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Research article

Solving Urysohn integral equations by common fixed point results in complex valued metric spaces

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Abstract: The purpose of this article is to investigate the existence of solutions for Urysohn integral equations. To achieve our objectives, we take advantage of common fixed point results for self-mappings satisfying a generalized contraction involving control functions of two variables in the context of complex valued metric spaces. We also supply a non-trivial example to show the validity of obtained results.

Keywords: common fixed point; Urysohn integral equations; closed ball; generalized contractions
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

In 2011, Azam et al. [1] gave the concept of a complex valued metric space (CVMS) as a special case of cone metric space. Since the concept to introduce complex valued metric spaces is designed to define rational expressions that cannot be defined in cone metric spaces and therefore several results of fixed point theory cannot be proved to cone metric spaces, so complex valued metric space form a special class of cone metric space. Actually, the definition of a cone metric space banks on the underlying Banach space which is not a division ring. However, we can study generalizations of many results of fixed point theory involving divisions in complex valued metric spaces. Moreover, this idea is also used to define complex valued Banach spaces [2] which offer a lot of scope for further investigation. Subsequently, Rouzkard et al. [3] proved some common fixed point results fulfilling rational inequalities in CVMS which generalize the chief results of Azam et al. [1]. Although, Klin-Eam et al. [4] extended the concept of CVMS and extended the main theorems of Azam et al. [1] and Rouzkard et al. [3]. Sintunavarat et al. [5] proved common fixed point results by putting control functions of one variable on the place of constants in contractive condition. Later on, Sitthikul et al. [6] extended the results of Sintunavarat et al. [5] by generalizing the control functions from one variable to two variables. Afterwards, Karuppiah et al. [7] obtained common coupled fixed point results for generalized rational type contractions in the background of complex valued metric spaces. For more details, we refer the readers to [8–18].

In this article, we obtain common fixed points of the contractive type mappings involving control functions of two variables with the conditions of contraction on a closed subset of CVMS. In this regard, we present some results which are more general than the results of Sitthikul et al. [6], Sintunavarat et al. [5], Rouzkard et al. [3] and Azam et al. [1] in complex valued metric spaces. We also supply a non trivial example to show the authenticity of our leading results.

2. Preliminaries

The conception of CVMS is given as follows:

Definition 1. *([1]) Let $\omega_1, \omega_2 \in \mathbb{C}$. A partial order $\precsim$ on $\mathbb{C}$ is defined in this way.*

$$\omega_1 \precsim \omega_2 \;\Leftrightarrow\; Re(\omega_1) \leqslant Re(\omega_2),\; Im(\omega_1) \leqslant Im(\omega_2).$$

It follows that

$$\omega_1 \precsim \omega_2$$

if one of these assertions is satisfied:

$$\begin{aligned}
&\text{(a) } \mathrm{Re}(\omega_1) &=& \;\mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2),\\
&\text{(b) } \mathrm{Re}(\omega_1) &<& \;\mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2),\\
&\text{(c) } \mathrm{Re}(\omega_1) &<& \;\mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) < \mathrm{Im}(\omega_2),\\
&\text{(d) } \mathrm{Re}(\omega_1) &=& \;\mathrm{Re}(\omega_2),\; \mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2).
\end{aligned}$$

Definition 2. *([1]) Let $\mathcal{P} \neq \emptyset$ and $\varphi : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ be a continuous mapping satisfying*

(i) $0 \precsim \varphi(o, \tau)$, for all $o, \tau \in \mathcal{P}$ and $\varphi(o, \tau) = 0$ if and only if $o = \tau$;
(ii) $\varphi(o, \tau) = \varphi(\tau, o)$ for all $o, \tau \in \mathcal{P}$;
(iii) $\varphi(o, \tau) \precsim \varphi(o, \nu) + \varphi(\nu, \tau)$, for all $o, \tau, \nu \in \mathcal{P}$,

then $(\mathcal{P}, \varphi)$ is said to be a CVMS. A point $o \in \mathcal{P}$ is said to be an interior point of $A \subseteq \mathcal{P}$, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(o, r) = \{\tau \in \mathcal{P} : \varphi(o, \tau) \prec r\} \subseteq A,$$

where $B(o, r)$ is an open ball. Then $\overline{B(o, r)} = \{\tau \in \mathcal{P} : \varphi(o, \tau) \preceq r\}$ is a closed ball.

Example 1. *([1]) Let $\mathcal{P} = [0, 1]$ and $o, \tau \in \mathcal{P}$. Define $\varphi : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ by*

$$\varphi(o, \tau) = \begin{cases} 0, & \text{if } o = \tau, \\ \frac{i}{2}, & \text{if } o \neq \tau. \end{cases}$$

Then $(\mathcal{P}, \varphi)$ is a CVMS.

Azam et al. [1] presented this result in CVMS.

Theorem 1. *([1]) Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L}_1,\mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist some constants $\ell_1,\ell_2 \in [0,1)$ with $\ell_1+\ell_2<1$ such that*

$$\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau) \leq \mu\varphi(o,\tau) + \ell_2\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},$$

for all $o,\tau\in\mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Rouzkard et al. [3] established this result.

Theorem 2. *([3]) Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L}_1,\mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist some constants $\ell_1,\ell_2,\ell_3 \in [0,1)$ with $\ell_1+\ell_2+\ell_3<1$ such that*

$$\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau) \leq \ell_1\varphi(o,\tau) + \ell_2\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)} + \ell_3\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},$$

for all $o,\tau\in\mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Sintunavarat et al. [5] proved the following result.

Theorem 3. *Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L}_1,\mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1,\varrho_2 : \mathcal{P}\to[0,1)$ such that*

(a) $\varrho_1(\mathcal{L}_1 o) \leq \varrho_1(o)$ and $\varrho_1(\mathcal{L}_2 o) \leq \varrho_1(o)$,
$\varrho_2(\mathcal{L}_1 o) \leq \varrho_2(o)$ and $\varrho_2(\mathcal{L}_2 o) \leq \varrho_2(o)$,
(b) $\varrho_1(o)+\varrho_2(o)<1$,
(c) $\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau) \leq \varrho_1(o)\varphi(o,\tau) + \varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}$,

for all $o,\tau\in\mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Lemma 1. *([1]) Let $(\mathcal{P},\varphi)$ be a CVMS and let $\{o_n\}\subseteq\mathcal{P}$. Then $\{o_n\}$ converges to o if and only if $|\varphi(o_n,o)|\to 0$ when $n\to\infty$.*

Lemma 2. *([1]) Let $(\mathcal{P},\varphi)$ be a CVMS and let $\{o_n\}\subseteq\mathcal{P}$. Then $\{o_n\}$ is Cauchy if and only if $|\varphi(o_n,o_{n+m})|\to 0$ when $n\to\infty$, for each $m\in\mathbb{N}$.*

3. Main results

Motivated with proposition proved by Sitthikul et al. [6], we state and prove the following proposition which is required in the proof of our main result.

Proposition 1. *Let $(\mathcal{P},\varphi)$ be a CVMS. Let $o_0\in\overline{B(o_0,r)}$. Define the sequence $\{o_n\}$ by*

$$o_{2n+1} = \mathcal{L}_1 o_{2n} \ \text{ and } \ o_{2n+2} = \mathcal{L}_2 o_{2n+1} \tag{3.1}$$

for all $n=0,1,2,\cdots$.

Assume that there exists $\varrho_1 : \mathcal{P}\times\mathcal{P}\to[0,1)$ satisfies

$$\varrho_1(\mathcal{L}_2\mathcal{L}_1 o,\tau) \leq \varrho_1(o,\tau) \ \text{ and } \varrho_1(o,\mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o,\tau)$$

for all $o,\tau\in\overline{B(o_0,r)}$. Then

$$\varrho_1(o_{2n},\tau) \leq \varrho_1(o_0,\tau) \ \text{ and } \varrho_1(o,o_{2n+1}) \leq \varrho_1(o,o_1)$$

for all $o,\tau\in\overline{B(o_0,r)}$ and $n=0,1,2,\cdots$.

