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Research article

Odd symmetry of ground state solutions for the Choquard system

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Abstract: This paper is dedicated to the following Choquard system:

$$\begin{cases} -\Delta u + u = \dfrac{2p}{p+q}(I_\alpha * |v|^q)|u|^{p-2}u, \\ -\Delta v + v = \dfrac{2q}{p+q}(I_\alpha * |u|^p)|v|^{q-2}v, \\ u(x) \to 0, \quad v(x) \to 0 \ \text{ as } |x| \to \infty, \end{cases}$$

where $N \geq 1$, $\alpha \in (0, N)$ and $\frac{N+\alpha}{N} < p,\ q < 2_*^\alpha$, in which 2_*^α denotes $\frac{N+\alpha}{N-2}$ if $N \geq 3$ and $2_*^\alpha := \infty$ if $N = 1,\ 2$. I_α is a Riesz potential. We obtain the odd symmetry of ground state solutions via a variant of Nehari constraint. Our results can be looked on as a partial generalization to results by Ghimenti and Schaftingen (Nodal solutions for the Choquard equation, J. Funct. Anal. 271 (2016), 107).

Keywords: Choquard system; odd symmetry; ground state solution; nonlocal Brézis-Lieb lemma
Mathematics Subject Classification: 35J20, 35J05, 35J60

1. Introduction

In this paper, we study the following Choquard system:

$$\begin{cases} -\Delta u + u = \dfrac{2p}{p+q}(I_\alpha * |v|^q)|u|^{p-2}u, \\ -\Delta v + v = \dfrac{2q}{p+q}(I_\alpha * |u|^p)|v|^{q-2}v, \\ u(x) \to 0, \quad v(x) \to 0 \ \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $u := u(x), v := v(x)$ in $H^1(\mathbb{R}^N)$ $(N \geq 1)$ are real valued functions. I_α: $\mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined at each point $x \in \mathbb{R}^N \backslash \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \text{ with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})}, \quad \alpha \in (0, N),$$

where Γ denotes the classical Gamma function and $*$ the convolution on the Euclidean space $\mathbb{R}^N$. When $p = q$ and $u(x) \equiv v(x)$, the system (1.1) is the following Choquard equation:

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Choquard Eq (1.2) is firstly appeared in a work by Pekar describing the quantum mechanics of a polaron at rest [21]. In the case of $N = 3$, $\alpha = 2$ and $p = 2$, Choquard described an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [10]. In the pioneering work [10], Lieb first proved the existence and uniqueness of positive solutions. Later Lions [12, 13] obtained the existence and multiplicity of solutions to (1.2). In 1996, Penrose proposed a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [15]. For more existence and qualitative properties of solutions to (1.2), we refer the reader to [1, 14, 16, 18–20] and references therein.

In recent years, there has been increasing attention to equations like (1.2) on the existence of positive solutions and ground states solutions. For the optimal range of parameters $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. Here and after, the 2_*^α denotes $\frac{N+\alpha}{N-2}$ if $N \geq 3$ and $2_*^\alpha := \infty$ if $N = 1, 2$. By using the concentration-compactness and Brézis-Lieb lemmas, Moroz and Schaftingen [17] showed the existence of ground state solutions for (1.2). Under symmetric assumptions on Ω, which is an unbounded smooth domain in $\mathbb{R}^N$, Clapp and Salazar [5] proved the existence of positive solutions of (1.2).

The Eq (1.2) with an additional perturbation of the following form:

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, \quad x \in \mathbb{R}^3 \tag{1.3}$$

has been studied in [2], where the existence of solutions is obtained for $N = 3$, $0 < \alpha < 1$, $p = 2$ and $4 \leq q < 6$. When $N = 3$, $\alpha = 2$, $p = 2$ and $q \in (2, 6)$, Vaira [24, 25] has obtained a positive radial ground state solution and further studied the nondegeneracy of the radial ground state solution for the special case $q = 3$. With the help of the mountain pass theorem and Pohožaev identity, Li et al. in [9] studied the Choquard Eq (1.3) with $\frac{N+\alpha}{N} < p < 2_*^\alpha$ and $2 < q < \frac{2N}{N-2}, N \geq 3$, where the existence of ground state solution of mountain pass type is obtained. Here, we would also like to mention the papers [7, 23, 30] for related topics.

For elliptic system of Choquard type, Chen and Liu [3] have obtained the existence of positive radial ground state solutions to

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v, \\ -\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \lambda u. \end{cases}$$

Yang et al. [29] have proved the existence of positive radial ground state solutions when p reaches the critical exponent. A slightly general version of system of Choquard type was studied by Xu et al. [28], where the authors have proven that the system of Choquard type admits a nontrivial vector solution under the Schwarz symmetrization method. In [3, 28, 29], all of them are positive solutions or ground state solutions of the linear coupled type Choquard system.

But we do not see any results to (1.1) in the case of $p \neq q$. The main purpose of the present paper is to study odd symmetry of ground state solutions to (1.1). According to Fubini theorem, we obtain the following symmetry property: for every $u, v \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |v|^q)|u|^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{A_\alpha |v(x)|^q |u(y)|^p}{|x-y|^{N-\alpha}} dxdy = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q.$$

Therefore for each function (u, v) in the Sobolev space $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we call $(u, v) \in H$ is a weak solution of (1.1) if for any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi_1 + u\varphi_1 - \frac{2p}{p+q}(I_\alpha * |v|^q)|u|^{p-2}u\varphi_1\Big) = 0$$

and

$$\int_{\mathbb{R}^N} \Big(\nabla v \nabla \varphi_2 + v\varphi_2 - \frac{2q}{p+q}(I_\alpha * |u|^p)|v|^{q-2}v\varphi_2\Big) = 0.$$

Hence there is a one-to-one correspondence between solutions of (1.1) and critical points of the following functional $\mathcal{I}$: $H \to \mathbb{R}$ defined by

$$\mathcal{I}(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2) - \frac{2}{p+q} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q.$$

For any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$ and $(u, v) \in H$, we compute the Gateaux derivative:

$$\begin{aligned}\langle \mathcal{I}'(u, v), (\varphi_1, \varphi_2)\rangle = \int_{\mathbb{R}^N} \Big(&\nabla u \nabla \varphi_1 + u\varphi_1 + \nabla v \nabla \varphi_2 + v\varphi_2 \\ &- \frac{2p}{p+q}(I_\alpha * |v|^q)|u|^{p-2}u\varphi_1 - \frac{2q}{p+q}(I_\alpha * |u|^p)|v|^{q-2}v\varphi_2\Big).\end{aligned}$$

