support vertex, and then $S \subseteq S(T) \subseteq S_1$. Since, $|S| = |S_1|$, $S = S(T) = S_1$, and $\gamma(T) = |S(T)|$.

The tree $T = (V, E)$ has one and only one strong support vertex, say, $v^* \in SS(T)$, and set $l_1, l_2, \cdots, l_t$ $(t \geq 2)$ for its leaves. Let $f = (V_0, V_1, V_2) = (V - S(T), S(T) \setminus \{v^*\}, \{v^*\})$. Then, it has no undefended vertex, as shown in Figure 2. Since $G[S(T)]$ is a tree, $\forall u, v \in S(T)$, there is no $u - v$ path that contains a vertex in $V - S(T)$ as the inner vertex. (Otherwise, if there is such a road P_1, since $G[S(T)]$ is the tree, the $u - v$ path P_2 exists in $G[S(T)]$. Obviously, $P_1 \neq P_2$, so T must contain cycles, which is a contradiction.) Since $S(T)$ is a dominating set, the tree T has one and only one strong support vertex. Then, $\forall\ u \in V - S(T) \cup \{l_1, l_2, \cdots, l_t\}$, u can only be adjacent to the only vertex v in $S(T) \setminus \{v^*\}$, and $d(u) = 1$. (Otherwise, if $d(u) \geq 2$, let $wu \in E(T)$. However, $w \notin S(T)$, and $w \notin N[S(T)]$. This contradicts $S(T)$ being a dominating set.) So, $\forall\ u \in V - S(T) \cup \{l_1, l_2, \cdots, l_t\}$, let $f_u = (V_0^*, V_1^*, V_2^*) = (V - S(T) \cup \{u\} \setminus \{v\}, S(T) \cup \{u\} \setminus \{v, v^*\}, \{v^*\})$. Obviously, no undefended vertices will be created. $l_1, l_2, \cdots, l_t$ $(t \geq 2)$ can be v^* security defense, so f is a WRDF of the tree T. Therefore, $\gamma_r(T) \leq w(f) = |S(T) \setminus \{v^*\}| + 2|\{v^*\}| = |S(T)| - 1 + 2 = |S(T)| + 1 = \gamma(T) + 1$.

From Lemma 3 and the fact that T has a strong support vertex, $\gamma(T) \neq \gamma_r(T)$. Again, by Lemma 1, know $\gamma(T) \leq \gamma_r(T)$, and then $\gamma_r(T) > \gamma(T)$, or $\gamma_r(T) \geq \gamma(T) + 1$. Hence, $\gamma_r(T) = \gamma(T) + 1$.

Theorem 6. Suppose the tree $T = (V, E)$ has only one strong support vertex (say, $v^* \in SS(T)$, and set $l_1, l_2, \cdots, l_s$ $(s \geq 2)$ for its leaves) and contains m (m is a positive integer) K_2 (let $u_i, v_i \in V$, $u_i v_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n} (t_i \geq 2, i = 1, 2, \cdots, n)$ (the set formed by their center vertices is C, and the set formed by their outer vertices is W) and the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $v_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m, v^*\}$, and $L(T) = \{v_1, v_2, \cdots, v_m\} \cup \{l_1, l_2, \cdots, l_s\}$.

(b) $\forall w \in W$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$.

(c) $\forall u \in S(T)$, except for its leaf, either u is adjacent to an outer vertex $w \in W$, or u is adjacent to other vertices in $S(T)$.

As shown in Figure 3, then, $\gamma_r(T) = \gamma(T) + 1$.

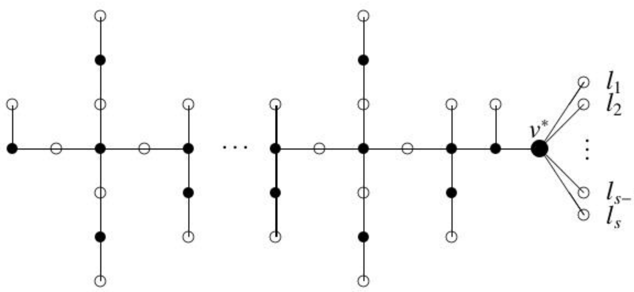

Figure 3. The constructions for the tree T. The large filled-in circles denote vertices in V_2, the small filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Proof. By Theorem 2, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$. Based on assumptions, the set formed by the center vertices of stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n} (t_i \geq 2, i = 1, 2, \cdots, n)$ is C. Then, obviously,

$S(T) \cup C$ is a dominating set of the tree T, and $S(T) \cap C = \emptyset$, so $\gamma(T) \leq |S(T) \cup C| = |S(T)| + |C|$. Based on assumptions, $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$, so $\gamma(T) \geq |S(T)| + |C|$. Hence, $\gamma(T) = |S(T)| + |C|$, and thus $S(T) \cup C$ is a $\gamma(T)$-set of the tree T.

The tree $T = (V, E)$ has one and only one strong support vertex, say, $v^* \in SS(T)$, and set $l_1, l_2, \cdots, l_s$ ($s \geq 2$) for its leaves. Let $f = (V_0, V_1, V_2) = (V - S(T) \cup C, S(T) \cup C \setminus \{v^*\}, \{v^*\})$, and then it has no undefended vertex, as shown in Figure 3. It is clear that the vertices in $V - S(T) \cup C$ consist of the set of all leaves $L(T)$ and the set of outer vertices W of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$, that is, $V - S(T) \cup C = L(T) \cup W$, and $L(T) \cap W = \emptyset$. $\forall\ v \in V - S(T) \cup C = L(T) \cup W$, if $v \in L(T) \setminus \{l_1, l_2, \cdots, l_s\}$, let $uv \in E$, and $u \in S(T)$. Then, obviously, $f_v = (V_0^*, V_1^*, V_2^*) = (L(T) \cup W \cup \{u\} \setminus \{v\}, S(T) \cup C \cup \{v\} \setminus \{u, v^*\}, \{v^*\})$ has no undefended vertex. If $v \in \{l_1, l_2, \cdots, l_s\}$, since $f(v^*) = 2$, then send a legion from v^* to v for security defense without creating undefended vertices. If $v \in W$, let $uv \in E$, and $u \in C$. Then, obviously, $f_v = (V_0^*, V_1^*, V_2^*) = (L(T) \cup W \cup \{u\} \setminus \{v\}, S(T) \cup C \cup \{v\} \setminus \{u, v^*\}, \{v^*\})$ has no undefended vertex. So, f is a WRDF of the tree T. Therefore, $\gamma_r(T) \leq w(f) = |S(T) \cup C \setminus \{v^*\}| + 2|\{v^*\}| = |S(T) \cup C| - 1 + 2 = |S(T)| + |C| + 1 = \gamma(T) + 1$.