Lemma 3. *Let* $\varrho_1, \varrho_2 : \mathcal{P}\times\mathcal{P}\to[0,1)$ *and* $o, \tau \in \overline{B(o_0, r)}$. *If* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$ *satisfy*

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_1(o, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_1 o) + \varrho_2(o, \mathcal{L}_1 o)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)}{1+\varphi(o, \mathcal{L}_1 o)},$$

$$\varphi(\mathcal{L}_1\mathcal{L}_2\tau, \mathcal{L}_2\tau) \leq \varrho_1(\mathcal{L}_2\tau, \tau)\varphi(\mathcal{L}_2\tau, \tau) + \varrho_2(\mathcal{L}_2\tau, \tau)\frac{\varphi(\mathcal{L}_2\tau, \mathcal{L}_1\mathcal{L}_2\tau)\varphi(\tau, \mathcal{L}_2\tau)}{1+\varphi(\mathcal{L}_2\tau, \tau)},$$

then

$$|\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)| \leq \varrho_1(o, \mathcal{L}_1 o)|\varphi(o, \mathcal{L}_1 o)| + \varrho_2(o, \mathcal{L}_1 o)|\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)|.$$

$$|\varphi(\mathcal{L}_1\mathcal{L}_2\tau, \mathcal{L}_2\tau)| \leq \varrho_1(\mathcal{L}_2\tau, \tau)|\varphi(\mathcal{L}_2\tau, \tau)| + \varrho_2(\mathcal{L}_2\tau, \tau)|\varphi(\mathcal{L}_2\tau, \mathcal{L}_1\mathcal{L}_2\tau)|.$$

Proof. We can write

$$\begin{aligned}
|\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)| &\leq \left|\varrho_1(o, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_1 o) + \varrho_2(o, \mathcal{L}_1 o)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)}{1+\varphi(o, \mathcal{L}_1 o)}\right| \\
&\leq \varrho_1(o, \mathcal{L}_1 o)|\varphi(o, \mathcal{L}_1 o)| + \varrho_2(o, \mathcal{L}_1 o)\left|\frac{\varphi(o, \mathcal{L}_1 o)}{1+\varphi(o, \mathcal{L}_1 o)}\right||\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)| \\
&\leq \varrho_1(o, \mathcal{L}_1 o)|\varphi(o, \mathcal{L}_1 o)| + \varrho_2(o, \mathcal{L}_1 o)|\varphi(\mathcal{L}_1 o, \mathcal{L}_2\mathcal{L}_1 o)|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|\varphi(\mathcal{L}_1\mathcal{L}_2\tau, \mathcal{L}_2\tau)| &\leq \left|\varrho_1(\mathcal{L}_2\tau, \tau)\varphi(\mathcal{L}_2\tau, \tau) + \varrho_2(\mathcal{L}_2\tau, \tau)\frac{\varphi(\mathcal{L}_2\tau, \mathcal{L}_1\mathcal{L}_2\tau)\varphi(\tau, \mathcal{L}_2\tau)}{1+\varphi(\mathcal{L}_2\tau, \tau)}\right| \\
&\leq \varrho_1(\mathcal{L}_2\tau, \tau)|\varphi(\mathcal{L}_2\tau, \tau)| + \varrho_2(\mathcal{L}_2\tau, \tau)\left|\frac{\varphi(\tau, \mathcal{L}_2\tau)}{1+\varphi(\mathcal{L}_2\tau, \tau)}\right||\varphi(\mathcal{L}_2\tau, \mathcal{L}_1\mathcal{L}_2\tau)| \\
&\leq \varrho_1(\mathcal{L}_2\tau, \tau)|\varphi(\mathcal{L}_2\tau, \tau)| + \varrho_2(\mathcal{L}_2\tau, \tau)|\varphi(\mathcal{L}_2\tau, \mathcal{L}_1\mathcal{L}_2\tau)|.
\end{aligned}$$

□

Theorem 4. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist the mappings* $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P}\times\mathcal{P}\to[0,1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o, \tau)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_2(o, \tau)$ and $\varrho_2(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_2(o, \tau)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_3(o, \tau)$ and $\varrho_3(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_3(o, \tau)$,
(b) $\varrho_1(o, \tau) + \varrho_2(o, \tau) + \varrho_3(o, \tau) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2\tau) \leq \varrho_1(o, \tau)\varphi(o, \tau) + \varrho_2(o, \tau)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\tau, \mathcal{L}_2\tau)}{1+\varphi(o, \tau)} + \varrho_3(o, \tau)\frac{\varphi(\tau, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_2\tau)}{1+\varphi(o, \tau)}, \quad (3.2)$$

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1-\lambda)|r|, \quad (3.3)$$

where $\lambda = \frac{\varrho_1(o_0, o_1)}{1-\varrho_2(o_0, o_1)} < 1$, then there exists a unique point $o^* \in \overline{B(o_0, r)}$ such that $\mathcal{L}_1 o^* = \mathcal{L}_2 o^* = o^*$.

Proof. For the arbitrary point o_0 in $\overline{B(o_0, r)}$, define the sequence $\{o_n\}$ by

$$o_{2n+1} = \mathcal{L}_1 o_{2n} \text{ and } o_{2n+2} = \mathcal{L}_2 o_{2n+1}$$

for all $n = 0, 1, 2, ...$ Now by inequality (3.3) and the fact that $0 \leq \lambda < 1$, we have

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq |r|.$$

It yields that $o_1 \in \overline{B(o_0, r)}$. Let $o_2, o_3, ..., o_j \in \overline{B(o_0, r)}$. It is enough to show that $o_j + 1 \in \overline{B(o_0, r)}$. First suppose that j is even, then we can write $j = 2k$ also $j + 1 = 2k + 1$. Now by the inequality (3.2), we have

$$\begin{aligned}
\varphi(o_{2k+1}, o_{2k}) &= \varphi(\mathcal{L}_1\mathcal{L}_2 o_{2k-1}, \mathcal{L}_2 o_{2k-1}) \\
&\leq \varrho_1(\mathcal{L}_2 o_{2k-1}, o_{2k-1})\varphi(\mathcal{L}_2 o_{2k-1}, o_{2k-1}) \\
&\quad + \varrho_2(\mathcal{L}_2 o_{2k-1}, o_{2k-1}) \frac{\varphi(\mathcal{L}_2 o_{2k-1}, \mathcal{L}_1\mathcal{L}_2 o_{2k-1})\, \varphi(o_{2k-1}, \mathcal{L}_2 o_{2k-1})}{1 + \varphi(\mathcal{L}_2 o_{2k-1}, o_{2k-1})} \\
&\quad + \varrho_3(\mathcal{L}_2 o_{2k-1}, o_{2k-1}) \frac{\varphi(o_{2k-1}, \mathcal{L}_1\mathcal{L}_2 o_{2k-1})\, \varphi(\mathcal{L}_2 o_{2k-1}, \mathcal{L}_2 o_{2k-1})}{1 + \varphi(\mathcal{L}_2 o_{2k-1}, o_{2k-1})},
\end{aligned}$$

which implies

$$\begin{aligned}
\varphi(o_{2k+1}, o_{2k}) &\leq \varrho_1(o_{2k}, o_{2k-1})\varphi(o_{2k-1}, o_{2k}) + \varrho_2(o_{2k}, o_{2k-1}) \frac{\varphi(o_{2k}, o_{2k+1})\, \varphi(o_{2k-1}, o_{2k})}{1 + \varphi(o_{2k-1}, o_{2k})} \\
&\quad + \varrho_3(o_{2k}, o_{2k-1}) \frac{\varphi(o_{2k-1}, o_{2k+1})\, \varphi(o_{2k}, o_{2k})}{1 + \varphi(\mathcal{L}_2 o_{2k}, o_{2k-1})} \\
&= \varrho_1(o_{2k}, o_{2k-1})\varphi(o_{2k-1}, o_{2k}) + \varrho_2(o_{2k}, o_{2k-1}) \frac{\varphi(o_{2k}, o_{2k+1})\, \varphi(o_{2k-1}, o_{2k})}{1 + \varphi(o_{2k-1}, o_{2k})}.
\end{aligned}$$

