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Research article

Consecutive integers in the form $a^x + y^b$

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Abstract: Let a, b and k be integers greater than 1. For a tuple of k consecutive integers sorted in ascending order, denoted by T_k, call T_k a nice k-tuple if each integer of T_k is a sum of two powers of the form $a^x + y^b$ and a perfect k-tuple if each integer of T_k is a sum of two perfect powers of the form $a^x + y^b$, respectively. Let $N_k(a, b)$ be the number of nice k-tuples and $\widetilde{N}_k(a, b)$ be the number of perfect k-tuples. For a given (a, b), it is quite interesting to find out $N_k(a, b)$ and $\widetilde{N}_k(a, b)$. In 2020, Lin and Cheng obtained the formula for $N_k(2, 2)$. The main goal of this paper is to establish the formulas for $N_k(a, b)$ and $\widetilde{N}_k(a, b)$. Actually, by using the method of modulo coverage together with some elementary techniques, the formulas for $\widetilde{N}_k(2, 2)$, $\widetilde{N}_k(3, 2)$ and $N_k(3, 2)$ are derived.

Keywords: Diophantine equations; consecutive integers; sum of two powers
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1. Introduction

Catalan's conjecture, one of the famous classical problems in number theory, was first enunciated by Catalan [1] in 1844. It states that the equation

$$x^p - y^q = 1$$

has no solutions in positive integers x and y, other than $3^2 - 2^3 = 1$, where p and q are different prime numbers. In 1976, by applying the Gelfond-Baker method Tijdeman [2] succeeded in solving Catalan's conjecture (see Mignotte [3] for an excellent survey of developments). Thirty years later, the conjecture was re-proved by Mihuailesc [4], who used completely different approaches with the theory of cyclotomic fields. Also, in a series of papers in the 1930s and 1940s, some scholars (see, for example, [5–8]) studied the solutions to the general equation

$$a^x - b^y = c, \tag{1.1}$$

where a, b and c are fixed positive integers. Furthermore, in 1936 Pillai conjectured that the number of positive integer solutions (a, b, x, y), with $x \geq 2, y \geq 2$, to (1.1) is finite, which is still open for all $c > 1$.

Let n be an integer. We say n is a *power* if $n = x^y$ for some nonnegative integers x and y, and a *perfect power* if $n = u^v$ for some integers u and v with $u \geq 2$ and $v \geq 2$. Then, Pillai's conjecture amounts to saying that the distance between two consecutive terms in the sequence of all perfect powers tends to infinity. In particular, Catalan's conjecture is equivalent to the statement that no two consecutive integers are perfect powers, other than 2^3 and 3^2. Note that there are no four consecutive integers with all of them being perfect powers, since any set of four consecutive integers must contain one integer congruent to 2 modulo 4 which cannot be a perfect power. Are there three consecutive integers with all of them being perfect powers? In 1962, Ko [9], by supplying a sufficient and necessary condition for the equation $x^p - y^q = 1$ to be solvable with positive integers x and y, showed that no three consecutive integers are powers of other positive integers.

Naturally, one shall ask, for a given positive integer k with $k \geq 2$, if there exist k consecutive integers such that each of them is a sum of two powers (or two perfect powers)? If exist, how many such k-tuples are there? In this paper, we concentrate the investigation on consecutive integers in a fixed form and would like to give an answer to this question. Let a and b be integers no less than 2. Let $T_k = (t_1, t_2, \cdots, t_k)$ be a k-tuple of k consecutive integers, where $t_{i+1} - t_i = 1$ for any $1 \leq i \leq k-1$. We call T_k a *nice k-tuple with type* (a, b) if each integer of T_k is a sum of two powers of the form $a^x + y^b$ and a *perfect k-tuple with type* (a, b) if each integer of T_k is a sum of two perfect powers of the form $a^x + y^b$. Let $N_k(a, b)$ be the number of nice k-tuples with type (a, b) and $\widetilde{N}_k(a, b)$ the number of perfect k-tuples with type (a, b). It is interesting to study the formulas for $N_k(a, b)$ and $\widetilde{N}_k(a, b)$. In 2020, Lin and Cheng [10] obtained the formula for $N_k(2, 2)$. In this paper, by using the method of modulo coverage together with some elementary techniques, we present the formulas for $\widetilde{N}_k(2, 2)$, $\widetilde{N}_k(3, 2)$ and $N_k(3, 2)$. To be more precise, we report the main results as follows.