From Lemma 3 and the fact that T has a strong support vertex, $\gamma(T) \neq \gamma_r(T)$. Again, by Lemma 1, know $\gamma(T) \leq \gamma_r(T)$, and then $\gamma_r(T) > \gamma(T)$, or $\gamma_r(T) \geq \gamma(T) + 1$. Hence, $\gamma_r(T) = \gamma(T) + 1$.

Theorem 7. Suppose the tree $T = (V, E)$ contains a star $K_{1,t}^*$ ($t \geq 3$) (its center vertex is denoted as v^*; its outer vertex is denoted as $w_1, w_2, \cdots, w_t$ ($t \geq 3$), among them, there is at least one vertex that is a leaf of the tree T, denoted $w_1, w_2, \cdots, w_s$; and at least two outer vertices are adjacent to the support vertex of the tree T, denoted $w_{s+1}, w_{s+2}, \cdots, w_t$ ($t \geq 3$)) and contains m (m is a positive integer) K_2 (let $u_i, v_i \in V$, $u_i v_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$ (the set formed by their center vertices is C, and the set formed by their outer vertices is W), the subtree $T - K_{1,t}^*$ does not contain any strong support vertices, and the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $v_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m, v^*\}$, and $L(T) = \{v_1, v_2, \cdots, v_m\} \cup \{w_1, w_2, \cdots, w_s\}$.

(b) $\forall w \in W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\}$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$.

(c) $\forall u \in S(T) \setminus \{v^*\}$, except for its leaf, either u is adjacent to an outer vertex $w \in W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\}$, or u is adjacent to other vertices in $S(T) \setminus \{v^*\}$.

As shown in Figure 4, then, $\gamma_r(T) = \gamma(T) + 1$.

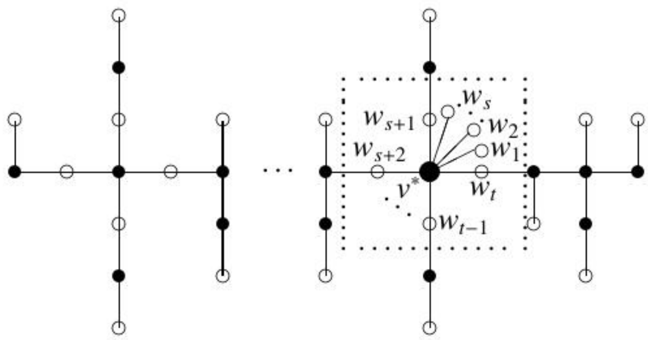

Figure 4. The constructions for the tree T. The large filled-in circles denote vertices in V_2, the small filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Proof. By Theorem 2, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$. Based on assumptions, the set formed by the center vertices of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n} (t_i \geq 2, i = 1, 2, \cdots, n)$ is C. Then, obviously, $S(T) \cup C$ is a dominating set of the tree T, and $S(T) \cap C = \emptyset$, so $\gamma(T) \leq |S(T) \cup C| = |S(T)| + |C|$. Based on assumptions, $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$, so $\gamma(T) \geq |S(T)| + |C|$. Hence, $\gamma(T) = |S(T)| + |C|$, and thus $S(T) \cup C$ is a $\gamma(T)$-set of the tree T.

Let $f = (V_0, V_1, V_2) = (V - S(T) \cup C, S(T) \cup C \setminus \{v^*\}, \{v^*\})$. Then, it has no undefended vertex, as shown in Figure 4. Obviously, the vertices in $V - S(T) \cup C$ consist of three parts: (i) the set of all leaves $L(T)$ (where $w_1, w_2, \cdots, w_s \in L(T)$); (ii) the set of outer vertices W of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}$ $(t_i \geq 2, i = 1, 2, \cdots, n)$; (iii) the set of outer vertices $\{w_{s+1}, w_{s+2}, \cdots, w_t\}$ of the star $K^*_{1,t}$ $(t \geq 3)$, that is, $V - S(T) \cup C = L(T) \cup W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\}$, and $L(T) \cap W = \emptyset$, $L(T) \cap \{w_{s+1}, w_{s+2}, \cdots, w_t\} = \emptyset$, $W \cap \{w_{s+1}, w_{s+2}, \cdots, w_t\} = \emptyset$. $\forall\ v \in V - S(T) \cup C = L(T) \cup W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\}$, if $v \in L(T) \setminus \{w_1, w_2, \cdots, w_s\}$, let $uv \in E$, and $u \in S(T)$. Then, obviously, $f_v = (V^*_0, V^*_1, V^*_2) = (L(T) \cup W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\} \cup \{u\} \setminus \{v\}, S(T) \cup C \cup \{v\} \setminus \{u, v^*\}, \{v^*\})$ has no undefended vertex. If $v \in \{w_1, w_2, \cdots, w_t\}$, since $f(v^*) = 2$, then send a legion from v^* to v for security defense without creating undefended vertices. If $v \in W$, let $uv \in E$, and $u \in C$. Then, obviously, $f_v = (V^*_0, V^*_1, V^*_2) = (L(T) \cup W \cup \{w_{s+1}, w_{s+2}, \cdots, w_t\} \cup \{u\} \setminus \{v\}, S(T) \cup C \cup \{v\} \setminus \{u, v^*\}, \{v^*\})$ has no undefended vertex. So, f is a WRDF of the tree T. Therefore, $\gamma_r(T) \leq w(f) = |S(T) \cup C \setminus \{v^*\}| + 2|\{v^*\}| = |S(T) \cup C| - 1 + 2 = |S(T)| + |C| + 1 = \gamma(T) + 1$.

Since at least one outer vertex of the star $K^*_{1,t}$ $(t \geq 3)$ is a leaf of the tree T, then for any weak Roman dominating function $f = (V_0, V_1, V_2)$ of the tree T, $f(N[v^*]) \geq 2$. (Otherwise, if $f(N[v^*]) \leq 1$, then $f(v^*) = 1$, and $f(w_1) = 0$. Also, since the outer vertex w_{s+1} is adjacent to the support vertex of the tree T, send a legion from v^* to w_{s+1} for security defense, such that w_1 is an undefended vertex, which is a contradiction.) On the other hand, $\{v^*\} \succ N(v^*)$, so $\gamma_r(T) \geq \gamma(T) + 1$. Hence, $\gamma(T) = \gamma_r(T) + 1$.