Then, $(u, v) \in H$ is a solution of (1.1) if and only if

$$\langle \mathcal{I}'(u, v), (\varphi_1, \varphi_2)\rangle = 0.$$

In view of the Hardy-Littlewood-Sobolev inequality (see [11, Theorem 4.3]), which states that if $s \in (1, \frac{N}{\alpha})$ then for every $v \in L^s(\mathbb{R}^N)$, $I_\alpha * v \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} \leq C\Big(\int_{\mathbb{R}^N} |v|^s\Big)^{\frac{N}{N-\alpha s}}, \tag{1.4}$$

and of the classical Sobolev embedding, the action functional $\mathcal{I}$ is well defined and continuously differentiable whenever $\frac{N+\alpha}{N} < p,\ q < 2_*^\alpha$. Denote

$$\mathcal{P}(u, v) := \langle \mathcal{I}'(u, v), (u, v)\rangle = 0$$

and set

$$\mathcal{N}_0 = \{(u, v) \in H \backslash \{(0, 0)\} \mid \mathcal{P}(u, v) = 0\}.$$

We know that $\mathcal{N}_0$ is a Nahari manifold which can be a natural constraint to find critical point of the functional $\mathcal{I}$ on H. A nontrivial solution $(u, v) \in H$ of (1.1) is called a ground state if

$$\mathcal{I}(u, v) = c_0 := \inf_{(u,v)\in \mathcal{N}_0} \mathcal{I}(u, v).$$

Motivated by the results in [4], we consider the Sobolev space of odd functions

$$H_{\text{odd}} := H^1_{\text{odd}}(\mathbb{R}^N) \times H^1_{\text{odd}}(\mathbb{R}^N),$$

where

$$H^1_{\text{odd}}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \mid u(x', -x_N) = -u(x', x_N) \ \text{ a.e. } \ (x', x_N) \in \mathbb{R}^N\}.$$

Define the odd Nehari manifold

$$\mathcal{N}_{\text{odd}} = \mathcal{N}_0 \cap H_{\text{odd}}$$

and the corresponding level $c_{\text{odd}} = \inf\limits_{\mathcal{N}_{\text{odd}}} \mathcal{I}$.

Our main result is that this level c_{odd} is achieved.

Theorem 1.1. *If $p,\ q \in \left(\frac{N+\alpha}{N}, 2^\alpha_*\right)$ and $p \neq q$, then there exists a solution $(u, v) \in H_{\text{odd}}$ to the system* (1.1) *such that $\mathcal{I}(u, v) = c_{\text{odd}}$.*

Remark 1.2. The question we are interested in is whether the sign-changing solution of system (1.1) is odd, thus consistent with the sign-changing solution of Theorem 1.1, and it is not even known whether the sign-changing solution has axial symmetry with Theorem 1.1. We consider these issues as further research questions.

Remark 1.3. Since $\mathcal{N}_{\text{odd}} = \mathcal{N}_0 \cap H_{\text{odd}} \subset \mathcal{N}_0$ we have $c_{\text{odd}} \geq c_0$. We not know whether $c_0 = c_{\text{odd}}$.

Remark 1.4. Compared with related results, there are some differences and difficulties in our proofs: Firstly, by using the mountain pass theorem, the authors in [4] proved the existence of ground state solution. Here, we use the method of Nehari manifold (see Section 2) for our proof. Moreover, we generalize the Brézis-Lieb lemma for the nonlocal term $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q$ (see Lemma 2.5).

Secondly, different from [6, 8, 26], in this paper, we have to overcome the difficulty caused by the nonlinear coupled Choquard term $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q$. Finally, in [3, 28, 29], they all obtain the positive solutions or ground state solutions of the Choquard system with $p = q$ and $u(x) = v(x)$, and these solutions have radial symmetry. However, we prove that the positive ground state solution of the Choquard system (1.1) with $p \neq q$ and $u(x) \neq v(x)$ has odd symmetry.

The rest of the paper is organized as follows. In Section 2, we set the variational framework for system (1.1) and some preliminary results. Section 3 is devoted to showing the existence of ground state solutions to system (1.1) by using minimizing arguments on odd Nehari set and then prove the Theorem 1.1.

2. Preliminaries and the proof Theorem 1.1

Throughout this paper, $\|u\|_{H^1}$ and $|u|_r$ denote the usual norm of $H^1(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ for $r > 1$, respectively. Let $\|(u, v)\|^2 := \|u\|^2_{H^1} + \|v\|^2_{H^1}$. For convenience, C and C_i ($i = 1, 2, \ldots$) denote (possibly different) positive constants, $\int_{\mathbb{R}^N} g$ denotes the integral $\int_{\mathbb{R}^N} g(z)dz$. The "$\rightarrow$" and "$\rightharpoonup$" denote strong convergence and weak convergence, respectively.

Lemma 2.1. *If (u, v) is a weak solution of* (1.1)*, then (u, v) satisfies $P(u, v) = 0$, where*

$$\begin{aligned} P(u, v) : = & \frac{N-2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) + \frac{N}{2} \int_{\mathbb{R}^N} (|u|^2 + |v|^2) \\ & - \frac{2(N+\alpha)}{p+q} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q. \end{aligned}$$

Proof. The proof is standard, so we omit the details here. □

Let $t \in \mathbb{R}^+$ and $(u, v) \in H_{\text{odd}}$, one has

$$\mathcal{I}(tu, tv) = \frac{1}{2}t^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2) - \frac{2}{p+q} t^{p+q} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q.$$

Denote

$$h(t) := \mathcal{I}(tu, tv).$$

Since

$$p + q > \frac{2(N+\alpha)}{N} > 2,$$

we see that $h(t) > 0$ for $t > 0$ small enough and $h(t) \to -\infty$ as $t \to +\infty$, which implies that $h(t)$ attains its maximum.

Lemma 2.2. *Let θ_1 and θ_2 be positive constants. For $t \geq 0$, we define*

$$h(t) := \theta_1 t^2 - \theta_2 t^{p+q}.$$

Then h has a unique critical point which corresponds to its maximum.