It yields

$$\begin{aligned}
|\varphi(o_{2k+1}, o_{2k})| &\leq \left| \varrho_1(o_{2k}, o_{2k-1})\varphi(o_{2k-1}, o_{2k}) + \varrho_2(o_{2k}, o_{2k-1})\varphi(o_{2k}, o_{2k+1}) \frac{\varphi(o_{2k-1}, o_{2k})}{1 + \varphi(o_{2k-1}, o_{2k})} \right| \\
&\leq \varrho_1(o_{2k}, o_{2k-1})\, |\varphi(o_{2k-1}, o_{2k})| + \varrho_2(o_{2k}, o_{2k-1})\, |\varphi(o_{2k}, o_{2k+1})| \frac{|\varphi(o_{2k-1}, o_{2k})|}{|1 + \varphi(o_{2k-1}, o_{2k})|}.
\end{aligned}$$

Using Proposition 1 and the fact that $\frac{|\varphi(o_{2k-1}, o_{2k})|}{|1+\varphi(o_{2k-1}, o_{2k})|} < 1$ in above inequality, we have

$$\begin{aligned}
|\varphi(o_{2k+1}, o_{2k})| &\leq \varrho_1(o_0, o_{2k-1})|\varphi(o_{2k}, o_{2k-1})| + \varrho_2(o_0, o_{2k-1})|\varphi(o_{2k}, o_{2k+1})| \\
&\leq \varrho_1(o_0, o_1)|\varphi(o_{2k}, o_{2k-1})| + \varrho_2(o_0, o_1)|\varphi(o_{2k}, o_{2k+1})|
\end{aligned}$$

which implies that

$$|\varphi(o_{2k+1}, o_{2k})| \leq \frac{\varrho_1(o_0, o_1)}{1 - \varrho_2(o_0, o_1)} |\varphi(o_{2k}, o_{2k-1})|. \tag{3.4}$$

Similarly, if j is odd, then we can write $j = 2k + 1$ and $j + 1 = 2k + 2$. Now by inequality (3.2), we have

$$\begin{aligned}
\varphi(o_{2k+2}, o_{2k+1}) &= \varphi(\mathcal{L}_2\mathcal{L}_1 o_{2k}, \mathcal{L}_1 o_{2k}) \\
&= \varphi(\mathcal{L}_1 o_{2k}, \mathcal{L}_2\mathcal{L}_1 o_{2k}) \\
&\leq \varrho_1(o_{2k}, \mathcal{L}_1 o_{2k})\varphi(o_{2k}, \mathcal{L}_1 o_{2k}) \\
&\quad +\varrho_2(o_{2k}, \mathcal{L}_1 o_{2k})\frac{\varphi(o_{2k}, \mathcal{L}_1 o_{2k})\,\varphi(\mathcal{L}_1 o_{2k}, \mathcal{L}_2\mathcal{L}_1 o_{2k})}{1+\varphi(o_{2k}, \mathcal{L}_1 o_{2k})} \\
&\quad +\varrho_3(o_{2k}, \mathcal{L}_1 o_{2k})\frac{\varphi(\mathcal{L}_1 o_{2k}, \mathcal{L}_1 o_{2k})\,\varphi(o_{2k}, \mathcal{L}_2\mathcal{L}_1 o_{2k})}{1+\varphi(o_{2k}, \mathcal{L}_1 o_{2k})}, \\
&= \varrho_1(o_{2k}, o_{2k+1})\varphi(o_{2k}, o_{2k+1}) \\
&\quad +\varrho_2(o_{2k}, o_{2k+1})\frac{\varphi(o_{2k}, o_{2k+1})\,\varphi(o_{2k+1}, o_{2k+2})}{1+\varphi(o_{2k}, o_{2k+1})}.
\end{aligned}$$

It implies

$$\begin{aligned}
|\varphi(o_{2k+2}, o_{2k+1})| &\leq \left|\begin{array}{c} \varrho_1(o_{2k}, o_{2k+1})\varphi(o_{2k}, o_{2k+1}) \\ +\varrho_2(o_{2k}, o_{2k+1})\frac{\varphi(o_{2k}, o_{2k+1})}{1+\varphi(o_{2k}, o_{2k+1})}\varphi(o_{2k+1}, o_{2k+2}) \end{array}\right| \\
&\leq \varrho_1(o_{2k}, o_{2k+1})\,|\varphi(o_{2k}, o_{2k+1})| \\
&\quad +\varrho_2(o_{2k}, o_{2k+1})\frac{|\varphi(o_{2k}, o_{2k+1})|}{|1+\varphi(o_{2k}, o_{2k+1})|}\,|\varphi(o_{2k+1}, o_{2k+2})|\,.
\end{aligned}$$

□

Using Proposition 1 and the fact that $\frac{|\varphi(o_{2k}, o_{2k+1})|}{|1+\varphi(o_{2k}, o_{2k+1})|} < 1$ in above inequality, we have

$$|\varphi(o_{2k+2}, o_{2k+1})| \leq \varrho_1(o_0, o_1)\,|\varphi(o_{2k}, o_{2k+1})| + \varrho_2(o_0, o_1)\,|\varphi(o_{2k+1}, o_{2k+2})|\,,$$

implies that

$$|\varphi(o_{2k+2}, o_{2k+1})| \leq \frac{\varrho_1(o_0, o_1)}{1-\varrho_2(o_0, o_1)}|\varphi(o_{2k+1}, o_{2k})|. \tag{3.5}$$

Since $\lambda = \frac{\varrho_1(o_0, o_1)}{1-\varrho_2(o_0, o_1)} < 1$, then by (3.4) and (3.5), we conclude that

$$|\varphi(o_{j+1}, o_j)| \leq \lambda|\varphi(o_j, o_{j-1})| \tag{3.6}$$

for all $j \in \mathbb{N}$. Therefore we have

$$|\varphi(o_{j+1}, o_j)| \leq \lambda|\varphi(o_j, o_{j-1})| \leq \lambda^2|\varphi(o_{j-1}, o_{j-2})| \leq \cdots \leq \lambda^j|\varphi(o_1, o_0)| \tag{3.7}$$

for all $j \in \mathbb{N}$. Now by triangle inequality and inequality(3.7), we have

$$\begin{aligned}
|\varphi(o_{j+1}, o_0)| &\leq |\varphi(o_{j+1}, o_j)| + ... + |\varphi(o_1, o_0)| \\
&\leq \lambda^j|\varphi(o_1, o_0)| + \lambda^{j-1}|\varphi(o_1, o_0)| .... + |\varphi(o_1, o_0)| \\
&\leq |\varphi(o_1, o_0)|(\lambda^j + \lambda^{j-1} + ... + 1) \\
&\leq \frac{(1-\lambda^{j+1})}{1-\lambda}|\varphi(o_1, o_0)|.
\end{aligned}$$

By inequality (3.3), we have

$$|\varphi(o_{j+1}, o_0)| \leq \frac{(1-\lambda^{j+1})}{1-\lambda}(1-\lambda)|r| \leq |r|,$$

gives $o_{j+1} \in \overline{B(o_0, r)}$. Thus $o_n \in \overline{B(o_0, r)}$, for all $n \in \mathbb{N}$. Now, by inequality (3.2) and the inequality (3.7), we have

$$|\varphi(o_{n+1}, o_n)| \leq \lambda^n |\varphi(o_0, o_1)|$$

for all $n \in \mathbb{N}$. Now for $m > n$, we have

$$\begin{aligned}
|\varphi(o_n, o_m)| &\leq |\varphi(o_n, o_{n+1})| + |\varphi(o_{n+1}, o_{n+2})| + ... + |\varphi(o_{m-1}, o_m)| \\
&\leq |\varphi(o_1, o_0)|(\lambda^n + \lambda^{n+1} + \lambda^{m-1} + ... + 1) \\
&\leq \frac{\lambda^n}{1-\lambda}|\varphi(o_0, o_1)| \to 0,
\end{aligned}$$

as $n \to \infty$. It implies that the sequence $\{o_n\}$ is a Cauchy sequence in $\overline{B(o_0, r)}$. As $\overline{B(o_0, r)}$ is closed set in $\mathcal{P}$ and $(\mathcal{P}, \varphi)$ is complete. So $\left(\overline{B(o_0, r)}, \varphi\right)$ is complete. Thus there exists $o^/ \in \overline{B(o_0, r)}$ such that $o_n \to o^/$ as $n \to \infty$.