Theorem 8. Suppose the tree $T = (V, E)$ does not contain any strong support vertices, contains m (m is a positive integer) K_2 (let $u_i, l_i \in V$, $u_i l_i \in E, i = 1, 2, \cdots, m$) and has and only has one of the following seven cases: (1) K_1; (2) P_2; (3) P_4; (4) P_5; (5) P_6; (6) P_7; (7) P_9. Further, suppose the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $l_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{l_1, l_2, \cdots, l_m\}$.

(b) Both ends of the seven cases: (1) K_1; (2) P_2; (3) P_4; (4) P_5; (5) P_6; (6) P_7; (7) P_9 are adjacent to a unique support vertex $u \in S(T)$.

(c) $\forall u \in S(T)$, except for its leaf, either u is adjacent to one of the endpoints of the seven cases: (1) K_1; (2) P_2; (3) P_4; (4) P_5; (5) P_6; (6) P_7; (7) P_9, or u is adjacent to other vertices in $S(T)$.

As shown in Figure 5, then, $\gamma_r(T) = \gamma(T) + 1$.

Proof. By Theorem 2, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$. Then, divided into seven cases, the proof is as follows:

(1) If the tree T has and only has the K_1, as shown in Figure 5(T_1), based on assumptions, obviously, $S(T_1)$ can dominate the tree T_1, then $S \subseteq S(T_1)$. Also, $S(T_1) \subseteq S$, so $S = S(T_1)$, and $\gamma(T_1) = |S(T_1)|$.

Let $f = (V_0, V_1, V_2) = (V - S(T_1) \cup \{v_1\}, S(T_1) \cup \{v_1\}, \emptyset)$. Then, it has no undefended vertex. Since the tree T_1 does not contain any strong support vertices, both ends of K_1 are adjacent to a unique support vertex, and $S(T_1)$ is a dominating set of T_1, $\forall u \in V - S(T_1) \cup \{v_1\} = L(T_1)$, u can only be adjacent to the only vertex v in $S(T_1)$, and $d(u) = 1$. Let $f_u = (V^*_0, V^*_1, V^*_2) = (V - S(T_1) \cup \{v_1, u\} \setminus \{v\}, S(T_1) \cup \{v_1, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex, so f is a WRDF of the tree T_1. Therefore,

$\gamma_r(T_1) \leq w(f) = |S(T_1) \cup \{v_1\}| = |S(T_1)| + 1 = \gamma(T_1) + 1$.

Since the leaves of T_1 one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(u_1) = 2$, or $f(u_2) = 2$, as shown in Figure 5(T_1). So, $\gamma_r(T_1) \geq |S(T_1)| + 1 = \gamma(T_1) + 1$. Therefore, $\gamma_r(T_1) = \gamma(T_1) + 1$.

(2) If the tree T has and only has the path P_2, as shown in Figure 5(T_2), the proof of (1) is the same.

(3) If the tree T has and only has the path P_6, as shown in Figure 5(T_5), based on assumptions, $S(T_5) \cup \{v_2, v_5\}$ can dominate the tree T_5, so $\gamma(T_5) \leq |S(T_5) \cup \{v_2, v_5\}| = |S(T_5)| + 2$. v_1 is adjacent to u_1, and v_6 is adjacent to u_2. $u_1, u_2 \in S(T_5)$, and from Lemma 4, $\gamma(P_4) = \lceil \frac{4}{3} \rceil = 2$. So, $\gamma(T_5) \geq |S(T_5)| + 2$. Therefore, $\gamma(T_5) = |S(T_5)| + 2$.