Theorem 1.1. *Let k be a positive integer with $k \geq 2$. The following statements are true.*

(a) Let $\widetilde{N}_k(2, 2)$ and $\widetilde{N}_k(3, 2)$ be the numbers of perfect k-tuples with type $(2, 2)$ and $(3, 2)$, respectively. Then, we have

$$\widetilde{N}_k(2,2) = \begin{cases} +\infty, & \text{if } k = 2, \\ 0, & \text{if } k \geq 3. \end{cases}, \quad \widetilde{N}_k(3,2) = \begin{cases} +\infty, & \text{if } k = 2, \\ 0, & \text{if } k \geq 3. \end{cases}$$

(b) Let $N_k(3, 2)$ be the number of nice k-tuples with type $(3, 2)$, and we have

$$N_k(3,2) = \begin{cases} +\infty, & \text{if } 2 \leq k \leq 3, \\ 3, & \text{if } k = 4, \\ 1, & \text{if } k = 5, \\ 0, & \text{if } k \geq 6. \end{cases}$$

Moreover, the only 3 nice 4-tuples with type $(3, 2)$ are $(1, 2, 3, 4)$, $(2, 3, 4, 5)$ and $(25, 26, 27, 28)$, and the only nice 5-tuple with type $(3, 2)$ is $(1, 2, 3, 4, 5)$.

The paper is organized as follows. First, in Section 2, we present some lemmas that will be used to prove Theorem 1.1. Particularly, by figuring out the 3-adic representation for one of the variables, we show that a Diophantine equation $3^{2x} + 2 \cdot 3^x + 9 - 12 \cdot 9^y = (24z)^2$ in nonnegative integers x, y, z has no solution, other than $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (2, 1, 0)$. In Section 3, the proof of Theorem 1.1 is given. Finally, in Section 4, two further research problems are raised for the interested.

2. Lemmas

In this section, some useful lemmas are presented. In fact, we first give the results for $2^x + y^2 \pmod 8$, $3^x + y^2 \pmod 8$ and $3^x + y^2 \pmod 9$, and then we determine the solvability of a Diophantine equation.

Lemma 2.1. *Let m and n be positive integers of the forms $m = 3^x + y^2$ and $n = 2^u + v^2$ with x, y, u and v being nonnegative integers. The following statements are true.*

(a) If $u \geq 2$, then the set of the possible remainders of n modulo 8 is $\{0, 1, 4, 5\}$.

(b) The set of the possible remainders of m modulo 8 is $\{1, 2, 3, 4, 5, 7\}$. Precisely,

- *(i) $m \equiv 1 \pmod 8$ if and only if $x \equiv 0 \pmod 2$ and $y \equiv 0 \pmod 4$;*
- *(ii) $m \equiv 2 \pmod 8$ if and only if $x \equiv 0 \pmod 2$ and $y \equiv \pm 1 \pmod 4$;*
- *(iii) $m \equiv 3 \pmod 8$ if and only if $x \equiv 1 \pmod 2$ and $y \equiv 0 \pmod 4$;*
- *(iv) $m \equiv 4 \pmod 8$ if and only if $x \equiv 1 \pmod 2$ and $y \equiv \pm 1 \pmod 4$;*
- *(v) $m \equiv 5 \pmod 8$ if and only if $x \equiv 0 \pmod 2$ and $y \equiv 2 \pmod 4$; and*
- *(vi) $m \equiv 7 \pmod 8$ if and only if $x \equiv 1 \pmod 2$ and $y \equiv 2 \pmod 4$.*

(c) The set of the possible remainders of m modulo 9 equals $\{0, 1, 2, 3, 4, 5, 7, 9\}$. More concretely,

- *(i) $m \equiv 0 \pmod 9$ if and only if $x \geq 2$ and $y \equiv 0 \pmod 3$;*
- *(ii) $m \equiv 1 \pmod 9$ if and only if $x = 0$ and $y \equiv 0 \pmod 3$, or $x = 1$ and $y \equiv \pm 4 \pmod 9$, or $x \geq 2$ and $y \equiv \pm 1 \pmod 9$;*
- *(iii) $m \equiv 2 \pmod 9$ if and only if $x = 0$ and $y \equiv \pm 1 \pmod 9$;*
- *(iv) $m \equiv 3 \pmod 9$ if and only if $x = 1$ and $y \equiv 0 \pmod 3$;*
- *(v) $m \equiv 4 \pmod 9$ if and only if $x = 1$ and $y \equiv \pm 1 \pmod 9$, or $x \geq 2$ and $y \equiv \pm 2 \pmod 9$;*
- *(vi) $m \equiv 5 \pmod 9$ if and only if $x = 0$ and $y \equiv \pm 2 \pmod 9$;*
- *(vii) $m \equiv 7 \pmod 9$ if and only if $x = 1$ and $y \equiv \pm 2 \pmod 9$, or $x \geq 2$ and $y \equiv \pm 4 \pmod 9$; and*
- *(viii) $m \equiv 8 \pmod 9$ if and only if $x = 0$ and $y \equiv \pm 4 \pmod 9$.*

In particular, if $x \geq 2$, then the remainder of m modulo 9 runs over $\{0, 1, 4, 7\}$.