Next, we show that $o^/$ is a fixed point of $\mathcal{L}_1$. By (3.2) and Proposition 1, we have

$$\begin{aligned}
\varphi(o^/, \mathcal{L}_1 o^/) &\precsim \varphi(o^/, \mathcal{L}_2 o_{2n+1}) + \varphi(\mathcal{L}_2 o_{2n+1}, \mathcal{L}_1 o^/) \\
&= \varphi(o^/, o_{2n+2}) + \varphi(\mathcal{L}_1 o^/, \mathcal{L}_2 o_{2n+1}) \\
&\precsim \varphi(o^/, o_{2n+2}) + \varrho_1(o^/, o_{2n+1})\varphi(o^/, o_{2n+1}) + \varrho_2(o^/, o_{2n+1})\frac{\varphi(o^/, \mathcal{L}_1 o^/)\varphi(o_{2n+1}, \mathcal{L}_2 o_{2n+1})}{1+\varphi(o^/, o_{2n+1})} \\
&\quad +\varrho_3(o^/, o_{2n+1})\frac{\varphi(o_{2n+1}, \mathcal{L}_1 o^/)\varphi(o^/, \mathcal{L}_2 o_{2n+1})}{1+\varphi(o^/, o_{2n+1})} \\
&\precsim \varphi(o^/, o_{2n+2}) + \varrho_1(o^/, o_1)\varphi(o^/, o_{2n+1}) + \varrho_2(o^/, o_1)\frac{\varphi(o^/, \mathcal{L}_1 o^/)\varphi(o_{2n+1}, o_{2n+2})}{1+\varphi(o^/, o_{2n+1})} \\
&\quad \varrho_3(o^/, o_1)\frac{\varphi(o_{2n+1}, \mathcal{L}_1 o^/)\varphi(o^/, o_{2n+2})}{1+\varphi(o^/, o_{2n+1})},
\end{aligned}$$

letting $n \to \infty$, we have

$$\varphi(o^/, \mathcal{L}_1 o^/) = 0$$

and hence $o^/ = \mathcal{L}_1 o^/$. We also show that $o^/$ is a fixed point of $\mathcal{L}_2$. By (3.2) and Proposition 1, we have

$$\begin{aligned}
\varphi(o^/, \mathcal{L}_2 o^/) &\precsim \varphi(o^/, \mathcal{L}_1 o_{2n}) + \varphi(\mathcal{L}_1 o_{2n}, \mathcal{L}_2 o^/) \\
&\precsim \varphi(o^/, o_{2n+1}) + \varrho_1(o_{2n}, o^/)\varphi(o_{2n}, o^/) + \varrho_2(o_{2n}, o^/)\frac{\varphi(o_{2n}, \mathcal{L}_1 o_{2n})\varphi(o^/, \mathcal{L}_2 o^/)}{1+\varphi(o_{2n}, o^/)} \\
&\quad +\varrho_3(o_{2n}, o^/)\frac{\varphi(o^/, \mathcal{L}_1 o_{2n})\varphi(o_{2n}, \mathcal{L}_2 \tau)}{1+\varphi(o_{2n}, o^/)} \\
&\precsim \varphi(o^/, o_{2n+1}) + \varrho_1(o_0, o^/)\varphi(o_{2n}, o^/) + \varrho_2(o_0, o^/)\frac{\varphi(o_{2n}, o_{2n})\varphi(o^/, \mathcal{L}_2 o^/)}{1+\varphi(o_{2n}, o^/)} \\
&\quad +\varrho_3(o_0, o^/)\frac{\varphi(o^/, o_{2n+1})\varphi(o_{2n}, \mathcal{L}_2 \tau)}{1+\varphi(o_{2n}, o^/)},
\end{aligned}$$

letting $n \to \infty$, we have

$$\varphi(o^{/}, \mathcal{L}_2 o^{/}) = 0$$

and hence $o^{/} = \mathcal{L}_2 o^{/}$. Therefore $o^{/}$ is a common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$. Now assume that there is $o^* \in \overline{B(o_0, r)}$ is another fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$, then $o^* = \mathcal{L}_1 o^* = \mathcal{L}_2 o^*$ and $o^{/} \neq o^*$. Now by (3.2) , we have

$$\begin{aligned}
\varphi(o^{/}, o^*) &= \varphi(\mathcal{L}_1 o^{/}, \mathcal{L}_2 o^*) \\
&\precsim \varrho_1(o^{/}, o^*)\varphi(o^{/}, o^*) + \varrho_2(o^{/}, o^*)\frac{\varphi(o^{/}, \mathcal{L}_1 o^{/})\varphi(o^*, \mathcal{L}_2 o^*)}{1 + \varphi(o^{/}, o^*)} + \varrho_3(o^{/}, o^*)\frac{\varphi(o^*, \mathcal{L}_1 o^{/})\varphi(o^{/}, \mathcal{L}_2 o^*)}{1 + \varphi(o^{/}, o^*)} \\
&= \varrho_1(o^{/}, o^*)\varphi(o^{/}, o^*) + \varrho_3(o^{/}, o^*)\frac{\varphi(o^*, o^{/})\varphi(o^{/}, o^*)}{1 + \varphi(o^{/}, o^*)},
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\varphi(o^{/}, o^*)\right| &\leq \left|\varrho_1(o^{/}, o^*)\varphi(o^{/}, o^*) + \varrho_3(o^{/}, o^*)\varphi(o^{/}, o^*)\frac{\varphi(o^{/}, o^*)}{1 + \varphi(o^{/}, o^*)}\right| \\
&\leq \varrho_1(o^{/}, o^*)\left|\varphi(o^{/}, o^*)\right| + \varrho_3(o^{/}, o^*)\left|\varphi(o^{/}, o^*)\right| \frac{\left|\varphi(o^{/}, o^*)\right|}{\left|1 + \varphi(o^{/}, o^*)\right|}| \\
&\leq \varrho_1(o^{/}, o^*)\left|\varphi(o^{/}, o^*)\right| + \varrho_3(o^{/}, o^*)\left|\varphi(o^{/}, o^*)\right| \\
&\leq (\varrho_1(o^{/}, o^*) + \varrho_3(o^{/}, o^*))\left|\varphi(o^{/}, o^*)\right|.
\end{aligned}$$

Since $\varrho_1(o^{/}, o^*) + \varrho_3(o^{/}, o^*) < 1$, we have $|\varphi(o^{/}, o^*)| = 0$. Thus $o^{/} = o^*$.

Corollary 1. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and* $\mathcal{L} : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist mappings* $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P} \times \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}\tau) \leq \varrho_1(o, \tau)$,
$\varrho_2(\mathcal{L}o, \tau) \leq \varrho_2(o, \tau)$ and $\varrho_2(o, \mathcal{L}\tau) \leq \varrho_2(o, \tau)$,
$\varrho_3(\mathcal{L}o, \tau) \leq \varrho_3(o, \tau)$ and $\varrho_3(o, \mathcal{L}\tau) \leq \varrho_3(o, \tau)$,
(b) $\varrho_1(o, \tau) + \varrho_2(o, \tau) + \varrho_3(o, \tau) < 1$,
(c) $\varphi(\mathcal{L}o, \mathcal{L}\tau) \precsim \varrho_1(o, \tau)\varphi(o, \tau) + \varrho_2(o, \tau)\frac{\varphi(o, \mathcal{L}o)\varphi(\tau, \mathcal{L}\tau)}{1+\varphi(o, \tau)} + \varrho_3(o, \tau)\frac{\varphi(\tau, \mathcal{L}o)\varphi(o, \mathcal{L}\tau)}{1+\varphi(o, \tau)}$,

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}o_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \frac{\varrho_1(o_0, o_1)}{1 - \varrho_2(o_0, o_1)} < 1$. Then $\mathcal{L}$ has a unique fixed point.