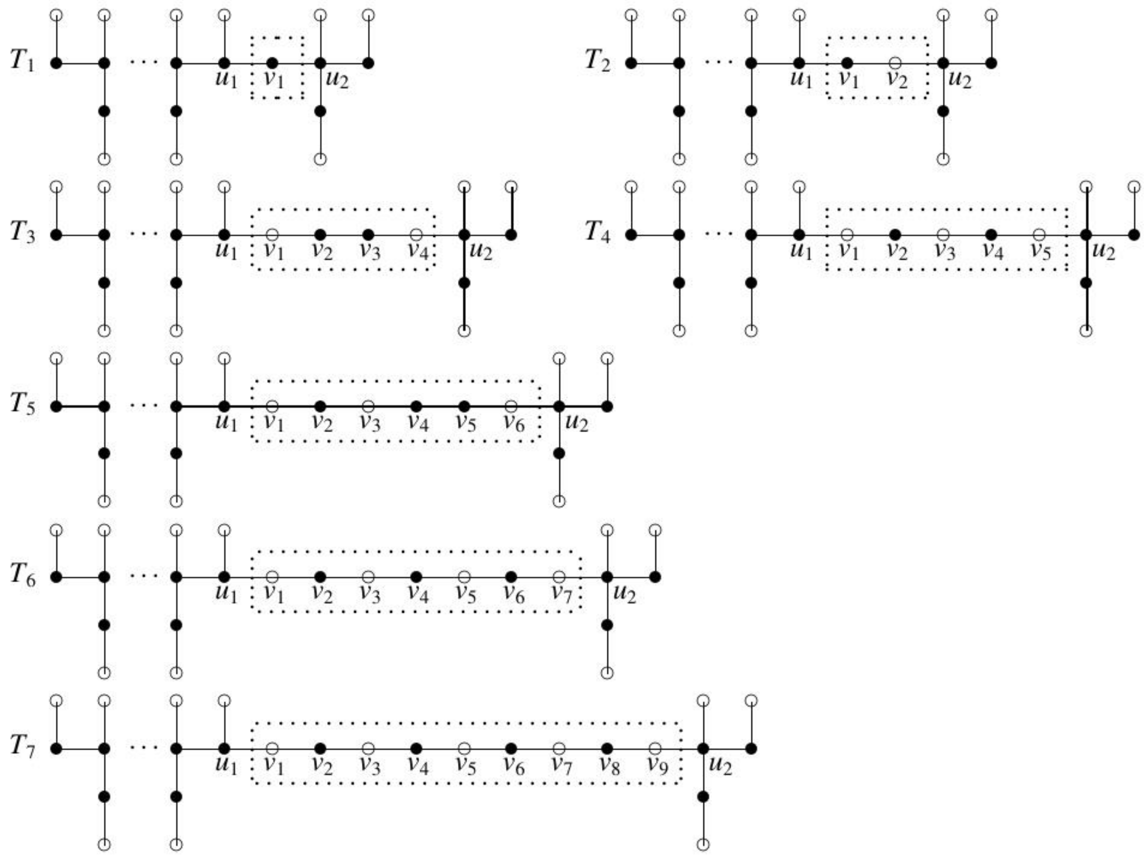

Figure 5. The constructions for the tree T. The filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Let $f = (V_0, V_1, V_2) = (V - S(T_5) \cup \{v_2, v_4, v_5\}, S(T_5) \cup \{v_2, v_4, v_5\}, \emptyset)$. Then, it has no undefended vertex. The tree T_5 does not contain any strong support vertices, both ends of P_6 are adjacent to a unique support vertex, and $S(T_5) \cup \{v_2, v_5\}$ is a dominating set of T_5. Then, $\forall u \in V - S(T_5) \cup \{v_2, v_4, v_5\} = L(T_5) \cup \{v_1, v_3, v_6\}$, if $u \in L(T_5)$, u can only be adjacent to the only vertex v in $S(T_5)$ and $d(u) = 1$. Let $f_u = (V_0^*, V_1^*, V_2^*) = (V - S(T_5) \cup \{v_2, v_4, v_5, u\} \setminus \{v\}, S(T_5) \cup \{v_2, v_4, v_5, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u = v_1$, let $f_u = (V_0^*, V_1^*, V_2^*) = (V - S(T_5) \cup \{v_1, v_4, v_5\}, S(T_5) \cup \{v_1, v_4, v_5\}, \emptyset)$, and then, obviously, f_u has no undefended vertex. If $u = v_3$, let $f_u = (V_0^*, V_1^*, V_2^*) = (V - S(T_5) \cup$

$\{v_2, v_3, v_5\}, S(T_5) \cup \{v_2, v_3, v_5\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u = v_6$, let $f_u = (V_0^*, V_1^*, V_2^*) = (V - S(T_5) \cup \{v_2, v_4, v_6\}, S(T_5) \cup \{v_2, v_4, v_6\}, \emptyset)$, and then, obviously, f_u has no undefended vertex. So, f is a WRDF of the tree T_5. Therefore, $\gamma_r(T_5) \le w(f) = |S(T_5) \cup \{v_2, v_4, v_5\}| = |S(T_5)| + 3 = \gamma(T_5) + 1$.

Since the leaves of T_5 one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(v_2) = 1$, or $f(u_1) = 2$. The same if the vertex v_6 for security defense, either $f(v_6) = 1$, or $f(v_5) = 1$, or $f(u_2) = 2$, as shown in Figure 5(T_5). That is, the vertex v_1 cannot be securely defended by u_1, and the vertex v_6 cannot be securely defended by u_2, unless u_1 or u_2 are stationed in two legions. From Lemma 2, $\gamma_r(P_6) = \lceil \frac{3\times 6}{7} \rceil = 3$, so $\gamma_r(T_5) \ge |S(T_5)| + 3 = \gamma(T_5) + 1$. Therefore, $\gamma_r(T_5) = \gamma(T_5) + 1$.

(4) If the tree T has and only has the path P_4, P_5, P_7, P_9, as shown in Figure 5(T_3, T_4, T_6, T_7), the proof of (3) is the same.

Note 1. Suppose the tree $T = (V, E)$ does not contain any strong support vertices, contains m (m is a positive integer) K_2 (let $u_i, l_i \in V$, $u_il_i \in E, i = 1, 2, \cdots, m$) and has only one of the following three cases: (1) P_3; (2) P_8; (3) P_n ($n \ge 10$). Suppose also that the following conditions are satisfied: (a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $l_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{l_1, l_2, \cdots, l_m\}$. ($b$) Both ends of P_3, P_8, P_n ($n \ge 10$) are adjacent to a unique support vertex $u \in S(T)$. (c) $\forall u \in S(T)$, except for its leaf, either u is adjacent to one of the endpoints of P_3, P_8, P_n ($n \ge 10$), or u is adjacent to other vertices in $S(T)$. Then, $\gamma_r(T) \ne \gamma(T) + 1$.

In fact, if the tree T has and only has the path P_3, then P_3 is also star $K_{1,2}$. According to Lemma 5, $\gamma_r(T) = \gamma(T) \ne \gamma(T) + 1$. If the tree T has and only has the path P_8, as shown in Figure 6, based on assumptions, $S(T) \cup \{v_3, v_6\}$ can dominate the tree T, so $\gamma(T) \le |S(T) \cup \{v_3, v_6\}| = |S(T)| + 2$. v_1 is adjacent to u_1, v_8 is adjacent to u_2, and $u_1, u_2 \in S(T)$. From Lemma 4, $\gamma(P_6) = \lceil \frac{6}{3} \rceil = 2$. So, $\gamma(T) \ge |S(T)| + 2$. Therefore, $\gamma(T) = |S(T)| + 2$. Since the leaves of T one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(v_2) = 1$, or $f(u_1) = 2$. The same if the vertex v_8 for security defense, either $f(v_8) = 1$, or $f(v_7) = 1$, or $f(u_2) = 2$. That is, the vertex v_1 cannot be securely defended by u_1, and the vertex v_8 cannot be securely defended by u_2, unless u_1 or u_2 are stationed in two legions. From Lemma 2, $\gamma_r(P_8) = \lceil \frac{3\times 8}{7} \rceil = 4$, so $\gamma_r(T) \ge |S(T)| + 4 = \gamma(T) + 2 > \gamma(T) + 1$. If the tree T has and only has the path P_n ($n \ge 10$), the same is true for the path P_8 above. $\gamma(T) = |S(T)| + \lceil \frac{n-2}{3} \rceil$, and from Lemma 2, $\gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$. Then, when $n \ge 10$, $\gamma_r(T) \ge |S(T)| + \lceil \frac{3n}{7} \rceil = \gamma(T) + \lceil \frac{3n}{7} \rceil - \lceil \frac{n-2}{3} \rceil \ge \gamma(T) + 2 > \gamma(T) + 1$.

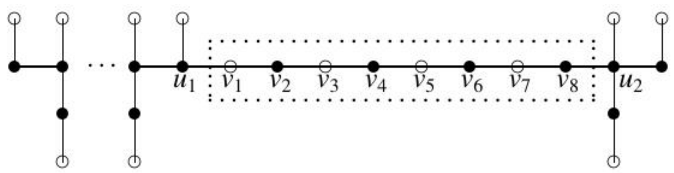

Figure 6. The constructions for the tree T. The filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Theorem 9. Suppose the tree $T = (V, E)$ does not contain any strong support vertices, contains m (m is a positive integer) K_2 (let $u_i, l_i \in V$, $u_il_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$ (the set formed by their center vertices is C, and the set formed by their outer vertices is W) and has only one of the following seven cases: (1) K_1; (2) P_2; (3) P_4; (4) P_5; (5) P_6; (6) P_7; (7) P_9. Further, also suppose the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $l_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{l_1, l_2, \cdots, l_m\}$.

(b) $\forall w \in W$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$.