Proof. With some direct computations, the results are immediate. □

Lemma 2.2. *The Diophantine equation*

$$3^{2x} + 2 \cdot 3^x + 9 = 4 \cdot 3^{2y+1} + (24z)^2 \tag{2.1}$$

in nonnegative integers x, y and z does not have any solutions except for $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (2, 1, 0)$.

Proof. Suppose that (x, y, z) is a solution to Eq (2.1) with $(x, y, z) \neq (0, 0, 0), (2, 1, 0)$. First of all, one easily checks that (x, y, z) with $0 \leq x \leq 2$ cannot be the solution to Eq (2.1). So, one lets $x \geq 3$, $y \geq 0$ and $z \geq 0$ in the following. One then claims that

$$y + 2 \leq x \leq 2y. \tag{2.2}$$

In fact, if $y \geq x - 1$, one then checks that

$$(24z)^2 = 3^{2x} + 2 \cdot 3^x + 9 - 4 \cdot 3^{2y+1} \leq -\frac{1}{3} \cdot 3^{2x} + 2 \cdot 3^x + 9 = -\frac{1}{3}(3^x + 3)(3^x - 9) < 0$$

since $x \geq 3$, a contradiction. So, the first inequality of (2.2) holds. Next, from (2.1) one finds that

$$(3^x + 1 - 24z)(3^x + 1 + 24z) = 4 \cdot 3^{2y+1} - 8 > 0.$$

This implies $3^x + 1 - 24z > 0$. Note that $3^x + 1 - 24z \equiv 2 \pmod 4$, and $3^x + 1 - 24z \equiv 1 \pmod 3$. This implies that $3^x + 1 - 24z \geq 10$. Therefore,

$$10 \cdot 3^x < (3^x + 1 - 24z)(3^x + 1 + 24z) = 4 \cdot 3^{2y+1} - 8 < 4 \cdot 3^{2y+1},$$

which implies that $x < 2y + \log_3(\frac{6}{5}) = 2y + 0.165\cdots$, that is, $x \leq 2y$ as desired. Then, it is immediate from (2.2) that $x \geq 4$. So, we only need to prove that Eq (2.1) has no solutions (x, y, z) for $x \geq 4$, which will be done in what follows. Let $x \geq 4$, and rewrite (2.1) as

$$3^{2x-2} + 2 \cdot 3^{x-2} - 4 \cdot 3^{2y-1} = (8z + 1)(8z - 1). \tag{2.3}$$

By (2.2), one has $x - 2 < 2y - 1 < 2x - 2$. Taking remainders of modulo 3^{x-2} both sides of (2.3), one derives that

$$(8z + 1)(8z - 1) \equiv 0 \pmod{3^{x-2}}.$$

It then follows that $8z + \lambda \equiv 0 \pmod{3^{x-2}}$ for some $\lambda \in \{\pm 1\}$. Let

$$8z = t_1 \cdot 3^{x-2} - \lambda \tag{2.4}$$

for a positive integer t_1, and then (2.3) becomes

$$3^x + 2 - 4 \cdot 3^{2y-x+1} = t_1(t_1 \cdot 3^{x-2} - 2\lambda). \tag{2.5}$$

It is also noted that $2y - x + 1 < x - 2 < x$. By taking remainders of modulo 3^{2y-x+1} both sides of (2.5), one then has that $2t_1\lambda \equiv -2 \pmod{3^{2y-x+1}}$, i.e., $t_1\lambda \equiv -1 \pmod{3^{2y-x+1}}$. So, one may write

$$t_1\lambda = t_2 \cdot 3^{2y-x+1} - 1, \tag{2.6}$$

where t_2 is an integer having the same sign as λ. Putting (2.6) into (2.5), one deduces that

$$3^{2x-2y-1} - 4 = t_1^2 \cdot 3^{2x-2y-3} - 2t_2, \tag{2.7}$$

which implies that $t_2 \equiv 2 \pmod{3^{2x-2y-3}}$. Now, let

$$t_2 = t_3 \cdot 3^{2x-2y-3} + 2, \tag{2.8}$$

where either $t_3 = 0$ or t_3 is an integer having the same sign as t_2. Substituting it into (2.7), we then obtain $t_1^2 - 2t_3 = 9$. Suppose $t_3 = 0$. One then finds $t_1 = 3$, $t_2 = 2$ and $\lambda = 1$. By (2.6), we arrive at $3 = 2 \cdot 3^{2y-x+1} - 1 \geq 5$, a contradiction. It then follows that $\frac{x_3}{\lambda}$ is a positive integer. Now, putting (2.8) and (2.6) into (2.4), we have that

$$8z = \frac{t_3}{\lambda} \cdot 3^{2x-4} + \frac{2}{\lambda} \cdot 3^{2y-1} - 3^{x-2} - \lambda. \tag{2.9}$$