Proof. We already know that h has a maximum. For $t \geq 0$, we compute directly that derivatives of h:

$$h'(t) = 2\theta_1 t - (p+q)\theta_2 t^{p+q-1}.$$

Since $h'(t) \to -\infty$ as $t \to +\infty$ and is positive for $t > 0$ small, we obtain that there is $t > 0$ such that $h'(t) = 0$. The uniqueness of the critical point of h follows from the fact that the equation

$$h'(t) = 2\theta_1 t - (p+q)\theta_2 t^{p+q-1} = 0$$

has a unique positive solution $(\frac{2\theta_1}{(p+q)\theta_2})^{\frac{1}{p+q-2}}$. □

Lemma 2.3. *Suppose that $u,\ v \in H^1_{\text{odd}}(\mathbb{R}^N)\backslash\{0\}$, then there is a unique $\tilde{t} := t(u, v) > 0$ such that h attains its maximum at $\tilde{t}$ and*

$$c_{\text{odd}} = \inf_{(u,v)\in H_{\text{odd}}} \max_{t>0} \mathcal{I}(tu, tv).$$

Moreover, if $\mathcal{P}(u, v) < 0$, then $\tilde{t} \in (0, 1)$.

Proof. For every $u,\ v \in H^1_{\text{odd}}(\mathbb{R}^N)\backslash\{0\}$ and any $t > 0$, we consider

$$\begin{aligned} h(t) &= \mathcal{I}(tu, tv) \\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2) - \frac{2}{p+q} t^{p+q} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q. \end{aligned}$$

From Lemma 2.2, h has a unique critical point $\tilde{t} > 0$ corresponding to its maximum, i.e.,

$$h(\tilde{t}) = \max_{t>0} h(t),\ h'(\tilde{t}) = 0,$$

and hence $\mathcal{P}(\tilde{t}u, \tilde{t}v) = 0$ and $(\tilde{t}u, \tilde{t}v) \in \mathcal{N}_{\text{odd}}$. If $\mathcal{P}(u, v) < 0$, one has

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2) - 2\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q < 0,$$

$$\tilde{t}^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2) - 2\tilde{t}^{p+q} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q = 0,$$

then

$$\left(\tilde{t}^{p+q} - \tilde{t}^2\right) \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2) < 0,$$

which implies that $\tilde{t} < 1$. This finishes the proof. □

Lemma 2.4. *The $\mathcal{N}_{\text{odd}}$ is a C^1 manifold and every critical point of $\mathcal{I}|_{\mathcal{N}_{\text{odd}}}$ is a critical point of $\mathcal{I}$ in H_{odd}.*

Proof. According to Lemma 2.3, we know $\mathcal{N}_{\text{odd}} \neq \emptyset$.
Claim 1. $\mathcal{N}_{\text{odd}}$ is bounded away from zero. For any $(u, v) \in \mathcal{N}_{\text{odd}}$, by using $\mathcal{P}(u, v) = 0$, the semigroup property of the Riesz potential $I_\alpha = I_{\frac{\alpha}{2}} * I_{\frac{\alpha}{2}}$ [11, Theorem 5.9 and Corollary 5.10], Cauchy-Schwarz inequality, (1.4), Sobolev and Young inequalities, one has

$$\begin{aligned}
\|(u, v)\|^2 &= \mathcal{P}(u, v) + 2\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q \\
&= 2\int_{\mathbb{R}^N} (I_{\frac{\alpha}{2}} * |u|^p)(I_{\frac{\alpha}{2}} * |v|^q) \\
&\leq 2\Big(\int_{\mathbb{R}^N} (I_{\frac{\alpha}{2}} * |u|^p)^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} (I_{\frac{\alpha}{2}} * |v|^q)^2\Big)^{\frac{1}{2}} \\
&= 2\Big(\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} (I_\alpha * |v|^q)|v|^q\Big)^{\frac{1}{2}} \\
&\leq C_1\Big(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}}\Big(\int_{\mathbb{R}^N} |v|^{\frac{2Nq}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}} \\
&\leq C_2\|u\|_{H^1}^p\|v\|_{H^1}^q \\
&\leq C\|(u, v)\|^{\frac{p+q}{2}}.
\end{aligned}$$

Hence, there is $C' > 0$ such that $\|(u, v)\| \geq C'$. This proves that $\mathcal{N}_{\text{odd}}$ is bounded away from zero.
Claim 2. $c_{\text{odd}} > 0$. Since $\mathcal{N}_{\text{odd}} = \mathcal{N}_0 \cap H_{\text{odd}} \subset \mathcal{N}_0$ we have $c_{\text{odd}} \geq c_0$. For any $(u, v) \in \mathcal{N}_0$,

$$\begin{aligned}
\mathcal{I}(u, v) &= \mathcal{I}(u, v) - \frac{1}{p+q}\mathcal{P}(u, v) \\
&= \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2) \\
&> 0,
\end{aligned}$$

we obtain $c_0 > 0$.
Claim 3. The $\mathcal{N}_{\text{odd}}$ is a C^1 manifold. Since $\mathcal{P}(u, v)$ is a C^1 functional, in order to prove $\mathcal{N}_{\text{odd}}$ is a

C^1 manifold, it suffices to prove that $\mathcal{P}'(u,v) \neq 0$ for all $(u,v) \in \mathcal{N}_{\text{odd}}$. Suppose on the contrary that $\mathcal{P}'(u,v) = 0$ for some $(u,v) \in \mathcal{N}_{\text{odd}}$, then (u,v) satisfies

$$\begin{cases} -\Delta u + u = p(I_\alpha * |v|^q)|u|^{p-2}u, \\ -\Delta v + v = q(I_\alpha * |u|^p)|v|^{q-2}v, \end{cases}$$

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = p \int_{\mathbb{R}^N} (I_\alpha * |v|^q)|u|^p,$$

and

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) = q \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q.$$

Therefore,

$$\begin{aligned} 0 &= \mathcal{P}(u,v) \\ &= \|(u,v)\|^2 - 2\int_{\mathbb{R}^N} (I_\alpha * |u|^p|v|^q) \\ &= (p+q-2)\int_{\mathbb{R}^N} (I_\alpha * |u|^p|v|^q) > 0, \end{aligned}$$

which is a contradiction. Hence $\mathcal{P}'(u,v) \neq 0$ for any $(u,v) \in \mathcal{N}_{\text{odd}}$.

