AIMS Mathematics, 8(8): 17539–17584.
DOI: 10.3934/math.2023896
Received: 21 March 2023
Revised: 25 April 2023
Accepted: 08 May 2023
Published: 22 May 2023

http://www.aimspress.com/journal/Math

Research article

A novel bivariate Lomax-G family of distributions: Properties, inference, and applications to environmental, medical, and computer science data

Aisha Fayomi[1], **Ehab M. Almetwally**[2,*] **and Maha E. Qura**[3]

[1] Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia, Email: afayomi@kau.edu.sa

[2] Department of Statistical, Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt, Email: ehab.metwaly@deltauniv.edu.eg

[3] Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt, Email: maha.qura@fcom.bu.edu.eg

* **Correspondence:** ehab.metwaly@deltauniv.edu.eg.

Abstract: This paper presents a novel family of bivariate continuous Lomax generators known as the BFGMLG family, which is constructed using univariate Lomax generator (LG) families and the Farlie Gumbel Morgenstern (FGM) copula. We have derived several structural statistical properties of our proposed bivariate family, such as marginals, conditional distribution, conditional expectation, product moments, moment generating function, correlation, reliability function, and hazard rate function. The paper also introduces four special submodels of the new family based on the Weibull, exponential, Pareto, and log-logistic baseline distributions. The study establishes metrics for local dependency and examines the significant characteristics of the proposed bivariate model. To provide greater flexibility, a multivariate version of the continuous FGMLG family are suggested. Bayesian and maximum likelihood methods are employed to estimate the model parameters, and a Monte Carlo simulation evaluates the performance of the proposed bivariate family. Finally, the practical application of the proposed bivariate family is demonstrated through the analysis of four data sets.

Keywords: bivariate Lomax-G family of distributions; Lomax-G family of distributions; multivariate distributions; dependence concepts; skewed distribution; Bayesian estimation
Mathematics Subject Classification: 62H10, 34A12, 62F15

1. Introduction

The Lomax distribution is a heavy-tailed probability model defined by scale and shape parameters, and has various real-world applications in fields like business, medicine, engineering, biology and

finance. In recent years, adding extra shape parameters to the basic distribution has resulted in the development of new univariate continuous distributions. Cordeiro et al. [1] introduced the Lomax-G (LG) family of distributions, a continuous univariate family based on the Lomax distribution, with two additional positive parameters, α and β. The cumulative density function (cdf) of this family is derived by

$$F_{LG}(x;\alpha,\beta,\zeta) = 1 - \left(\frac{\beta}{\beta - \log\left[1 - G(x;\zeta)\right]}\right)^{\alpha}, \tag{1.1}$$

where $G(x;\zeta)$ is the baseline cdf and ζ is a vector of parameters, $(1 \times k), k = 1, 2, 3, ...$ The survival function (SF) and the probability density function (pdf) of the LG family are given by

$$S_{LG}(x;\alpha,\beta,\zeta) = \left(\frac{\beta}{\beta - \log\left[1 - G(x;\zeta)\right]}\right)^{\alpha}, \tag{1.2}$$

and

$$f_{LG}(x;\alpha,\beta,\zeta) = \alpha\beta^{\alpha}\frac{g(x;\zeta)}{\left[1 - G(x;\zeta)\right]\left[\beta - \log\left(1 - G(x;\zeta)\right)\right]^{\alpha+1}}, \quad \alpha,\beta > 0, \tag{1.3}$$

where $g(x;\zeta)$ is the baseline pdf. Bivariate distributions were proposed and studied by many authors, and they have found extensive use in the fields such as insurance, finance, economics, risk management, hydrology, environment, management science, operations research, reliability, survival analysis, engineering, medical sciences, and others. Recently, new bivariate distributions have been constructed using classical univariate distributions based on different copula functions and Marshall-Olkin methodology.

Using the copula function, Vaidyananthan et al. [2] proposed a bivariate Lindley distribution using Morgenstern approach. Baharith et al. [3] introduced two bivariate Pareto Type II distributions; one is derived from copula and the other through a mixture and copula. Peres et al. [4] proposed a bivariate model based on a defective Gompertz distribution and a Clayton copula function to capture dependence between the lifetimes. Almetwally and Muhammed [5] introduced a new bivariate Fréchet distribution using Farlie-Gumbel-Morgenstern (FGM) and Ali-Mikhail-Haq (AMH) AMH copula functions, and discussed their properties. Peres et al. [6] used bivariate standard Weibull lifetime distributions with different copula functions for real data applications. Zhao et al. [7] presented Farlie-Gumble-Morgenstern bivariate Lomax-Claim distribution. Haj Ahmad et al. [8] introduced bivariate modified extended exponential based on FGM. Qura et al. [9] obtained Bivariate power Lomax distribution based on FGM copula. El-Sherpieny et al. [10] introduced Bivariate Weibull-G Family Based on FGM Copula Function and discussed their statistical properties.

Using the Marshall-Olkin technique, Muhammed et al. [11] proposed a bivariate inverse Weibull (BIW) distribution, characterized by inverse Weibull marginals. Eliwa and El-Morshedy [12] proposed the bivariate Gumbel-G family, a new class of bivariate distributions based on univariate Gumbel-G families. Alotaibi et al. [13] developed a new bivariate exponentiated half logistic distribution with explicit forms for its joint probability density function and cumulative distribution function. El-Sherpieny et al. [14] discussed accelerated life testing for bivariate distributions based on progressive censored samples with random removal.

To understand and motivate the construction of our BFGMLG family, it is important to first examine the fundamentals of copulas. The Sklar theorem, which was established by Sklar in 1959 [15] and is

central to the theory of copulas, states that multivariate distributions can be created by using copula functions that can be derived from the joint distribution function of two or more marginal univariate distributions (Nelsen [16]). In the range [0, 1], consider a random vector $F(X) = (F_1(X_1), ..., F_d(X_d)) = (U_1, ..., U_d)$ that follows a d-variate copula with d uniform marginal distributions. Let θ represent the d-variate copula's parameter vector and $(F_1(x_1), ..., F_d(x_d)) \in [0, 1]^d$, the respective copula is a function $C : [0; 1]^d \to [0; 1]$ that satisfies

$$\begin{aligned} C_d(F_1(x_1), ..., F_d(x_d); \theta) &= C_\theta(F_1(x_1), ..., F_d(x_d)) = C_\theta(u_1, ..., u_d) \\ &= P\left[F_1(X_1) \leq F_1(x_1), \mathbf{...}, F_d(X_d) \leq F_d(x_d)\right]. \end{aligned} \tag{1.4}$$

An element of θ is referred to as a dependence parameter. The joint density function of $F(X)$ is denoted by c_d and its formula is:

$$c_d(F(x_1), ..., F(x_d); \theta) = \frac{\partial^d}{\partial F(x_1), ..., \partial F(x_d)} C_d(F(x_1), ..., F(x_d); \theta). \tag{1.5}$$

Then, the joint cdf of X, denoted by F_d is obtained by

$$F_d(x_1, ..., x_d) = C_d(F_1(X_1), ..., F_d(X_d); \theta) = C_\theta(F_1(X_1), ..., F_d(X_d)); \quad x \in \mathbb{R}^d, \tag{1.6}$$

and the joint pdf, denoted by f_d, is obtained by

$$f_d(x_1, ..., x_d) = c_d(F_1(x_1), ..., F_d(x_d); \theta) \prod_{j=1}^{d} f_j(x_j); \quad x \in \mathbb{R}^d, \tag{1.7}$$

where $f_j(x_j), j = 1, ..., d$ are the marginal density functions, and $c_d(F(x_1), ..., F(x_d); \theta)$ being the derivative of order d of (1.6) with respect to $x_1, ..., x_d$. When the random variables are independent, $c_d(F(x_1), ..., F(x_d); \theta) = 1$. For the bivariate case, $(d = 2)$, a function $C[0, 1]^2 \to [0, 1]$ is considered a bivariate copula if it satisfies the conditions $C(0, u) = C(u, 0) = 0$, $C(1, u) = C(u, 1) = u$, and $C(v_2, u_2) - C(v_2, u_1) - C(v_1, u_2) + C(v_1, u_1) \geq 0$ for all $u, v \in [0; 1]$, $0 \leq v_1 \leq v_2 \leq 1$ and $0 \leq u_1 \leq u_2 \leq 1$.

For a bivariate distribution, the joint cdf is given as

$$F(x_1, x_2) = C_\theta(F_1(x_1), F_2(x_2)). \tag{1.8}$$

The density of the associated joint is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) c_\theta(F_1(x_1), F_2(x_2)). \tag{1.9}$$

For building a broad class of multivariate distributions based on marginals from various families, the copula approach offers a potent tool. Through a copula in which the dependence structure and marginals are separately specified, any joint distribution function may be represented. A good source on copulas can be found at Nelsen [16] and Joe [17].

One of the most well-known parametric families of copulas, the Farlie Gumbel Morgenstern (FGM) copula was discussed by Gumbel [18]. The FGM copula and its density are presented as

$$C(u_1, u_2) = u_1\ u_2\ \left(1 + \theta(1-u_1)(1-u_2)\right), -1 \leq \theta \leq 1, \tag{1.10}$$

and

$$c(u_1, u_2) = 1 + \theta(1-2u_1)(1-2u_2) \tag{1.11}$$

respectively. As a result, the FGM copula is simple and adaptable when handling the construction of bivariate distributions with complex marginal distributions with regards to functions. We use it in our study to creat Bivariate Lomax generator family, which we have dubbed the BFGMLG family.

The cdf and pdf of FGM copula are represented as follows:

$$C(F(x_1;\zeta_1), F(x_2;\zeta_2)) = F(x_1;\zeta_1)F(x_2;\zeta_2)\left(1 + \theta\left[(1-F(x_1;\zeta_1))\left(1-F(x_2;\zeta_2)\right)\right]\right), \tag{1.12}$$

and

$$c(F(x_1;\zeta_1), F(x_2;\zeta_2)) = 1 + \theta\left[(1-2F(x_1;\zeta_1))\left(1-2F(x_2;\zeta_2)\right)\right]; \quad \theta \in [-1,1]. \tag{1.13}$$

Our motivation for proposing this article is to:

(1) Construct a new bivariate continuous family of distributions, namely the BFGMLG family, that can effectively model bivariate continuous data with heavy-tailed and skewed distributions, which is useful in a variety of applications.

(2) Address the lack of existing distributions in modeling certain random bivariate phenomena, and provide a more comprehensive modeling approach.

(3) Generate various special bivariate models and realize all sorts of hazard rate functions (hrfs), which can provide more accurate and detailed information about the phenomena being studied.

(4) Build a multivariate FGMLG family, namely the MFGMLG family, that can fit multivariate data and provide a more comprehensive modeling approach for more complex phenomena.

(5) Meet the growing demands of applied fields by providing a more flexible and powerful tool for modeling heavy-tailed dependence structures in environmental, medical and computer science applications.

Our proposed BFGMLG family provides a new approach to modeling non-Gaussian and heavy-tailed dependence structures in environmental, medical and computer science applications. The univariate Lomax generator has been shown to be a good fit for modeling survival data with heavy tails, and the Farlie Gumbel Morgenstern (FGM) copula has been used successfully to model non-Gaussian dependence structures. By combining the univariate Lomax generator and the Farlie Gumbel Morgenstern (FGM) copula, we have created a more powerful and flexible tool for modeling bivariate continuous data bivariate continuous data with heavy-tailed and skewed distributions, which is useful in a variety of applications.

This paper is organized as follows: In Section 2, we introduce a new family of bivariate lomax generator using the univariate lomax generator family and the FGM copula function. In Section 3, we present some new submodels from the general class. We derive some properties of BFGMLG family including, marginal distributions, conditional distributions, regression function, moment generating function and product moments in Section 4. In Section 5, We present the reliability and some concepts

of dependence for our proposed bivariate family. In Section 6, we introduce a multivariate FGMLG family. Section 7 discusses the methods for estimating model parameters that are unknown, including Bayesian estimation and maximum likelihood. In Section 8, the performance of the estimators is thoroughly evaluated through a Monte Carlo simulation study. The use of real data sets and their interpretations are discussed in Section 9, followed by the presentation of conclusions.

2. BFGMLG family

Using any copula function, the joint cdf and pdf of the bivariate LG family are defined as follows

$$F(x_1, x_2) = C\left(1 - \left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}, 1 - \left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2}\right) \tag{2.1}$$

and

$$\begin{aligned} f_{BFGMLG}(x_1, x_2) =& \alpha_1\beta_1^{\alpha_1}\frac{g(x_1;\zeta_1)}{\left[1 - G(x_1;\zeta_1)\right]\left[\beta_1 - \log\left(1 - G(x_1;\zeta_1)\right)\right]^{\alpha_1+1}} \\ & \alpha_2\beta_2^{\alpha_2}\frac{g(x_2;\zeta_2)}{\left[1 - G(x_2;\zeta_2)\right]\left[\beta_2 - \log\left(1 - G(x_2;\zeta_2)\right)\right]^{\alpha_2+1}} \\ & c\left(1 - \left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}, 1 - \left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2}\right), \end{aligned} \tag{2.2}$$

we derive the joint cdf and pdf of the BFGMLG family using the FGM copula function, as defined in Eqs (1.12) and (1.13) and the bivariate LG family based on any copula function, as outlined in Eqs (2.1) and (2.2) as follows:

$$\begin{aligned} F_{BFGMLG}(x_1, x_2) =& \left[1 - \left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}\right]\left[1 - \left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2}\right] \\ & \left[1 + \theta\left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2}\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} f_{BFGML-G}(x_1, x_2) =& \alpha_1\beta_1^{\alpha_1}\frac{g(x_1;\zeta_1)}{\left[1 - G(x_1;\zeta_1)\right]\left[\beta_1 - \log\left(1 - G(x_1;\zeta_1)\right)\right]^{\alpha_1+1}} \\ & \alpha_2\beta_2^{\alpha_2}\frac{g(x_2;\zeta_2)}{\left[1 - G(x_2;\zeta_2)\right]\left[\beta_2 - \log\left(1 - G(x_2;\zeta_2)\right)\right]^{\alpha_2+1}} \\ & \left[1 + \theta\left(2\left[\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right]^{\alpha_1} - 1\right)\left(2\left[\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right]^{\alpha_2} - 1\right)\right]. \end{aligned} \tag{2.4}$$

Sreelakshm [19] introduced the relationship between copulas and reliability copulas which is described as follows:

$$S(x_1, x_2) = 1 - F(x_1) - F(x_2) + C(F(x_1), F(x_2)). \tag{2.5}$$

Based in Eq (2.5), The FGM survival function can be found as follows:

$$S(x_1, x_2) = 1 - F(x_1) - F(x_2) + F(x_1)F(x_2)\left[1 + \theta\left(1 - F(x_1)\right)\left(1 - F(x_2)\right)\right].$$

The following is the survival function for BFGMLG family:

$$\begin{aligned} S_{BFGMLG}(x_1, x_2) = &\left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2} \\ &\left[1 + \theta\left(1 - \left(\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right)^{\alpha_1}\right)\left(1 - \left(\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right)^{\alpha_2}\right)\right]. \end{aligned} \tag{2.6}$$

3. Special BFGMLG distributions

We presented four special models of the BFGMLG family of distributions in this section. When the cdf $G(x)$ and pdf $g(x)$ have simple analytic expressions, the pdf (2.4) will be most feasible. Taking the baseline distributions, we focus on providing four sub-models of this family: Weibull (W), Log-Logistic (LL) and Pareto (Pa). Table 1 shows the cdf and pdf of the baseline models. The BFGMLG family is very adaptable in its sub-models and can approach various bivariate distributions by altering its parameters.

Table 1. baseline models in cdf and pdf.

Model	Cdf: $G(x;\zeta)$	Pdf: $g(x;\zeta)$
Weibull	$1 - exp\left[-(bx)^a\right]$; x > 0	$ab^a x^{a-1} exp\left[-(bx)^a\right]$
Exponential	$1 - e^{-(bx)}$; x > 0	$b\ e^{-(bx)}$
Pareto	$1 - (1+x)^{-b}$; x > 0	$b\ (1+x)^{-(1+b)}$
Log-Logistic	$1 - \left[1 + \left(\frac{x}{a}\right)^b\right]^{-1}$; x > 0	$\frac{b}{a^b}\ x^{b-1}\left[1 + \left(\frac{x}{a}\right)\right]^{-2}$

3.1. BFGMLG-Weibull (BFGMLGW) distribution

The cdf and pdf of the BFGMLGW distribution are obtained by using the LG family and the Weibull distribution to obtain the LG-Weibull (LGW) distribution as follows:

$$\begin{aligned} F_{BFGMLGW}(x_1, x_2) = &\left[1 - \left(\frac{\beta_1}{\beta_1 + (b_1 x_1)^{a_1}}\right)^{\alpha_1}\right]\left[1 - \left(\frac{\beta_2}{\beta_2 + (b_2 x_2)^{a_2}}\right)^{\alpha_2}\right] \\ &\left[1 + \theta\left(\frac{\beta_1}{\beta_1 + (b_1 x_1)^{a_1}}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + (b_2 x_2)^{a_2}}\right)^{\alpha_2}\right], \end{aligned} \tag{3.1}$$

$$\begin{aligned} f_{BFGMLGW}(x_1, x_2) = &\left[\alpha_1 \beta_1^{\alpha_1} a_1 b_1^{a_1} \frac{x_1^{a_1 - 1}}{\left[\beta_1 + (b_1 x_1)^{a_1}\right]^{\alpha_1 + 1}}\right]\left[\alpha_2 \beta_2^{\alpha_2} a_2 b_2^{a_2} \frac{x_2^{a_2 - 1}}{\left[\beta_2 + (b_2 x_2)^{a_2}\right]^{\alpha_2 + 1}}\right] \\ &\left\{1 + \theta\left[2\left(\frac{\beta_1}{\beta_1 + (b_1 x_1)^{a_1}}\right)^{\alpha_1} - 1\right]\left[2\left(\frac{\beta_2}{\beta_2 + (b_2 x_2)^{a_2}}\right)^{\alpha_2} - 1\right]\right\}, \end{aligned} \tag{3.2}$$

- When $a_1 = a_2 = 1$, we obtain a new bivariate FGM Lomax exponential (BFGMLE), which is an univariate distribution of Lomax exponential.

- When $\beta_1 = \beta_2 = 1$ in addition to $b_1 = b_2 = 1$, we obtain a new bivariate of two-parameter Burr distribution (BFGMB), which is an univariate Burr distribution has been introduced by Burr [20].
- When $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = b_1 = b_2 = 1$, we obtain a new bivariate log-logistic distribution (BFGMLL), which is an univariate one-parameter log-logistic distribution has been presented by Bain [21] (In economics, this is known as the one-parameter Fisk distribution).
- When $\alpha_1 = \alpha_2 = a_1 = a_2 = b_1 = b_2 = 1$, we obtain a new bivariate Pareto type II (BFGMP), which is an univariate Pareto type II distribution has been introduced and studied by Lomax [22]

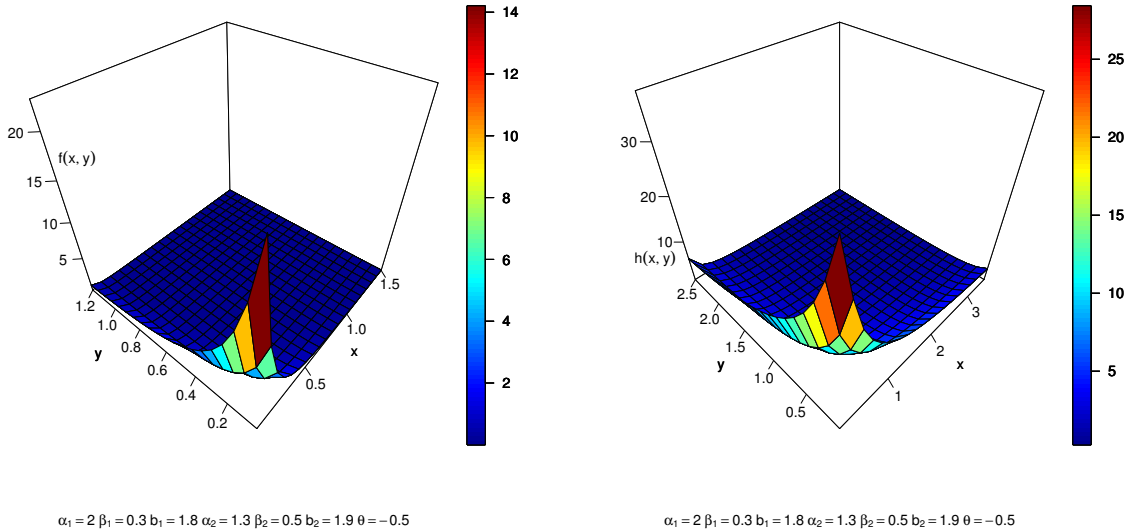
3.2. BFGMLG-exponential (BFGMLGE) distribution

The cdf and pdf of the BFGMLGE distribution are produced by obtaining LG-exponential (LGE) distribution using the LG family and exponential distribution a follows:

$$F_{BFGMLE}(x_1, x_2) = \left[1 - \left(\frac{\beta_1}{\beta_1 + b_1 x_1}\right)^{\alpha_1}\right]\left[1 - \left(\frac{\beta_2}{\beta_2 + b_2 x_2}\right)^{\alpha_2}\right] \left[1 + \theta\left(\frac{\beta_1}{\beta_1 + b_1 x_1}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + b_2 x_2}\right)^{\alpha_2}\right], \tag{3.3}$$

$$f_{BFGMLE}(x_1, x_2) = \left[\frac{\alpha_1 \beta_1^{\alpha_1} b_1}{[\beta_1 + b_1 x_1]^{\alpha_1+1}}\right]\left[\frac{\alpha_2 \beta_2^{\alpha_2} b_2}{[\beta_2 + b_2 x_2]^{\alpha_2+1}}\right] \left\{1 + \theta\left[2\left(\frac{\beta_1}{\beta_1 + b_1 x_1}\right)^{\alpha_1} - 1\right]\left[2\left(\frac{\beta_2}{\beta_2 + b_2 x_2}\right)^{\alpha_2} - 1\right]\right\}. \tag{3.4}$$

Figures 1–3 discussed three shapes of joint density and joint hazard BFGMLE distribution. These 3-dimensional figures indicates that BFGMLE distribution have different shapes.