Note that $\frac{x_3}{\lambda}$ is a positive integer, and $\lambda \in \{\pm 1\}$. It follows from (2.9) that

$$8z \geq 3^{2x-6} + 2\sum_{k=0}^{2x-7} 3^k.$$

Then, one deduces that

$$\begin{aligned} 3^{2x-2} + 2 \cdot 3^{x-2} - 4 \cdot 3^{2y-1} + 1 &< 3^{2x-2} + 2 \cdot 3^{x-2} + 1 \\ &< \left(3^{2x-6} + 2\sum_{k=0}^{2x-7} 3^k\right)^2 \\ &\leq (8z)^2 \end{aligned}$$

for any $x \geq 6$. This means that Eq (2.1) has no integral solution (x, y, z) with $x \geq 6$. Let $(4, y, z)$ be a solution to (2.1), and then by (2.2) one has $y = 2$. This implies that $z^2 = 10$, a contradiction. Similarly, one can confirm that $(5, y, z)$ cannot be a solution to Eq (2.1). Therefore, Eq (2.1) has no integral solution (x, y, z) with $x \geq 4$.

This finishes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all, we prove Item (a). Let k be an integer with $k \geq 2$. From Lemma 2.1 we know that there do not exist any perfect k-tuples with types $(2, 2)$ and $(3, 2)$ if $k \geq 3$, that is, $\widetilde{N}_k(2, 2) = \widetilde{N}_k(3, 2) = 0$ for any $k \geq 3$. Now, let us compute $\widetilde{N}_2(2, 2)$, that is, the number of solutions to the Diophantine equation

$$2^x + y^2 + 1 = 2^u + v^2 \tag{3.1}$$

in integers $x, y, u, v \geq 2$. For any nonnegative integer k, one observes that

$$2^k \equiv M_k \pmod{10}, \tag{3.2}$$

where

$$M_k = \begin{cases} 2, & \text{if } k \equiv 1 \pmod 4, \\ 4, & \text{if } k \equiv 2 \pmod 4, \\ 8, & \text{if } k \equiv 3 \pmod 4, \\ 6, & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Let x be any positive integer with $x \equiv 1 \pmod 4$ and $x \geq 9$, and take $u = x - 6$. It then follows from (3.2) that

$$505 \leq 2^x - 2^u + 1 \equiv 5 \pmod{10}.$$

So, we can write

$$2^x - 2^u + 1 = 2^x - 2^{x-6} + 1 = 5 \cdot \Delta_x.$$

Clearly, $\Delta_x = \frac{2^x - 2^{x-6}+1}{5}$ is an odd integer no less than 101 depending on x. If one lets $v = \frac{\Delta_x+5}{2}$ and $y = \frac{\Delta_x-5}{2}$, then (3.1) is satisfied. It follows that

$$\left(2^x + \left(\frac{\frac{2^x-2^{x-6}+1}{5} - 5}{2}\right)^2, 2^{x-6} + \left(\frac{\frac{2^x-2^{x-6}+1}{5} + 5}{2}\right)^2\right) \tag{3.3}$$

is indeed a perfect 2-tuple with type $(2, 2)$. Note that

$$x \mapsto 2^x + \left(\frac{\frac{2^x-2^{x-6}+1}{5} - 5}{2}\right)^2$$

is a one-to-one map from $\mathbb{N}$ to itself. Thus, there are infinitely many perfect 2-tuples with type $(2, 2)$ in the form (3.3). This implies that $\widetilde{N}_2(2, 2) = +\infty$. For the purpose of deriving $\widetilde{N}_2(3, 2)$, one needs to consider another Diophantine equation

$$3^m + y^n + 1 = 2^k + s^2 \tag{3.4}$$

in integers $m, n, k, s \geq 2$. For any positive integer z, it is easy to see that

$$(m, n, s, t) = \left(4z, \frac{\frac{3^{4z}-3^{4z-1}+1}{5} - 5}{2}, 4z - 1, \frac{\frac{3^{4z}-3^{4z-1}+1}{5} + 5}{2}\right)$$

is a solution to (3.4). That is to say,

$$\left(3^{4z} + \left(\frac{\frac{3^{4z}-3^{4z-1}+1}{5} - 5}{2}\right)^2, 3^{4z-1} + \left(\frac{\frac{3^{4z}-3^{4z-1}+1}{5} + 5}{2}\right)^2\right) \tag{3.5}$$

is a perfect 2-tuple with type $(3, 2)$. Also, one checks that

$$z \mapsto 3^{4z} + \left(\frac{\frac{3^{4z}-3^{4z-1}+1}{5} - 5}{2}\right)^2$$

is one-to one from $\mathbb{N}$ to itself as well. So, there are infinitely many perfect 2-tuples with type $(3, 2)$ as the form in (3.5). It follows that $\widetilde{N}_2(3, 2) = +\infty$.