(c) Both ends of K_1, P_2, P_4, P_5, P_6, P_7, P_9 are adjacent to a unique support vertex $u \in S(T)$.

(d) $\forall u \in S(T)$, except for its leaf, either u is adjacent to an outer vertex $w \in W$, or u is adjacent to one of the endpoints of K_1, P_2, P_4, P_5, P_6, P_7, P_9, or u is adjacent to other vertices in $S(T)$.

As shown in Figure 7, then, $\gamma_r(T) = \gamma(T) + 1$.

Proof. By Theorem 2, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$. Then, divided into seven cases, the proof is as follows:

(1) If the tree T has only the K_1, as shown in Figure 7(T_1), then, obviously, $S(T_1) \cup C(T_1)$ is a dominating set of tree T_1, and $S(T_1) \cap C(T_1) = \emptyset$, so $\gamma(T_1) \leq |S(T_1) \cup C(T_1)| = |S(T_1)| + |C(T_1)|$. Based on assumptions, $\forall\, v \in C(T_1)$, v is at least 2 away from every vertex in $S(T_1)$, so $\gamma(T_1) \geq |S(T_1)| + |C(T_1)|$. Hence, $\gamma(T_1) = |S(T_1)| + |C(T_1)|$, and thus $S(T_1) \cup C(T_1)$ is a $\gamma(T_1)$-set of the tree T_1.

Let $f = (V_0, V_1, V_2) = (V - S(T_1) \cup C(T_1) \cup \{v_1\}, S(T_1) \cup C(T_1) \cup \{v_1\}, \emptyset)$, and then it has no undefended vertex. It is clear that the vertices in $V - S(T_1) \cup C(T_1) \cup \{v_1\}$ consist of the set of all leaves $L(T_1)$ and the set of outer vertices $W(T_1)$ of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$, that is, $V - S(T_1) \cup C(T_1) \cup \{v_1\} = L(T_1) \cup W(T_1)$, and $L(T_1) \cap W(T_1) = \emptyset$. $\forall u \in V - S(T_1) \cup C(T_1) \cup \{v_1\} = L(T_1) \cup W(T_1)$, if $u \in L(T_1)$, since the tree T_1 does not contain any strong support vertices, let $uv \in E(T_1)$ and $v \in S(T_1)$, and let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_1) \cup W(T_1) \cup \{v\} \setminus \{u\}, S(T_1) \cup C(T_1) \cup \{v_1, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u \in W(T_1)$, let $uv \in E(T_1)$ and $v \in C(T_1)$, and let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_1) \cup W(T_1) \cup \{v\} \setminus \{u\}, S(T_1) \cup C(T_1) \cup \{v_1, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. So, f is a WRDF of the tree T_1. Therefore, $\gamma_r(T_1) \leq w(f) = |S(T_1) \cup C(T_1) \cup \{v_1\}| = |S(T_1) \cup C(T_1)| + 1 = |S(T_1)| + |C(T_1)| + 1 = \gamma(T_1) + 1$.

The leaves of T_1 one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex. $\forall v \in C(T_1)$, v is at least 2 away from every vertex in $S(T_1)$, so the vertex v requires at least one legion for security defense, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(u_1) = 2$, or $f(u_2) = 2$, as shown in Figure 7(T_1). So, $\gamma_r(T_1) \geq |S(T_1) \cup C(T_1)| + 1 = |S(T_1)| + |C(T_1)| + 1 = \gamma(T_1) + 1$. Therefore, $\gamma_r(T_1) = \gamma(T_1) + 1$.

(2) If the tree T has only the path P_2, as shown in Figure 7(T_2), the proof of (1) is the same.

(3) If the tree T has only the path P_4, as shown in Figure 7(T_3), based on assumptions, $S(T_3) \cup C(T_3) \cup \{v_2\}$ can dominate the tree T_3, and $S(T_3) \cap C(T_3) = \emptyset$, so $\gamma(T_3) \leq |S(T_3) \cup C(T_3) \cup \{v_2\}| = |S(T_3)| + |C(T_3)| + 1$. Based on assumptions, $\forall\, v \in C(T_3)$, v is at least 2 away from every vertex in $S(T_3)$. Since v_1 is adjacent to u_1, v_4 is adjacent to u_2 and $u_1, u_2 \in S(T_3)$ and from Lemma 4, $\gamma(P_2) = \lceil \frac{2}{3} \rceil = 1$. So, $\gamma(T_3) \geq |S(T_3)| + |C(T_3)| + 1$. Therefore, $\gamma(T_3) = |S(T_3)| + |C(T_3)| + 1$, and thus $S(T_3) \cup C(T_3) \cup \{v_2\}$ is a $\gamma(T_3)$-set of the tree T_3.