$\alpha_1 = 2\ \beta_1 = 0.3\ b_1 = 1.8\ \alpha_2 = 1.3\ \beta_2 = 0.5\ b_2 = 1.9\ \theta = -0.5$ $\alpha_1 = 2\ \beta_1 = 0.3\ b_1 = 1.8\ \alpha_2 = 1.3\ \beta_2 = 0.5\ b_2 = 1.9\ \theta = -0.5$

Figure 1. 3-dimension of joint density of BFGMLE.

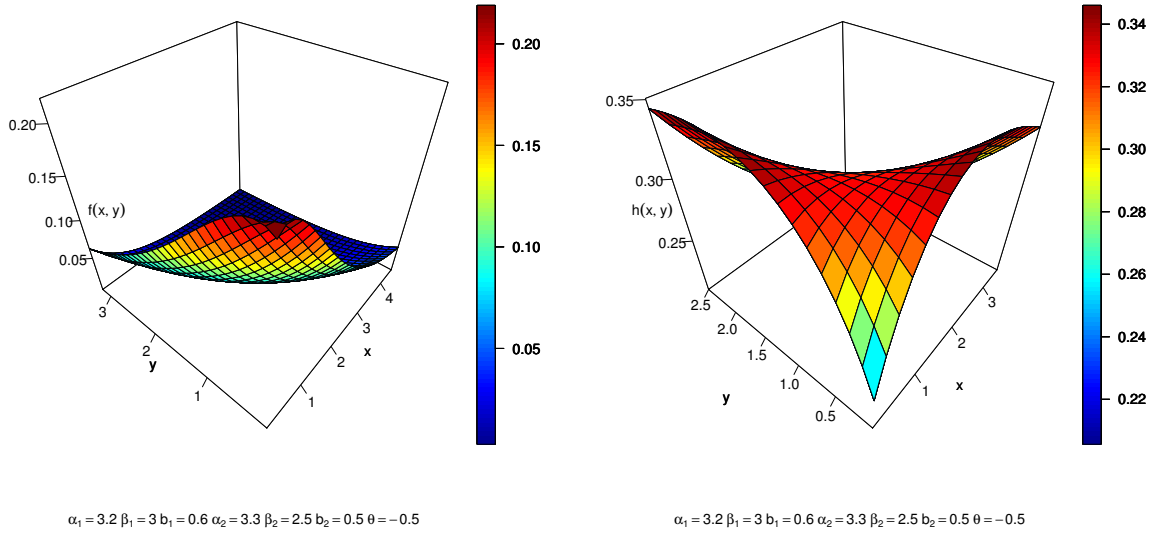


Figure 2. 3-dimension of joint density of BFGMLE.

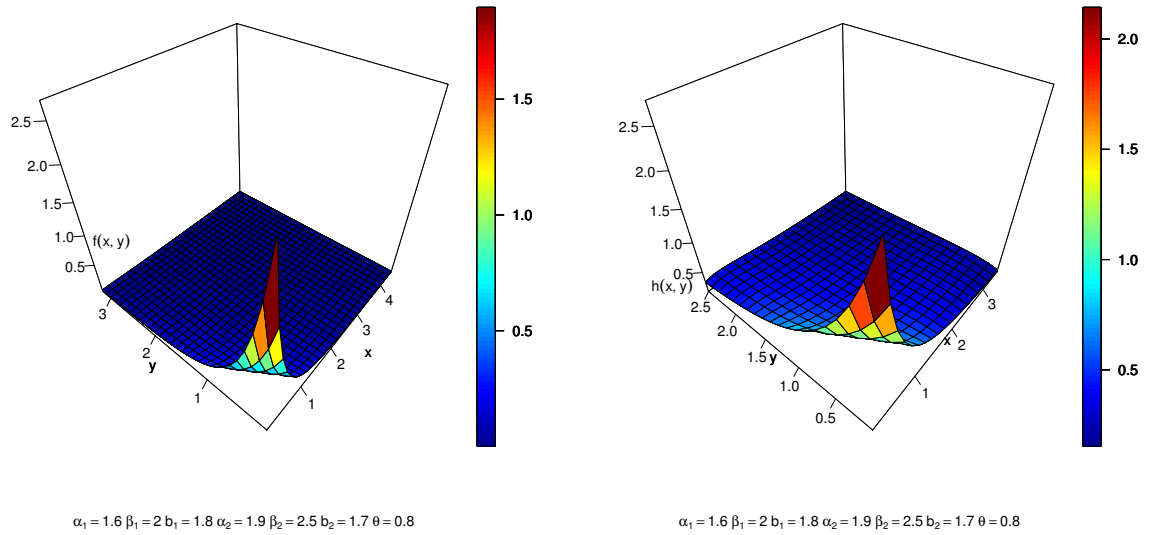


Figure 3. 3-dimension of joint density of BFGMLE.

3.3. BFGMLG-Pareto (BFGMLGP) distribution

The cdf and pdf of the BFGMLGP distribution are produced by obtaining LG-Exponential (LGP) distribution using the LG family and Exponential distribution a follows:

$$F_{BFGMLGP}(x_1, x_2) = \left[1 - \left(\frac{\beta_1}{\beta_1 + b_1 \log(1 + x_1)}\right)^{\alpha_1}\right]\left[1 - \left(\frac{\beta_2}{\beta_2 + b_2 \log(1 + x_2)}\right)^{\alpha_2}\right] \left[1 + \theta\left(\frac{\beta_1}{\beta_1 + b_1 \log(1 + x_1)}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + b_2 \log(1 + x_2)}\right)^{\alpha_2}\right], \tag{3.5}$$

$$
\begin{aligned}
f_{BFGMLGP}(x_1, x_2) =& \alpha_1 \beta_1^{\alpha_1} \frac{b_1}{(1+x_1)\left[\beta_1 + b_1 \log(1+x_1)\right]^{\alpha_1+1}} \\
& \alpha_2 \beta_2^{\alpha_2} \frac{b_2}{(1+x_2)\left[\beta_2 + b_2 \log(1+x_2)\right]^{\alpha_2+1}} \\
& \left\{1 + \theta \left[2\left(\frac{\beta_1}{\beta_1 + b_1 \log(1+x_1)}\right)^{\alpha_1} - 1\right]\left[2\left(\frac{\beta_2}{\beta_2 + b_2 \log(1+x_2)}\right)^{\alpha_2} - 1\right]\right\}.
\end{aligned} \tag{3.6}
$$

3.4. BFGMLG-log-logistic (BFGMLGLL) distribution

The cdf and pdf of the BFGMLLL distribution are obtained by using the LG family and the Normal distribution to obtain the LG-Weibull (LGLL) distribution as follows:

$$
\begin{aligned}
F_{BFGMLGLL}(x_1, x_2) =& \left[1 - \left(\frac{\beta_1}{\beta_1 + \log\left(1 + \left(\frac{x_1}{a_1}\right)^{b_1}\right)}\right)^{\alpha_1}\right]\left[1 - \left(\frac{\beta_2}{\beta_2 + \log\left(1 + \left(\frac{x_2}{a_2}\right)^{b_2}\right)}\right)^{\alpha_2}\right] \\
& \left[1 + \theta \left(\frac{\beta_1}{\beta_1 + \log\left(1 + \left(\frac{x_1}{a_1}\right)^{b_1}\right)}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + \log\left(1 + \left(\frac{x_2}{a_2}\right)^{b_2}\right)}\right)^{\alpha_2}\right],
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
f_{BFGMLGLL}(x_1, x_2) =& \frac{\alpha_1 \beta_1^{\alpha_1} b_1}{a_1^{b_1}} \frac{x_1^{b_1-1}}{\left[1 + \left(\frac{x_1}{a_1}\right)^{b_1}\right]\left[\beta_1 + \log\left(1 + \left(\frac{x_1}{a_1}\right)^{b_1}\right)\right]^{\alpha_1+1}} \\
& \frac{\alpha_2 \beta_2^{\alpha_2} b_2}{a_2^{b_2}} \frac{x_2^{b_2-1}}{\left[1 + \left(\frac{x_2}{a_2}\right)^{b_2}\right]\left[\beta_2 + \log\left(1 + \left(\frac{x_2}{a_2}\right)^{b_2}\right)\right]^{\alpha_2+1}} \\
& \left\{1 + \theta \left[2\left(\frac{\beta_1}{\beta_1 + \log\left(1 + \left(\frac{x_1}{a_1}\right)^{b_1}\right)}\right)^{\alpha_1} - 1\right]\right. \\
& \left.\left[2\left(\frac{\beta_2}{\beta_2 + \log\left(1 + \left(\frac{x_2}{a_2}\right)^{b_2}\right)}\right)^{\alpha_2} - 1\right]\right\}.
\end{aligned}
$$

4. Properties of BFGMLG family

In this section, we introduce some properties of BFGMLG family such as the marginal distributions and its linear representation, conditional distributions, moment generating function and product moments.

4.1. The Marginal distributions

The marginal cdfs of the joint BFGMLG family mentioned in Eq (2.3) can be represented as follows

$$F_{LG}(x_i;\alpha_i,\beta_i,\zeta_i) = 1-\left(\frac{\beta_i}{\beta_i-\log\left[1-G(x_i;\zeta_i)\right]}\right)^{\alpha_i}, \quad \alpha_i,\beta_i>0, \quad i=1,2. \tag{4.1}$$

The marginal density functions of the joint BFGMLG family stated in Eq (2.4), are LG family marginals and are given as:

$$f(x_i;\alpha_i,\beta_i,\zeta_i) = \alpha_i\beta_i^{\alpha_i}\frac{g(x_i;\zeta_i)}{\left[1-G(x_i;\zeta_i)\right]\left[\beta_i-\log\left(1-G(x_i;\zeta_i)\right)\right]^{\alpha_i+1}}, \quad \alpha_i,\beta_i>0. \tag{4.2}$$

We present a useful linear representation for the marginal cdfs of the BFGMLG family. Using power series, expanding the logarithmic function and an equation by Gradshteyn et al. [23] for a power series raised to a positive integer n, we get

$$F_{LG}(x_i;\alpha_i,\beta_i,\zeta_i) = \sum_{k,q\geq 0}^{\infty}\upsilon_{k,q}^{(i)}W_{k+q}^{(i)}(x_i,\zeta_i); \quad i=1,2, \tag{4.3}$$

where $W_{k+q}^{(i)}(x_i,\zeta_i) = G^{k+q}(x_i,\zeta_i)$ represents the cdf of the exponentiated-G (exp-G) family of distributions, with a power parameter $(k+q)$ and $\upsilon_{k,q}^{(i)} = (-1)^k d_{k,q}\alpha_i^{(k)}/(\beta_i^k k!)$ with $d_{k,0}=1$ and (for $q\geq 1$) $d_{k,q} = q^{-1}\sum_{h=1}^{q}\frac{[h(k+1)-q]}{h+1}d_{k,q-h}$.
Also, the marginal pdfs for the BFGMLG family can be expressed in a linear representation as follows

$$f_{LG}(x_i;\alpha_i,\beta_i,\zeta_i) = \sum_{\substack{k,q\geq 0\\ k+q\geq 1}}^{\infty}\upsilon_{k,q}^{(i)}w_{k+q}^{(i)}(x_i,\zeta_i); \quad i=1,2, \tag{4.4}$$

where $w_{k+q}^{(i)}(x_i,\zeta_i) = (k+q)g(x_i,\zeta_i)G^{k+q-1}(x_i,\zeta_i)$ represents the pdf of the exp-G family of distributions with a power parameter of $(k+q)$.

4.2. The conditional distributions

The conditional probability distribution, cumulative distribution function, and survival function of X_i given $X_j = x_j$ are presented for $i,j=1,2$ where $i\neq j$ as follows.

The conditional probability distribution of X_i given $X_j = x_j$ is

$$\begin{aligned} f(x_i\mid x_j) =& \frac{\alpha_i}{\beta_i}\frac{g(x_i;\zeta_i)}{\left[1-G(x_i;\zeta_i)\right]}\omega_i^{\frac{1}{\alpha_i}+1}(x_i;\alpha_i,\beta_i,\zeta_i) \\ & \left\{1+\theta\left[2\omega_i(x_i;\alpha_i,\beta_i,\zeta_i)-1\right]\left[2\omega_j(x_j;\alpha_j,\beta_j,\zeta_j)-1\right]\right\}, \end{aligned} \tag{4.5}$$

where $\omega_j(z_j;\alpha_j,\beta_j,\zeta_j) = \left(\frac{\beta_j}{\beta_j-\log\left[1-G(z_j;\zeta_j)\right]}\right)^{\alpha_j}$ and $j=1,2$, z is vector of x_1 and x_2. The conditional cdf of X_i given $Y_j = y_j$ is

$$\begin{aligned} F(x_i\mid x_j) &= P(X_i\leq x_i\mid Y_j=y_j) \\ &= (1-\omega_i(x_i;\alpha_i,\beta_i,\zeta_i))\left\{1+\theta\,\omega_i(x_i;\alpha_i,\beta_i,\zeta_i)\left[2\omega_j(x_j;\alpha_j,\beta_j,\zeta_j)-1\right]\right\}. \end{aligned} \tag{4.6}$$

The conditional survival of X_i given $X_j = x_j$ is

$$\begin{aligned} S(x_i \mid x_j) &= P(X_i > x_i \mid X_j = x_j) \\ &= \omega_i(x_i;\alpha_i,\beta_i,\zeta_i)\left\{1-\theta\ [1-\omega_i(x_i;\alpha_i,\beta_i,\zeta_i)]\left[2\omega_j(x_j;\alpha_j,\beta_j,\zeta_j)-1\right]\right\}. \end{aligned} \tag{4.7}$$

By (4.6), we can generate a bivariate sample of the LG family using the conditional approach

(1) Generate U and V independently from a uniform(0, 1) distribution.

(2) Set $x_1 = Q_{LG}(u) = G^{-1}\left\{1 - e^{\beta_1\left[1-(1-u)^{-1}/\alpha_1\right]}\right\}$.

(3) Use numerical analysis such as Newton-Raphson to find x_2 by setting $F(x_2 \mid x_1) = V$ in (4.6).

(4) Repeat 1-3 (n) times to get (x_{1i}, x_{2i}), $i = 1, 2, \ldots, n$.

4.3. Regression function, moment generating function and product moments

In this subsection, we introduce the regression function, moment generating function, and product moments.

4.3.1. Regression function

Before introducing the regression function of X_i given $X_j = x_j$, let's first examine the rth moment and the probability weighted moments (PWMs) of X_i when $X_i \sim LG(x_i;\alpha_i,\beta_i,\zeta_i)$, where $i = 1, 2$. The rth moment of X_i, denoted by $\mu_i^{\prime(r)}$, can be written as $\mu_i^{\prime(r)} = \int_{-\infty}^{+\infty} x_i^r f(x_i;\zeta_i)dx_i$. By using Eq (4.4), we obtain

$$\mu_i^{\prime(r)} = \sum_{\substack{k,q\geq 0\\ k+q\geq 1}}^{\infty} \upsilon_{k,q}^{(i)} E(X_{i,k+q}^r); \quad i = 1, 2. \tag{4.8}$$

Here, $X_{i,k+q}^r$ is a random variable with the cdf of $W_{k+q}^{(i)}(x_i,\zeta_i)$ and pdf of $w_{k+q}^{(i)}(x_i,\zeta_i)$. The expectation of $X_{i,k+q}^r$ is given by $E(X_{i,k+q}^r) = \frac{r_i!}{\lambda_i^{r_i}} \sum_{l_i=0}^{k_i+q_i-1} \frac{(-1)^{l_i}}{(l_i+1)^{r_i+1}}\binom{k_i+q_i-1}{l_i}$. Setting $r = 1$ in Eq (4.8), we obtain the mean of X_i, denoted by μ_i', for $i = 1, 2$. We can also obtain PWMs, which are mainly used to estimate parameters for a distribution whose inverse cannot be expressed explicitly. The (n, s)th PWM of X_i is denoted by $\eta_{(n,s)}^{(i)}$ and can be expressed as $\eta_{(n,s)}^{(i)} = E\left[X_i^n F^s(X_i)\right] = \int_{-\infty}^{+\infty} x_i^n F^s(x_i,\zeta_i) f(x_i,\zeta_i)dx_i$. By using Eq (4.4) and $F^s(x_i,\zeta_i) = \sum_{k+q\geq 0} \varphi_{s,k+q}^{(i)} G^{k+q}(x_i,\zeta_i)$; where $\varphi_{s,k+q}^{(i)} = ((k_i+q_i)\upsilon_{s,0}^{(i)})^{-1} \sum_{h_i=1}^{k_i+q_i}[h_i(s_i+1)-(k_i+q_i)]\ \upsilon_{h_i}^{(i)}\ \varphi_{s_i,(k_i+q_i-h_i)}$, we obtain

$$\eta_{(n,s)}^{(i)} = \sum_{k_i+q_i\geq 0} \sum_{\substack{k_i,q_i\geq 0\\ k_i+q_i\geq 1}} (k_i+q_i)\ \upsilon_{k,q}^{(i)}\ \varphi_{s,k+q}^{(i)}\ \Xi_n^{(i)}(2k+2q-1); \quad i = 1, 2, \tag{4.9}$$

where $\Xi_n(m) = \int_0^1 Q_G(u)^n u^m du$. Setting $n = 1$ and $s = 1$ in Eq (4.9), we obtain $\eta^{(i)} = E\left[X_i F(X_i,\zeta_i)\right]$ of X_i, $i = 1, 2$. In BFGMLG family, the regression function of X_i given $X_j = x_j$ or the conditional expectation of X_i given $X_j = x_j$ is calculated using the conditional density of X_i given $X_j = x_j$ in (4.5), as follows:

$$E(X_i \mid X_j = x_j) = \mu_i'\left[1+\theta-2\theta\ F(x_j;\alpha_j,\beta_j,\zeta_j) + \Omega_i\left(4\theta\ F(x_j;\alpha_j,\beta_j,\zeta_j) - 2\theta\right)\right], \tag{4.10}$$

where $i, j = 1, 2$ and $i \neq j$, and $\Omega_i = \eta^{(i)}/\mu_i' = E\left[X_i F(X_i)\right]/E[X_i]$.

4.3.2. Moment generating function

Let (X_1, X_2) represent a random variable with pdf defined in Eq (1.3). The moment generating function of (X_1, X_2) is then obtained by,

$$M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1}e^{t_2X_2}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(t_1)^{n_1}}{n_1!}\frac{(t_2)^{n_2}}{n_2!}\mu'_{n_1}\mu'_{n_2}\left[1+\theta\right.$$
$$\left.-2\theta\Omega_2^{n_2} - 2\theta\Omega_1^{n_1} + 4\theta\Omega_1^{n_1}\Omega_2^{n_2}\right], \tag{4.11}$$

where $\Omega_i^{n_i} = \eta_{n_i}^{(i)}/\mu_i'^{n_i} = E\left[X_i^{n_i}F(X_i)\right]/E[X_i^{n_i}]$ for $i = 1, 2$.

4.3.3. Product moments

If the distribution of the random variable (X_1, X_2) follows the BFGMLG family, then the rth and sth joint moments around zero, denoted by $\mu_i'^{rs}$, can be expressed as follows:

$$\mu_i'^{rs} = E(X_1^rX_2^s) = \mu_1'^r\mu_2'^s\left[1+\theta-2\theta\Omega_2^s-2\theta\Omega_1^r+4\theta\Omega_1^r\Omega_2^s\right], \tag{4.12}$$

The covariance and correlation coefficient (ρ) between X_1 and X_2 can be calculated from Eq (4.12) as follows:

$$cov(X_1,X_2) = \mu_1'\,\mu_2'\,\theta\left[1-2\Omega_2-2\Omega_1+4\Omega_1\Omega_2\right], \tag{4.13}$$

and

$$\rho(X_1,X_2) = \frac{\theta\left[1-2\Omega_2-2\Omega_1+4\Omega_1\Omega_2\right]}{\sqrt{\frac{\mu_1'^{(2)}}{(\mu_1')^2}-1}\sqrt{\frac{\mu_2'^{(2)}}{(\mu_2')^2}-1}}. \tag{4.14}$$

where $\mu_i'^{(2)} = \int_{-\infty}^{\infty}x_i^2\;g(x_i;\zeta_i)dx_i$. It can be observed that when $\theta = 0$, ρ becomes 0, indicating that X_1 and X_2 are independent.