Next, we turn our attention to the proof of item (b). Let k be any positive integer with $k \geq 2$. First, it is immediate that $N_2(3, 2) = +\infty$ since $N_2(3, 2) \geq \widetilde{N}_2(3, 2) = +\infty$.

Second, let $k = 3$ and y be a nonnegative integer. Then, $(y^2 + 3^0, y^2 + 2, y^2 + 3^1)$ is a nice 3-tuple with type $(3, 2)$ if and only if the equation

$$y^2 + 2 = u^2 + 3^v \tag{3.6}$$

in nonnegative integers y, u and v has at least one solution. Note that

$$y^2 + 2 = u^2 + 3^v \Leftrightarrow y^2 - u^2 = 3^v - 2 \Leftrightarrow (y + u)(y - u) = 3^v - 2.$$

Then, one may take $y + u = 3^v - 2$ and $y - u = 1$ and infer that

$$(y, u, v) = \left(\frac{3^v - 1}{2}, \frac{3^v - 3}{2}, v\right)$$

is a solution to (3.6) for any positive integer v. It then follows that

$$\left(\left(\frac{3^v - 1}{2}\right)^2 + 3^0, \left(\frac{3^v - 3}{2}\right)^2 + 3^v, \left(\frac{3^v - 1}{2}\right)^2 + 3^1\right)$$

is a nice 3-tuple with type $(3, 2)$ for any positive integer v. Together with $v \mapsto \left(\frac{3^v-1}{2}\right)^2 + 3^0$ being injective from $\mathbb{Z}^+$ to itself, this implies that there exist infinitely many nice 3-tuples with type $(3, 2)$, i.e., $N_3(3, 2) = +\infty$.

Third, let $k = 4$. Let $(A, A + 1, A + 2, A + 3)$ be a nice 4-tuple with type $(3, 2)$, and $A = 3^{x_0} + y_0^2$ with x_0, y_0 being nonnegative integers. From Item (a) of Lemma 2.1, one knows that

$$(A, A + 1, A + 2, A + 3) \equiv (1, 2, 3, 4) \text{ or } (2, 3, 4, 5) \pmod 8.$$

Case 1. $(A, A + 1, A + 2, A + 3) \equiv (1, 2, 3, 4) \pmod 8$. By Item (b-i) of Lemma 2.1, we have that x_0 is even, and $y_0 \equiv 0 \pmod 4$. Now, we split all possible values of x_0 into the following subcases.

Subcase 1.1. $x_0 = 0$. Item (c) of Lemma 2.1 tells us that only three kinds of results for $(A, A + 1, A + 2, A + 3)$ modulo 9 would happen, that is,

$$(A, A + 1, A + 2, A + 3) \equiv (1, 2, 3, 4), \text{ or } (2, 3, 4, 5), \text{ or } (8, 0, 1, 2) \pmod 9,$$

which will be handled one by one in what follows.

Subcase 1.1.1. $(A, A + 1, A + 2, A + 3) \equiv (1, 2, 3, 4) \pmod 9$. Let

$$A + 1 = y_0^2 + 2 = y_1^2 + 3^{x_1},$$

which is congruent to 2 modulo 9. It follows from Item (c-iii) of Lemma 2.1 that $x_1 = 0$. Then, we have

$$y_1^2 - y_0^2 = 1.$$

This gives us that $y_1 = 1$ and $y_0 = 0$. So, $(A, A + 1, A + 2, A + 3) = (1, 2, 3, 4)$. It is indeed a nice 4-tuple with type $(3, 2)$ since $(1, 2, 3, 4) = (0^2 + 3^0, 1^2 + 3^0, 0^2 + 3^1, 1^2 + 3^1)$.

Subcase 1.1.2. $(A, A + 1, A + 2, A + 3) \equiv (2, 3, 4, 5) \pmod 9$. Let $A + 1 = y_1^2 + 3^{x_1}$, that is,

$$y_0^2 + 2 = y_1^2 + 3^{x_1}. \tag{3.7}$$

Clearly, $y_1^2 + 3^{x_1} \equiv 3 \pmod 9$. By Item (c-iv) of Lemma 2.1, one has $x_1 = 1$. Then, (3.7) becomes $(y_0 + y_1)(y_0 - y_1) = 1$, i.e., $y_0 = 1$ and $y_1 = 0$, contradicting $y_0 \equiv 0 \pmod 4$. Therefore, in this subcase there are no nice 4-tuples with type $(3, 2)$.