Claim 4. Every critical point of $\mathcal{I}|_{\mathcal{N}_{\text{odd}}}$ is a critical point of $\mathcal{I}$ in H_{odd}. If (u,v) is a critical point of $\mathcal{I}|_{\mathcal{N}_{\text{odd}}}$, i.e., $(\mathcal{I}|_{\mathcal{N}_{\text{odd}}})'(u,v) = 0$ and $(u,v) \in \mathcal{N}_{\text{odd}}$. Thanks to the Lagrange multiplier rule, there exists $\rho \in \mathbb{R}$ such that

$$\mathcal{I}'(u,v) = \rho\mathcal{P}'(u,v),$$

i.e.,

$$0 = \langle \mathcal{I}'(u,v),(u,v)\rangle = \rho\langle \mathcal{P}'(u,v),(u,v)\rangle.$$

According to $\mathcal{P}'(u,v) = 0$ and its corresponding Pohožaev identity we get

$$\begin{aligned} \langle \mathcal{P}'(u,v),(u,v)\rangle &= \left(2 - \frac{(p+q)(N-2)}{N+\alpha}\right)\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \left(2 - \frac{(p+q)N}{N+\alpha}\right)\int_{\mathbb{R}^N} (|u|^2 + |v|^2) \\ &\leq \left(2 - \frac{(p+q)(N-2)}{N+\alpha}\right)\|(u,v)\|^2, \end{aligned}$$

by Claim 1, we know $\langle \mathcal{P}'(u,v),(u,v)\rangle \neq 0$, we deduce $\rho = 0$, and then $\mathcal{I}'(u,v) = 0$. □

Our main tool is the following Brézis-Lieb lemma for the nonlocal term $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q$.

Lemma 2.5. *Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. If $u_n \to u$ and $v_n \to v$ a.e in $\mathbb{R}^N$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|v_n|^q - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q = \lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)|v_n - v|^q.$$

Proof. For $n = 1, 2, \ldots$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|v_n|^q - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)|v_n - v|^q \\ = \int_{\mathbb{R}^N} (I_\alpha * (|u_n|^p - |u_n - u|^p))|v_n|^q + \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)(|v_n|^q - |v_n - v|^q).\end{aligned}$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, by [17, Lemma 2.5] with $q = p$ and $r = \frac{2Np}{N+\alpha}$, one has

$$\int_{\mathbb{R}^N} (|u_n|^p - |u_n - u|^p - |u|^p)^{\frac{2N}{N+\alpha}} \to 0 \text{ as } n \to \infty,$$

which means

$$|u_n|^p - |u_n - u|^p \to |u|^p \text{ in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Since the Riesz potential is a linear bounded map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, by the Hardy-Littlewood-Sobolev inequality (1.4), this implies that

$$I_\alpha * (|u_n|^p - |u_n - u|^p) \to I_\alpha * |u|^p$$

in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Since $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, by $|v_n|^q \rightharpoonup |v|^q$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * (|u_n|^p - |u_n - u|^p))|v_n|^q \to \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q \text{ as } n \to \infty.$$

Similarly, according to $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, by [17, Lemma 2.5] with $r = \frac{2Nq}{N+\alpha}$, one has

$$|v_n|^q - |v_n - v|^q \to |v|^q \quad \text{in} \quad L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Since $|u_n - u|^p \rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, by the Riesz potential is a linear bounded map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and the Hardy-Littlewood-Sobolev inequality (1.4), this implies that $I_\alpha * |u_n - u|^p \rightharpoonup 0$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)(|v_n|^q - |v_n - v|^q) \to 0 \text{ as } n \to \infty.$$

This proves the lemma. □

Lemma 2.6. *If $c_{\text{odd}} < 2c_0$, c_{odd} is achieved at some $(u, v) \in \mathcal{N}_{\text{odd}}$.*

Proof. Let $(u_n, v_n) \in \mathcal{N}_{\text{odd}}$ so that $\mathcal{I}(u_n, v_n) \to c_{\text{odd}}$. We first show that $\{(u_n, v_n)\}$ is bounded in H_{odd}. For n large enough, we get

$$\begin{aligned} c_{\text{odd}} + o_n(1) &\geq \mathcal{I}(u_n, v_n) - \frac{1}{p+q}\mathcal{P}(u_n, v_n) \\ &= \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + |v_n|^2). \end{aligned} \tag{2.1}$$

Then, there exist the subsequence of $\{u_n\}, \{v_n\}$ (still denoted by $\{u_n\}, \{v_n\}$) such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. This implies in particular that $\{|u_n|^p\}$ and $\{|v_n|^q\}$ are bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, $p, q \in (\frac{N+\alpha}{N}, 2_*^\alpha)$.

Claim 5. We claim $v \neq 0$. We show that there exists $R > 0$ such that

$$\liminf_{n\to\infty} \int_{D_R} |v_n|^{\frac{2Nq}{N+\alpha}} > 0, \tag{2.2}$$

where the set $D_R \subset \mathbb{R}^N$ is the infinite slab $D_R = \mathbb{R}^{N-1} \times [-R, R]$. Suppose by contradiction that for each $R > 0$,

$$\liminf_{n\to\infty} \int_{D_R} |v_n|^{\frac{2Nq}{N+\alpha}} = 0.$$

Define

$$(\omega_n, v_n) := (\chi_{\mathbb{R}^{N-1}\times(0,\infty)} u_n, \chi_{\mathbb{R}^{N-1}\times(0,\infty)} v_n)$$

and

$$(\tilde{\omega}_n, \tilde{v}_n) := (\chi_{\mathbb{R}^{N-1}\times(-\infty,0)} u_n, \chi_{\mathbb{R}^{N-1}\times(-\infty,0)} v_n).$$

Since $(u_n, v_n) \in H_{\mathrm{odd}}$, we have $(\omega_n, v_n) \in H_{\mathbb{R}^{N-1}\times(0,\infty)} \subset H$ and $(\tilde{\omega}_n, \tilde{v}_n) \in H_{\mathbb{R}^{N-1}\times(-\infty,0)} \subset H$. We now compute

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q \leq 2 &\int_{\mathbb{R}^N}\int_{D_R} I_\alpha(x-y)|\omega_n(y)|^p|\tilde{v}_n(x)|^q \\ &+ \int_{\mathbb{R}^N\setminus D_R}\int_{\mathbb{R}^N\setminus D_R} I_\alpha(x-y)|\omega_n(y)|^p|\tilde{v}_n(x)|^q.\end{aligned}$$