Proof. Take $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ in Theorem 4. □

Corollary 2. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist mappings* $\varrho_1, \varrho_2 : \mathcal{P} \times \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o, \tau)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_2(o, \tau)$ and $\varrho_2(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_2(o, \tau)$,
(b) $\varrho_1(o, \tau) + \varrho_2(o, \tau) < 1$,

(c) $\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \precsim \varrho_1(o,\tau)\varphi(o,\tau) + \varrho_2(o,\tau)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}$,

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{\varrho_1(o_0,o_1)}{1-\varrho_2(o_0,o_1)} < 1$. Then there exists a unique point $o^* \in \overline{B(o_0, r)}$ such that $\mathcal{L}_1 o^* = \mathcal{L}_2 o^* = o^*$.

Proof. Take $\varrho_3(o,\tau) = 0$ in Theorem 4. □

Corollary 3. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist mappings* $\varrho_1, \varrho_3 : \mathcal{P} \times \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_1(o,\tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o,\tau)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_3(o,\tau)$ and $\varrho_3(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_3(o,\tau)$,
(b) $\varrho_1(o,\tau) + \varrho_3(o,\tau) < 1$,
(c) $\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \precsim \varrho_1(o,\tau)\varphi(o,\tau) + \varrho_3(o,\tau)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}$,

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \varrho_1(o_0, o_1) < 1$. Then there exists a unique point $o^* \in \overline{B(o_0, r)}$ such that $\mathcal{L}_1 o^* = \mathcal{L}_2 o^* = o^*$.

Proof. Take $\varrho_2(o,\tau) = 0$ in Theorem 4. □

4. Deduced results

Theorem 5. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist the mappings* $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}_1 o) \leq \varrho_1(o)$ and $\varrho_1(\mathcal{L}_2 o) \leq \varrho_1(o)$,
$\varrho_2(\mathcal{L}_1 o) \leq \varrho_2(o)$ and $\varrho_2(\mathcal{L}_2 o) \leq \varrho_2(o)$,
$\varrho_3(\mathcal{L}_1 o) \leq \varrho_3(o)$ and $\varrho_3(\mathcal{L}_2 o) \leq \varrho_3(o)$,
(b) $\varrho_1(o) + \varrho_2(o) + \varrho_3(o) < 1$,
(c) $\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \leq \varrho_1(o)\varphi(o,\tau) + \varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)} + \varrho_3(o)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}$,

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{\varrho_1(o_0)}{1-\varrho_2(o_0)} < 1$, then there exists a unique point $o^* \in \overline{B(o_0, r)}$ such that $\mathcal{L}_1 o^* = \mathcal{L}_2 o^* = o^*$.

Proof. Define $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P}\times\mathcal{P} \to [0, 1)$ by

$$\varrho_1(o,\tau) = \varrho_1(o), \quad \varrho_2(o,\tau) = \varrho_2(o) \quad \text{and} \quad \varrho_3(o,\tau) = \varrho_3(o)$$

for all $o, \tau \in \overline{B(o_0, r)}$. Then for all $o, \tau \in \overline{B(o_0, r)}$, we have

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) = \varrho_1(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_1(\mathcal{L}_1 o) \leq \varrho_1(o) = \varrho_1(o,\tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) = \varrho_1(o) = \varrho_1(o,\tau)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o, \tau) = \varrho_2(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_2(\mathcal{L}_1 o) \leq \varrho_2(o) = \varrho_2(o,\tau)$ and $\varrho_2(o, \mathcal{L}_1\mathcal{L}_2\tau) = \varrho_2(o) = \varrho_2(o,\tau)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o, \tau) = \varrho_3(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_3(\mathcal{L}_1 o) \leq \varrho_3(o) = \varrho_3(o,\tau)$ and $\varrho_3(o, \mathcal{L}_1\mathcal{L}_2\tau) = \varrho_3(o) = \varrho_3(o,\tau)$,

(b) $\varrho_1(o,\tau)+\varrho_2(o,\tau)+\varrho_3(o,\tau)=\varrho_1(o)+\varrho_2(o)+\varrho_3(o)<1$,
(c)

$$\begin{aligned}\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau) &\leq \varrho_1(o)\varphi(o,\tau)+\varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}+\varrho_3(o)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}\\ &= \varrho_1(o,\tau)\varphi(o,\tau)+\varrho_2(o,\tau)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}+\varrho_3(o,\tau)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},\end{aligned}$$

(d) $\lambda=\frac{\varrho_1(o_0,o_1)}{1-\varrho_2(o_0,o_1)}=\frac{\varrho_1(o_0)}{1-\varrho_2(o_0)}<1$.
By Theorem 4, $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point. □

Remark 1. *Condition (a) and (b) of Theorem* 4 *can be weakened by the following condition*

$$\varrho_1(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_1(o),\ \ \varrho_2(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_2(o)\ \text{ and } \varrho_3(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_3(o)$$

for all $o,\tau\in\overline{B(o_0,r)}$*. So, it will be interesting to present the following result in this context.*

Theorem 6. *Let* $(\mathcal{P},\varphi)$ *be a complete CVMS and let* $\mathcal{L}_1,\mathcal{L}_2:\overline{B(o_0,r)}\to\mathcal{P}$*. If there exist the mappings* $\varrho_1,\varrho_2,\varrho_3:\mathcal{P}\to[0,1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_1(o)$ and $\varrho_1(\mathcal{L}_1\mathcal{L}_2 o)\leq\varrho_1(o)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_2(o)$ and $\varrho_2(\mathcal{L}_1\mathcal{L}_2 o)\leq\varrho_2(o)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_3(o)$ and $\varrho_3(\mathcal{L}_1\mathcal{L}_2 o)\leq\varrho_3(o)$,
(b) $\varrho_1(o)+\varrho_2(o)+\varrho_3(o)<1$,
(c) $\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau)\leq\varrho_1(o)\varphi(o,\tau)+\varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}+\varrho_3(o)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}$,

for all $o_0,o,\tau\in\overline{B(o_0,r)}$, $0\prec r\in\mathbb{C}$ and

$$|\varphi(o_0,\mathcal{L}_1 o_0)|\leq(1-\lambda)|r|,$$

where $\lambda=\frac{\varrho_1(o_0)}{1-\varrho_2(o_0)}<1$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Proof. Define $\varrho_1,\varrho_2,\varrho_3:\mathcal{P}\times\mathcal{P}\to[0,1)$ by

$$\varrho_1(o,\tau)=\varrho_1(o),\quad \varrho_2(o,\tau)=\varrho_2(o)\quad\text{and}\quad\varrho_3(o,\tau)=\varrho_3(o).$$