Subcase 1.1.3. $(A, A+1, A+2, A+3) \equiv (8, 0, 1, 2) \pmod 9$. Let $A + 3 = y_3^2 + 3^{x_3}$. On the one hand, $y_3^2 + 3^{x_3} \equiv 4 \pmod 8$. It follows from Item (b-iv) of Lemma 2.1 that

$$x_3 \equiv 1 \pmod 2. \tag{3.8}$$

On the other hand, one notes that $y_3^2 + 3^{x_3} \equiv 2 \pmod 9$. By Item (c-iii) of Lemma 2.1, one then derives that $x_3 = 0$, a contradiction with (3.8). Thus, there does not exist any nice 4-tuple with type $(3, 2)$ in the subcase.

Subcase 1.2. $x_0 \geq 2$ is an even number. By Item (c) of Lemma 2.1, one checks that

$$(A, A+1, A+2, A+3) \equiv (0, 1, 2, 3), \text{ or } (1, 2, 3, 4), \text{ or } (7, 8, 0, 1) \pmod 9.$$

Then, we have the following discussions.

Subcase 1.2.1. $(A, A+1, A+2, A+3) \equiv (0, 1, 2, 3) \pmod 9$. Let $A + 2 = 3^{x_2} + y_2^2$. Note that

$$3^{x_2} + y_2^2 \equiv 3 \pmod 8, \text{ and } 3^{x_2} + y_2^2 \equiv 2 \pmod 9.$$

It then follows from Items (b-iii) and (c-iii) of Lemma 2.1 that $x_2 \equiv 1 \pmod 2$ and $x_2 = 0$, a contradiction. So, in this subcase we have no nice 4-tuples with type $(3, 2)$.

Subcase 1.2.2. $(A, A+1, A+2, A+3) \equiv (1, 2, 3, 4) \pmod 9$. First, applying Item (c-ii) of Lemma 2.1 to the fact $3^{x_0} + y_0^2 \equiv 1 \pmod 9$, we know that

$$y_0 \equiv \pm 1 \pmod 9. \tag{3.9}$$

Next, we let

$$A + 1 = y_1^2 + 3^{x_1}, \tag{3.10}$$

which is congruent to 2 modulo 9. By Item (c-iii) of Lemma 2.1, one then has $x_1 = 0$ and $y_1 \equiv \pm 1 \pmod 9$. Putting $x_1 = 0$ into (3.10), one has $y_1^2 - y_0^2 = 3^{x_0}$, i.e.,

$$(y_1 + y_0)(y_1 - y_0) = 3^{x_0}. \tag{3.11}$$

If $x_0 = 2$, then (3.11) implies that

$$(y_1, y_0) = (3, 0), \text{ or } (5, 4),$$

which contradicts (3.9). In the following, let $x_0 \geq 4$. By (3.11), one may let $y_1 + y_0 = 3^t$ and $y_1 - y_0 = 3^{x_0 - t}$ with $t \leq x_0$ being a nonnegative integer. Note that $x_0 \geq 4$ and $y_0 \neq 0$. Then, $y_0 = \frac{3^t - 3^{x_0 - t}}{2}$ with $2 \leq \frac{x_0}{2} < t \leq x_0$. Write $y_0 = \frac{3^t - 1}{2} - \frac{3^{x_0 - t} - 1}{2}$, and one computes that

$$y_0 = \sum_{k=0}^{t-1} 3^k - \sum_{j=0}^{x_0 - t - 1} \equiv 0 \pmod 9 \text{ if } \frac{x_0}{2} + 1 \leq t \leq x_0 - 2, \tag{3.12}$$

$$y_0 = \sum_{k=0}^{t-1} 3^k - 1 \equiv 3 \pmod 9 \text{ if } t = x_0 - 1, \text{ and} \tag{3.13}$$

$$y_0 = \sum_{k=0}^{t-1} 3^k \equiv 4 \pmod 9 \text{ if } t = x_0. \tag{3.14}$$

Obviously, all the results of (3.12)–(3.14) contradict with (3.9). Hence, we have no nice 4-tuples with type $(3, 2)$ in this subcase.