By definition of the region D_R we have, if $\beta \in (\alpha, N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q &\leq 2\int_{D_R} (I_\alpha * |u_n|^p)|v_n|^q + \int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{2R}} I_\alpha) * |u_n|^p)|v_n|^q \\ &\leq 2\int_{D_R} (I_\alpha * |u_n|^p)|v_n|^q + \frac{C}{R^{\beta-\alpha}}\int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{2R}} I_\beta) * |u_n|^p)|v_n|^q.\end{aligned}$$

By using the semigroup property of the Riesz potential $I_\alpha = I_{\frac{\alpha}{2}} * I_{\frac{\alpha}{2}}$, Cauchy-Schwarz inequality, (1.4), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|v|^q &\leq \Big(\int_{\mathbb{R}^N} (I_{\frac{\alpha}{2}} * |u|^p)^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} (I_{\frac{\alpha}{2}} * |v|^q)^2\Big)^{\frac{1}{2}} \\ &\leq C_1\Big(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}}\Big(\int_{\mathbb{R}^N} |v|^{\frac{2Nq}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}}.\end{aligned} \tag{2.3}$$

Using (2.3) and the classical Sobolev inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q &\leq C_2\Big(\int_{D_R} |u_n|^{\frac{2Np}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}}\Big(\int_{D_R} |v_n|^{\frac{2Nq}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}} \\ &\quad + \frac{C}{R^{\beta-\alpha}}\int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{2R}} I_\beta) * |u_n|^p)|v_n|^q \\ &\leq C_3\Big(\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2)\Big)^{\frac{p}{2}}\Big(\int_{D_R} |v_n|^{\frac{2Nq}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}} \\ &\quad + \frac{C_4}{R^{\beta-\alpha}}\Big(\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2)\Big)^{\frac{p}{2}}\Big(\int_{D_R} |v_n|^{\frac{2Nq}{N+\alpha}}\Big)^{\frac{N+\alpha}{2N}},\end{aligned}$$

from which, and as $\{(u_n, v_n)\}$ is bounded in the space H, we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q = 0. \tag{2.4}$$

Similarly, one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * |\tilde{\omega}_n|^p)|v_n|^q = 0. \tag{2.5}$$

For each $n \in \mathbb{N}$, we fix $t_n \in (0, \infty)$ so that $(t_n\omega_n, t_nv_n) \in \mathcal{N}_{\text{odd}}$ or, equivalently,

$$\begin{aligned} t_n^{p+q-2} &= \frac{\displaystyle\int_{\mathbb{R}^N}(|\nabla\omega_n|^2 + |\omega_n|^2 + |\nabla v_n|^2 + |v_n|^2)}{\displaystyle 2\int_{\mathbb{R}^N}(I_\alpha * |\omega_n|^p)|v_n|^q} \\ &= \frac{\displaystyle\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |u_n|^2 + |\nabla v_n|^2 + |v_n|^2)}{\displaystyle 2\int_{\mathbb{R}^N}((I_\alpha * |u_n|^p)|v_n|^q - (I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q - (I_\alpha * |\tilde{\omega}_n|^p)|v_n|^q)}. \end{aligned} \tag{2.6}$$

For every $n \in \mathbb{N}$, we have

$$\mathcal{I}(t_nu_n, t_nv_n) = 2\mathcal{I}(t_n\omega_n, t_nv_n) - \frac{2t_n^{p+q}}{p+q}\int_{\mathbb{R}^N}((I_\alpha * |\omega_n|^p)|\tilde{v}_n|^q + (I_\alpha * |\tilde{\omega}_n|^p)|v_n|^q).$$

By (2.3)–(2.6), in view of Lemma 2.1, we note that $\lim\limits_{n\to\infty} t_n = 1$ and thus according to (2.4) and (2.5) again we conclude

$$\begin{aligned} c_{\text{odd}} &= \lim_{n\to\infty}\mathcal{I}(u_n, v_n) = \lim_{n\to\infty}\mathcal{I}(t_nu_n, t_nv_n) \\ &= 2\lim_{n\to\infty}\mathcal{I}(t_n\omega_n, t_nv_n) \\ &\geq 2c_0, \end{aligned}$$

in contradiction with the assumption $c_{\text{odd}} < 2c_0$ of the Lemma. We can now fix $R > 0$ such that (2.2) holds. We take a function $\eta \in C^\infty(\mathbb{R}^N)$ such that $\text{supp}\eta \subset D_{3R/2}$, $\eta = 1$ on D_R, $\eta \leq 1$ on $\mathbb{R}^N$ and $\nabla\eta$ in $L^\infty(\mathbb{R}^N)$. By using the inequality [22, (3.4)],

$$\begin{aligned} \int_{D_R}|v_n|^{\frac{2Nq}{N+\alpha}} &\leq \int_{\mathbb{R}^N}|\eta v_n|^{\frac{2Nq}{N+\alpha}} \\ &\leq C\Big(\sup_{a\in\mathbb{R}^N}\int_{B_{R/2}(a)}|\eta v_n|^{\frac{2Nq}{N+\alpha}}\Big)^{1-\frac{N+\alpha}{Nq}}\int_{\mathbb{R}^N}(|\nabla(\eta v_n)|^2 + |\eta v_n|^2) \\ &\leq C_1\Big(\sup_{a\in\mathbb{R}^{N-1}\times\{0\}}\int_{B_{2R}(a)}|v_n|^{\frac{2Nq}{N+\alpha}}\Big)^{1-\frac{N+\alpha}{Nq}}\int_{\mathbb{R}^N}(|\nabla v_n|^2 + |v_n|^2). \end{aligned}$$

Since the sequence $\{(u_n, v_n)\}$ is bounded in H_{odd}, we deduce from (2.2) that there exists a sequence of points $\{a_n\}$ in the hyperplane $\mathbb{R}^{N-1} \times \{0\}$ such that

$$\liminf_{n\to\infty}\int_{B_{2R}(a_n)}|v_n|^{\frac{2Nq}{N+\alpha}} > 0.$$

By the Rellich theorem, $v_n \to v$ in $L_{loc}^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)$ and then $v \neq 0$.

Claim 6. The infimum of $\mathcal{I}|_{\mathcal{N}_{\text{odd}}}$ is achieved.

We claim that $(u, v) \in \mathcal{N}_{\text{odd}}$. Indeed, if $(u, v) \notin \mathcal{N}_{\text{odd}}$, we will discuss it in two cases: $\mathcal{P}(u, v) < 0$ and $\mathcal{P}(u, v) > 0$.