Then for all $o,\tau\in\overline{B(o_0,r)}$, we have
(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o,\tau)=\varrho_1(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_1(o)=\varrho_1(o,\tau)$ and $\varrho_1(o,\mathcal{L}_1\mathcal{L}_2\tau)=\varrho_1(o)=\varrho_1(o,\tau)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o,\tau)=\varrho_2(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_2(o)=\varrho_2(o,\tau)$ and $\varrho_2(o,\mathcal{L}_1\mathcal{L}_2\tau)=\varrho_2(o)=\varrho_2(o,\tau)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o,\tau)=\varrho_3(\mathcal{L}_2\mathcal{L}_1 o)\leq\varrho_3(o)=\varrho_3(o,\tau)$ and $\varrho_3(o,\mathcal{L}_1\mathcal{L}_2\tau)=\varrho_3(o)=\varrho_3(o,\tau)$,
(b) $\varrho_1(o,\tau)+\varrho_2(o,\tau)+\varrho_3(o,\tau)=\varrho_1(o)+\varrho_2(o)+\varrho_3(o)<1$,
(c)

$$\begin{aligned}\varphi(\mathcal{L}_1 o,\mathcal{L}_2\tau) &\leq \varrho_1(o)\varphi(o,\tau)+\varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}+\varrho_3(o)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}\\ &= \varrho_1(o,\tau)\varphi(o,\tau)+\varrho_2(o,\tau)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)}+\varrho_3(o,\tau)\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},\end{aligned}$$

(d) $\lambda=\frac{\varrho_1(o_0,o_1)}{1-\varrho_2(o_0,o_1)}=\frac{\varrho_1(o_0)}{1-\varrho_2(o_0)}<1$.
By Theorem 4, $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point. □

Corollary 4. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist some constants* $\ell_1, \ell_2, \ell_3 \in [0, 1)$ *with* $\ell_1 + \ell_2 + \ell_3 < 1$ *such that*

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \preceq \ell_1 \varphi(o, \tau) + \ell_2 \frac{\varphi(o, \mathcal{L}_1 o)\, \varphi(\tau, \mathcal{L}_2 \tau)}{1 + \varphi(o, \tau)} + \ell_3 \frac{\varphi(\tau, \mathcal{L}_1 o)\, \varphi(o, \mathcal{L}_2 \tau)}{1 + \varphi(o, \tau)},$$

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \frac{\ell_1}{1-\ell_2} < 1$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Proof. Define $\varrho_1, \varrho_2, \varrho_3 : o \to [0, 1)$ by

$$\varrho_1(\cdot) = \ell_1, \quad \varrho_2(\cdot) = \ell_2 \quad \text{and} \quad \varrho_3(\cdot) = \ell_3$$

in the Theorem 6. □

Corollary 5. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$. *If there exist some constants* $\ell_1, \ell_2 \in [0, 1)$ *with* $\ell_1 + \ell_2 < 1$ *such that*

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \preceq \ell_1 \varphi(o, \tau) + \ell_2 \frac{\varphi(o, \mathcal{L}_1 o)\, \varphi(\tau, \mathcal{L}_2 \tau)}{1 + \varphi(o, \tau)},$$

for all $o_0, o, \tau \in \overline{B(o_0, r)}$, $0 \prec r \in \mathbb{C}$ and

$$|\varphi(o_0, \mathcal{L}_1 o_0)| \leq (1 - \lambda)|r|$$

where $\lambda = \frac{\ell_1}{1-\ell_2} < 1$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Now if we expand the closed ball $\overline{B(o_0, r)}$ to the whole space $\mathcal{P}$, we obtain this result.

Corollary 6. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. *If there exist the mappings* $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P} \times \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2 \mathcal{L}_1 o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}_1 \mathcal{L}_2 \tau) \leq \varrho_1(o, \tau)$,
$\varrho_2(\mathcal{L}_2 \mathcal{L}_1 o, \tau) \leq \varrho_2(o, \tau)$ and $\varrho_2(o, \mathcal{L}_1 \mathcal{L}_2 \tau) \leq \varrho_2(o, \tau)$,
$\varrho_3(\mathcal{L}_2 \mathcal{L}_1 o, \tau) \leq \varrho_3(o, \tau)$ and $\varrho_3(o, \mathcal{L}_1 \mathcal{L}_2 \tau) \leq \varrho_3(o, \tau)$,
(b) $\varrho_1(o, \tau) + \varrho_2(o, \tau) + \varrho_3(o, \tau) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \preceq \varrho_1(o, \tau)\, \varphi(o, \tau) + \varrho_2(o, \tau) \frac{\varphi(o, \mathcal{L}_1 o)\, \varphi(\tau, \mathcal{L}_2 \tau)}{1 + \varphi(o, \tau)} + \varrho_3(o, \tau) \frac{\varphi(\tau, \mathcal{L}_1 o)\, \varphi(o, \mathcal{L}_2 \tau)}{1 + \varphi(o, \tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Corollary 7. *Let* $(\mathcal{P}, \varphi)$ *be a complete CVMS and let* $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. *If there exist the mappings* $\varrho_1, \varrho_2 : \mathcal{P} \times \mathcal{P} \to [0, 1)$ *such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o, \tau)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_2(o, \tau)$ and $\varrho_2(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_2(o, \tau)$,
(b) $\varrho_1(o, \tau) + \varrho_2(o, \tau) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2\tau) \leq \varrho_1(o, \tau)\varphi(o, \tau) + \varrho_2(o, \tau)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\tau, \mathcal{L}_2\tau)}{1 + \varphi(o, \tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Corollary 8. *Let $(\mathcal{P}, \varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1 : \mathcal{P}\times\mathcal{P}\to[0, 1)$ such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o, \tau) \leq \varrho_1(o, \tau)$ and $\varrho_1(o, \mathcal{L}_1\mathcal{L}_2\tau) \leq \varrho_1(o, \tau)$,
(b) $\varrho_1(o, \tau) < 1$,
(c) $\varphi(\mathcal{L}_1 o, \mathcal{L}_2\tau) \leq \varrho_1(o, \tau)\varphi(o, \tau)$,
for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Now if we expand the closed ball $\overline{B(o_0, r)}$ to the whole space $\mathcal{P}$ in Theorem 5, we obtain this result.

Corollary 9. *Let $(\mathcal{P}, \varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P}\to[0, 1)$ such that*

(a) $\varrho_1(\mathcal{L}_1 o) \leq \varrho_1(o)$ and $\varrho_1(\mathcal{L}_2 o) \leq \varrho_1(o)$,
$\varrho_2(\mathcal{L}_1 o) \leq \varrho_2(o)$ and $\varrho_2(\mathcal{L}_2 o) \leq \varrho_2(o)$,
$\varrho_3(\mathcal{L}_1 o) \leq \varrho_3(o)$ and $\varrho_3(\mathcal{L}_2 o) \leq \varrho_3(o)$,
(b) $\varrho_1(o) + \varrho_2(o) + \varrho_3(o) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2\tau) \leq \varrho_1(o)\varphi(o, \tau) + \varrho_2(o)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\tau, \mathcal{L}_2\tau)}{1 + \varphi(o, \tau)} + \varrho_3(o)\frac{\varphi(\tau, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_2\tau)}{1 + \varphi(o, \tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Now if we expand the closed ball $\overline{B(o_0, r)}$ to the whole space $\mathcal{P}$ in Theorem 6, we obtain this result.

Theorem 7. *Let $(\mathcal{P}, \varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P}\to[0, 1)$ such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_1(o)$ and $\varrho_1(\mathcal{L}_1\mathcal{L}_2 o) \leq \varrho_1(o)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_2(o)$ and $\varrho_2(\mathcal{L}_1\mathcal{L}_2 o) \leq \varrho_2(o)$,
$\varrho_3(\mathcal{L}_2\mathcal{L}_1 o) \leq \varrho_3(o)$ and $\varrho_3(\mathcal{L}_1\mathcal{L}_2 o) \leq \varrho_3(o)$,
(b) $\varrho_1(o) + \varrho_2(o) + \varrho_3(o) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2\tau) \leq \varrho_1(o)\varphi(o, \tau) + \varrho_2(o)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\tau, \mathcal{L}_2\tau)}{1 + \varphi(o, \tau)} + \varrho_3(o)\frac{\varphi(\tau, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_2\tau)}{1 + \varphi(o, \tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Corollary 10. *Let $(\mathcal{P}, \varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1, \varrho_2 : \mathcal{P}\to[0, 1)$ such that*