Subcase 1.2.3. $(A, A+1, A+2, A+3) \equiv (7, 8, 0, 1) \pmod 9$. In this subcase, $3^{x_0} + y_0^2 \equiv 7 \pmod 9$, which gives us $y_0 \equiv \pm 4 \pmod 9$. Let $A + 1 = 3^{x_1} + y_1^2$, i.e.,

$$3^{x_0} + y_0^2 + 1 = 3^{x_1} + y_1^2. \tag{3.15}$$

Note that $3^{x_1} + y_1^2 \equiv 8 \pmod 9$. By Item (c-viii) of Lemma 2.1, one has $x_1 = 0$. Then, (3.15) can be simplified to $y_1^2 - y_0^2 = 3^{x_0}$. If $x_0 = 2$, one then deduces that $y_1 = 5$ and $y_0 = 4$. This gives a nice 4-tuple $(A, A+1, A+2, A+3) = (25, 26, 27, 28)$ with type $(3, 2)$, since $(25, 26, 27, 28) = (4^2 + 3^2, 5^2 + 3^0, 0^2 + 3^3, 5^2 + 3^1)$. Now, let $x_0 \geq 4$. As in Subcase 1.2.2, (3.12)–(3.14) can also be derived. Note that $y_0 \equiv \pm 4 \pmod 9$. It then follows that only (3.14) would occur among (3.12)–(3.14). So, we have

$$y_0 = \frac{3^{x_0} - 1}{2} \text{ and } y_1 = \frac{3^{x_0} + 1}{2},$$

implying that

$$(A, A+1, A+2, A+3) = \left(\left(\frac{3^{x_0} - 1}{2}\right)^2 + 3^{x_0}, \left(\frac{3^{x_0} + 1}{2}\right)^2 + 3^0, A+2, \left(\frac{3^{x_0} + 1}{2}\right)^2 + 3^1\right).$$

Let $A + 2 = 3^{x_2} + y_2^2$. We know that $3^{x_2} + y_2^2 \equiv 0 \pmod 9$ and $3^{x_2} + y_2^2 \equiv 3 \pmod 8$. It follows from Items (b-iii) and (c-i) of Lemma 2.1 that $y_2 \equiv 0 \pmod{12}$ and $x_2 \equiv 1 \pmod 2$. Write $y_2 = 12t$ and $x_2 = 2k + 1$ with $t \geq 0$ and $k \geq 1$ being integers. Hence, we have that $(A, A+1, A+2, A+3)$ is a nice 4-tuple with type $(3, 2)$ if and only if the equation

$$\left(\frac{3^{x_0} - 1}{2}\right)^2 + 3^{x_0} + 2 = (12t)^2 + 3^{2k+1} \tag{3.16}$$

in nonnegative integers x_0, t, k with $x_0 \geq 4$ being even and $k \geq 1$ has at least one solution. However, Lemma 2.2 tells that (3.16) has no solutions. So, $(A, A+1, A+2, A+3)$ is not a nice 4-tuple with type $(3, 2)$ in this subcase.

Next, we discuss the second case.

Case 2. $(A, A+1, A+2, A+3) \equiv (2, 3, 4, 5) \pmod 8$. In this case, $A = 3^{x_0} + y_0^2 \equiv 2 \pmod 8$. From Item (b-ii) of Lemma 2.1, one has $x_0 \equiv 0 \pmod 2$ and $y_0 \equiv \pm 4 \pmod 4$. Then, the following subcases are considered.

Subcase 2.1. $x_0 = 0$. Item (c) of Lemma 2.1 tells us that

$$(A, A+1, A+2, A+3) \equiv (1, 2, 3, 4), (2, 3, 4, 5) \text{ or } (8, 0, 1, 2) \pmod 9.$$

First, we assume that $(A, A+1, A+2, A+3) \equiv (1, 2, 3, 4) \pmod 9$. If let $A + 1 = 3^{x_1} + y_1^2$, then $3^{x_1} + y_1^2 \equiv 2 \pmod 9$. It follows from Item (c-iii) of Lemma 2.1 that $x_1 = 0$. So, one derives that $y_1^2 - y_0^2 = 1$, which implies that $y_0 = 0$. This is impossible since $y_0 \equiv \pm 1 \pmod 4$.

Second, assume that $(A, A+1, A+2, A+3) \equiv (2, 3, 4, 5) \pmod 9$. Let $A + 1 = 3^{x_1} + y_1^2$, which is 3 modulo 9. Then, by Item (c-iv) of Lemma 2.1, one has $x_1 = 1$. Now, putting $x_1 = 1$ and $x_0 = 0$ into $3^{x_0} + y_0^2 + 1 = 3^{x_1} + y_1^2$, one derives $y_1 = 0$ and $y_0 = 1$. This is to say $(A, A+1, A+2, A+3) = (2, 3, 4, 5)$, which is really a nice 4-tuple with type $(3, 2)$ since $(2, 3, 4, 5) = (3^0 + 1^2, 3^1 + 0^2, 3^1 + 1^2, 3^0 + 2^2)$.