5. Reliability and dependence

A bivariate random vector (X_1, X_2) with joint density $f(x_1, x_2)$ and survival function $S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ has a bivariate hazard rate function, as stated by Basu [24], given by:

$$h(x_1,x_2) = \frac{f(x_1,x_2)}{S(x_1,x_2)}. \tag{5.1}$$

The hazard rate function of BFGMPLG family is

$$\begin{aligned} h_{BFGMLG}(x_1,x_2) =& \frac{\alpha_1\;g(x_1;\zeta_1)}{\left[1-G(x_1;\zeta_1)\right]\left[\beta_1-\log\left(1-G(x_1;\zeta_1)\right)\right]} \\ & \frac{\alpha_2\;g(x_2;\zeta_2)}{\left[1-G(x_2;\zeta_2)\right]\left[\beta_2-\log\left(1-G(x_2;\zeta_2)\right)\right]} \\ & \frac{1+\theta\left[2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1)-1\right]\left[2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2)-1\right]}{1+\theta\left[1-\omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\right]\left[1-\omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\right]}. \end{aligned} \tag{5.2}$$

5.1. Hazard gradient functions

Consider a bivariate random vector (X_1, X_2) with joint density $f(x_1, x_2)$ and survival function $S(x_1, x_2)$, then, as stated by Johnson et al. [25], the bivariate hazard rate function in vector form is given by

$$h(x_1, x_2) = \left(\frac{-\partial \ln S(x_1, x_2)}{\partial x_1}, \frac{-\partial \ln S(x_1, x_2)}{\partial x_2}\right), \tag{5.3}$$

For FGM copula, Vaidyanathan et al. [2] introduced $\frac{-\partial \ln S(x_1,x_2)}{\partial x_1}$ as follows:

$$\frac{-\partial \ln S(x_1, x_2)}{\partial x_1} = h(x_1)\left[1 - \left([1 - F(x_1)]^{-1}\left[(\theta F(x_2))^{-1} + 1\right] - 1\right)^{-1}\right]. \tag{5.4}$$

From (2.6), we get

$$\begin{aligned}\frac{-\partial \ln S(x_1, x_2)}{\partial x_1} =& \frac{\alpha_1\ g(x_1;\zeta_1)}{[1 - G(x_1;\zeta_1)]\left[\beta_1 - \log\left(1 - G(x_1;\zeta_1)\right)\right]} \\ & \left\{1 - \frac{\theta\ \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\left[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\right]}{1 + \theta\left[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\right]\left[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\right]}\right\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}\frac{-\partial \ln S(x_1, x_2)}{\partial x_2} =& \frac{\alpha_2\ g(x_2;\zeta_2)}{[1 - G(x_2;\zeta_2)]\left[\beta_2 - \log\left(1 - G(x_2;\zeta_2)\right)\right]} \\ & \left\{1 - \frac{\theta\ \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\left[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\right]}{1 + \theta\left[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\right]\left[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\right]}\right\}.\end{aligned} \tag{5.6}$$

By substituting the above expressions in (5.3), the vector hazard rate function of BFGMPLG family is obtained. The Eqs (5.5) and (5.6) have two terms: The first term is the hazard rate of the univariate Lomax family distributions, and the second term is a positive increasing function for positive θ and is a negative decreasing function for negative θ.

Also, the conditional hazard rate function $h(x_1 \mid X_2 = x_2)$ of X_1 given $X_2 = x_2$ and $h(x_2 \mid X_1 = x_1)$ of X_2 given $X_1 = x_1$ for the BFGMLG family are

$$\begin{aligned}h(x_1 \mid X_2 = x_2) =& \frac{\alpha_1}{\beta_1}\frac{g(x_1;\zeta_1)}{[1 - G(x_1;\zeta_1)]}\omega_1^{\frac{1}{\alpha_1}}(x_1;\alpha_1,\beta_1,\zeta_1) \\ & \left\{\frac{1 + \theta\left[2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2) - 1\right]}{1 - \theta\ \left[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)\right]\left[2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2) - 1\right]}\right\},\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}h(x_2 \mid X_1 = x_1) =& \frac{\alpha_2}{\beta_2}\frac{g(x_2;\zeta_2)}{[1 - G(x_2;\zeta_2)]}\omega_2^{\frac{1}{\alpha_2}}(x_2;\alpha_2,\beta_2,\zeta_2) \\ & \left\{\frac{1 + \theta\left[2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2) - 1\right]}{1 - \theta\ \left[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)\right]\left[2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1) - 1\right]}\right\}.\end{aligned} \tag{5.8}$$

In reliability theory and lifetime data analysis, the concept of random variable dependence is extremely useful. Covariance and product moment correlation are traditional methods for determining the degree of dependence between two variables. In addition to these traditional measures, several other concepts of new dependence have been suggested in the literature. In this subsections, we will study various measures of dependence for the BFGMLG family and discuss their important properties.

5.2. Positive quadrant dependence

Definition 5.1. *Let $(X_1; X_2)$ be a bivariate random vector with distribution and marginals $F(x_1, x_2)$, $F(x_1)$ and $F(x_2)$, respectively. $(X_1; X_2)$ is considered to be positive quadrant dependent (PQD) if*

$$F(x_1, x_2) \geq F(x_1)F(x_2) \qquad \text{for } x_1 \text{ and } x_2,$$

or, equivalently, if

$$S(x_1, x_2) \geq S(x_1)S(x_2) \qquad \text{for } x_1 \text{ and } x_2$$

where $S(x_1, x_2)$, $S(x_1)$ and $S(x_2)$ symbolise the joint and marginals survival functions. If the reverse inequality holds, the random vector $(X_1; X_2)$ is negative quadrant dependent (NQD) (Lehmann [26] and Nelsen [16]).

Proposition 5.1. *Let $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$. Then the BFGMLG family is PQD (NQD) for a positive (negative) value of θ.*

Proof. From Eq (2.6), the marginal survival functions $S(x_1)$ and $S(x_2)$ are easy to obtain. One can quickly determine that $S(x_1, x_2) \geq (\leq) S(x_1)S(x_2)$, which relates to the PQD (NQD) of the BFGMLG family using joint and marginal survival function.

Remark 5.1. *X_1 and X_2 are positively (negatively) correlated if $Cov(X_1; X_2) \geq 0$ (≤ 0), respectively. Therefore, for the BFGMLG family, $Cov(X_1, X_2) \geq 0$ (≤ 0) is a direct consequence of the PQD (NQD) property, respectively.*

5.3. Regression dependence

Compared to PQD, regression dependence is a stronger concept of dependence.

Definition 5.2. *$F(x_1, x_2)$ is positively regression dependent if (Nelsen [16]):*

$$P(X_2 > x_2 \mid X_1 = x_1) \text{ is increasing in } x_1 \text{ for all values of } x_2.$$

Proposition 5.2. *Let $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$ with cdf function $F(x_1, x_2)$. Then, $F(x_1, x_2)$ in (2.3) is positively regression dependent.*

Proof. The conditional survival function $P(X_2 > x_2 \mid X_1 = x_1)$ of X_2 on $X_1 = x_1$ is given in Eq (4.7). On differentiation with respect to x_1, we obtain:

$$\frac{\partial}{\partial x_1} P(X_2 > x_2 \mid X_1 = x_1) = \left\{ \frac{2\theta\alpha_1\beta_1 g(x_1; \zeta_1)\omega_1^{(1-\frac{1}{\alpha_1})}(x_1; \alpha_1, \beta_1, \zeta_1)}{[1 - G(x_1; \zeta_1)]} \right.$$
$$\left. \frac{\omega_2(x_2; \alpha_2, \beta_2, \zeta_2)\ [1 - \omega_2(x_2; \alpha_2, \beta_2, \zeta_2)]}{[\beta_1 - \log(1 - G(x_1; \zeta_1))]^2} \right\} \geq 0. \tag{5.9}$$

We start with a local dependence function to establish the TP2 property of the BFGMLG family.

5.4. A local dependence function $\gamma(x_1, x_2)$

Holland and Wang [27] introduced a local dependence function (x_1, x_2), to study the dependence between random variables X_1 and X_2, and defined it as follows:

$$\gamma(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \ln f(x_1, x_2). \tag{5.10}$$

This dependence function, is an effective tool for investigating the totally positive of order 2 (TP2) property of a bivariate distribution. Holland and Wang [27], and Balakrishnan and Lai [28] investigated the detailed properties of $\gamma(x_1, x_2)$.

Proposition 5.3. *Let* $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$*. Then, its local dependence function is*

$$\begin{aligned}\gamma(x_1, x_2) =& \alpha_1 \beta_1^{\alpha_1} \frac{g(x_1; \zeta_1)}{[1 - G(x_1; \zeta_1)]\left[\beta_1 - \log\left(1 - G(x_1; \zeta_1)\right)\right]^{\alpha_1 + 1}} \\ & \alpha_2 \beta_2^{\alpha_2} \frac{g(x_2; \zeta_2)}{[1 - G(x_2; \zeta_2)]\left[\beta_2 - \log\left(1 - G(x_2; \zeta_2)\right)\right]^{\alpha_2 + 1}} \\ & \frac{4\,\theta}{\left[1 + \theta\left(2\left[\frac{\beta_1}{\beta_1 - \log\left[1 - G(x_1;\zeta_1)\right]}\right]^{\alpha_1} - 1\right)\left(2\left[\frac{\beta_2}{\beta_2 - \log\left[1 - G(x_2;\zeta_2)\right]}\right]^{\alpha_2} - 1\right)\right]^2}.\end{aligned} \tag{5.11}$$

It is worth nothing, that when $\theta = 0$, then $\gamma(x_1, x_2) = 0$, implying that X_1 and X_2 are independent. Holland and Wang [27] and Nelsen [16] demonstrated that a bivariate density $f(x_1, x_2)$ has the TP2 (totally positive of order 2) property if and only if $\gamma(x_1, x_2) \geq 0$ and has the TN2 (totally negative of order 2) property if and only if $\gamma(x_1, x_2) < 0$. The final result for the TP2 property of the BFGMLG family is as follows:

Proposition 5.4. *Let* $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$ *with density* $f(x_1, x_2)$ *defined in (2.4). Then,* $f(x_1, x_2)$ *has the TP2 property if* $\theta \geq 0$ *and the TN2 property if* $\theta < 0$*.*

It is worth noting that TP2 is a stronger concept of dependence than other well-known forms of dependence such as stochastically increasing (SI), right-tail increasing (RTI), association, and positive quadrant dependence (PQD). It has been established that TP2 implies these other forms of dependence, as shown by Nelsen [16] and Balakrishnan and Lai [28]. As a result, the BFGMLG family has all of these dependence properties for $0 \leq \theta \leq 1$.

5.5. Clayton-Oakes association measure

Based on the survival function and its dervatives , Clayton [29] and Oakes [30] defined a local dependence function as:

$$l(x_1, x_2) = \frac{f(x_1, x_2)\, S(x_1, x_2)}{S_1(x_1, x_2)\, S_2(x_1, x_2)} \tag{5.12}$$

where $S_1(x_1, x_2) = \frac{\partial}{\partial x_1} S(x_1, x_2)$ and $S_2(x_1, x_2) = \frac{\partial}{\partial x_2} S(x_1, x_2)$.

Proposition 5.5. *Let* $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$*, then,*

$$l(x_1, x_2) = \frac{\{1 + \theta\,[2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1) - 1]\,[2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2) - 1]\}}{\{1 + \theta\ [1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)]\ [1 - 2\omega_1(x_1;\alpha_1,\beta_1,\zeta_1)]\}} \frac{\{1 + \theta\,[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)]\,[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)]\}}{\{1 + \theta\ [1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)]\ [1 - 2\omega_2(x_2;\alpha_2,\beta_2,\zeta_2)]\}}, \tag{5.13}$$

It is simple to demonstrate that for $l(x_1, x_2) = 1$, the random variables X_1 and X_2 are independent. From Eq (5.13), the random variables X_1 and X_2 are independent for $\theta = 0$. For $0 < \theta < 1$, we have $l(x_1, x_2) > 1$, implying that (X_1, X_2) is right corner set increasing (RCSI) (Nelsen [16]).

5.6. Conditional probability measure $\psi(x_1, x_2)$

By using conditional probability, Anderson et al. [31] defined a measure of association for random vector (X_1, X_2) $\psi(x_1, x_2)$ as:

$$\psi(x_1, x_2) = \frac{P(X_1 > x_1 \mid X_2 > x_2)}{P(X_1 > x_1)} = \frac{S\,(x_1, x_2)}{S\,(x_1, 0)\ S\,(0, x_2)}. \tag{5.14}$$

From Eq (5.14), it is evident that $\psi(x_1, x_2) = 1$ if and only if X_1 and X_2 are independent. Moreover, if $\psi(x_1, x_2) > 1$ for all (x_1, x_2), then (X_1, X_2) is positively quadrant dependent (PQD). It is also worth noting that $l(x_1, x_2) > 1$ implies $\psi(x_1, x_2) > 1$. The BFGMLG family exhibits the following property.

Proposition 5.6. *Let $(X_1, X_2) \sim BFGMLG(\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$, then,*

$$\psi(x_1, x_2) = 1 + \theta\,[1 - \omega_1(x_1;\alpha_1,\beta_1,\zeta_1)]\,[1 - \omega_2(x_2;\alpha_2,\beta_2,\zeta_2)]\,. \tag{5.15}$$

We can see from Eq (5.15) that when $\theta = 0$, we get $\psi(x_1, x_2) = 1$. As a result, X_1 and X_2 are independent. Similarly, X_1 and X_2 are PQD when $\theta > 0$.

5.7. BFGMLG family and dependence measures

The product moments correlation is a measure of linear dependence that can produce misleading results, even when the dependence is strong for non-elliptical random variables. Copula-based measures of concordance can capture non-linear dependence and are widely regarded as the superior alternative to linear correlation. In this section, we introduce some measures of dependence based on copulas for the BFGMLG family, such as Spearman's rho (ρ_c), Gini's gamma (γ_c), and the measure of regression dependence $r(X_1, X_2)$, which are defined in Nelsen [32] and Popović et al. [33].

5.8. Spearman's rho (ρ_c)

The proportional to the probability of concordance minus the probability of discordance for two vectors is denoted as Spearman's rho (ρ_c). For copula terms, Spearman's (ρ_c) is defined as:

$$\rho_c = 12 \int_0^1 \int_0^1 C\,\{F(x_1), F(x_2)\}\ dF(x_1)\ dF(x_2)\ - 3.$$

In case of the FGM copula, Spearman's rho (ρ_{c_θ}) is $\frac{\theta}{3}$ and means that $\rho_{c_\theta} \in [\frac{-1}{3}, \frac{1}{3}]$.

5.9. Gini's gamma (γ_c)

Nelsen [32] defined Gini's measure of association for X_1 and X_2, denoted by γ_c, as follows:

$$\gamma_c = 4\int_0^1 \{C[F(x_1), F(x_1)] + C[F(x_1), 1 - F(x_1)]\}\ dF(x_1)\ - 2.$$

In case of FGM copula, Gini's gamma γ_c is $\gamma_{c_\theta} = \frac{4}{15}\theta$ and this means that $\gamma_{c_\theta} \in [\frac{-4}{15}, \frac{4}{15}]$.

5.10. A measure of regression dependence

Dette et al. [34] proposed a measure of regression dependence between two random variables X_1 and X_2, defined in terms of the copula C. This measure quantifies the strength of the relationship between the variables and is given by

$$r(X_1, X_2) = 6\int_0^1\int_0^1 \left(\frac{\partial}{\partial F(x_1)}C(F(x_1), F(x_2))\right)^2 dF(x_1)dF(x_2) - 2. \tag{5.16}$$

For the FGM copula, a measure of regression dependence between two random variables X_1 and X_2 is given by $\left(\frac{\theta^2}{30} - \frac{\theta}{4}\right)$ and this means that $r \in [0.216, 0.283]$.

6. Multivariate FGMLG family

In this section, we discuss the multivariate of the FGMLG family. The multivariate aspect of the FGMLG family is important for modeling the dependence structure of multiple variables simultaneously. The joint distribution function of the d-variate FGM copula, denoted by C_d, is defined by Johnson and Kotz [35] as follows:

$$C_\theta(F_1(x_1), ..., F_d(x_d)) = \prod_{j=1}^{d} F_j(x_j)\left(1 + \sum_{m=2}^{d}\sum_{1\le j_1<...<j_m\le d}\theta_{j_1,...,j_m}\left(1 - F_{j_1}(x_{j_1})\right)...\left(1 - F_{j_m}(x_{j_m})\right)\right), \tag{6.1}$$

where $(F_1(x_1), ..., F_d(x_d)) \in [0,1]^d$ and $\theta_{j_1,...,j_m} \in \theta$ is a parameter. The dependence parameters in the d-variate FGM copula are $\sum_{j=2}^{d}\binom{d}{j} = 2^d - d - 1$. The formula for the joint density function of $F(X)$, denoted by c_d , is as follows:

$$c_\theta(F(x_1), ..., F(x_d)) = 1 + \sum_{m=2}^{d}\sum_{1\le j_1<...<j_m\le d}\theta_{j_1,...,j_m}\left(1 - 2F_{j_1}(x_{j_1})\right)...\left(1 - 2F_{j_m}(x_{j_m})\right). \tag{6.2}$$

By using Eq (4.1) and (1.6), the joint cdf of the multivariate FGM lomax generator family, denoted by (MFGMLG) family, is

$$\begin{aligned} F_d(x_1, ..., x_d; \alpha_j, \beta_j, \theta_j, \zeta_j) = &\left\{\prod_{j=1}^{d} 1 - \left(\frac{\beta_j}{\beta_j - \log\left[1 - G_j(x_j;\zeta_j)\right]}\right)^{\alpha_j}\right\} \\ &\left(1 + \sum_{m=2}^{d}\sum_{1\le j_1<...<j_m\le d}\theta_{j_1,...,j_m}\left(\frac{\beta_{j_1}}{\beta_{j_1} - \log\left[1 - G_{j_1}(x_{j_1};\zeta_{j_1})\right]}\right)^{\alpha_{j_1}}...\left(\frac{\beta_{j_m}}{\beta_{j_m} - \log\left[1 - G_{j_m}(x_{j_m};\zeta_{j_m})\right]}\right)^{\alpha_{j_m}}\right). \end{aligned} \tag{6.3}$$

The joint density function of the MFGMLG family of distributions by using Eqs (4.2) and (1.7) is

$$f_d(x_1,...,x_d) = \prod_{j=1}^{d} \alpha_j \beta_j^{\alpha_j} \frac{g_j(x_j;\zeta_j)}{\left[1-G_j(x_j;\zeta_j)\right]\left[\beta_j - \log\left(1-G_j(x_j;\zeta_j)\right)\right]^{\alpha_j+1}} \left\{1 + \sum_{m=2}^{d} \sum_{1\le j_1<...<j_m\le d} \theta_{j_1,...,j_m} \left(2\left(\frac{\beta_{j_1}}{\beta_{j_1} - \log\left[1-G_{j_1}(x_{j_1};\zeta_{j_1})\right]}\right)^{\alpha_{j_1}} - 1\right)\cdots \cdots \left(2\left(\frac{\beta_{j_m}}{\beta_{j_m} - \log\left[1-G_{j_m}(x_{j_m};\zeta_{j_m})\right]}\right)^{\alpha_{j_m}} - 1\right)\right\}.$$

7. Methods for estimating parameters

In this section, we introduce two estimation methods for the parameters $\Theta = (\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$ of the bivariate continuous FGMLG family: Maximum likelihood estimation (MLE) and Bayesian estimation.

7.1. Maximum likelihood estimation

In this subsection, we study the problem of determining the maximum likelihood estimators (MLEs) of the unknown parameters of the BFGMLG family distributions using a random sample. We assume that $\{(X_{11}, X_{21}), ..., (X_{1n}, X_{2n})\}$ is a random bivariate sample from BFGMLG $\sim (\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$. For a sample of size n, the log-likelihood function is given by:

$$\begin{aligned}
\ln L =& n\ln\alpha_1 + n\alpha_1\ln\beta_1 + \sum_{i=1}^{n}\ln g(x_{1i},\zeta_1) - (\alpha_1+1)\sum_{i=1}^{n}\ln\left[\beta_1 - \log\left(1-G(x_{1i};\zeta_1)\right)\right] \\
&- \sum_{i=1}^{n}\ln\left[1-G(x_{1i};\zeta_1)\right] + n\ln\alpha_2 + n\alpha_2\ln\beta_2 + \sum_{i=1}^{n}\ln g(x_{2i},\zeta_2) \\
&- \sum_{i=1}^{n}\ln\left[1-G(x_{2i};\zeta_2)\right] - (\alpha_2+1)\sum_{i=1}^{n}\ln\left[\beta_2 - \log\left(1-G(x_{2i};\zeta_2)\right)\right] \\
&+ \sum_{i=1}^{n}\ln\left[1+\theta\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1)-1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right)-1\right]\right].
\end{aligned} \tag{7.1}$$

By differentiating (7.1) with respect to $\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2$, and θ, and equating the resulting expressions to zero, we obtain the normal equations as follows:

$$\begin{aligned}
\frac{\partial L(\Theta)}{\partial\alpha_j} =& \frac{n}{\alpha_j} + n\ln\beta_j - \sum_{i=1}^{n}\ln\left[\beta_j - \log\left(1-G(x_{ji};\zeta_j)\right)\right] \\
&+ \sum_{i=1}^{n}\frac{2\theta\left(2\omega_{li}\left(x_{li};\alpha_l,\beta_l,\zeta_l\right)-1\right)\omega_{ji}\left(x_{ji};\alpha_j,\beta_j,\zeta_j\right)\ln\omega_{ji}^{\frac{1}{\alpha_j}}\left(x_j i;\alpha_j,\beta_j,\zeta_j\right)}{1+\theta\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1)-1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right)-1\right]} = 0,
\end{aligned} \tag{7.2}$$

$$\frac{\partial L(\Theta)}{\partial \beta_j} = \frac{n\alpha_j}{\beta_j} - (\alpha_j + 1)\sum_{i=1}^{n}\left[\beta_j - \log\left(1 - G(x_{ji};\zeta_j)\right)\right]^{-1} - \sum_{i=1}^{n}\left\{\frac{2\theta\alpha_j\left(2\omega_{li}\left(x_{li};\alpha_l,\beta_l,\zeta_l\right) - 1\right)\omega_{ji}^{\left(1-\frac{1}{\alpha_j}\right)}\left(x_{ji};\alpha_j,\beta_j,\zeta_j\right)}{1 + \theta\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right) - 1\right]} \frac{\log\left(1 - G(x_{ji};\zeta_j)\right)}{\left[\beta_j - \log\left(1 - G(x_{ji};\zeta_j)\right)\right]^2}\right\} = 0,$$

$$\frac{\partial L(\Theta)}{\partial \zeta_j} = \sum_{i=1}^{n}\frac{\phi(x_{ji},\zeta_j)}{g(x_{ji},\zeta_j)} - (\alpha_j + 1)\sum_{i=1}^{n}\frac{\Phi(x_{ji};\zeta_j)}{\left(1 - G(x_{ji};\zeta_j)\right)\left[\beta_j - \log\left(1 - G(x_{ji};\zeta_j)\right)\right]} - \sum_{i=1}^{n}\left\{\frac{2\theta\alpha_j\left(2\omega_{li}\left(x_{li};\alpha_l,\beta_l,\zeta_l\right) - 1\right)\omega_{ji}^{\left(1-\frac{1}{\alpha_j}\right)}\left(x_{ji};\alpha_j,\beta_j,\zeta_j\right)}{1 + \theta\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right) - 1\right]} \frac{\beta_j\Phi(x_{ji};\zeta_j)}{\left(1 - G(x_{ji};\zeta_j)\right)\left[\beta_j - \log\left(1 - G(x_{ji};\zeta_j)\right)\right]^2}\right\} = 0,$$

$$\frac{\partial L(\Theta)}{\partial \theta} = \sum_{i=1}^{n}\frac{\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right) - 1\right]}{1 + \theta\left[2\omega_{1i}(x_{1i};\alpha_1,\beta_1,\zeta_1) - 1\right]\left[2\omega_{2i}\left(x_{2i};\alpha_2,\beta_2,\zeta_2\right) - 1\right]} = 0, \tag{7.3}$$

where $\Phi(x_{ji};\zeta_j) = \frac{\partial g(x_{ji};\zeta_j)}{\partial \zeta_j}$, $\Phi(x_{ji};\zeta_j) = \frac{\partial G(x_{ji};\zeta_j)}{\partial \zeta_j}$ and $j = (1,2)$, $l = (1,2)$; $j \neq l$, (for example, $j = 1$ then $l = 2$).