Case 1: $\mathcal{P}(u, v) < 0$. By Lemma 2.3, there exists $t \in (0, 1)$ such that $(tu, tv) \in \mathcal{N}_{\text{odd}}$, it follows from $(u_n, v_n) \in \mathcal{N}_{\text{odd}}$ and Fatou's lemma that

$$\begin{aligned}
c_{\text{odd}} &= \liminf_{n\to+\infty}\left(\mathcal{I}(u_n, v_n) - \frac{1}{p+q}\mathcal{P}(u_n, v_n)\right) \\
&= \frac{p+q-2}{2(p+q)} \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + |v_n|^2) \\
&\geq \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2) \\
&> \frac{p+q-2}{2(p+q)} t^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2) \\
&= \mathcal{I}(tu, tv) - \frac{1}{p+q}\mathcal{P}(tu, tv) \\
&\geq c_{\text{odd}},
\end{aligned}$$

which is a contradiction.

Case 2: $\mathcal{P}(u, v) > 0$. We define

$$\xi_n := u_n - u, \gamma_n := v_n - v.$$

Using Brézis-Lieb lemma [27, Lemma 1.32] and Lemma 2.5, we may obtain

$$\mathcal{I}(u_n, v_n) = \mathcal{I}(u, v) + \mathcal{I}(\xi_n, \gamma_n) + o_n(1) \tag{2.7}$$

and

$$\mathcal{P}(u_n, v_n) = \mathcal{P}(u, v) + \mathcal{P}(\xi_n, \gamma_n) + o_n(1). \tag{2.8}$$

Then

$$\limsup_{n\to\infty} \mathcal{P}(\xi_n, \gamma_n) < 0.$$

By Lemma 2.3, there exists $t_n \in (0, 1)$ such that $(t_n\xi_n, t_n\gamma_n) \in \mathcal{N}_{\text{odd}}$. Furthermore, one has

$$\limsup_{n\to\infty} t_n < 1,$$

otherwise, along a subsequence, $t_n \to 1$ and then

$$\mathcal{P}(\xi_n, \gamma_n) = \mathcal{P}(t_n\xi_n, t_n\gamma_n) + o_n(1) = o_n(1),$$

which is a contradiction. For n large enough, it follows from $(u_n, v_n) \in \mathcal{N}_{\text{odd}}$, (2.7) and (2.8) that

$$\begin{aligned}
c_{\text{odd}} + o_n(1) &= \mathcal{I}(u_n, v_n) - \frac{1}{p+q}\mathcal{P}(u_n, v_n) \\
&= \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |\nabla \xi_n|^2 + |\nabla \gamma_n|^2 + |u|^2 + |v|^2 + |\xi_n|^2 + |\gamma_n|^2) \\
&> \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2) + \frac{p+q-2}{2(p+q)} t_n^2 \int_{\mathbb{R}^N} (|\nabla \xi_n|^2 + |\nabla \gamma_n|^2 + |\xi_n|^2 + |\gamma_n|^2) \\
&= \mathcal{I}(u, v) - \frac{1}{p+q}\mathcal{P}(u, v) + \mathcal{I}(t_n\xi_n, t_n\gamma_n) - \frac{1}{p+q}\mathcal{P}(t_n\xi_n, t_n\gamma_n) \\
&= \mathcal{I}(t_n\xi_n, t_n\gamma_n) + \frac{p+q-2}{2(p+q)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2),
\end{aligned}$$

which is also a contradiction.

Therefore, $(u, v) \in \mathcal{N}_{\text{odd}}$ and then (u, v) is a minimizer of $\mathcal{I}|_{\mathcal{N}_{\text{odd}}}$. □

It remains now to establish the strict inequality $c_{\text{odd}} < 2c_0$.

Lemma 2.7. $c_{\text{odd}} < 2c_0$.

Proof. Motivated by the Proposition 2.4 in [4]. We give a detailed proof. It is easy to prove that Choquard system (1.1) has a least action solution on the usual Nehari manifold. More precisely, there exists $0 \neq \omega, \nu \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\mathcal{I}'(\omega, \nu) = 0 \quad \text{and} \quad \mathcal{I}(\omega, \nu) = \inf_{\mathcal{N}_0} \mathcal{I}.$$

We take a function $\eta \in C_c^2(\mathbb{R}^N)$ such that $\eta = 1$ on B_1, $0 \leq \eta \leq 1$ on $\mathbb{R}^N$ and supp $\eta \subset B_2$ and we define for each $R > 0$ the function $\eta_R \in C_c^2(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ by $\eta_R(x) = \eta(x/R)$, η_R is even in x. We define the function u_R: $\mathbb{R}^N \to \mathbb{R}$ for each $x = (x', x_N) \in \mathbb{R}^N$ by

$$u_R(x) = (\eta_R\omega)(x', x_N - 2R) - (\eta_R\omega)(x', -x_N - 2R),$$

$$v_R(x) = (\eta_R\nu)(x', x_N - 2R) - (\eta_R\nu)(x', -x_N - 2R).$$

It is clear that $(u_R, v_R) \in H_{\text{odd}}$. Note that

$$\langle \mathcal{I}'((u_R)_{t_R}, (v_R)_{t_R}), ((u_R)_{t_R}, (v_R)_{t_R}) \rangle = 0,$$

if and only if $t_R \in (0, \infty)$ satisfies

$$t_R^{p+q-2} = \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u_R|^2 + |u_R|^2 + |\nabla v_R|^2 + |v_R|^2)}{\displaystyle 2\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|v_R|^q}.$$

Such a t_R always exists and

$$\mathcal{I}(t_Ru_R, t_Rv_R) = \frac{\left(\frac{1}{2} - \frac{2}{p+q}\right)\left(\displaystyle\int_{\mathbb{R}^N} (|\nabla u_R|^2 + |u_R|^2 + |\nabla v_R|^2 + |v_R|^2)\right)^{\frac{p+q}{p+q-2}}}{\left(2\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|v_R|^q\right)^{\frac{2}{p+q-2}}}.$$