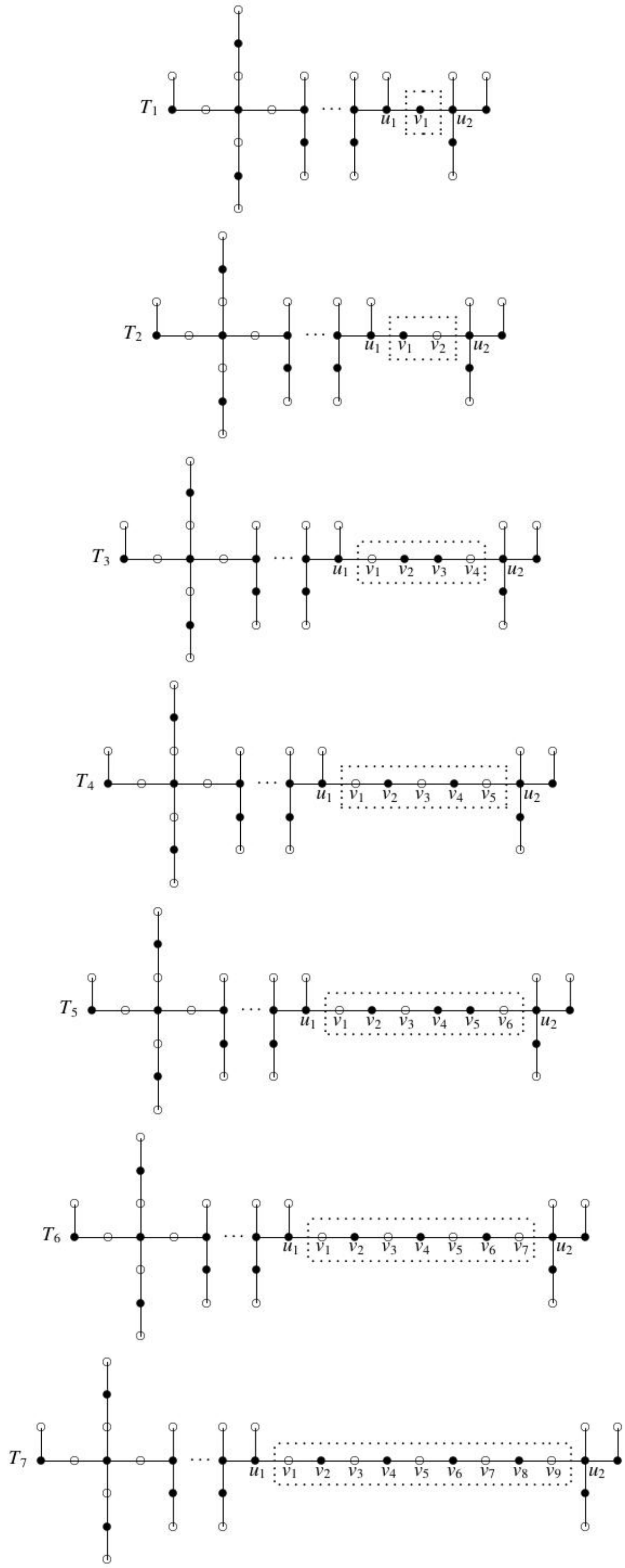

Figure 7. The constructions for the tree T. The filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Let $f = (V_0, V_1, V_2) = (V - S(T_3) \cup C(T_3) \cup \{v_2, v_3\}, S(T_3) \cup C(T_3) \cup \{v_2, v_3\}, \emptyset)$, and then it has no undefended vertex. Obviously, the vertices in $V - S(T_3) \cup C(T_3) \cup \{v_2, v_3\}$ consist of three parts: (i) the set of all leaves $L(T_3)$; (ii) the set of outer vertices $W(T_3)$ of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$; (iii) the set $\{v_1, v_4\}$, that is, $V - S(T_3) \cup C(T_3) \cup \{v_2, v_3\} = L(T_3) \cup W(T_3) \cup \{v_1, v_4\}$ and $L(T_3) \cap W(T_3) = \emptyset$, $L(T_3) \cap \{v_1, v_4\} = \emptyset$, $W(T_3) \cap \{v_1, v_4\} = \emptyset$. $\forall u \in V - S(T_3) \cup C(T_3) \cup \{v_2, v_3\} = L(T_3) \cup W(T_3) \cup \{v_1, v_4\}$, if $u \in L(T_3)$, since the tree T_3 does not contain any strong support vertices, let $uv \in E(T_3)$ and $v \in S(T_3)$, and let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_3) \cup W(T_3) \cup \{v_1, v_4, v\} \setminus \{u\}, S(T_3) \cup C(T_3) \cup \{v_2, v_3, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u \in W(T_3)$, let $uv \in E(T_3)$ and $v \in C(T_3)$, and let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_3) \cup W(T_3) \cup \{v_1, v_4, v\} \setminus \{u\}, S(T_3) \cup C(T_3) \cup \{v_2, v_3, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u = v_1$, let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_3) \cup W(T_3) \cup \{v_2, v_4\}, S(T_3) \cup C(T_3) \cup \{v_1, v_3\}, \emptyset)$, and then, obviously, f_u has no undefended vertex. If $u = v_4$, let $f_u = (V_0^*, V_1^*, V_2^*) = (L(T_3) \cup W(T_3) \cup \{v_1, v_3\}, S(T_3) \cup C(T_3) \cup \{v_2, v_4\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. So f is a WRDF of the tree T_3. Therefore, $\gamma_r(T_3) \leq w(f) = |S(T_3) \cup C(T_3) \cup \{v_2, v_3\}| = |S(T_3)| + |C(T_3)| + 2 = \gamma(T_3) + 1$.

The leaves of T_3 one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex. $\forall v \in C(T_3)$, v is at least 2 away from every vertex in $S(T_3)$, so the vertex v requires at least one legion for security defense, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(v_2) = 1$, or $f(u_1) = 2$. The same if the vertex v_4 for security defense, either $f(v_4) = 1$, or $f(v_3) = 1$, or $f(u_2) = 2$, as shown in Figure 7(T_3). That is, the vertex v_1 cannot be securely defended by u_1, and the vertex v_4 cannot be securely defended by u_2, unless u_1 or u_2 are stationed in two legions. From Lemma 2, $\gamma_r(P_4) = \lceil \frac{3\times 4}{7} \rceil = 2$, so $\gamma_r(T_3) \geq |S(T_3)| + |C(T_3)| + 2 = \gamma(T_3) + 1$. Therefore, $\gamma_r(T_3) = \gamma(T_3) + 1$.

(4) If the tree T has only the path P_5, P_6, P_7, P_9, as shown in Figure 7(T_4, T_5, T_6, T_7), the proof of (3) is the same.

Note 2. Suppose the tree $T = (V, E)$ does not contain any strong support vertices, contains m (m is a positive integer) K_2 (let $u_i, l_i \in V$, $u_i l_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$ (the set formed by their center vertices is C, and the set formed by their outer vertices is W), and has only one of the following three cases: (1) P_3; (2) P_8; (3) P_n ($n \geq 10$). Further suppose also that the following conditions are satisfied: (a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $l_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{l_1, l_2, \cdots, l_m\}$. ($b$) $\forall w \in W$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$. (c) Both ends of P_3, P_8, P_n ($n \geq 10$) are adjacent to a unique support vertex $u \in S(T)$. (d) $\forall u \in S(T)$, except for its leaf, either u is adjacent to an outer vertex $w \in W$, or u is adjacent to one of the endpoints of P_3, P_8, P_n ($n \geq 10$), or u is adjacent to other vertices in $S(T)$. Then, $\gamma_r(T) \neq \gamma(T) + 1$.