The non-linear normal equations involved in finding the MLEs of the unknown parameters are complex and manually solving them can be time-consuming and inconvenient. Therefore, computational methods are used to determine the MLEs of the parameters. In addition, based on large sample theory and ML estimates of the asymptotic distribution, we can obtain approximate confidence intervals (CI) for the parameters $\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2$, and θ. Specifically, the parameter vector $\Theta = (\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2, \theta)$ follows a multivariate normal distribution with mean Θ and covariance matrix $I^{-1}(\Theta)$, where I^{-1} is the inverse of the observed information matrix obtained as:

$$I^{-1}(\hat{\Theta}) = -E\begin{bmatrix} I_{\hat{\alpha_1}\hat{\alpha_1}} & & & & & & \\ I_{\hat{\beta_1}\hat{\alpha_1}} & I_{\hat{\beta_1}\hat{\beta_1}} & & & & & \\ I_{\hat{\zeta_1}\hat{\alpha_1}} & I_{\hat{\zeta_1}\hat{\beta_1}} & I_{\hat{\zeta_1}\hat{\zeta_1}} & & & & \\ I_{\hat{\alpha_2}\hat{\alpha_1}} & I_{\hat{\alpha_2}\hat{\beta_1}} & I_{\hat{\alpha_2}\hat{\zeta_1}} & I_{\hat{\alpha_2}\hat{\alpha_2}} & & & \\ I_{\hat{\beta_2}\hat{\alpha_1}} & I_{\hat{\beta_2}\hat{\beta_1}} & I_{\hat{\beta_2}\hat{\zeta_1}} & I_{\hat{\beta_2}\hat{\alpha_2}} & I_{\hat{\beta_2}\hat{\beta_2}} & & \\ I_{\hat{\zeta_2}\hat{\alpha_1}} & I_{\hat{\zeta_2}\hat{\beta_1}} & I_{\hat{\zeta_2}\hat{\zeta_1}} & I_{\hat{\zeta_2}\hat{\alpha_2}} & I_{\hat{\zeta_2}\hat{\beta_2}} & I_{\hat{\zeta_2}\hat{\zeta_2}} & \\ I_{\hat{\theta}\hat{\alpha_1}} & I_{\hat{\theta}\hat{\beta_1}} & I_{\hat{\theta}\hat{\zeta_1}} & I_{\hat{\theta}\hat{\alpha_2}} & I_{\hat{\theta}\hat{\beta_2}} & I_{\hat{\theta}\hat{\zeta_2}} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}^{-1}. \tag{7.4}$$

$I(\Theta)$ is the second derivatives of the natural logarithm of the likelihood function evaluated at $\hat{\Theta} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\zeta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\zeta}_2, \hat{\theta})$. As a result, for the parameters $\alpha_1, \beta_1, \zeta_1, \alpha_2, \beta_2, \zeta_2$ and θ for $j = 1, 2$, $100(1-\gamma)$ % approximate CI are given as: $\hat{\alpha}_j \pm Z_{\gamma/2}\sqrt{I_{\hat{\alpha}_j\hat{\alpha}_j}}$, $\hat{\beta}_j \pm Z_{\gamma/2}\sqrt{I_{\hat{\beta}_j\hat{\beta}_j}}$, $\hat{\zeta}_j \pm Z_{\gamma/2}\sqrt{I_{\hat{\zeta}_j\hat{\zeta}_j}}$ and $\hat{\theta} \pm Z_{\gamma/2}\sqrt{I_{\hat{\theta}\hat{\theta}}}$, where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right tail probability $\frac{\gamma}{2}$.

7.2. Bayesian estimation

Before discussing how a Bayesian technique could estimate a population parameter, it is crucial to understand one key difference between frequentest and Bayesian statisticians. The difference is whether a statistician considers a parameter to be an unknowable constant or a random variable. An estimator or decision rule used in estimating theory and decision theory that minimises the posterior expected value of a loss function is referred to as a Bayes estimator, also known as a Bayes action (i.e., the posterior expected loss). In other words, it maximises the posterior expectation of the utility function. The following steps are often followed when using the Bayesian technique with a bivariate model based on the FGM copula for inference:

(1) The joint independent prior distribution is $\Pi(\Theta) = \prod_{j=1}^{p-1} \pi_j(\Theta_j)$ by selecting the independent prior distributions $\pi_j(\Theta_j);\ j = 1, ..., p-1$ for all parameters, where $p = length(\Theta)$.

(2) For the copula parameter $\pi_j(\theta)$, select the independent prior distribution where $-1 \leq \theta \leq 1$.

(3) By obtaining the likelihood function for the joint statistical model $L(x, y|\Theta)$ that reflects our perceptions of X and Y under the assumption of Θ.

(4) Do the joint posterior distribution calculation. Applying the Bayes law of conditional probabilities, we obtain $\Pi(\Theta|x, y)$ as

$$\Pi(\Theta|x, y) = \frac{\Pi(\Theta)\ L(x, y|\Theta)}{\int_{\Theta_1} ... \int_{\Theta_j} \Pi(\Theta)\ L(x, y|\Theta) d\Theta_1 ... d\Theta_j}.$$

(5) For $\Theta_j;\ j = 1, ..., p$, obtain the proportionate posterior distribution.

(6) Use Gibbs sampling or the Metropolis-Hastings (MH) algorithm to numerically analyse Bayesian estimate using Markov chain Monte Carlo (MCMC).

(7) Pick out loss functions that are symmetric and asymmetric.

To learn more about Bayesian algorithms, consult the citations provided by Suzuki et al. [36] and Louzada et al. [37]. We employed informative prior as independent gamma distributions in the parameter vector Θ. We used prior distributions that are not informative, such as $uniform(w_7, q_7)$; $-1 < \theta < 1$, for the copula parameter. The independent joint prior density function of Θ for the BFGMLG family can be stated as follows:

$$\Pi(\Theta) \propto \alpha_1^{w_1-1} \beta_1^{w_2-1} \zeta_1^{w_3-1} \alpha_2^{w_4-1} \beta_2^{w_5-1} \zeta_2^{w_6-1} \frac{1}{q_7 - w_7} e^{-(q_1\alpha_1 + q_2\beta_1 + q_3\zeta_1 + q_4\alpha_2 + q_5\beta_2 + q_6\zeta_2)}. \tag{7.5}$$

The estimate and variance-covariance matrix of the MLE approach can be used to find the appropriate hyper-parameters of the independent joint prior. The estimated hyper-parameters can be expressed as by equating the mean and variance of the gamma priors.

$$w_j = \frac{\left[\frac{1}{L}\sum_{i=1}^{L} \hat{\Theta}_j^i\right]^2}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1,$$

$$q_j = \frac{\frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1.$$

where L denotes the quantity of iterations. The calculated hyper-parameter for the copula parameter is denoted by

$$w_7 = \sqrt{\frac{3}{L-1}\sum_{i=1}^{L}\left[\hat{\theta}^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i\right]^2} - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i,$$

$$q_7 = \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i - \sqrt{\frac{3}{L-1}\sum_{i=1}^{L}\left[\hat{\theta}^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\theta}^i\right]^2}.$$

The likelihood function of the BFGMLG family yields the joint posterior density function of Θ, which is as follows.

$$\begin{aligned}\Pi(\Theta|x_1, x_2) =& \alpha_1^{n+w_1-1}\beta_1^{w_2+n\alpha_1-1}\zeta_1^{n+w_3-1}\prod_{i=1}^{n}\frac{g(x_{1i};\zeta_1)e^{-(q_1\alpha_1+q_2\beta_1+q_3\zeta_1)}}{[1-G(x_{1i};\zeta_1)]\left[\beta_1 - \log\left(1-G(x_{1i};\zeta_1)\right)\right]^{\alpha_1+1}} \\ & \alpha_2^{n+a_4-1}\beta_2^{n\alpha_2+a_5-1}\zeta_2^{n+w_6-1}\prod_{i=1}^{n}\frac{g(x_{2i};\zeta_2)e^{-(\alpha_2+q_5\beta_2+q_6\zeta_2)}}{[1-G(x_{2i};\zeta_2)]\left[\beta_2 - \log\left(1-G(x_{2i};\zeta_2)\right)\right]^{\alpha_2+1}} \\ & \prod_{i=1}^{n}\left[1+\theta\left(2\left[\frac{\beta_1}{\beta_1 - \log\left[1-G(x_{1i};\zeta_1)\right]}\right]^{\alpha_1}-1\right)\left(2\left[\frac{\beta_2}{\beta_2 - \log\left[1-G(x_{2i};\zeta_2)\right]}\right]^{\alpha_2}-1\right)\right].\end{aligned} \tag{7.6}$$

By using the most common of symmetric loss function, which is a squared error loss function. The Bayes estimators of $\tilde{\Theta}$ based on squared error loss function is given by

$$\begin{aligned}S(\tilde{\Theta}) &= E\left(\tilde{\Theta}-\Theta\right)^2 \\ &= \int_0^\infty ... \int_0^\infty \int_{-1}^{1}\left(\tilde{\Theta}-\Theta\right)^2 \Pi(\Theta|x,y)d\Theta_1...d\Theta_5.\end{aligned} \tag{7.7}$$

It is noteworthy that the integrals provided by Eq (7.7) cannot be computed directly. Because of this, we apply the MCMC to determine an approximated integral value. An essential subclass of MCMC approaches are Gibbs sampling and more general Metropolis within Gibbs samplers. The two most popular MCMC method examples are the MH algorithm and Gibbs sampling. Similar to acceptance-rejection sampling, the MH method assumes that a candidate value can be produced from a proposal distribution that is a normal distribution for each iteration of the process. Gibbs sampling measurements include.

7.3. Highest posterior density

The approach of Chen and Shao [38] was extensively used to construct the HPD intervals of unknown benefit distribution parameters in highest posterior density (HPD) intervals for Bayesian estimate. For instance, a 95% HPD interval can be constructed using the lower and upper percentiles of the MCMC sample results as the two ends, respectively. The following is how Bayes, reliable intervals of the Θ parameters are obtained:

(1) Arrange Θ_j; $j = 1, ..., 5$ as $\alpha_l^{[1]} < \alpha_l^{[2]} < ... < \alpha_l^{[L]}$, $\beta_l^{[1]} < \beta_l^{[2]} < ... < \beta_l^{[L]}$, $\zeta_l^{[1]} < \zeta_l^{[2]} < ... < \zeta_l^{[L]}$, and $\theta^{[1]} < \theta^{[2]} < ... < \theta^{[L]}$ where $l = 1, 2$, and L is the length of MCMC generated.

(2) The 95% symmetric credible intervals of $\alpha_1, \beta_1, \alpha_2, \beta_2$ and θ become $\left(\alpha_l^{L_{250/10000}}, \alpha_l^{L_{9750/10000}}\right)$, $\left(\beta_l^{L_{250/10000}}, \beta_l^{L_{9750/10000}}\right)$, $\left(\zeta_l^{L_{250/10000}}, \zeta_l^{L_{9750/10000}}\right)$ and $\left(\theta^{L_{250/10000}}, \theta^{L_{9750/10000}}\right)$.

8. Simulation

For comparing the likelihood and Bayesian estimation methods, MCMC simulation studies were performed. The results are presented in Table 2 $\alpha_1 = 1.6, \beta_1 = 2, b_1 = 1.8, \alpha_2 = 1.9, \beta_2 = 2.5, b_2 = 1.7$, Table 3 $\alpha_1 = 0.5, \beta_1 = 1.2, b_1 = 1.2, \alpha_2 = 0.8, \beta_2 = 0.5, b_2 = 1.3$, Table 4 $\alpha_1 = 3.2, \beta_1 = 3, b_1 = 0.6, \alpha_2 = 3.3, \beta_2 = 2.5, b_2 = 0.5$, and Table 5 $\alpha_1 = 2, \beta_1 = 0.3, b_1 = 1.8, \alpha_2 = 1.3, \beta_2 = 0.5, b_2 = 1.9$. Based on the bias, mean squared errors (MSE), and length of confidence intervals (LCI), numerical assessments were carried out. First, 5000 samples of the BFGM LGE model were created. For computational time of Bayesian estimation, we simulated 12000 MCMC samples and ignored the first 2000 iterations as burn-in. It should be noted that all estimating methods work better when $n \Rightarrow +\infty$, as shown by Tables 2–5. The best guess is a value with a lower range of numerical assessments. It should be emphasised that the MLE and Bayesian approaches are generally suggested for statistical modelling and applications; this assessment, as shown in Tables 2–5, is mostly based on a thorough simulation study, and the simulation, as is well known, comes before the application on real data. Furthermore, despite the efficiency of likelihood method, and the Bayesian approach remains the most effective and reliable of likelihood method, but all likelihood methods are effective. Although using simulation to compare various estimation methods is not prohibited, in this section we use simulation studies to evaluate them rather than compare them. However, since real data are frequently used to compare various estimation methods, we decided to present four applications for this particular use.

The following conclusions can be drawn from Table 2–5:

- The proposed estimates of the parameters for the BFGM LGE distribution performed well, which is the key general observation.
- The results of Tables show that the BFGM LGE distribution is stable since the range of bias, and MSE for seven parameters of the BFGM LGE distribution is fairly modest.
- As the sample size increases, we occasionally observe a decrease in the bias, MSE, and LCI for all estimations.
- This indicates that for high sample sizes, several estimating methodologies yield correct bias and MSE findings.
- Due to the gamma information, the Bayes estimates of the BFGM LGE distribution's parameters behaved more predictably than the other estimates. Regarding HPD credible intervals, the same statement might be made.
- The Bayesian estimation approach is the most accurate way to estimate the BFGM LGE distribution parameter.
- When increasing the value of the copula parameter θ, the estimated MSE will become as low as possible, knowing that the maximum correlation for the data based on FGM copula is from –0.3333 to 0.3333.

Table 2. MLE and Bayesian for $\alpha_1 = 1.6, \beta_1 = 2, b_1 = 1.8, \alpha_2 = 1.9, \beta_2 = 2.5, b_2 = 1.7$.

θ		0.5						-0.5					
		MLE			Bayesian			MLE			Bayesian		
n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
40	α_1	0.6642	2.5631	5.7131	0.0618	0.1894	1.6150	0.6374	2.5308	5.7165	0.0658	0.2322	1.5180
	β_1	0.0914	0.2767	2.0318	-0.0668	0.2347	2.0232	0.0798	0.2744	2.0303	-0.0543	0.2364	2.0032
	b_1	-0.1782	0.4246	2.4581	-0.0139	0.3391	2.1761	-0.1569	0.4143	2.4481	-0.0162	0.3239	2.1852
	α_2	0.8666	3.7683	6.8125	0.0237	0.2075	1.6963	0.7872	3.2469	6.3570	0.0861	0.2667	1.7660
	β_2	0.0405	0.2302	1.8750	-0.0405	0.2141	1.7384	0.0391	0.2258	1.8573	-0.0276	0.2046	2.0520
	b_2	-0.1679	0.5383	5.7131	0.0200	0.2877	1.6150	-0.1547	0.5114	5.7165	-0.0284	0.2557	1.5180
	θ	0.0313	0.5964	2.0318	-0.0799	0.1861	2.2316	-0.0100	0.4295	2.0303	0.0665	0.1740	2.3239
100	α_1	0.1752	0.3612	2.2548	0.0064	0.0491	0.8035	0.2046	0.4470	2.4965	0.0175	0.0422	0.7744
	β_1	0.0343	0.1065	1.2729	-0.0330	0.1010	1.2207	0.0429	0.1129	1.3070	-0.0228	0.1027	1.2420
	b_1	-0.0625	0.1490	1.4936	-0.0117	0.0895	1.1471	-0.0727	0.1585	1.5350	-0.0141	0.0928	1.2017
	α_2	0.3277	0.9488	3.5976	0.0017	0.0577	0.9101	0.3041	0.8366	3.3831	-0.0019	0.0540	0.8953
	β_2	0.0303	0.0953	1.2046	-0.0120	0.0911	1.1253	0.0182	0.1039	1.2623	-0.0421	0.0911	1.2087
	b_2	-0.0893	0.2283	2.2548	-0.0030	0.0807	0.8035	-0.0749	0.2409	2.4965	0.0035	0.0903	0.7744
	θ	0.0136	0.0893	1.2729	-0.0142	0.0592	1.2207	0.0209	0.0877	1.3070	0.0201	0.0613	1.2420
150	α_1	0.1062	0.2098	1.7477	0.0047	0.0162	0.4926	0.1267	0.2186	1.7653	-0.0016	0.0155	0.4721
	β_1	0.0150	0.0816	1.1188	-0.0031	0.0287	0.6601	0.0316	0.0794	1.0979	-0.0163	0.0296	0.6836
	b_1	-0.0331	0.1073	1.2783	-0.0069	0.0285	0.6710	-0.0511	0.1078	1.2720	0.0023	0.0281	0.6623
	α_2	0.1991	0.4197	2.4178	-0.0074	0.0186	0.5191	0.2074	0.5468	2.7837	0.0070	0.0182	0.5398
	β_2	0.0173	0.0703	1.0373	-0.0057	0.0298	0.6699	0.0123	0.0731	1.0596	0.0010	0.0304	0.6850
	b_2	-0.0581	0.1631	1.7477	0.0102	0.0256	0.4926	-0.0500	0.1709	1.7653	-0.0036	0.0248	0.4721
	θ	-0.0045	0.0574	1.1188	-0.0051	0.0252	0.6601	0.0259	0.0571	1.0979	0.0112	0.0244	0.6836

Table 3. MLE and Bayesian for $\alpha_1 = 0.5, \beta_1 = 1.2, b_1 = 1.2, \alpha_2 = 0.8, \beta_2 = 0.5, b_2 = 1.3$.