The proposition will follow once we have established that for some $R > 0$,

$$\frac{\Big(\int_{\mathbb{R}^N}(|\nabla u_R|^2 + |u_R|^2 + |\nabla v_R|^2 + |v_R|^2)\Big)^{\frac{p+q}{p+q-2}}}{\Big(2\int_{\mathbb{R}^N}(I_\alpha * |u_R|^p)|v_R|^q\Big)^{\frac{2}{p+q-2}}} < 2\frac{\Big(\int_{\mathbb{R}^N}(|\nabla \omega|^2 + |\omega|^2 + |\nabla \nu|^2 + |\nu|^2)\Big)^{\frac{p+q}{p+q-2}}}{\Big(2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q\Big)^{\frac{2}{p+q-2}}}. \tag{2.9}$$

Observe that, by construction of the function (u_R, v_R)

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * |\eta_R\omega|^p)|\eta_R\nu|^q &= \int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q - 2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)(1-\eta_R^q)|\nu|^q \\ &\quad + \int_{\mathbb{R}^N}(I_\alpha * (1-\eta_R^p)|\omega|^p)(1-\eta_R^q)|\nu|^q \\ &\quad + \int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q(\eta_R^p - \eta_R^q) \\ &\geq \int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q - 2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)(1-\eta_R^q)|\nu|^q.\end{aligned}$$

For the first term, without losing generality, we may suppose $p \geq q$, one has

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * |\eta_R\omega|^p)|\eta_R\nu|^q &= \int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q - 2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)(1-\eta_R^q)|\nu|^q \\ &\quad + \int_{\mathbb{R}^N}((I_\alpha * (1-\eta_R^p)|\omega|^p)(1-\eta_R^q)|\nu|^q + (I_\alpha * |\omega|^p)|\nu|^q(\eta_R^p - \eta_R^q)) \\ &\geq \int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q - 2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)(1-\eta_R^q)|\nu|^q.\end{aligned}$$

By the asymptotic properties of $(I_\alpha * |\omega|^p)|\nu|^q$ [17, Theorem 4], we obtain

$$\lim_{|x|\to\infty}\frac{(I_\alpha * |\omega|^p)|\nu|^q}{I_\alpha(x)} = \int_{\mathbb{R}^N}|\omega|^p|\nu|^q,$$

so

$$2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)(1-\eta_R^q)|\nu|^q \leq C\int_{\mathbb{R}^N\setminus B_R}\frac{|\omega|^p|\nu|^q}{|x|^{N-\alpha}}.$$

Thus

$$\int_{\mathbb{R}^N}(I_\alpha * |u_R|^p)|v_R|^q \geq 2\int_{\mathbb{R}^N}(I_\alpha * |\omega|^p)|\nu|^q + \frac{2A_\alpha}{(4R)^{N-\alpha}}\int_{B_R}|\omega|^p|\nu|^q - C\int_{\mathbb{R}^N\setminus B_R}\frac{|\omega|^p|\nu|^q}{|x|^{N-\alpha}}.$$

Since

$$p+q > \frac{N+\alpha}{N} > 2,$$

according to (ω, ν) decays exponentially at infinity, we may obtain

$$\int_{\mathbb{R}^N\setminus B_R}\frac{|\omega|^p|\nu|^q}{|x|^{N-\alpha}} = o\Big(\frac{1}{R^{N-\alpha}}\Big).$$

Thus
$$\int_{\mathbb{R}^N}(I_\alpha*|u_R^p|)|v_R|^q \geq 2\int_{\mathbb{R}^N}(I_\alpha*|\omega|^p)|v|^q+\frac{2A_\alpha}{(4R)^{N-\alpha}}\int_{B_R}|\omega|^p|v|^q+o\Big(\frac{1}{R^{N-\alpha}}\Big). \tag{2.10}$$
By using integration by parts, we have
$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u_R|^2+|u_R|^2+|\nabla v_R|^2+|v_R|^2) &= 2\int_{\mathbb{R}^N}(|\nabla(\eta_R\omega)|^2+|\eta_R\omega|^2+|\nabla(\eta_R v)|^2+|\eta_R v|^2)\\ &= 2\int_{\mathbb{R}^N}\eta_R^2(|\nabla\omega|^2+|\omega|^2+|\nabla v|^2+|v|^2)-2\int_{\mathbb{R}^N}\eta_R(\Delta\eta_R)(|\omega|^2+|v|^2)\\ &\leq 2\int_{\mathbb{R}^N}(|\nabla\omega|^2+|\omega|^2+|\nabla v|^2+|v|^2)+\frac{C}{R^2}\int_{B_{2R}\setminus B_R}(|\omega|^2+|v|^2).\end{aligned}$$

Thus
$$\int_{\mathbb{R}^N}(|\nabla u_R|^2+|u_R|^2+|\nabla v_R|^2+|v_R|^2) = 2\int_{\mathbb{R}^N}(|\nabla\omega|^2+|\omega|^2+|\nabla v|^2+|v|^2)+o\Big(\frac{1}{R^{N-\alpha}}\Big). \tag{2.11}$$
It follows from (2.10) and (2.11) that
$$\begin{aligned}\frac{\Big(\int_{\mathbb{R}^N}|\nabla u_R|^2+|u_R|^2\Big)^{\frac{p+q}{p+q-2}}}{\Big(2\int_{\mathbb{R}^N}(I_\alpha*|u_R|^p)|v_R|^q\Big)^{\frac{2}{p+q-2}}} &\leq 2\frac{\Big(\int_{\mathbb{R}^N}(|\nabla\omega|^2+|\omega|^2+|\nabla v|^2+|v|^2)+o\Big(\frac{1}{R^{N-\alpha}}\Big)\Big)^{\frac{p+q}{p+q-2}}}{\Big(2\int_{\mathbb{R}^N}(I_\alpha*|\omega|^p)|v|^q+\frac{2A_\alpha}{(4R)^{N-\alpha}}\int_{B_R}|\omega|^p|v|^q+o\Big(\frac{1}{R^{N-\alpha}}\Big)\Big)^{\frac{2}{p+q-2}}}\\ &\leq 2\frac{\Big(\int_{\mathbb{R}^N}(|\nabla\omega|^2+|\omega|^2+|\nabla v|^2+|v|^2)\Big)^{\frac{p+q}{p+q-2}}}{\Big(2\int_{\mathbb{R}^N}(I_\alpha*|\omega|^p)|v|^q\Big)^{\frac{2}{p+q-2}}}.\end{aligned}$$
The inequality (2.9) holds thus when R is large enough, and the conclusion follows. □

In this work, by using a variant of Nehari constraint, we obtain the odd symmetry of ground state solutions for Choquard system. Our results can be looked on as a partial generalization to some recent ones.