Finally, assume that $(A, A+1, A+2, A+3) \equiv (8, 0, 1, 2) \pmod 9$. In this case, $3^{x_0} + y_0^2 \equiv 8 \pmod 9$. It is implied from Item (c-viii) of Lemma 2.1 that

$$y_0 \equiv \pm 4 \pmod 9. \tag{3.17}$$

Let $A + 3 = 3^{x_3} + y_3^2$, which clearly is 2 modulo 9. Then, by Item (c-iii) of Lemma 2.1, one gets $x_3 = 0$. Substituting $x_0 = x_3 = 0$ into $3^{x_0} + y_0^2 + 3 = 3^{x_3} + y_3^2$, one deduces that $y_3^2 - y_0^2 = 3$. Consequently, $y_0 = 1$, and $y_3 = 2$, which contradicts (3.17).

SUBCASE 2.2. $x_0 \geq 2$ is an even number. By Item (c) of Lemma 2.1 we have that

$$(A, A+1, A+2, A+3) \equiv (0, 1, 2, 3), (1, 2, 3, 4) \text{ or } (7, 8, 0, 1) \pmod 9. \tag{3.18}$$

Now, one claims that all the congruences in (3.18) cannot happen, so there is not any nice 4-tuple with type $(3, 2)$ in this subcase. First, suppose $(A, A+1, A+2, A+3) \equiv (0, 1, 2, 3) \pmod 9$. Let $A + 2 = 3^{x_2} + y_2^2$, and we then know that $3^{x_2} + y_2^2 \equiv 4 \pmod 8$ and $3^{x_2} + y_2^2 \equiv 2 \pmod 9$. It follows from Items (b-iv) and (c-iii) of Lemma 2.1 that $x_2 \equiv 1 \pmod 2$ and $x_2 = 0$, a contradiction. Second, suppose $(A, A+1, A+2, A+3) \equiv (1, 2, 3, 4) \pmod 9$. Let $A + 1 = 3^{x_1} + y_1^2$. Note that $3^{x_1} + y_1^2 \equiv 3 \pmod 8$ and $3^{x_1} + y_1^2 \equiv 2 \pmod 9$. From Items (b-iii) and (c-iii) of Lemma 2.1, one derives that $x_1 \equiv 1 \pmod 2$ and $x_1 = 0$, a contradiction as well. Third, suppose $(A, A+1, A+2, A+3) \equiv (7, 8, 0, 1) \pmod 9$. If $A + 1 = 3^{x_1} + y_1^2$, one then finds that $3^{x_1} + y_1^2 \equiv 3 \pmod 8$ and $3^{x_1} + y_1^2 \equiv 8 \pmod 9$. From Items (b-iii) and (c-viii) of Lemma 2.1, it follows that $x_1 \equiv 1 \pmod 2$ and $x_1 = 0$, still a contradiction.

Combining all cases above, we have that there are only three nice 4-tuples with type $(3, 2)$. More precisely, all of the nice 4-tuples with type $(3, 2)$ are $(1, 2, 3, 4)$, $(2, 3, 4, 5)$ and $(25, 26, 27, 28)$. From this, we see that $(1, 2, 3, 4, 5)$ is the only nice 5-tuple with type $(3, 2)$, and there is no nice k-tuple with type $(3, 2)$ for any $k \geq 6$.

The proof of Theorem 1.1 is complete. □

4. Conclusions

The gaps in integer sequences are wide problems in number theory. The gap of primes $|p_n - p_{n+1}|$ is one of the most important topics in analytic Number Theory. In the field of Diophantine analysis, there are many open questions on the gap of the powers $|x^m - y^n|$. In this paper, we considered the gap $|(a^{x_1} + y_1^b) - (a^{x_2} + y_2^b)|$. In fact, we studied k-tuples of consecutive integers $(a_1, a_2, \cdots, a_k)$ such that each of them is the sum of powers. We used the method of modulo coverage together with some elementary techniques to present the formulas for $\widetilde{N}_k(2, 2)$, $\widetilde{N}_k(3, 2)$ and $N_k(3, 2)$. Note that in this paper we obtain that $\widetilde{N}_2(2, 2) = +\infty$ and $N_2(3, 2) = +\infty$. However, we neither give all perfect 2-tuples with types $(2, 2)$ and $(3, 2)$ nor present all nice k-tuples with type $(3, 2)$ when $k \geq 6$. It seems a little difficult to do this. Here, we post a problem as future research.

Problem 4.1. *Let a, b and k be integers with $k, a, b \geq 2$. Let $N_k(a, b)$ be the number of nice k-tuples with type (a, b).*

(A) Find (a, b) such that $N_k(a, b)$ can be completely determined.

(B) For fixed a, b and k, figure out the set of nice k-tuples with type (a, b) completely if $N_k(a, b) = +\infty$.

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Conflict of interest

There is no conflict of interest.

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