θ		0.5						-0.5					
		MLE			Bayesian			MLE			Bayesian		
n		Bias	MSE	LACI	Bias	MSE	LCCI	Bias	MSE	LACI	Bias	MSE	LCCI
40	α_1	0.0493	0.0263	0.6064	0.0329	0.0110	0.3321	0.0529	0.0291	0.6363	0.0419	0.0202	0.3301
	β_1	0.0482	0.0897	1.1592	0.0222	0.0824	1.0749	0.0350	0.0879	1.1543	-0.0010	0.0724	1.0832
	b_1	-0.0743	0.0965	1.1827	-0.0207	0.0823	1.0073	-0.0613	0.0926	1.1688	-0.0414	0.0825	1.0784
	α_2	0.1523	0.1993	1.6459	0.0934	0.0568	0.7018	0.1653	0.2367	1.7946	0.0989	0.0505	0.6934
	β_2	0.1584	0.1107	1.1472	0.1222	0.1026	1.0164	0.0840	0.0865	1.1051	0.0763	0.0794	1.0563
	b_2	-0.0709	0.0284	0.6064	-0.0862	0.0281	0.3321	-0.0372	0.0233	0.6363	-0.0130	0.0231	0.3301
	θ	-0.1969	0.3862	1.1592	-0.1401	0.1784	1.7491	0.1981	0.3824	1.1543	0.1576	0.1911	1.0083
100	α_1	0.0189	0.0072	0.3240	0.0113	0.0028	0.1950	0.0188	0.0075	0.3312	0.0105	0.0026	0.1875
	β_1	0.0245	0.0358	0.7362	-0.0085	0.0282	0.6092	0.0224	0.0408	0.7876	-0.0204	0.0371	0.6949
	b_1	-0.0361	0.0372	0.7428	-0.0271	0.0280	0.6116	-0.0356	0.0418	0.7898	-0.0340	0.0377	0.7030
	α_2	0.0433	0.0308	0.6674	0.0229	0.0117	0.3956	0.0747	0.0395	0.7224	0.0485	0.0161	0.4309
	β_2	0.0837	0.0339	0.6429	0.0640	0.0318	0.6687	0.0249	0.0281	0.6504	0.0037	0.0232	0.6388
	b_2	-0.0312	0.0062	0.3240	-0.0385	0.0051	0.1950	-0.0062	0.0050	0.3312	-0.0329	0.0042	0.1875
	θ	-0.2354	0.1483	0.7362	-0.1084	0.0726	1.0917	0.2461	0.1543	0.7876	0.0986	0.0694	1.0949
150	α_1	0.0105	0.0040	0.2455	0.0050	0.0014	0.1379	0.0101	0.0041	0.2480	0.0060	0.0015	0.1425
	β_1	0.0120	0.0255	0.6241	-0.0005	0.0247	0.6095	0.0134	0.0256	0.6257	0.0004	0.0246	0.6300
	b_1	-0.0198	0.0265	0.6342	-0.0049	0.0247	0.6333	-0.0206	0.0263	0.6306	-0.0136	0.0257	0.6443
	α_2	0.0301	0.0184	0.5181	-0.0068	0.0059	0.2884	0.0463	0.0206	0.5322	0.0192	0.0071	0.3130
	β_2	0.0799	0.0243	0.5249	0.0343	0.0134	0.4284	0.0024	0.0166	0.5048	-0.0146	0.0122	0.4182
	b_2	-0.0298	0.0042	0.2455	-0.0157	0.0040	0.1379	0.0032	0.0027	0.2480	0.0008	0.0026	0.1425
	θ	-0.2273	0.1132	0.6241	-0.0402	0.0258	0.6095	0.2368	0.1207	0.6257	0.0465	0.0283	0.6300

Table 4. MLE and Batesian for $\alpha_1 = 3.2, \beta_1 = 3, b_1 = 0.6, \alpha_2 = 3.3, \beta_2 = 2.5, b_2 = 0.5$.

			MLE			Bayesian				MLE			Bayesian		
θ	n		Bias	MSE	LACI	Bias	MSE	LCCI	θ	Bias	MSE	LACI	Bias	MSE	LCCI
-0.15	40	α_1	0.6833	3.4841	6.8204	-0.1041	0.4623	2.2573	0.5	0.7402	3.2912	6.4959	-0.0106	0.5286	2.4698
		β_1	-0.0263	0.0143	0.4574	-0.0125	0.0139	0.3885		-0.0180	0.0136	0.4523	0.0105	0.0134	0.4078
		b_1	0.0679	0.1654	1.5745	0.0973	0.0816	0.9472		0.0296	0.1481	1.5047	0.0858	0.0765	0.9481
		α_2	0.6594	2.9687	6.2505	-0.0324	0.5044	2.6858		0.6772	3.0802	6.3501	-0.0728	0.4937	2.6797
		β_2	-0.0145	0.0095	0.3790	0.0162	0.0041	0.2764		-0.0173	0.0102	0.3896	0.0173	0.0094	0.4057
		b_2	0.0320	0.1077	6.8204	0.0861	0.0779	2.2573		0.0379	0.1087	6.4959	0.0844	0.0642	2.4698
		θ	-0.0396	0.3552	0.4574	0.0185	0.1822	2.3885		0.0746	0.3478	0.4523	-0.0481	0.1809	2.4780
	100	α_1	0.3429	1.4142	4.4712	-0.0047	0.1052	1.2379		0.3705	1.5705	4.6953	-0.0391	0.1573	1.2966
		β_1	-0.0091	0.0035	0.2300	-0.0082	0.0021	0.2143		-0.0104	0.0049	0.2702	-0.0068	0.0031	0.2385
		b_1	0.0229	0.0635	0.9853	0.0272	0.0187	0.5007		0.0239	0.0738	1.0616	0.0329	0.0195	0.5208
		α_2	0.4194	1.6384	4.7485	0.0186	0.1558	1.2810		0.4265	1.5347	4.5616	-0.0038	0.1567	1.3404
		β_2	-0.0066	0.0034	0.2288	-0.0042	0.0029	0.2130		-0.0052	0.0032	0.2196	0.0057	0.0028	0.2040
		b_2	0.0090	0.0516	4.4712	0.0212	0.0142	1.2379		0.0072	0.0488	4.6953	0.0279	0.0148	1.2966
		θ	0.0010	0.0887	0.2300	0.0357	0.0676	1.4257		0.0148	0.0913	0.2702	-0.0191	0.0610	1.3085
	150	α_1	0.3785	1.1527	3.9450	0.0040	0.0355	0.6047		0.3417	1.1933	4.0693	-0.0020	0.0408	0.6865
		β_1	-0.0025	0.0026	0.2004	-0.0028	0.0020	0.1661		-0.0045	0.0028	0.2079	-0.0038	0.0026	0.1969
		b_1	-0.0065	0.0489	0.8676	0.0089	0.0067	0.3038		0.0024	0.0516	0.8907	0.0088	0.0071	0.3201
		α_2	0.2558	1.0589	3.9138	-0.0028	0.0335	0.7266		0.2570	1.0715	3.9326	-0.0134	0.0343	0.6677
		β_2	-0.0040	0.0016	0.1576	0.0049	0.0013	0.1466		-0.0062	0.0020	0.1748	0.0060	0.0017	0.1668
		b_2	0.0116	0.0323	3.9450	0.0104	0.0053	0.6047		0.0183	0.0361	4.0693	0.0153	0.0054	0.6865
		θ	0.0290	0.0614	0.2004	0.0185	0.0245	0.6605		0.0081	0.0577	0.2079	-0.0067	0.0250	0.6856
-0.5	40	α_1	0.7308	3.2146	6.4213	-0.0362	0.5308	2.4565	0.9	0.6092	2.9147	6.2624	-0.0316	0.4271	2.7672
		β_1	-0.0242	0.0184	0.5230	-0.0352	0.0144	0.5066		-0.0320	0.0237	0.5915	-0.0143	0.0214	0.4288
		b_1	0.0461	0.1705	1.6093	0.0855	0.0697	0.9645		0.0669	0.1828	1.6581	0.0629	0.0564	0.8594
		α_2	0.5884	2.8634	6.2225	-0.0365	0.5645	2.6689		0.5819	2.9359	6.3281	-0.1317	0.5600	2.8388
		β_2	-0.0206	0.0140	0.4568	-0.0028	0.0139	0.3686		-0.0286	0.0169	0.4979	-0.0672	0.0144	0.4647
		b_2	0.0455	0.1214	6.4213	0.0722	0.0632	2.4565		0.0633	0.1395	6.2624	0.0695	0.0581	2.7672
		θ	-0.0661	0.5414	0.5230	0.0673	0.1729	2.5661		0.2054	0.5363	0.5915	-0.0635	0.2254	2.4288
	100	α_1	0.4469	1.6378	4.7032	-0.0081	0.1372	1.3354		0.3991	1.5133	4.5691	-0.0081	0.1738	1.3429
		β_1	-0.0068	0.0042	0.2533	-0.0265	0.0041	0.2496		-0.0094	0.0047	0.2657	-0.0209	0.0031	0.2478
		b_1	0.0071	0.0700	1.0375	0.0204	0.0165	0.4827		0.0185	0.0730	1.0587	0.0280	0.0199	0.5014
		α_2	0.4426	1.6895	4.7932	-0.0258	0.1696	1.3443		0.3827	1.2969	4.2117	-0.0027	0.1207	1.2901
		β_2	-0.0054	0.0029	0.2091	-0.0166	0.0021	0.2305		-0.0043	0.0025	0.1973	0.0181	0.0021	0.1824
		b_2	0.0086	0.0481	4.7032	0.0288	0.0148	1.3354		0.0085	0.0420	4.5691	0.0324	0.0180	1.3429
		θ	0.0117	0.0828	0.2533	0.0235	0.0622	1.2958		0.0418	0.0981	0.2657	-0.0397	0.0573	1.2780
	150	α_1	0.3594	1.1807	4.0216	0.0002	0.0332	0.6673		0.2830	1.0412	3.8494	-0.0195	0.0524	0.7180
		β_1	-0.0034	0.0026	0.1989	0.0019	0.0031	0.1681		-0.0055	0.0026	0.1996	-0.0015	0.0025	0.1582
		b_1	-0.0012	0.0487	0.8654	0.0109	0.0063	0.2974		0.0100	0.0473	0.8532	0.0104	0.0058	0.2990
		α_2	0.2935	1.0624	3.8752	0.0031	0.0372	0.6906		0.2357	0.9354	3.6830	0.0034	0.0343	0.7426
		β_2	-0.0038	0.0021	0.1782	-0.0153	0.0021	0.6791		-0.0053	0.0021	0.1774	-0.0013	0.0020	0.1671
		b_2	0.0066	0.0339	4.0216	0.0069	0.0052	0.6673		0.0136	0.0341	3.8494	0.0059	0.0048	0.7180
		θ	-0.0031	0.0576	0.1989	0.0052	0.0225	0.6813		-0.0068	0.0589	0.1996	-0.0228	0.0215	0.5821

Table 5. MLE and Bayesian for $\alpha_1 = 2, \beta_1 = 0.3, b_1 = 1.8, \alpha_2 = 1.3, \beta_2 = 0.5, b_2 = 1.9$.

			MLE			Bayesian				MLE			Bayesian		
θ	n		Bias	MSE	LACI	Bias	MSE	LCCI	θ	Bias	MSE	LACI	Bias	MSE	LCCI
-0.15	40	α_1	1.0143	4.9522	7.7775	0.0841	0.2636	1.8706	0.5	0.9613	4.5273	7.4447	0.0663	0.2608	1.8408
		β_1	0.1733	0.1369	1.2833	0.0549	0.0338	0.5966		0.1669	0.1333	1.2733	0.0597	0.0382	0.6486
		b_1	-0.0526	0.0170	0.4681	0.0313	0.0137	0.3180		-0.0495	0.0160	0.4565	0.0333	0.0131	0.4332
		α_2	0.4559	1.1468	3.8048	0.1357	0.5264	1.7408		0.4318	1.2689	4.0805	0.1242	0.3579	1.4666
		β_2	0.1879	0.1805	1.4959	0.0913	0.1122	1.2078		0.1858	0.1922	1.5575	0.0962	0.1017	1.1277
		b_2	-0.0731	0.0367	7.7775	0.0067	0.0316	1.8706		-0.0764	0.0421	7.4447	-0.0400	0.0360	1.8408
		θ	-0.0316	0.2794	1.2833	-0.0422	0.1900	0.5966		0.0405	0.5765	1.2733	-0.0667	0.1829	0.6486
	100	α_1	0.3365	0.8459	3.3610	-0.0128	0.0851	1.0826		0.3609	0.9159	3.4763	0.0153	0.0836	1.1241
		β_1	0.0657	0.0327	0.6615	0.0163	0.0096	0.3313		0.0689	0.0333	0.6622	0.0180	0.0095	0.3557
		b_1	-0.0150	0.0021	0.1701	0.0150	0.0011	0.1305		-0.0158	0.0022	0.1732	-0.0057	0.0020	0.1329
		α_2	0.1245	0.1899	1.6397	0.0264	0.0488	0.8204		0.1391	0.1948	1.6426	0.0262	0.0475	0.8099
		β_2	0.0586	0.0481	0.8298	0.0302	0.0265	0.6300		0.0684	0.0501	0.8354	0.0232	0.0234	0.5434
		b_2	-0.0177	0.0060	3.3610	0.0158	0.0051	1.0826		-0.0206	0.0057	3.4763	-0.0248	0.0041	1.1241
		θ	0.0119	0.0961	0.6615	0.0169	0.0605	0.3313		-0.0042	0.0857	0.6622	-0.0291	0.0600	0.3557
	150	α_1	0.2656	0.5411	2.6936	0.0052	0.0262	0.6156		0.1982	0.5555	2.8178	-0.0020	0.0270	0.6374
		β_1	0.0529	0.0208	0.5273	0.0094	0.0031	0.2180		0.0375	0.0201	0.5360	0.0073	0.0032	0.2168
		b_1	-0.0107	0.0011	0.1241	0.0028	0.0010	0.1169		-0.0083	0.0012	0.1335	-0.0025	0.0012	0.1162
		α_2	0.0888	0.1088	1.2472	-0.0061	0.0123	0.4072		0.0693	0.0765	1.0503	0.0021	0.0170	0.4846
		β_2	0.0457	0.0311	0.6693	0.0039	0.0086	0.3643		0.0390	0.0237	0.5838	0.0065	0.0095	0.3752
		b_2	-0.0132	0.0031	2.6936	0.0065	0.0030	0.6156		-0.0106	0.0021	2.8178	-0.0142	0.0020	0.6374
		θ	0.0170	0.0590	0.5273	0.0015	0.0265	0.2180		0.0004	0.0614	0.5360	-0.0040	0.0243	0.2168
-0.5	40	α_1	0.8840	4.2754	7.3311	0.0703	0.2511	1.8356	0.9	0.7771	3.8798	7.1069	0.0690	0.2436	1.7781
		β_1	0.1515	0.1250	1.2532	0.0522	0.0338	0.6200		0.1412	0.1239	1.2660	0.0673	0.0327	0.5908
		b_1	-0.0451	0.0146	0.4397	0.0297	0.0136	2.2754		-0.0437	0.0158	0.4628	0.0419	0.0103	0.4219
		α_2	0.4555	1.4084	4.2979	0.1482	0.4878	1.4970		0.4095	1.0311	3.6485	0.0993	0.3547	1.5127
		β_2	0.1788	0.1880	1.5491	0.0781	0.0911	0.9930		0.1729	0.1801	1.5220	0.0793	0.0923	1.0526
		b_2	-0.0751	0.0411	7.3311	-0.0533	0.0338	1.8356		-0.0704	0.0354	7.1069	-0.0336	0.0307	1.7781
		θ	0.0050	0.3328	1.2532	0.0893	0.1663	0.6200		0.1092	0.6525	1.2660	-0.1034	0.2147	0.5908
	100	α_1	0.3469	0.9713	3.6179	0.0006	0.0860	1.1228		0.2859	0.9612	3.6823	0.0299	0.0703	0.9825
		β_1	0.0660	0.0344	0.6797	0.0145	0.0084	0.3432		0.0594	0.0358	0.7048	0.0258	0.0085	0.3382
		b_1	-0.0156	0.0024	0.1839	0.0010	0.0021	0.1728		-0.0148	0.0027	0.1942	0.0024	0.0021	0.1130
		α_2	0.1270	0.1732	1.5542	0.0158	0.0424	0.7388		0.1543	0.2115	1.7012	0.0429	0.0447	0.7300
		β_2	0.0642	0.0471	0.8134	0.0195	0.0251	0.5907		0.0774	0.0504	0.8271	0.0425	0.0323	0.6208
		b_2	-0.0186	0.0050	3.6179	-0.0178	0.0041	1.1228		-0.0250	0.0066	3.6823	-0.0315	0.0051	0.9825
		θ	0.0414	0.0874	0.6797	0.0273	0.0597	0.3432		0.0002	0.0900	0.7048	-0.0616	0.0618	0.3382
	150	α_1	0.1793	0.4412	2.5085	0.0057	0.0257	0.6040		0.2291	0.5390	2.7390	0.0075	0.0279	0.6210
		β_1	0.0360	0.0170	0.4923	0.0089	0.0030	0.2054		0.0425	0.0195	0.5218	0.0113	0.0033	0.2037
		b_1	-0.0073	0.0009	0.1118	-0.0160	0.0009	0.6565		-0.0089	0.0010	0.1218	0.0046	0.0009	0.1070
		α_2	0.0638	0.0831	1.1025	0.0001	0.0173	0.5019		0.0760	0.0951	1.1736	-0.0007	0.0148	0.4239
		β_2	0.0329	0.0250	0.6060	0.0049	0.0094	0.3650		0.0355	0.0275	0.6357	0.0053	0.0094	0.3515
		b_2	-0.0095	0.0024	2.5085	-0.0019	0.0022	0.6040		-0.0107	0.0027	2.7390	0.0005	0.0027	0.6210
		θ	0.0362	0.0594	0.4923	0.0128	0.0252	0.2054		0.0039	0.0530	0.5218	-0.0222	0.0229	0.2037

9. Application

To demonstrate the value of the suggested L-G family models, four applications from the environmental, medical and lifetime contexts are taken into consideration in this section. Table 6 discussed correlation dependence measure to confirm the data has correlation range from $\frac{-1}{3}$ to $\frac{1}{3}$.

Table 6. Correlation dependence measure.

	Environmental	Diabetic nephropathy	kidney patients	Computer series system
ρ_c	-0.1341	0.0418	0.0511	-0.0306

9.1. Environmental data

In this subsection, we take a look at a real-world bivariate data set on (X, Y) that includes data from 51 of the largest cities in the USA, with X standing for average precipitation (in millimetres) and Y standing for average maximum temperature (in degrees Celsius). The data created by the National Climatic Data Centre (NCDC) of the USA and made available on the website https://www.ncdc.noaa.gov have been replicated below. This is the bivariate data set in Table 7.

For the data description, we obtained Figure 4 to check outliers of these environmental data and described differentiates different categories as a scatter plot (strip), respectively. While Figure 5 discussed violin plot of these data which has been discussed to show peaks in the environmental data and visualize the distribution of numerical data. By Figures 4 and 5, we note that the data has right skewed shapes and hasn't symmetric ships.

Table 8 lists the estimates' results together with certain goodness-of-fit metrics, and Figures 6 and 7 show the estimated cdf with empirical cdf, pdf with histogram, and PP-plots for each sample and marginal distribution, respectively. We observe that these environmental data fit these LGW and LGLL distributions.

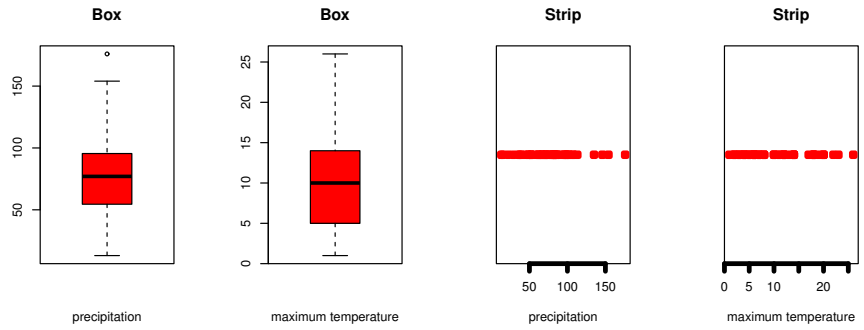


Figure 4. Box-plot and strip plot for environmental data.

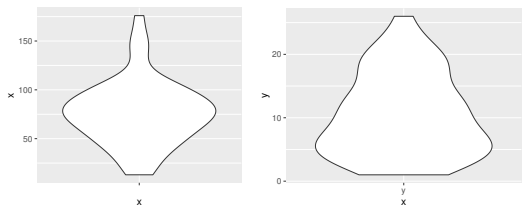


Figure 5. Violin plot for environmental data.

Table 7. Environmental data with average values.

No.	precipitation	maximum temperature	No.	precipitation	maximum temperature
1	99	12	27	108	10
2	61	17	28	135	18
3	86	7	29	102	6
4	113	13	30	48	10
5	96	5	31	66	23
6	99	2	32	90	7
7	83	12	33	22	20
8	57	2	34	72	4
9	80	6	35	176	7
10	79	4	36	107	6
11	75	5	37	84	12
12	70	14	38	83	10
13	15	6	39	37	20
14	62	2	40	67	3
15	87	4	41	83	12
16	95	18	42	36	3
17	81	4	43	49	18
18	71	19	44	39	18
19	44	5	45	102	14
20	13	14	46	66	14
21	52	20	47	154	8
22	97	8	48	72	6
23	146	11	49	63	22
24	52	26	50	83	11
25	52	1	51	77	8
26	29	3			

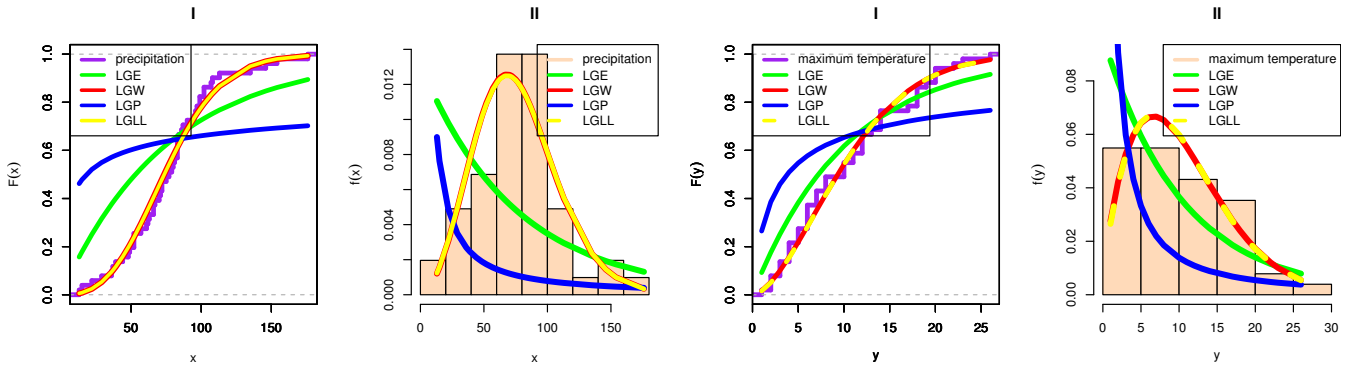


Figure 6. CDF and PDF estimated LG distributions: Environmental data.