(a) $\varrho_1(\mathcal{L}_2\mathcal{L}_1 o) \le \varrho_1(o)$ and $\varrho_1(\mathcal{L}_1\mathcal{L}_2 o) \le \varrho_1(o)$,
$\varrho_2(\mathcal{L}_2\mathcal{L}_1 o) \le \varrho_2(o)$ and $\varrho_2(\mathcal{L}_1\mathcal{L}_2 o) \le \varrho_2(o)$,
(b) $\varrho_1(o) + \varrho_2(o) < 1$,
(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \le \varrho_1(o)\varphi(o,\tau) + \varrho_2(o)\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Corollary 11. *([5]) Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L} : \mathcal{P} \to \mathcal{P}$. If there exist the mappings $\varrho_1, \varrho_2 : \mathcal{P} \to [0,1)$ such that*

(a) $\varrho_1(\mathcal{L} o) \le \varrho_1(o)$ and $\varrho_1(\mathcal{L} o) \le \varrho_1(o)$,
$\varrho_2(\mathcal{L} o) \le \varrho_2(o)$ and $\varrho_2(\mathcal{L} o) \le \varrho_2(o)$,
(b) $\varrho_1(o) + \varrho_2(o) < 1$,
(c)

$$\varphi(\mathcal{L} o, \mathcal{L} \tau) \le \varrho_1(o)\varphi(o,\tau) + \varrho_2(o)\frac{\varphi(o,\mathcal{L} o)\varphi(\tau,\mathcal{L}\tau)}{1+\varphi(o,\tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}$ has a unique fixed point.

Now if we expand the closed ball $\overline{B(o_0, r)}$ to the whole space $\mathcal{P}$ in result 4, we obtain a result which is main result of Rouzkard et al. [3].

Corollary 12. *([3]) Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist some constants $\ell_1, \ell_2, \ell_3 \in [0,1)$ with $\ell_1 + \ell_2 + \ell_3 < 1$ such that*

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \le \ell_1\varphi(o,\tau) + \ell_2\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)} + \ell_3\frac{\varphi(\tau,\mathcal{L}_1 o)\varphi(o,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Now we give a result which is main result of Azam et al. [1] from above result.

Corollary 13. *([1]) Let $(\mathcal{P},\varphi)$ be a complete CVMS and let $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$. If there exist some constants $\ell_1, \ell_2 \in [0,1)$ with $\ell_1 + \ell_2 < 1$ such that*

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \le \ell_1\varphi(o,\tau) + \ell_2\frac{\varphi(o,\mathcal{L}_1 o)\varphi(\tau,\mathcal{L}_2\tau)}{1+\varphi(o,\tau)},$$

for all $o, \tau \in \mathcal{P}$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ have a unique common fixed point.

Example 2. *Let*

$$\begin{aligned}\mathcal{P}_1 &= \{\omega \in \mathbb{C} : Re(\omega) \ge 0,\ Im(\omega) = 0\},\\ \mathcal{P}_2 &= \{\omega \in \mathbb{C} : Im(\omega) \ge 0,\ Re(\omega) = 0\},\end{aligned}$$

and let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Consider a metric $\varphi : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{C}$ as follows:

$$\varphi(\omega_1,\omega_2) = \begin{cases} \frac{2}{3}|o_1 - o_2| + \frac{i}{2}|o_1 - o_2|, & \text{if } \omega_1, \omega_2 \in \mathcal{P}_1 \\ \frac{1}{2}|\tau_1 - \tau_2| + \frac{i}{3}|\tau_1 - \tau_2|, & \text{if } \omega_1, \omega_2 \in \mathcal{P}_2 \\ \frac{2}{9}(o_1 + \tau_2) + \frac{i}{6}(o_1 + \tau_2), & \text{if } \omega_1 \in \mathcal{P}_1,\ \omega_2 \in \mathcal{P}_2 \\ \frac{i}{3}(o_2 + \tau_1) + \frac{2i}{9}(o_2 + \tau_1), & \text{if } \omega_1 \in \mathcal{P}_2,\ \omega_2 \in \mathcal{P}_1 \end{cases}$$

for $\omega_1 = o_1 + o_2 i$ *and* $\omega_2 = \tau_1 + \tau_2 i$. *Then* $(\mathcal{P}, \varphi)$ *is CVMS. Take* $o_0 = \frac{1}{2} + 0i$ *and* $r = \frac{1}{3} + \frac{1}{4}i$. *Then*

$$\overline{B(o_0, r)} = \begin{cases} \omega \in \mathbb{C} : 0 \leqslant Re(\omega) \leqslant 1,\ Im(\omega) = 0 & if\, \omega \in \mathcal{P}_1 \\ \omega \in \mathbb{C} : 0 \leqslant Im(\omega) \leqslant 1,\ Re(\omega) = 0 & if\, \omega \in \mathcal{P}_2 \end{cases}.$$

Define $\mathcal{L}_1, \mathcal{L}_2 : \overline{B(o_0, r)} \to \mathcal{P}$ as

$$\mathcal{L}_1\omega = \begin{cases} 0 + \frac{o}{3}i & \text{if } \omega \in \mathcal{P}_1 \text{ with } 0 \leqslant \text{Re}(\omega) \leqslant 1,\ \text{Im}(\omega) = 0 \\ \frac{4o}{5} + 0i & \text{if } \omega \in \mathcal{P}_1 \text{ with } \text{Re}(\omega) > 1,\ \text{Im}(\omega) = 0 \\ \frac{\tau}{4} + 0i & \text{if } \omega \in \mathcal{P}_2 \text{ with } 0 \leqslant \text{Im}(\omega) \leqslant 1,\ \text{Re}(\omega) = 0 \\ 0 + \frac{3\tau}{4}i & \text{if } \omega \in \mathcal{P}_2 \text{ with } \text{Im}(\omega) > 1,\ \text{Re}(\omega) = 0 \end{cases}.$$

$$\mathcal{L}_2\omega = \begin{cases} 0 + \frac{o}{5}i & \text{if } \omega \in \mathcal{P}_1 \text{ with } 0 \leq \text{Re}(\omega) \leq 1,\ \text{Im}(\omega) = 0 \\ \frac{5o}{6} + 0i & \text{if } \omega \in \mathcal{P}_1 \text{ withRe}(\omega) > 1,\ \text{Im}(\omega) = 0 \\ \frac{\tau}{8} + 0i & \text{if } \omega \in \mathcal{P}_2 \text{ with } 0 \leqslant \text{Im}(\omega) \leqslant 1,\ \text{Re}(\omega) = 0 \\ 0 + \dfrac{4\tau}{7}i & \text{if } \omega \in \mathcal{P}_2 \text{ with } \text{Im}(\omega) > 1,\ \text{Re}(\omega) = 0 \end{cases}.$$

Then the mappings $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the conditions (3.2) and (3.3) of our main Theorem 4 with $\varrho_1, \varrho_2, \varrho_3 : \mathcal{P}\times\mathcal{P} \to [0, 1)$ defined as follows

$$\varrho_1(\omega_1, \omega_2) = \begin{cases} \left|0 + \dfrac{o_1 + o_2 + \tau_1 + \tau_2}{16}i\right|, & \text{if } \omega_1, \omega_2 \in \overline{B(o_0, r)} \\ \frac{3}{4}, & \text{otherwise.} \end{cases}$$

$$\varrho_2(\omega_1, \omega_2) = \begin{cases} \left|0 + \dfrac{o_1 + o_2 + \tau_1 + \tau_2}{18}i\right|, & \text{if } \omega_1, \omega_2 \in \overline{B(o_0, r)} \\ \frac{1}{6}, & \text{otherwise.} \end{cases}$$

$$\varrho_3(\omega_1, \omega_2) = \begin{cases} \left|0 + \dfrac{o_1 + o_2 + \tau_1 + \tau_2}{17}i\right|, & \text{if } \omega_1, \omega_2 \in \overline{B(o_0, r)} \\ \frac{3}{50}, & \text{otherwise.} \end{cases}$$

Hence $\mathcal{L}_1$ and $\mathcal{L}_2$ have unique common fixed point $0 + 0i \in \overline{B(o_0, r)}$.