Proof of Theorem 1.1. From Lemma 2.6, we have a $(u,v)\in\mathcal{N}_{\rm odd}$ such that $\mathcal{I}(u,v)=c_{\rm odd}$. By Lemma 2.4, the (u,v) is a critical point of $\mathcal{I}$ and hence a solution to (1.1). We claim $u,v\neq 0$. From Lemma 2.6, we already know $v\neq 0$. Now we prove $u\neq 0$. Indeed, if $u=0$, then the second equation of (1.1) yields that $v=0$, then $(u,v)=(0,0)$, this is impossible by the Claim 1 in Lemma 2.4. The proof is complete.

3. Conclusions

In this work, by using a variant of Nehari constraint, we obtain the odd symmetry of ground state solutions for the Choquard system. Our results can be looked on as a partial generalization to some recent ones.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, *Math. Z.*, **248** (2004), 423–443. https://doi.org/10.1007/s00209-004-0663-y
2. J. Chen, B. Guo, Blow up solutions for one class of system of Pekar-Choquard type nonlinear Schrödinger equation, *Appl. Math. Comput.*, **186** (2007), 83–92. https://doi.org/10.1016/j.amc.2006.07.089
3. P. Chen, X. Liu, Ground states of linearly coupled systems of Choquard type, *Appl. Math. Lett.*, **84** (2018), 70–75. https://doi.org/10.1016/j.aml.2018.04.016
4. M. Ghimenti, J. V. Schaftingen, Nodal solutions for the Choquard equation, *J. Funct. Anal.*, **271** (2016), 107–135. https://doi.org/10.1016/j.jfa.2016.04.019
5. M. Clapp, D. Salazar, Positive and sign changing solutions to a nonlinear Choquard equation, *J. Math. Anal. Appl.*, **407** (2013), 1–15. https://doi.org/10.1016/j.jmaa.2013.04.081
6. C. Gui, H. Guo, On nodal solutions of the nonlinear Choquard equation, *Adv. Nonlinear Stud.*, **19** (2019), 677–691. https://doi.org/10.1515/ans-2019-2061
7. C. Gui, H. Guo, Nodal solutions of a nonlocal Choquard equation in a bounded domain, *Commun. Contemp. Math.*, **23** (2019), 1950067. https://doi.org/10.1142/S0219199719500676
8. Z. Huang, J. Yang, W. Yu, Multiple nodal solutions of nonlinear Choquard equations, *Electron. J. Differ. Equations*, **2017** (2017), 1–18.
9. X. Li, S. Ma, G. Zhang, Existence and qualitative properties of solutions for Choquard equations with a local term, *Nonlinear Anal.*, **45** (2019), 1–25. https://doi.org/10.1016/j.nonrwa.2018.06.007
10. E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.*, **57** (1977), 93–105. https://doi.org/10.1002/sapm197757293
11. E. Lieb, M. Loss, *Graduate studies in mathematics*, American Mathematical Society, 2001.
12. P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.*, **4** (1980), 1063–1072. https://doi.org/10.1016/0362-546X(80)90016-4

13. P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. H. Poincaré Anal. Non Linéaire.*, **1** (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30422-X

14. L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, **195** (2010), 455–467. https://doi.org/10.1007/s00205-008-0208-3

15. I. M. Moroz, R. Penrose, P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, *Class. Quantum Grav.*, **15** (1998), 2733–2742. https://doi.org/10.1088/0264-9381/15/9/019

16. V. Moroz, J. Schaftingen, Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains, *J. Differ Equations*, **254** (2013), 3089–3145. https://doi.org/10.1016/j.jde.2012.12.019

17. V. Moroz, J. Schaftingen, Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184. https://doi.org/10.1016/j.jfa.2013.04.007

18. V. Moroz, J. Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.*, **367** (2015), 6557–6579.

19. V. Moroz, J. Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, *Commun. Contemp. Math.*, **17** (2015), 1550005. https://doi.org/10.1142/S0219199715500054

20. V. Moroz, J. Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, **19** (2017), 773–813. https://doi.org/10.1007/s11784-016-0373-1

21. S. I. Pekar, *Untersuchung über die elektronentheorie der kristalle*, Akademie Verlag, 1954. https://doi.org/10.1515/9783112649305

22. J. V. Schaftingen, Interpolation inequalities between Sobolev and Morrey-Campanato spaces: a common gateway to concentration-compactness and Gagliardo-Nirenberg interpolation inequalities, *Port. Math.*, **71** (2014), 159–175. https://doi.org/10.4171/PM/1947

23. J. V. Schaftingen, J. Xia, Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent, *J. Math. Anal. Appl.*, **464** (2018), 1184–1202. https://doi.org/10.1016/j.jmaa.2018.04.047

24. G. Vaira, Ground states for Schrödinger-Poisson type systems, *Ric. Mat.*, **60** (2011), 263–297. https://doi.org/10.1007/s11587-011-0109-x

25. G. Vaira, Existence of bound states for Schrödinger-Newton type systems, *Adv. Nonlinear Stud.*, **13** (2013), 495–516. https://doi.org/10.1515/ans-2013-0214

26. T. Wang, H. Guo, Existence and nonexistence of nodal solutions for Choquard type equations with perturbation, *J. Math. Anal. Appl.*, **480** (2019), 123438. https://doi.org/10.1016/j.jmaa.2019.123438

27. M. Willem, *Minimax theorems*, Birkhäuser, 1996. https://doi.org/10.1007/978-1-4612-4146-1

28. N. Xu, S. Ma, R. Xing, Existence and asymptotic behavior of vector solutions for linearly coupled Choquard-type systems, *Appl. Math. Lett.*, **104** (2020), 106249. https://doi.org/10.1016/j.aml.2020.106249

29. M. Yang, J. C. D. Albuquerque, E. D. Silva, M. L. Silva, On the critical cases of linearly coupled Choquard systems, *Appl. Math. Lett.*, **91** (2018), 1–8. https://doi.org/10.1016/j.aml.2018.11.005

30. X. Zhong, C. Tang, Ground state sign-changing solutions for a class of subcritical Choquard equations with a critical pure power nonlinearity in $\mathbb{R}^N$, *Comput. Math. Appl.*, **76** (2018), 23–34. https://doi.org/10.1016/j.camwa.2018.04.001

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