Table 8. MLE of marginal models for environmental data.

			Etimates	SE	AIC	CAIC	BIC	HQIC	CVM	AD	KS	PVKS
LGE	x	α	38.9815	38.6657	551.8213	557.6168	552.3319	554.0359	0.1277	0.7508	0.3088	0.0001
		β	19.8679	21.6752								
		b	0.0067	0.0044								
	y	α	84.3428	138.8010	347.1617	352.9572	347.6723	349.3763	0.0656	0.4347	0.1829	0.0658
		β	23.4514	69.9253								
		b	0.0269	0.0719								
LGW	x	α	4.8013	5.6889	506.4401	514.1674	507.3096	509.3929	0.0678	0.4173	0.0721	0.9534
		β	5.5102	10.2114								
		b	2.8242	0.5020								
		a	0.0129	0.0075								
	y	α	77.5320	258.1893	332.2403	339.9676	333.1099	335.1932	0.0646	0.4144	0.0889	0.8151
		β	0.5124	1.7079								
		b	1.6684	0.1847								
		a	0.0043	0.0012								
LGP	x	α	107.8997	165.2554	689.1122	694.9077	689.6229	691.3269	0.3204	1.8797	0.4932	0.0000
		β	24.8648	133.6627								
		b	0.0543	0.3039								
	y	α	99.4600	145.5079	420.8929	426.6884	421.4036	423.1076	0.1461	0.9875	0.3806	0.0000
		β	13.0680	77.2472								
		b	0.0584	0.3380								
LGLL	x	α	75.6384	3.1556	506.4294	514.1567	507.2990	509.3822	0.0673	0.4159	0.0714	0.9574
		β	13.2360	6.1562								
		b	152.8863	39.1425								
		a	2.7976	0.5018								
	y	α	63.7445	11.6990	332.4175	340.1448	333.2871	335.3704	0.0653	0.4198	0.0910	0.7922
		β	1.4164	5.9881								
		b	107.8428	16.7536								
		a	1.6986	0.1939								

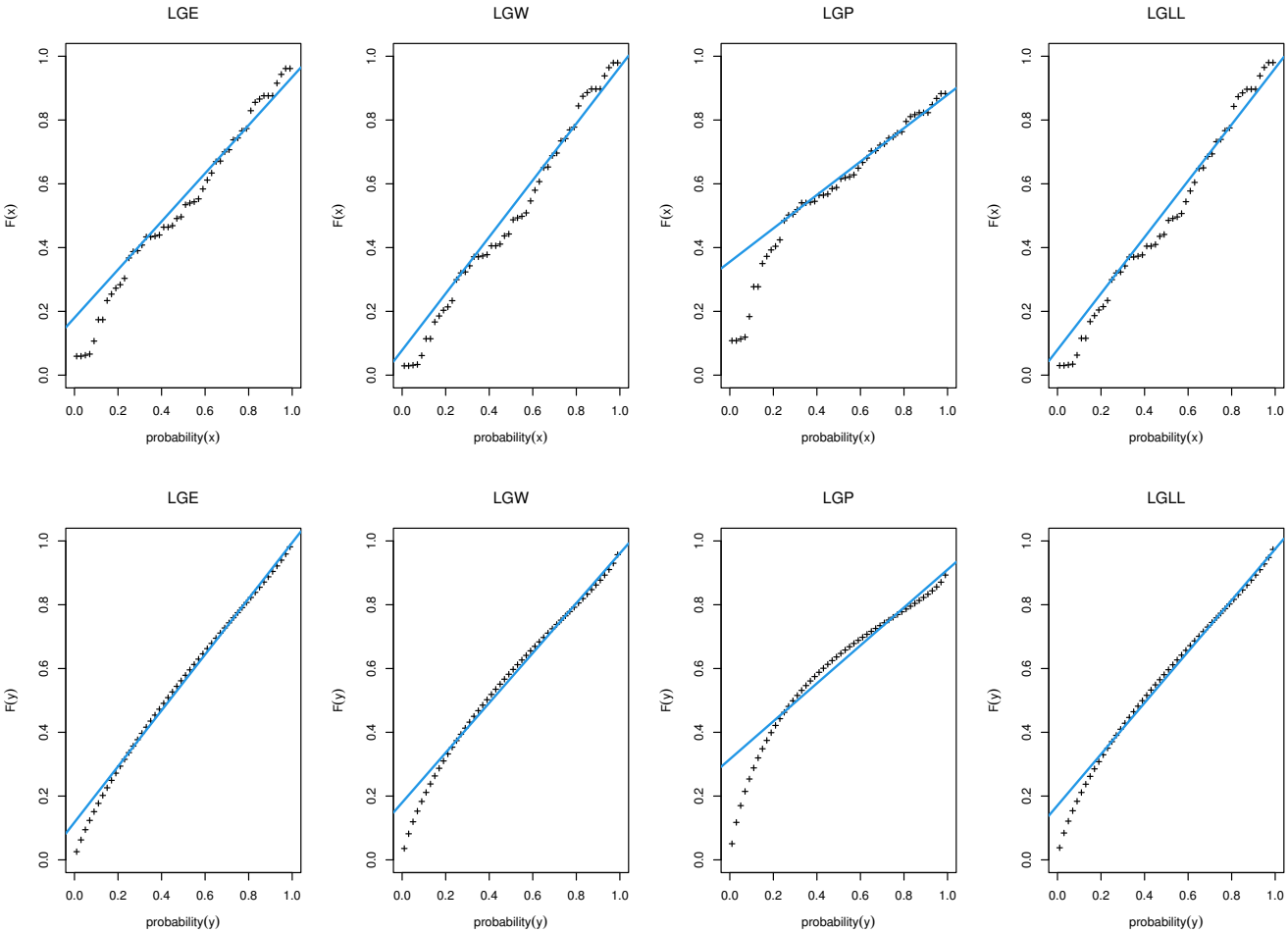


Figure 7. PP plot estimated LG distributions: Environmental data.

Tables 9 and 10 show the MLE and Bayesian estimation method, respectively for the parameters of L-G family models for two case only because the LGE and LGP distribution are not fitting for this data see Table 8. Table 9 discussed comparison of bivariate models based on bivariate L-G family model by using AIC, CIAC, BIC, HQIC, CVM, and AD measures. By these results, we conclude the BFGM LGW distribution is best models comparison another bivariate distribution as BFGM LGLL and bivariate FGM generalized half-logistic (BFGMGHL) by Hassan and Chesneau [39], according to AIC, CIAC, BIC, HQIC, CVM, and AD. When comparing Bayesian estimates and MLE, we note that Bayesian is better than MLE according to the value of SE.

Table 9. MLE of bivariate models for environmental data.

	BFGM LGW		BFGM LGLL		BFGMGHL	
	Etimates	SE	Etimates	SE	Etimates	SE
α_1	5.3868	8.1572	46.5772	951.1889	37.7384	16.0986
β_1	5.5366	12.0311	8.3576	194.8530	0.6771	0.3674
b_1	2.7920	0.5547	149.7490	176.4606		
a_1	0.0123	0.0074	2.8181	0.4991		
α_2	88.1141	307.7538	106.1346	401.4176	5.6825	4.3687
β_2	0.5653	1.9718	1.1013	6.1811	0.7623	0.7176
b_2	1.6910	0.1885	171.9876	418.5592		
a_2	0.0043	0.0012	1.6993	0.1928		
θ	-0.2846	0.4216	-0.2839	0.4207	0.2744	1.1290
AIC	840.2217		840.2770		872.5342	
CAIC	844.6120		844.6672		873.8675	
BIC	857.6082		857.6634		882.1933	
HQIC	846.8656		846.9209		876.2253	
CVM	7.6899		7.6927		7.7377	
AD	46.1279		46.1322		46.8422	

Table 10. Bayesian estimation of parameter for bivariate models: Environmental data.

		α_1	β_1	b_1	a_1	α_2	β_2	b_2	a_2	θ
BFGM LGW	mean	10.8227	5.2411	2.6673	0.0102	147.4914	0.9004	1.7212	0.0045	-0.6354
	sd	8.0436	3.2584	0.4678	0.0050	74.7410	0.4937	0.1800	0.0011	0.1927
BFGM LGLL	mean	99.3319	11.3700	205.0953	2.6880	168.2901	2.2241	155.1508	1.6927	0.1995
	sd	70.7136	7.6248	56.8707	0.3891	155.9503	1.3720	86.1843	0.1821	0.3673

9.2. Diabetic nephropathy

We have taken into account both serum creatinine (SrCr) and the length of diabetes in this subsection. Since the patients' diabetes was already known, we are calculating the complications that may result from it. Based on SrCr levels, the data has been divided into two groups: Those with diabetic nephropathy (DN) (SrCr $\geq$ 1.4mg/dl) and those without diabetic nephropathy (SrCr $<$ 1.4mg/dl). SrCr reports were provided for each patient from the 200 patients whose reports were available. From January 2012 to August 2013, the pathology reports of these patients were gathered from the path lab of Dr. Lal. This data, which includes the mean duration of diabetes for 132 individuals with types 2 diabetic nephropathy over various time intervals, was discussed by Grover et al. [40]. These data are:
Duration of diabetes: 7.4, 9, 10, 11, 12, 13, 13.75, 14.92, 15.8286, 16.9333, 18, 19, 20, 21, 22, 23, 24, 26, 26.6.
Serum Creatinine: 1.925, 1.5, 2, 1.6, 1.7, 1.7533, 1.54, 1.694, 1.8843, 1.8433, 1.832, 1.59, 1.7833, 1.2, 1.792, 1.5, 1.5033, 2, 2.14.

For the diabetic nephropathy data description, we obtained Figure 8 to check outliers of these diabetic nephropathy data and described differentiates different categories as a scatter plot (strip),

respectively. While Figure 9 discussed violin plot of these data which has been discussed to show peaks in the diabetic nephropathy data and visualize the distribution of numerical data. By Figures 8 and 9, we note that the data has right skewed shapes and hasn't symmetric ships.

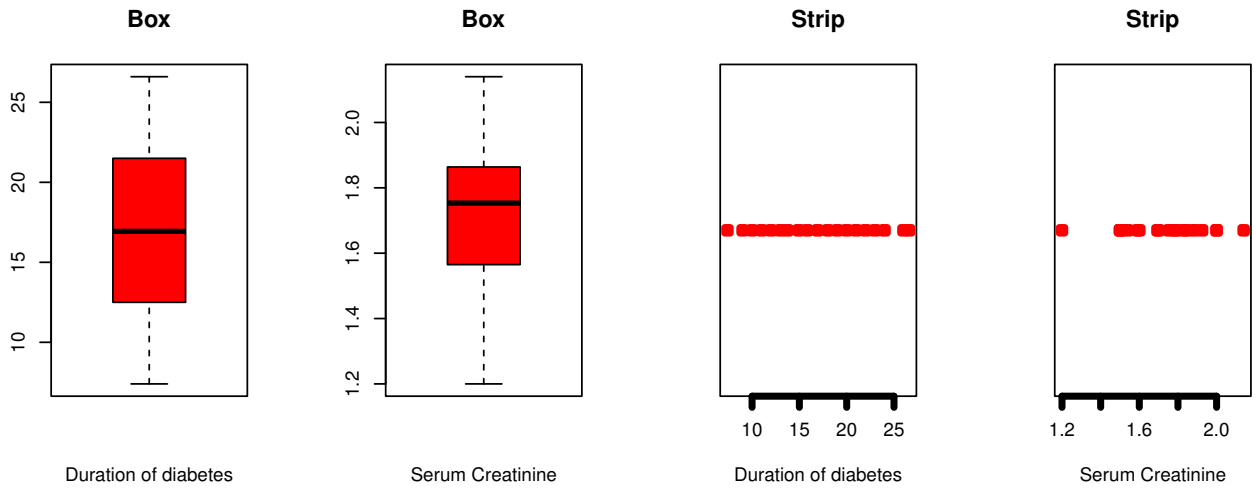


Figure 8. Box-plot and strip plot for diabetic nephropathy data.

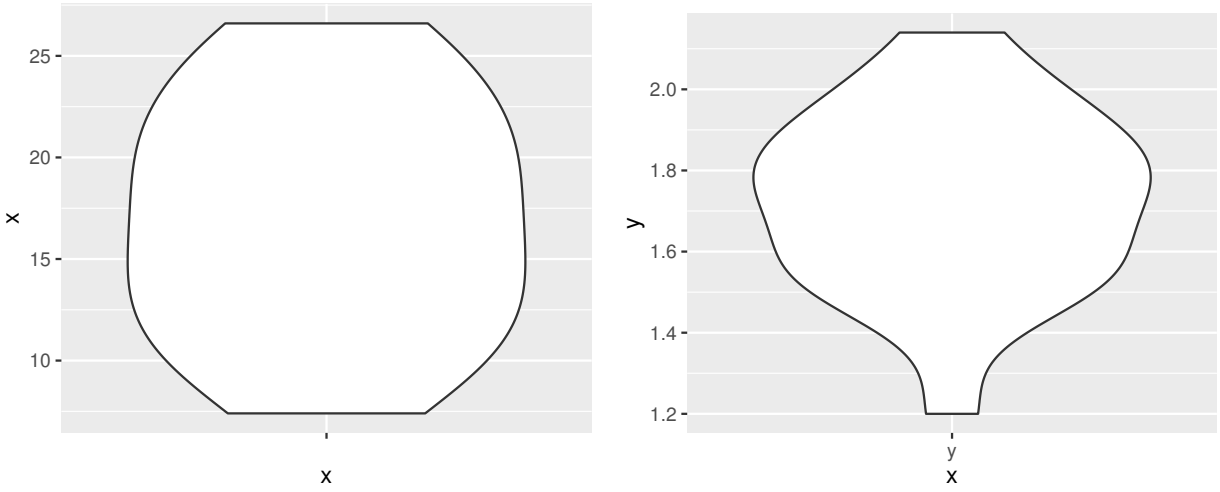


Figure 9. Violin plot for diabetic nephropathy data.

Table 11 lists the estimates' results together with certain goodness-of-fit metrics, and Figures 10 and 11 show the estimated cdf with empirical cdf, pdf with histogram, and PP-plots for each sample and marginal distribution, respectively. We observe that these diabetic nephropathy data fit these LGW and LGLL distributions.

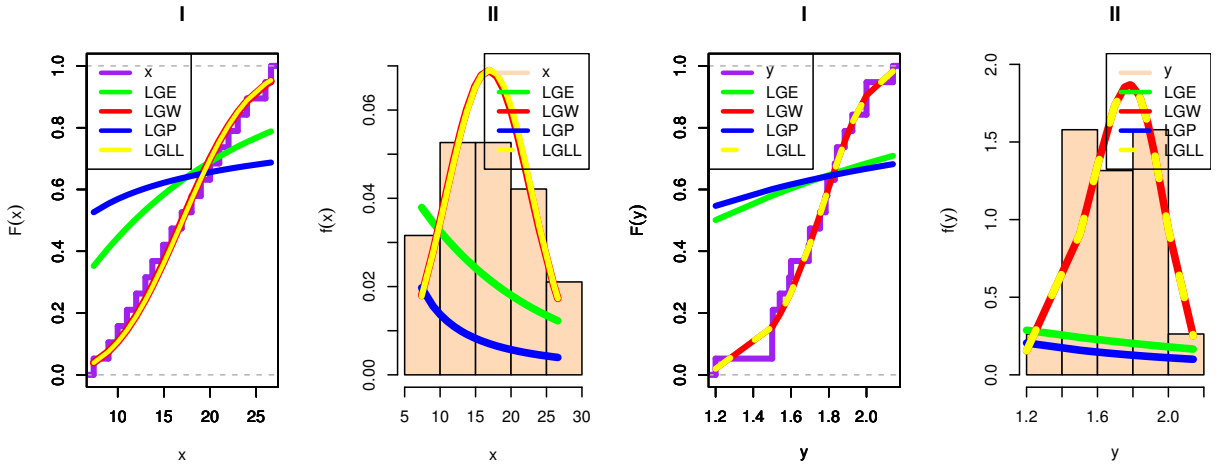


Figure 10. CDF and PDF estimated LG distributions: diabetic nephropathy data.

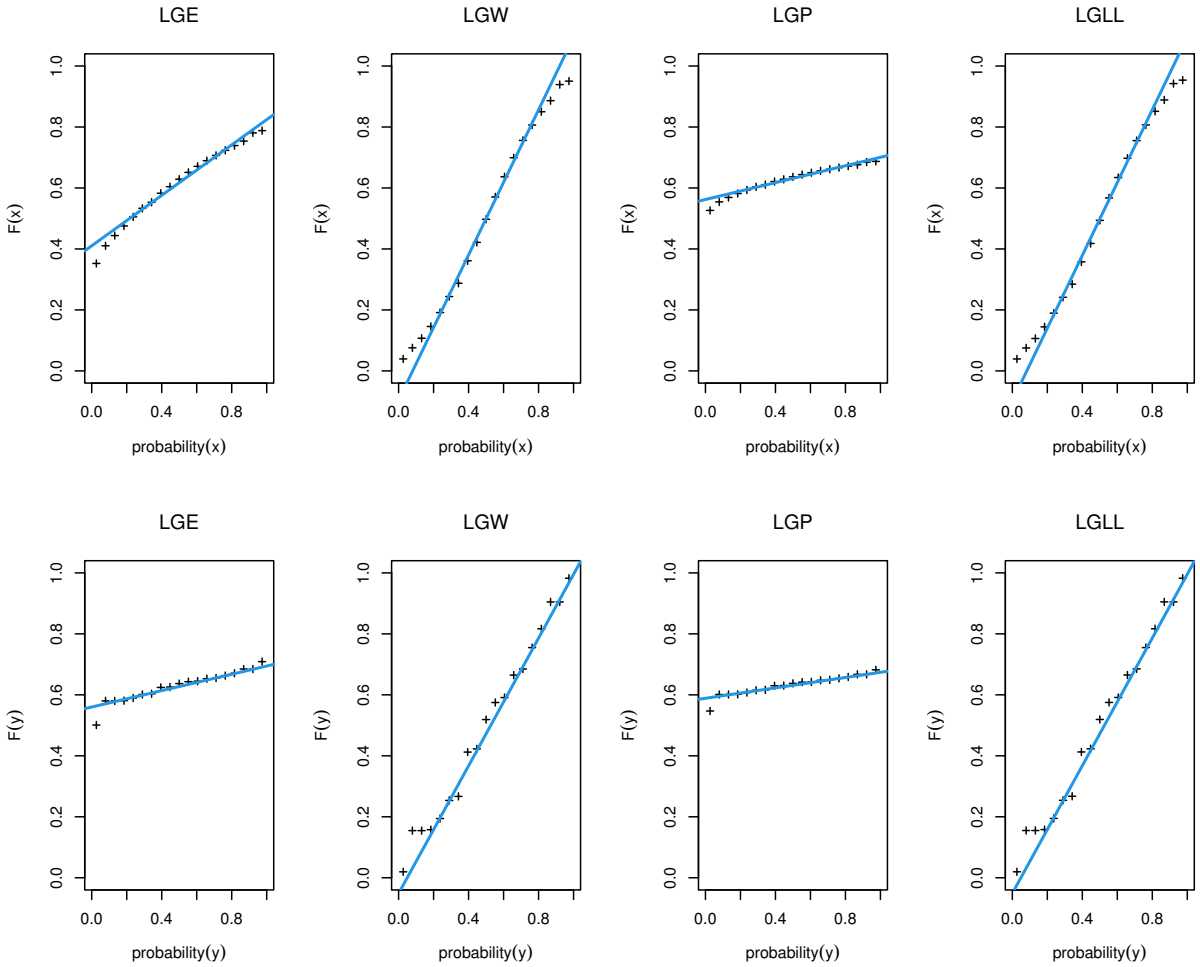


Figure 11. PP plot estimated LG distributions: diabetic nephropathy data.

Table 11. MLE of marginal models for diabetic nephropathy data.

			Etimates	SE	AIC	CAIC	BIC	HQIC	CVM	AD	KS	PVKS
LGE	x	α	83.4811	17.9433	151.9146	154.7479	153.5146	152.3941	0.0277	0.2028	0.3577	0.0111
		β	25.6754	19.5167								
		b	0.0181	0.0534								
	y	α	59.7456	21.5655	65.0366	67.8700	66.6366	65.5161	0.0334	0.2686	0.5274	0.0001
		β	9.0446	4.2133								
		b	0.0882	0.0156								
LGW	x	α	12.8529	27.4009	127.4632	131.2410	130.3204	128.1026	0.0214	0.1648	0.0810	0.9986
		β	4.2719	62.0344								
		b	3.4676	0.6752								
		a	0.0389	0.1645								
	y	α	8.0457	34.5386	4.1345	7.9122	6.9916	4.7738	0.0248	0.1915	0.1020	0.9890
		β	0.6660	0.3894								
		b	9.6559	3.2397								
		a	0.4286	0.2456								
LGP	x	α	92.9088	208.3350	191.6448	194.4781	193.2448	192.1243	0.0449	0.3106	0.5263	0.0000
		β	21.5151	286.7493								
		b	0.0816	1.0828								
	y	α	57.5959	18.1517	82.2695	85.1028	83.8695	82.7490	0.0390	0.3152	0.5485	0.0000
		β	5.6356	7.3131								
		b	0.0989	0.0316								
LGLL	x	α	39.3819	162.9695	127.3556	131.1334	130.2128	127.9950	0.0209	0.1616	0.0839	0.9976
		β	0.4879	2.3355								
		b	67.6007	93.1076								
		a	3.4403	0.6451								
	y	α	8.4545	42.1638	4.1357	7.9134	6.9928	4.7750	0.0248	0.1916	0.1024	0.9886
		β	0.0276	0.0910								
		b	3.2636	2.5462								
		a	9.6417	3.5093								

Tables 12 and 13 show the MLE and Bayesian estimation method, respectively for the parameters of L-G family models for two case only because the LGE and LGP distribution are not fitting for this data see Table 11. Table 12 discussed comparison of bivariate models based on bivariate L-G family model by using AIC, CIAC, BIC, HQIC, CVM, and AD measures. By these results, we conclude the BFGM LGLL distribution is best models comparison another bivariate distribution as BFGM LGW and bivariate FGM lomax-claim (BFGMLC) by Zho et al. [7], according to AIC, CIAC, BIC, HQIC, CVM, and AD. When comparing Bayesian estimates and MLE, we note that Bayesian is better than MLE according to the value of SE.