It is interesting to notice that contractiveness on the whole space $\mathcal{P}$ does not hold because if $\omega_1 = \omega_2 = \frac{4}{3} + 0i \notin \overline{B(o_0, r)}$, then

$$\begin{aligned} \varphi(\mathcal{L}_1\omega_1, \mathcal{L}_2\omega_2) &= \varphi\left(\frac{16}{15} + 0i, \frac{10}{9} + 0i\right) = \frac{4}{135} + \frac{1}{45}i > \frac{3}{4}(0 + 0i) + \frac{1}{6}(0.011 + 0.039i) \\ &\quad + \frac{3}{50}(0.011 + 0.039i) \\ &= \varrho_1(\omega_1, \omega_2)\varphi(\omega_1, \omega_2) + \varrho_2(\omega_1, \omega_2)\frac{\varphi(\omega_1, \mathcal{L}_1\omega_1)\varphi(\omega_2, \mathcal{L}_2\omega_2)}{1 + \varphi(\omega_1, \omega_2)} \\ &\quad + \varrho_3(\omega_1, \omega_2)\frac{\varphi(\omega_2, \mathcal{L}_1\omega_1)\varphi(\omega_1, \mathcal{L}_2\omega_2)}{1 + \varphi(\omega_1, \omega_2)}. \end{aligned}$$

So it is not necessary to obtain the common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ on the whole space.

5. Applications

Let $\mathcal{P} = C([a,b],\mathbb{R})$, $a > 0$ where $C[a,b]$ denotes the set of all real continuous functions defined on the closed interval $[a,b]$ and $\varphi : \mathcal{P}\times\mathcal{P} \to \mathbb{C}$ be defined in this way

$$\varphi(o,\tau) = \max_{t\in[a,b]} \|o(t) - \tau(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}$$

for all $o,\tau \in \mathcal{P}$ and $t \in [a,b]$. Then $(\mathcal{P},\varphi)$ is complete CVMS. Consider the Urysohn integral equations

$$o(t) = \int_a^b K_1(t,s,o(s))\varphi s + g(t), \tag{5.1}$$

$$o(t) = \int_a^b K_2(t,s,o(s))\varphi s + l(t), \tag{5.2}$$

where $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ and $g, l : [a,b] \to \mathbb{R}$ are continuous and $t \in [a,b]$.

Theorem 8. *Let $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ are such that $\mathfrak{L}_o(t), \mathfrak{M}_o(t) \in \mathcal{P}$ for each $o \in \mathcal{P}$, where*

$$\mathfrak{L}_o(t) = \int_a^b K_1(t,s,o(s))\varphi s, \quad \mathfrak{M}_o(t) = \int_a^b K_2(t,s,o(s))\varphi s,$$

for all $t \in [a,b]$. Suppose there exist $\varrho_1, \varrho_2, \varrho_3 : C([a,b],\mathbb{R}) \to [0,1)$ such that

(a) $\varrho_1(\mathfrak{L}_o + g) \le \varrho_1(o)$ and $\varrho_1(\mathfrak{M}_o + l) \le \varrho_1(o)$
$\varrho_2(\mathfrak{L}_o + g) \le \varrho_2(o)$ and $\varrho_2(\mathfrak{M}_o + l) \le \varrho_2(o)$
$\varrho_3(\mathfrak{L}_o + g) \le \varrho_3(o)$ and $\varrho_3(\mathfrak{M}_o + l) \le \varrho_3(o)$,
(b) $(\varrho_1 + \varrho_2 + \varrho_3)(o) < 1$,
(c)

$$\|\mathfrak{L}_o(t) - \mathfrak{M}_\tau(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a} \le \varrho_1(o)A(o,\tau)(t) + \varrho_2(o)B(o,\tau)(t) + \varrho_3(o)B(o,\tau)(t),$$

where

$$\begin{aligned} A(o,\tau)(t) &= \|o(t) - \tau(t)\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}, \\ B(o,\tau)(t) &= \frac{\|\mathfrak{L}_o(t) + g(t) - o(t)\|_\infty \|\mathfrak{M}_\tau(t) + l(t) - \tau(t)\|_\infty}{1 + \|o(t) - \tau(t)\|_\infty} \sqrt{1+a^2}e^{i\tan^{-1}a} \\ C(o,\tau)(t) &= \frac{\|\mathfrak{M}_\tau(t) + l(t) - o(t)\|_\infty \|\mathfrak{L}_o(t) + g(t) - \tau(t)\|_\infty}{1 + \|o(t) - \tau(t)\|_\infty} \sqrt{1+a^2}e^{i\tan^{-1}a} \end{aligned}$$

then the integral operators defined by (5.1) and (5.2) have a unique common solution.

Proof. Define continuous mappings $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{L}_1 o(t) = \mathfrak{L}_o(t) + g(t),$$

$$\mathcal{L}_2 o(t) = \mathfrak{M}_o(t) + g(t),$$

for all $t \in [a, b]$. Then

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) = \max_{t\in[a,b]} \|\mathfrak{L}_o(t) - \mathfrak{M}_\tau(t) + g(t) - l(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$\varphi(o, \mathcal{L}_1 o) = \max_{t\in[a,b]} \|\mathfrak{L}_o(t) + g(t) - o(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$\varphi(\tau, \mathcal{L}_2 \tau) = \max_{t\in[a,b]} \|\mathfrak{M}_\tau(t) + l(t) - \tau(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$\varphi(o, \mathcal{L}_2 \tau) = \max_{t\in[a,b]} \|\mathfrak{M}_\tau(t) + l(t) - o(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$\varphi(\tau, \mathcal{L}_1 o) = \max_{t\in[a,b]} \|\mathfrak{L}_o(t) + g(t) - \tau(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a}.$$

Then it very simple to show that for all $o, \tau \in \mathcal{P}$, we have

(a) $\varrho_1(\mathcal{L}_1 o) = \varrho_1(\mathfrak{L}_o + g) \preceq \varrho_1(o)$ and $\varrho_1(\mathcal{L}_2 o) = \varrho_1(\mathfrak{M}_o + l) \preceq \varrho_1(o)$,
$\varrho_2(\mathcal{L}_1 o) = \varrho_2(\mathfrak{L}_o + g) \preceq \varrho_2(o)$ and $\varrho_2(\mathcal{L}_2 o) = \varrho_2(\mathfrak{M}_o + l) \preceq \varrho_2(o)$,
$\varrho_3(\mathcal{L}_1 o) = \varrho_3(\mathfrak{L}_o + g) \preceq \varrho_3(o)$ and $\varrho_3(\mathcal{L}_2 o) = \varrho_3(\mathfrak{M}_o + l) \preceq \varrho_3(o)$,

(b) $(\varrho_1 + \varrho_2 + \varrho_3)(o) < 1$,

(c)

$$\varphi(\mathcal{L}_1 o, \mathcal{L}_2 \tau) \preceq \varrho_1(o)\varphi(o,\tau) + \varrho_2(o)\frac{\varphi(o, \mathcal{L}_1 o)\varphi(\tau, \mathcal{L}_2 \tau)}{1+\varphi(o,\tau)} + \varrho_3(o)\frac{\varphi(\tau, \mathcal{L}_1 o)\varphi(o, \mathcal{L}_2 \tau)}{1+\varphi(o,\tau)}.$$

□

Hence all the assumptions of Corollary 9 are satisfied and the integral equations (5.1) and (5.2) have a unique common solution.

6. Conclusions

This article is precised on the notion of complex valued metric space to establish common fixed points of two self mappings for generalized contractions involving control functions of two variables. A non-trivial example is also provided to show the validity of obtained results. At the end of this paper, we applied our result to discuss the solution of Urysohn integral equation. We believe that the established outcomes in this paper will set a contemporary connection for investigators.

Common fixed points of multivalued mappings and fuzzy mappings in the context of complex valued metric space can be interesting outline for the future work in this direction. Differential and integral inclusions can be investigated as applications of these results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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