In fact, if the tree T has only the path P_3, then P_3 is also star $K_{1,2}$, and according to Lemma 5, $\gamma_r(T) = \gamma(T) \neq \gamma(T)+1$. If the tree T has only the path P_8, as shown in Figure 8, based on assumptions, $S(T) \cup C(T) \cup \{v_3, v_6\}$ can dominate the tree T, and $S(T) \cap C(T) = \emptyset$, so $\gamma(T) \leq |S(T) \cup C(T) \cup \{v_3, v_6\}| = |S(T)| + |C(T)| + 2$. Based on assumptions, $\forall\ v \in C(T)$, v is at least 2 away from every vertex in $S(T)$. v_1 is adjacent to u_1, and v_8 is adjacent to u_2, and $u_1, u_2 \in S(T)$. With regards from Lemma 4, $\gamma(P_6) = \lceil \frac{6}{3} \rceil = 2$. So, $\gamma(T) \geq |S(T)| + |C(T)| + 2$. Therefore, $\gamma(T) = |S(T)| + |C(T)| + 2$. The leaves of T one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex. $\forall v \in C(T)$, v is at least 2 away from every vertex in $S(T)$, so the vertex v requires at least one legion for security defense, and if the vertex v_1 for security defense, either $f(v_1) = 1$, or $f(v_2) = 1$, or $f(u_1) = 2$. The same if the vertex v_8 for security defense, either $f(v_8) = 1$, or $f(v_7) = 1$, or $f(u_2) = 2$, as shown in Figure 8. That is, the vertex v_1 cannot be securely defended by u_1 and the vertex v_8 cannot be securely defended by u_2, unless u_1 or u_2 are stationed in two legions. From Lemma 2, $\gamma_r(P_8) = \lceil \frac{3\times 8}{7} \rceil = 4$, so $\gamma_r(T) \geq |S(T)| + |C(T)| + 4 = \gamma(T) + 2 > \gamma(T) + 1$. If the tree T has only the path P_n ($n \geq 10$), the same is true for the path P_8 above, $\gamma(T) = |S(T)| + |C(T)| + \lceil \frac{n-2}{3} \rceil$, and from Lemma 2, $\gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$. Then, when $n \geq 10$, $\gamma_r(T) \geq |S(T)| + |C(T)| + \lceil \frac{3n}{7} \rceil = \gamma(T) + \lceil \frac{3n}{7} \rceil - \lceil \frac{n-2}{3} \rceil \geq \gamma(T) + 2 > \gamma(T) + 1$.

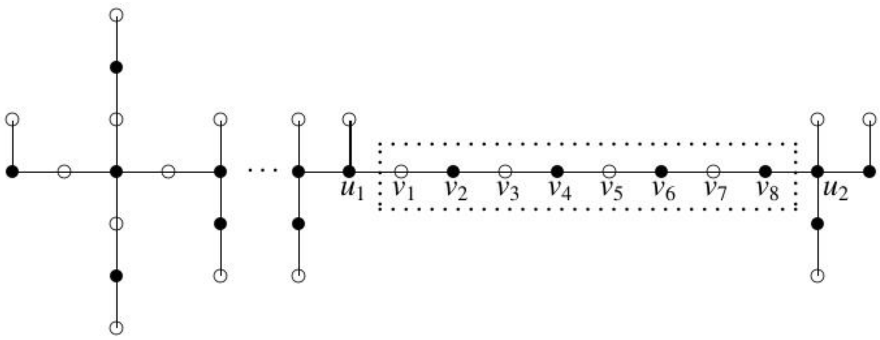

Figure 8. The constructions for the tree T. The filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Theorem 10. Suppose the tree $T = (V, E)$ does not contain any strong support vertices, contains m (m is a positive integer) K_2 (let $u_i, l_i \in V$, $u_i l_i \in E, i = 1, 2, \cdots, m$) and n (n is a positive integer) stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n} (t_i \geq 2, i = 1, 2, \cdots, n)$ and has only one connected branch from star $K^*_{1,t}$ ($t \geq 2$) and an outer vertex w^* of star $K^*_{1,t}$ ($t \geq 2$) connected to a vertex v^* (the set formed by the center vertices of all stars is C, and the set formed by the outer vertices is W). Further, suppose the following conditions are satisfied:

(a) Each K_2 has a vertex $u_i \in S(T)$ and another vertex $l_i \in L(T)$, $S(T) = \{u_1, u_2, \cdots, u_m\}$, and $L(T) = \{l_1, l_2, \cdots, l_m\}$.

(b) $\forall w \in W \backslash \{w^*\}$, there is a unique support vertex $u \in S(T)$, such that $wu \in E(T)$, and $\forall v \in C$, v is at least 2 away from every vertex in $S(T)$.

(c) $\exists u^* \in S(T)$, such that $u^* v^* \in E(T)$.

(d) $\forall u \in S(T) \backslash \{u^*\}$, except for its leaf, either u is adjacent to an outer vertex $w \in W \backslash \{w^*\}$, or u is adjacent to other vertices in $S(T)$.

As shown in Figure 9, then, $\gamma_r(T) = \gamma(T) + 1$.

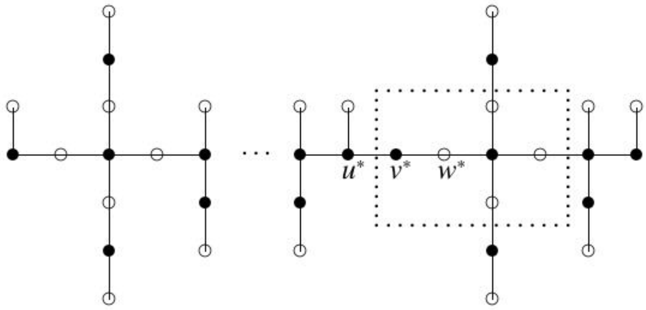

Figure 9. The constructions for the tree T. The filled-in circles denote vertices in V_1, and the empty circles denote vertices in V_0.

Proof. By Theorem 2, there is a $\gamma(T)$-set S such that $S(T) \subseteq S$. Based on assumptions, the set formed by the center vertices of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$ and the star $K^*_{1,t}$ $(t \geq 2)$ is $C(T)$. Then, obviously, $S(T) \cup C(T)$ is a dominating set of tree T, and $S(T) \cap C(T) = \emptyset$, so $\gamma(T) \leq |S(T) \cup C(T)| = |S(T)| + |C(T)|$. Based on assumptions, $\forall\ v \in C(T)$, v is at least 2 away from every vertex in $S(T)$, so $\gamma(T) \geq |S(T)| + |C(T)|$. Hence, $\gamma(T) = |S(T)| + |C(T)|$, and thus $S(T) \cup C(T)$ is a $\gamma(T)$-set of the tree T.