Table 12. MLE of bivariate models for Diabetic nephropathy data.

	BFGM LGW		BFGM LGLL		BFGMLC	
	Etimates	SE	Etimates	SE	Etimates	SE
α_1	32.6177	105.9016	77.5037	47.2668	32.3700	1.1932
β_1	10.6376	2.1516	0.2441	0.1904	0.0027	0.0013
b_1	3.4194	0.6425	103.1134	44.4474	0.5165	0.1517
a_1	0.0381	0.0152	3.4021	0.6322		
α_2	9.3182	373.1280	14.8761	9.5541	10.0287	0.5802
β_2	0.3185	515.7803	0.2144	0.1812	0.9969	0.3516
b_2	9.5726	21.9642	2.8331	1.0156	0.0000	0.0001
a_2	0.3893	67.9071	9.4468	1.9058		
θ	0.0538	0.6022	0.0547	0.5574	0.8882	0.0407
AIC	133.4597		133.4271		157.2159	
CAIC	153.4597		153.4271		167.3977	
BIC	141.9597		141.9270		163.8270	
HQIC	134.8982		134.8656		158.3348	
CVM	1.0015		0.9937		1.0773	
AD	5.2629		5.2112		6.2834	

Table 13. Bayesian estimation of parameter for bivariate models: Diabetic nephropathy data.

		α_1	β_1	b_1	a_1	α_2	β_2	b_2	a_2	θ
BFGM LGW 2-11	mean	39.1079	7.0735	3.2451	0.0317	11.2465	0.2944	9.7805	0.3569	-0.2001
	sd	28.2717	6.9918	0.5942	0.0174	7.8451	0.2773	2.9191	0.0709	0.0556
BFGM LGLL 2-11	mean	180.6559	0.8817	97.9884	3.2370	30.4859	0.9295	2.9327	9.1782	-0.0309
	sd	146.6881	0.4004	29.2129	0.4890	21.9932	0.6312	0.6546	1.7057	0.1032

9.3. Computer series system-simulated data

The data was sourced from Oliveira et al. [41]. A processor and memory make up the $n = 50$ simulated rudimentary computer series systems in the data set. If both parts of the system function properly, the computer operates. Let's say the system is experiencing a latent deteriorating process. The degeneration advances quickly over a brief period of time (in hours). It makes the system more vulnerable to shocks, making it possible for a deadly shock to randomly destroy the first, second, or

both components. The independence presumption could not be accurate because a deadly shock can simultaneously kill both components, so we used FGM copula to discussed this problem. The data set is given as follows:
Processor lifetime: 1.9292 3.6621 3.6621 3.6621 1.0833 1.0833 0.3309 0.3309 0.5784 0.5520 1.9386 2.1000 0.9867 0.9867 1.3989 2.3757 3.5202 2.3364 0.8584 4.3435 1.1739 1.3482 3.0935 2.1396 1.3288 0.1115 0.8503 0.1955 0.4614 3.3887 0.1181 5.0533 1.6465 0.9096 1.7494 0.1058 0.1058 0.9938 5.7561 5.7561 0.6270 0.7947 0.5079 2.5913 2.5372 1.1917 1.5254 1.0986 1.0051 1.3640.
Memory lifetime: 3.9291 0.0026 0.0026 0.0026 3.3059 3.3059 0.3309 0.3309 1.8795 0.5520 4.0043 2.0513 0.9867 0.9867 4.1268 2.7953 1.4095 0.1624 1.9556 1.0001 3.3857 1.9705 3.0935 2.1548 0.9689 0.1115 2.8578 0.1955 0.8584 1.9796 0.0884 2.3238 2.0197 0.6214 2.3643 0.1058 0.1058 1.7689 0.3212 0.3212 1.7289 0.7947 5.3535 2.5913 2.4923 0.0801 4.4088 1.0986 1.0051 1.3640.

The computer data was presented as a series system, and the likelihood function should ideally be based on the likelihood for series system data. However, in our analysis, we used a different approach by modeling the time to the first failure as a continuous distribution. Although this approach is not specific to series systems, it can still provide useful insights into the data. For the data description, we obtained Figure 12 to check outliers of these data and described differentiates different categories as a scatter plot (strip), respectively. While Figure 13 discussed violin plot of these data which has been discussed to show peaks in the data and visualize the distribution of numerical data. By Figures 12 and 13, we note that the data has right skewed shapes and hasn't symmetric ships.

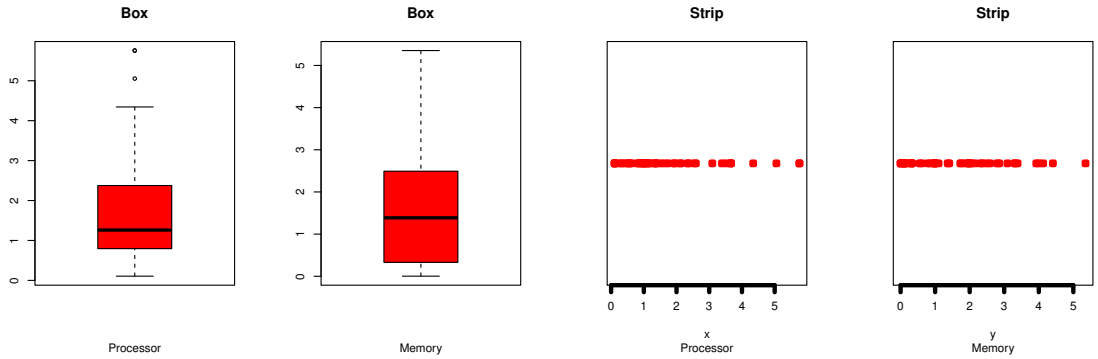


Figure 12. Box-plot and strip plot for computer series system data.

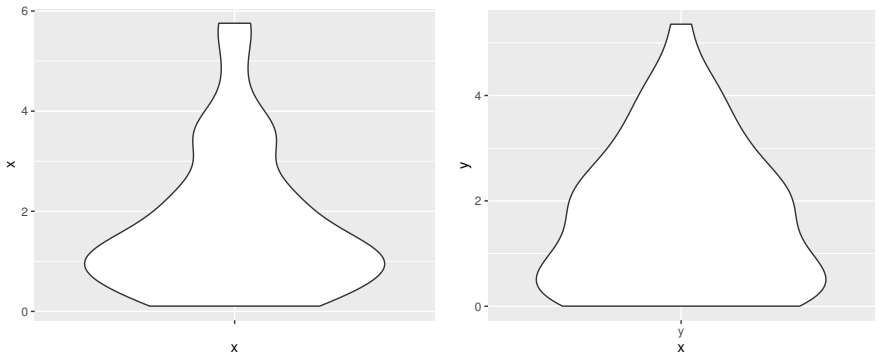


Figure 13. Violin plot for computer series system data.

Table 14 discussed MLE estimator of marginal parameters with standard error (SE), also different measures of goodness of fit as Akaike information criterion (AIC), the corrected AIC (CAIC), the Bayesian information criterion (BIC), the Hannan-Quinn information criterion (HQIC), the Cramer-von Mises (CVM), and the Anderson–Darling (AD), and Kolmogorov-Smirnov (KS) statistics with P-value (PVKS) for all competitive models as Lomax G exponential (LGE), Lomax G Weibull (LGW), Lomax G Pareto (LGP), and Lomax G- Log-Logistic (LGLL). Table 14 lists the estimates' results together with certain goodness-of-fit metrics, and Figures 14 and 15 show the estimated cdf with empirical cdf, pdf with histogram, and PP-plots for each sample and marginal distribution, respectively. We observe that these data fit these distributions.

Table 14. MLE of marginal models for computer series system data.

			Etimates	SE	AIC	CAIC	BIC	HQIC	CVM	AD	KS	PVKS
LGE	x	α	76.4346	13.9515	161.8862	167.6223	162.4080	164.0706	0.0363	0.2786	0.1275	0.3909
		β	10.6443	9.2913								
		b	0.0805	0.6494								
	y	α	146.6252	5.1560	155.1517	160.8877	155.6734	157.3360	0.1634	1.0349	0.1334	0.3355
		β	119.7771	24.5216								
		b	0.5025	0.4872								
LGW	x	α	33.4734	124.1500	160.9341	167.5022	161.8230	163.8465	0.0352	0.2728	0.0712	0.9616
		β	5.5518	75.6627								
		b	1.2336	0.1543								
		a	0.1274	1.3690								
	y	α	5.7086	3.7167	159.6049	167.2530	160.4938	162.5174	0.2204	1.3799	0.1418	0.2669
		β	1.4932	19.8523								
		b	0.9397	0.1168								
		a	0.1677	2.2953								
LGP	x	α	75.3043	9.2516	183.3328	189.0688	183.8545	185.5171	0.1028	0.7927	0.2445	0.0051
		β	6.8581	8.8796								
		b	0.1036	0.0915								
	y	α	8.6340	12.6090	173.7290	179.4651	174.2508	175.9133	0.4277	2.5704	0.1968	0.0416
		β	1.5722	98.6888								
		b	0.2294	14.0499								
LGLL	x	α	1495.255	354.2560	160.8776	168.5257	161.7665	163.7900	0.0407	0.2845	0.0736	0.9492
		β	1.5659	0.9854								
		b	523.6691	95.5166								
		a	1.2167	0.1060								
	y	α	5164.399	165.1982	156.2214	163.8695	157.1103	159.1339	0.1787	1.1297	0.1443	0.2487
		β	1.0828	0.8522								
		b	19561.244	780.7736								
		a	0.8980	0.1082								

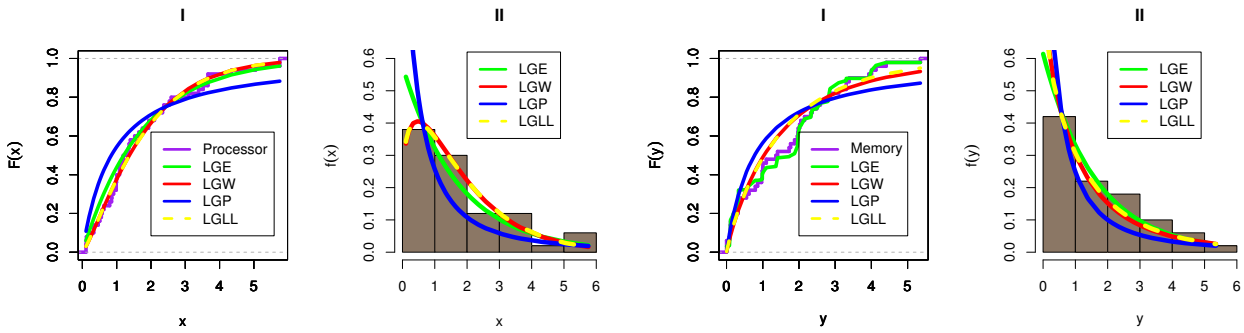


Figure 14. Estimated LG distributions: Computer series system data.

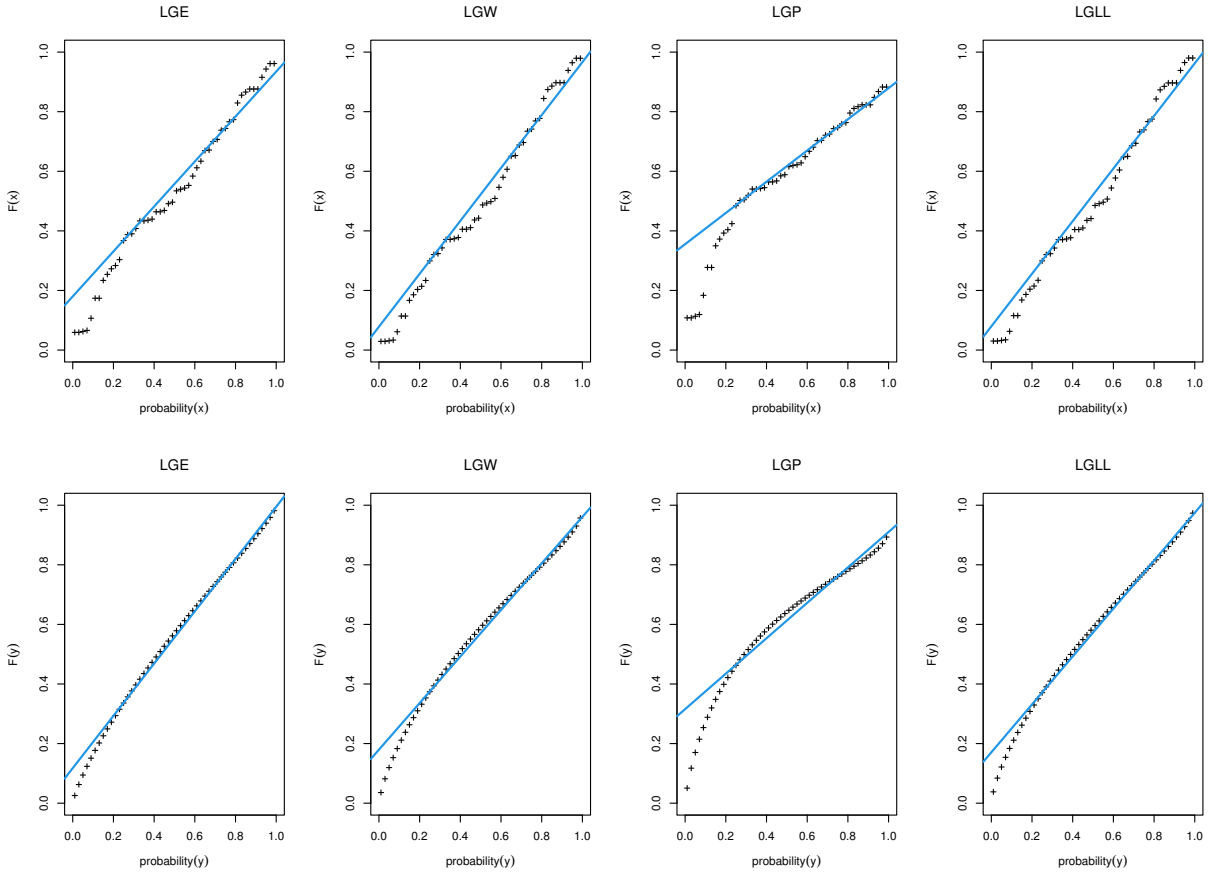


Figure 15. Estimated LG distributions: Computer series system data.

Tables 15 and 16 show the MLE and Bayesian estimation method, respectively for the parameters of L-G family models. Table 15 discussed comparison of bivariate models based on bivariate L-G family model by using AIC, CIAC, BIC, HQIC, CVM, and AD measures. By these results, we conclude the BFGM LGE distribution is best models comparison another bivariate distribution according to AIC, CIAC, BIC, HQIC, CVM, and AD. When comparing Bayesian estimates and MLE, we note that Bayesian is better than MLE according to the value of SE.

Table 15. MLE for computer series system data.

	BFGM LGE		BFGM LGW		BFGM LGP		BFGM LGLL		BFGM LC	
	Etimates	SE	Etimates	SE	Etimates	SE	Etimates	SE	Etimates	SE
α_1	147.3693	78.5271	35.4338	18.6493	160.7897	52.5459	370.201	31.369	134.2571	56.9368
β_1	116.6487	60.0803	5.7763	8.7089	78.2561	15.1095	0.694	15.312	0.0045	0.0018
b_1	0.4715	1.9090	1.2079	0.1668	0.5808	55.1790	346.102	34.266	825.2748	19.1517
a_1			0.1263	1.3683			1.193	0.138		
α_2	146.6066	51.5343	189.6576	86.5617	137.4092	87.5797	5165.707	10.040	68.9989	56.7898
β_2	119.7495	62.7662	40.6490	78.2630	29.7471	9.0236	1.000	2.907	0.0095	0.0078
b_2	0.5164	2.3986	0.9069	0.1084	0.2856	7.6191	19561.19	610.643	369.3705	256.5926
a_2			0.1179	1.4935			0.906	0.108		
θ	0.4987	0.3314	0.4139	0.3435	0.8581	0.3257	0.410	0.344	0.5054	0.3286
AIC	317.0232		317.9571		351.2878		317.8398		319.6854	
CAIC	319.6898		322.4571		353.9545		322.3398		322.3521	
BIC	330.4073		335.1653		364.6720		335.0480		335.1696	
HQIC	322.1199		324.5100		356.3846		324.3928		324.7822	
CVM	7.1673		7.6779		7.9260		7.6778		7.7252	
AD	40.5523		43.7650		43.5415		43.7399		43.7716	

Table 16. Bayesian estimation of parameter for bivariate models: Computer series system.

		α_1	β_1	b_1	a_1	α_2	β_2	b_2	a_2	θ
BFGM LGE 2-11	mean	87.6009	119.8282	1.0653		58.1685	178.0698	2.0726		0.5010
	sd	50.7412	33.6634	0.6837		32.0613	39.2821	0.6151		0.2948
BFGM LGW 2-11	mean	67.1512	24.2494	1.1909	0.2751	82.8430	105.1435	0.9167	0.7610	0.4701
	sd	13.9465	8.0916	0.1446	0.1337	39.4208	67.3779	0.1111	0.3451	0.3640
BFGM LGLL 2-11	mean	144.3689	1.4465	111.7554	1.1397	1272.2821	0.5740	19369.5048	0.8174	1.0769
	sd	30.2474	0.2892	27.4467	0.0988	9.5403	0.4079	99.9004	0.0462	0.2149
BFGM LGP 2-11	mean	93.9844	237.0822	2.8319		99.9061	40.9085	0.5557		0.5869
	sd	31.2070	13.6399	2.7624		42.2993	13.3844	0.2808		0.2531

9.4. Kidney patients data

The data set presented by McGilchrist and Aisbett [42]. This information shows how often infections return in kidney patients. Let's say x and y are the first and second recurrence times, respectively. The data presented for 30 patients as follows:
First recurrence time: 8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130, 17, 185, 292, 22, 15.
Second recurrence time: 16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25,362, 24, 66, 46, 40, 201, 156, 30, 25, 26, 4, 117, 114, 159, 108.

In order to check for outliers in the data for kidney patients and to distinguish between different groups, we obtained Figure 16 and represented it as a scatter plot (strip), respectively. While Figure 17 discussed a violin plot of these data that was used to illustrate data peaks and depict the distribution of kidney patient data. We can see from Figures 16 and 17 that the kidney patients data exhibits right-skewed shapes and non-symmetric ships. For all competitive models, including LGE, LGW, LGP, and LGLL, Table 17 addressed the MLE estimator of marginal parameters with SE as well as several measures of goodness of fit, including AIC, CAIC, BIC, HQIC, CVM, AD, and KS statistics with PVKS. Figures 18 and 19 display the estimated cdf with empirical cdf, pdf with histogram, and PP-plots for each sample and marginal distribution, respectively. Table 17 displays the estimates' results along with various goodness-of-fit measures. We note that the kidney patients data are consistent with these distributions.

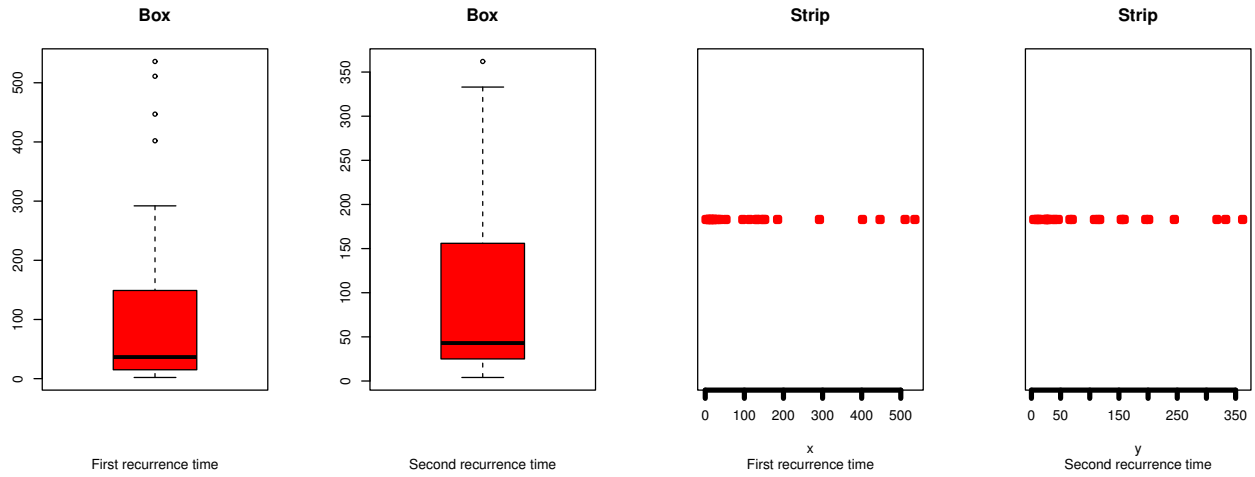


Figure 16. Box-plot and strip plot for kidney patients data.

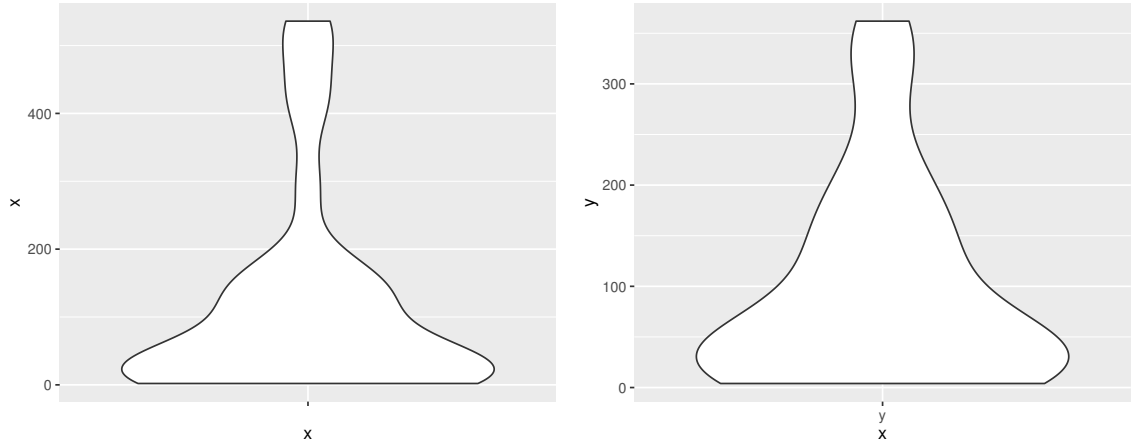


Figure 17. Violin plot for kidney patients data.

Table 17. MLE of marginal models: Kidney patients.

			Etimates	SE	AIC	CAIC	BIC	HQIC	CVM	AD	KS	PVKS
LGE	x	α	1.6241	0.9281	348.9320	353.1356	349.8551	350.2768	0.1174	0.6549	0.1341	0.6533
		β	2.0743	1.4259								
		b	0.0212	0.0733								
	y	α	10.3515	9.5644	341.6516	345.8551	342.5746	342.9963	0.1102	0.6092	0.1532	0.4821
		β	32.3030	12.0899								
		b	0.0347	0.1593								
LGW	x	α	2.4795	5.6282	350.8655	356.4702	352.4655	352.6585	0.1206	0.6784	0.1368	0.6288
		β	2.7580	1.8529								
		b	0.9084	0.3785								
		a	0.0161	0.0424								
	y	α	0.7994	0.6637	345.2155	350.8203	346.8155	347.0085	0.0908	0.5437	0.1308	0.6842
		β	1.6446	1.9677								
		b	1.4834	0.5208								
		a	0.0339	0.1628								
LGP	x	α	21.0619	19.2427	385.4904	389.6940	386.4135	386.8352	0.0715	0.4710	0.3752	0.0004
		β	6.3487	3.3107								
		b	0.0771	0.7684								
	y	α	3.2661	1.2152	397.0708	401.2743	397.9938	398.4155	0.0717	0.4834	0.3724	0.0005
		β	1.5504	1.0915								
		b	0.1219	0.0916								
LGLL	x	α	175.4646	75.3074	350.8699	356.4747	352.4699	352.6630	0.1222	0.6920	0.1389	0.6088
		β	54.4813	46.1912								
		b	285.4688	181.4908								
		a	0.8753	0.4343								
	y	α	8.8665	8.1862	347.1499	352.7547	348.7499	348.9429	0.0900	0.5838	0.1530	0.4838
		β	14.8596	12.0452								
		b	26.9063	13.6426								
		a	1.7673	0.5273								

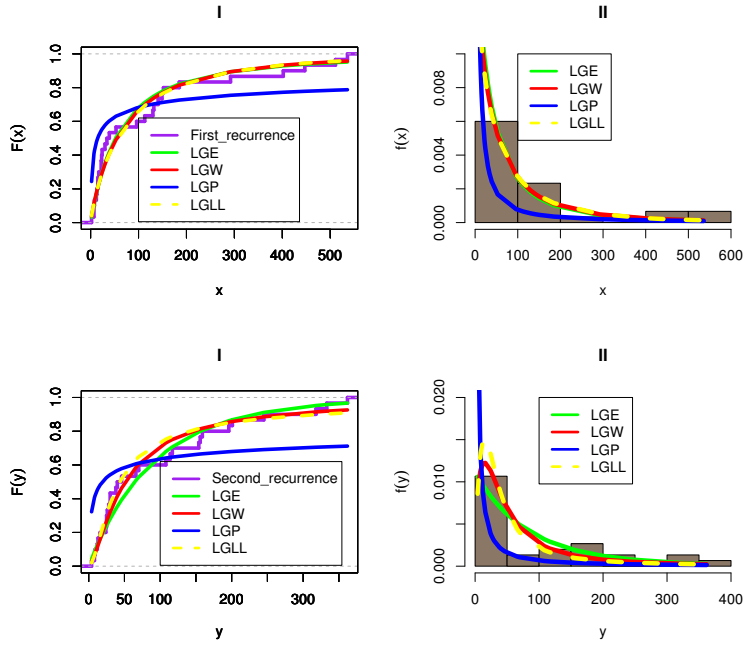


Figure 18. Estimated LG distributions: Kidney patients.

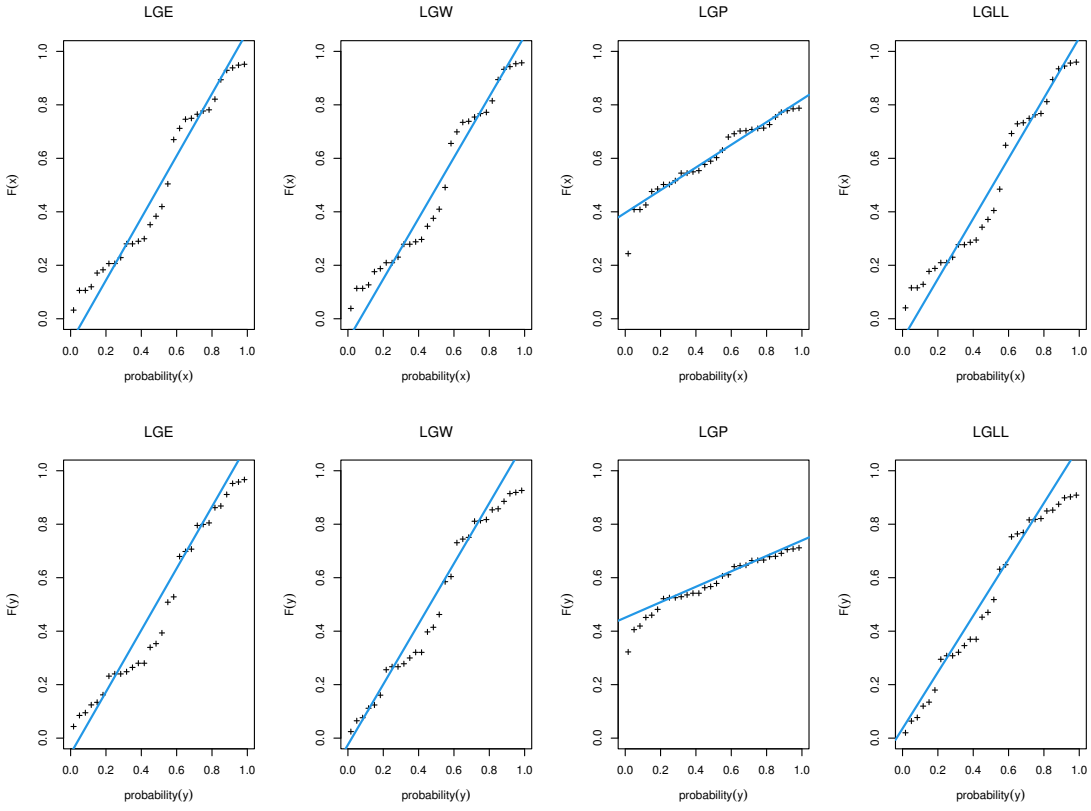


Figure 19. Estimated LG distributions: Kidney patients.

Tables 18 and 19 show the MLE and Bayesian estimation method, respectively for the parameters of L-G family models. Table 18 discussed comparison of bivariate models based on bivariate L-G family model by using AIC, CIAC, BIC, HQIC, CVM, and AD measures. By these results, we conclude the BFGM LGE distribution is best models comparison anther bivariate distribution according to AIC, CIAC, BIC, HQIC, CVM, and AD. When comparing Bayesian estimates and MLE, we note that Bayesian is better than MLE according to the value of SE.

Table 18. MLE of bivariate models: Kidney patients.

	BFGM LGE		BFGM LGW		BFGM LGLLog		BFGM LGP		BFGM LC	
	Etimates	SE	Etimates	SE	Etimates	SE	Etimates	SE	Etimates	SE
α_1	1.6301	0.9353	3.6680	1.0296	175.4646	38.1556	146.0044	82.5462	1.3445	0.7207
β_1	46.4487	26.254	2.1477	1.3569	54.4813	42.5162	118.4694	48.3222	0.0155	0.0157
b_1	0.4731	0.9140	0.8571	0.4542	285.4688	128.5598	0.2064	0.0141	14.1703	11.4864
a_1			0.0070	0.0084	0.8753	0.4986				
α_2	8.8120	2.2553	10.3885	3.7640	8.8665	4.4967	143.3950	1.7083	5.3623	1.3412
β_2	111.0747	56.380	13.9674	11.2431	14.8596	12.1520	129.6081	74.3266	0.0024	0.0005
b_2	0.1434	2.6325	0.9747	0.1967	30.4746	25.9423	0.2260	0.2090	17.7349	14.8967
a_2			0.0153	0.0317	1.7673	1.2649				
θ	0.4032	0.3765	0.3763	0.2927	0.0610	0.0513	0.3610	0.2778	0.3963	0.4530
AIC	690.0815		695.8045		699.6651		774.6638		690.9825	
CAIC	694.9724		704.8045		708.6651		779.7547		696.1334	
BIC	701.6899		708.4153		712.2759		784.4722		712.3589	
HQIC	695.0193		699.8388		703.6994		777.8016		699.5733	
CVM	4.0697		4.0564		4.0277		4.1747		4.2653	
AD	27.9690		26.8935		25.7854		27.6079		27.9913	

Table 19. Bayesian estimation of parameter for bivariate models: kidney patients data.

		α_1	β_1	b_1	a_1	α_2	β_2	b_2	a_2	θ
BFGM LGE	mean	1.7050	212.2050	2.5467		24.6862	256.5568	0.1274		0.0811
	sd	0.8656	16.6110	0.3316		7.6916	70.5392	0.0942		0.2436
BFGM LGW	mean	4.0559	2.1217	0.8666	0.0074	4.8123	24.6371	0.9755	0.0874	0.6470
	sd	0.9888	1.1917	0.1684	0.0073	3.4960	10.3928	0.1856	0.0316	0.2745
BFGM LGLL	mean	23.2650	13.4323	0.1703		3.2662	1.5206	0.1106		0.0889
	sd	8.0363	3.0128	0.0471		0.0123	0.8806	0.0617		0.0580
BFGM LGP	mean	159.9208	39.1830	285.6908	0.8706	8.4728	13.8619	30.8455	1.7398	0.0419
	sd	7.7641	2.9656	0.7591	0.0162	1.0027	0.7175	0.5174	0.0338	0.0034

10. Conclusions

In conclusion, the BFGMLG family proposed in this study represents a new approach to modeling bivariate continuous data with skewed and heavy-tailed distributions, which is useful in a variety of applications. The BFGMLG family is based on the FGM copula function and univariate LG family to handle non-normality in the data. The family includes several distributions such as bivariate exponential LG distribution, bivariate Weibull LG distribution, bivariate Pareto LG distribution, and

bivariate log-logistic distribution, which are all derived based on the FGM copula function. This family has been shown to have various structural statistical properties and can model local dependence as well as concepts such as PQD, TP2, SI, and RTI. Additionally, the proposed family has been extended to the multivariate BFGMLG family to model and support multivariate data. The performance of the proposed family has been evaluated using two estimation methods, ML and Bayesian estimation, and Bayesian estimation has been found to offer the best performance. The application of the proposed family to four different data sets in the fields of Environmental, computer and medicine sciences has demonstrated its superiority over competing bivariate probability models such as BFGM Lomax Claim and bivariate FGM generalized half-logistic. Overall, the BFGMLG family has shown great potential for use in various fields, and future work will focus on studying the properties of its multivariate version family.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. G. M. Cordeiro, E. M. Ortega, B. V. Popović, R. R. Pescim, The Lomax generator of distributions: Properties, minification process and regression model, *Appl. Math. Comput.*, **247** (2014), 465–486. https://doi.org/10.1016/j.amc.2014.09.004
2. V. S. Vaidyanathan, A. S. Varghese, Morgenstern type bivariate lindley distribution, *Stat. Optim. Inf. Comput.*, **4** (2016), 132–146. https://doi.org/10.19139/soic.v4i2.183
3. L. Baharith, H. Alzahrani, New bivariate Pareto type II models, *Entropy*, **21** (2019), 473. https://doi.org/10.3390/e21050473
4. M. V. Peres, R. P. Oliveira, J. A. Achcar, E. Z. Martinez, The Bivariate defective Gompertz distribution based on Clayton Copula with applications to medical data, *Aust. J. Stat.*, **51** (2022), 144–168. https://doi.org/10.17713/ajs.v51i2.1285
5. E. M. Almetwally, H. Z. Muhammed, On a bivariate Frechet distribution, *J. Stat. Appl. Proba.*, **9** (2020), 1–21.
6. M. V. Perres, J. A. Achcar, E. Z. Martinez, Bivariate lifetime models in presence of cure fraction: A comparative study with many different copula functions, *Heliyon*, **6** (2020), e03961. https://doi.org/10.1016/j.heliyon.2020.e03961
7. J. Zhao, H. Faqiri, Z. Ahmad, W. Emam, M. Yusuf, A. M. Sharawy, The Lomax-Claim model: Bivariate extension and applications to financial data, *Complexity*, 2021, 1–17. https://doi.org/10.1155/2021/9993611
8. H. H. Ahmad, E. M. Almetwally, D. A. Ramadan, Investigating the relationship between processor and memory reliability in data science: A bivariate model approach, *Mathematics*, **11** (2023), 2142. https://doi.org/10.3390/math11092142

9. M. E. Qura, A. Fayomi, M. Kilai, E. M. Almetwally, Bivariate power Lomax distribution with medical applications, *Plos One*, **18** (2023), e0282581. https://doi.org/10.1371/journal.pone.0282581
10. E. S. A. El-Sherpieny, E. M. Almetwally, U. Z. Muhammed, Bivariate Weibull-G family based on copula function: Properties, Bayesian and non-Bayesian estimation and applications, *Stat. Optim. Inf. Comput.*, **10** (2022), 678–709. https://doi.org/10.19139/soic-2310-5070-1129
11. H. Z. Muhammed, Bivariate inverse Weibull distribution, *J. Stat. Comput. Simul.*, **86** (2016), 2335–2345. https://doi.org/10.1080/00949655.2015.1110585
12. M. S. Eliwa, M. El-Morshedy, Bivariate Gumbel-G family of distributions: Statistical properties, Bayesian and non-Bayesian estimation with application, *Ann. Data Sci.*, **6** (2019), 39–60. https://doi.org/10.1007/s40745-018-00190-4
13. R. M. Alotaibi, H. R. Rezk, I. Ghosh, S. Dey, Bivariate exponentiated half logistic distribution: Properties and application, *Commun. Stat.-Theor. M.*, **50** (2021), 6099–6121. https://doi.org/10.1080/03610926.2020.1739310
14. E. S. A. El-Sherpieny, H. Z. Muhammed, E. M. Almetwally, Accelerated life testing for bivariate distributions based on progressive censored samples with random removal, *J. Stat. Appl. Probab.*, **11** (2022), 203–223.
15. A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publications de l'Institut de statistique de l'Université de Paris, **8** (1959), 229–231.
16. R. B. Nelsen, *An introduction to copulas*, 2 Eds., Springer Science Business Media, 2006.
17. H. Joe, *Multivariate models and dependence concepts*, 2 Eds., New York: Chapman and Hall, 1997. https://doi.org/10.1201/9780367803896
18. E. J. Gumbel, Bivariate exponential distributions, *J. Am. Stat. Assoc.*, **55** (1960), 698–707. https://doi.org/10.1080/01621459.1960.10483368
19. N. Sreelakshmi, An introduction to copula-based bivariate reliability concepts, *Commun. Stat.-Theor. M.*, **47** (2018), 996–1012. https://doi.org/10.1080/03610926.2017.1316396
20. I. W. Burr, Cumulative frequency functions, *Ann. Math. Stat.*, **13** (1942), 215–232. https://doi.org/10.1214/aoms/1177731607
21. L. J. Bain, Analysis for the linear failure rate life-testing distribution, *Technometrics*, **16** (1974), 551–559. https://doi.org/10.1080/00401706.1974.10489237
22. K. S. Lomax, Business failures: Another example of the analysis of failure, *J. Am. Stat. Assoc.*, **49** (1954), 847–852. https://doi.org/10.1080/01621459.1954.10501239
23. I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, 7 Eds., San Diego: Academic Press, 2007.
24. A. Basu, Bivariate failure rate, *J. Am. Stat. Assoc.*, **66** (1971), 103–104. https://doi.org/10.1080/01621459.1971.10482228
25. N. L. Johnson, S. Kotz, A vector multivariate hazard rate, *J. Multivariate Anal.*, **5** (1975), 53–66. https://doi.org/10.1016/0047-259X(75)90055-X
26. E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.*, **37** (1966), 1137–1153. https://doi.org/10.1214/aoms/1177699260
27. W. Holland, Y. J. Wang, Dependence function for continuous bivariate densities, *Commun. Stat.-Theor. M.*, **16** (1987), 863–876. https://doi.org/10.1080/03610928708829408

28. N. Balakrishnan, C. D. Lai, *Continuous bivariate distributions*, 2 Eds., New York: Springer Science Business Media, 2009.
29. D. G. Clayton, A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence, *Biometrika*, **65** (1978), 141–151. https://doi.org/10.1093/biomet/65.1.141
30. D. Oakes, Bivariate survival models induced by frailties, *J. Am. Stat. Assoc.*, **84** (1989), 487–493. https://doi.org/10.1080/01621459.1989.10478795
31. J. E. Anderson, T. A. Louis, N. V. Holm, B. Harvald, Time-dependent association measures for bivariate survival distributions, *J. Am. Stat. Assoc.*, **87** (1992), 641–650. https://doi.org/10.1080/01621459.1992.10475263
32. R. B. Nelsen, Concordance and Gini's measure of association, *J. Nonparametr. Stat.*, **9** (1998), 227–238. https://doi.org/10.1080/10485259808832744
33. B. V. Popović, M. M. Ristić, A. İ. Genç, Dependence properties of multivariate distributions with proportional hazard rate marginals, *Appl. Math. Model.*, **77** (2020), 182–198. https://doi.org/10.1016/j.apm.2019.07.030
34. H. Dette, K. F. Siburg, P. A. Stoimenov, A copula-based non-parametric measure of regression dependence, *Scand. J. Stat.*, **40** (2013), 21–41. https://doi.org/10.1111/j.1467-9469.2011.00767.x
35. N. L. Johnson, S. Kotz, On some generalized Farlie-Gumbel-Morgenstern distributions, *Commun. Stat.*, **4** (1975), 415–427. https://doi.org/10.1080/03610917508548400
36. A. K. Suzuki, F. Louzada-Neto, V. G. Cancho, G. D. Barriga, The FGM bivariate lifetime copula model: A bayesian approach, *Adv. Appl. Stat.*, **21** (2011), 55–76.
37. F. Louzada, A. K. Suzuki, V. G. Cancho, The FGM long-term bivariate survival copula model: Modeling, Bayesian estimation, and case influence diagnostics, *Commun. Stat.-Theor. M.*, **42** (2013), 673–691. https://doi.org/10.1080/03610926.2012.725147
38. M. H. Chen, Q. M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, *J. Comput. Graph. Stat.*, **8** (1999), 69–92. https://doi.org/10.1080/10618600.1999.10474802
39. M. K. Hassan, C. Chesneau, Bivariate generalized half-logistic distribution: Properties and its application in household financial affordability in KSA, *Math. Comput. Appl.*, **27** (2022), 72. https://doi.org/10.3390/mca27040072
40. G. Grover, A. Sabharwal, J. Mittal, Application of multivariate and bivariate normal distributions to estimate duration of diabetes, *Int. J. Stat. Appl.*, **4** (2014), 46–57.
41. R. P. Oliveira, J. A. Achcar, J. Mazucheli, W. Bertoli, A new class of bivariate Lindley distributions based on stress and shock models and some of their reliability properties, *Reliab. Eng. Syst.*, **211** (2021), 107528. https://doi.org/10.1016/j.ress.2021.107528
42. C. A. McGilchrist, C. W. Aisbett, Regression with frailty in survival analysis, *Biometrics*, **47** (1991), 461–466. https://doi.org/10.2307/2532138

AIMS Press