Let $f = (V_0, V_1, V_2) = (V - S(T) \cup C(T) \cup \{v^*\}, S(T) \cup C(T) \cup \{v^*\}, \emptyset)$, and then it has no undefended vertex, as shown in Figure 9. Obviously, the vertices in $V - S(T) \cup C(T) \cup \{v^*\}$ consist of two parts: (i) the set of all leaves $L(T)$; (ii) the set of outer vertices $W(T)$ of the stars $K_{1,t_1}, K_{1,t_2}, \cdots, K_{1,t_n}(t_i \geq 2, i = 1, 2, \cdots, n)$ and the star $K^*_{1,t}$ $(t \geq 2)$, that is, $u \in V - S(T) \cup C(T) \cup \{v^*\} = L(T) \cup W(T)$ and $L(T) \cap W(T) = \emptyset$. $\forall u \in V - S(T) \cup C(T) \cup \{v^*\} = L(T) \cup W(T)$, if $u \in L(T)$, since the tree T does not contain any strong support vertices, let $uv \in E(T)$ and $v \in S(T)$, and let $f_u = (V^*_0, V^*_1, V^*_2) = (L(T) \cup W(T) \cup \{v\} \setminus \{u\}, S(T) \cup C(T) \cup \{v^*, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. If $u \in W(T)$, let $uv \in E(T)$ and $v \in C(T)$, and let $f_u = (V^*_0, V^*_1, V^*_2) = (L(T) \cup W(T) \cup \{v\} \setminus \{u\}, S(T) \cup C(T) \cup \{v^*, u\} \setminus \{v\}, \emptyset)$. Then, obviously, f_u has no undefended vertex. So, f is a WRDF of the tree T. Therefore, $\gamma_r(T) \leq w(f) = |S(T) \cup C(T) \cup \{v^*\}| = |S(T)| + |C(T)| + 1 = \gamma(T) + 1$.

The leaves of T one-to-one correspond to their support vertices, and each leaf requires at least one legion for security defense or security defense by its support vertex. $\forall v \in C(T)$, v is at least 2 away from every vertex in $S(T)$, so the vertex v requires at least one legion for security defense, and if the vertex v^* for security defense, either $f(v^*) = 1$, or $f(w^*) = 1$, or $f(u^*) = 2$, as shown in Figure 9, so $\gamma_r(T) \geq |S(T) \cup C(T)| + 1 = |S(T)| + |C(T)| + 1 = \gamma(T) + 1$. Therefore, $\gamma_r(T) = \gamma(T) + 1$.

4. Conclusions

In this paper, we give some sufficient conditions for a tree to have its weak Roman domination number be equal to its domination number plus 1 ($\gamma_r(T) = \gamma(T) + 1$) by recursion and construction, such as Theorems 6–10. Conversely, we hold that the trees T satisfying $\gamma_r(T) = \gamma(T) + 1$ only have the characteristics of Theorems 6–10, which is subject to further verification. We will further investigate some characteristics of trees in which the weak Roman domination number is equal to the domination number plus k ($\gamma_r(T) = \gamma(T) + k$).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank the associate editor and the reviewers for their useful feedback that improved this paper.

This work is supported partly by the National Natural Science Foundation (NNSF) of China under Grants 62073122 and 61203050, the Plan of Key Scientific Research Projects of Colleges and Universities in Henan Province (No. 22A880007), the National Natural Science Foundation of Henan (No. 202300410343), the Postgraduate Education Reform and Quality Improvement Project of Henan Province (No. YJS2022ZX34), the Soft Science Research Project of 2023 Science and Technology Development Plan of Henan Province (No. 232400410210) and the Higher Education Teaching Reform research and practice project in Henan Province (No. 2019SJGLX735).

Conflict of interest

We declare no conflict of interest.

References

1. I. Stewart, Defend the Roman Empire, *Sci. Am.*, **281** (1999), 136–138. https://doi.org/10.1038/scientificamerican1299-136
2. E. J. Cockayne, P. A. Jr. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, *Disc. Math.*, **278** (2004), 11–22. https://doi.org/10.1016/j.disc.2003.06.004
3. A. Klobučar, I. Puljić, Roman domination number on cardinal product of paths and cycles, *Croat. Oper. Res. Rev.*, **6** (2015), 71–78. https://doi.org/10.17535/crorr.2015.0006
4. M. A. Henning, S. T. Hedetniemi, Defending the Roman Empire-A new strategy, *Discrete Math.*, **266** (2003), 239–251. https://doi.org/10.1016/S0012-365X(02)00811-7
5. R. Hernández-Ortiz, L. P. Montejano, J. A. Rodríguez-Velázquez, Weak Roman domination in rooted product graphs, *AIMS Math.*, **6** (2021), 3641–3653. https://doi.org/10.3934/math.2021217
6. J. Yang, J. L. Song, Trees T with weak Roman domination number being equal to domination number, *Mathematics in Practice and Theory* (in Chinese), **43** (2013), 134–140.
7. J. Yang, Z. Q. Li, Some properties of weak Roman graph and weak Roman domination in graphs, *Chin. J. of Eng. Math.* (in Chinese), **39** (2022), 621–630.
8. F. N. Pour, H. A. Ahangar, M. Chellali, S. M. Sheikholeslami, Global triple Roman dominating function, *Discrete Appl. Math.*, **314** (2022), 228–237. https://doi.org/10.1016/j.dam.2022.02.015
9. Z. Li, A note on the bounds of Roman domination numbers, *AIMS Math.*, **6** (2021), 3940–3946. https://doi.org/10.3934/math.2021234
10. M. Chellali, T. W. Haynes, S. T. Hedetniemi, Roman and total domination, *Quaest. Math.*, **38** (2015), 749–757. https://doi.org/10.2989/16073606.2015.1015647

11. Z. Du, A. A. S. A. Jamri, R. Hasni, D. A. Mojdeh, Maximal first Zagreb index of trees with given Roman domination number, *AIMS Math.*, **7** (2022), 11801–11812. https://doi.org/10.3934/math.2022658

12. S. C. García, A. C. Martínez, I. G. Yero, Quasi-total Roman domination in graphs, *Results Math.*, **74** (2019), 173. https://doi.org/10.1007/s00025-019-1097-5

13. P. R. L. Pushpam, S. Padmapriea, Global Roman domination in graphs, *Discrete Appl. Math.*, **200** (2016), 176–185. https://doi.org/10.1016/j.dam.2015.07.014

14. H. A. Ahangar, M. A. Henning, V. Samodivkin, I. G. Yero, Total Roman domination in graphs, *Appl. Anal. Discr. Math.*, **10** (2016), 501–517. https://doi.org/10.2298/AADM160802017A

15. J. Amjadi, S. M. Sheikholeslami, M. Soroudi, Nordhaus-Gaddum bounds for total Roman domination, *J. Comb. Optim.*, **35** (2018), 126–133. https://doi.org/10.1007/s10878-017-0158-5

16. G. Gunther, B. L. Hartnell, L. Markus, D. Rall, Graphs with unique minimum domination sets, *Congressus Numerantium*, **101** (1994), 55–63.

17. Z. Gu, J. X. Meng, Z. Zhang, J. E. Wan, Some upper bounds related with domination number, *J. Oper. Res. Soc. China*, **1** (2013), 217–225. https://doi.org/10.1007/s40305-013-0012-0