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Research article

Action of projections on Banach algebras

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Abstract: Let $\mathcal{A}$ be a Banach algebra and $n > 1$, a fixed integer. The main objective of this paper is to talk about the commutativity of Banach algebras via its projections. Precisely, we prove that if $\mathcal{A}$ is a prime Banach algebra admitting a continuous projection $\mathcal{P}$ with image in $\mathcal{Z}(\mathcal{A})$ such that $\mathcal{P}(a^n) = a^n$ for all $a \in \mathcal{G}$, the nonvoid open subset of $\mathcal{A}$, then $\mathcal{A}$ is commutative and $\mathcal{P}$ is the identity mapping on $\mathcal{A}$. Apart from proving some other results, as an application we prove that, a normed algebra is commutative iff the interior of its center is non-empty. Furthermore, we provide some examples to show that the assumed restrictions cannot be relaxed. Finally, we conclude our paper with a direction for further research.

Keywords: commutativity; continuity; open subset; Banach algebra; prime Banach algebra
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1. Introduction

This research has been motivated by the work's of Ali-Khan [2] and Khan [9]. All over this paper unless otherwise stated, $\mathcal{A}$ denotes a Banach algebra with the center $\mathcal{Z}(\mathcal{A})$, $\mathcal{M}$ be a closed linear subspace of $\mathcal{A}$ and $Int(\mathcal{Z}(\mathcal{A}))$, the interior of $\mathcal{Z}(\mathcal{A})$. For any $a, b \in \mathcal{A}$, the symbol [a,b] will denote the commutator, $ab - ba$ and $a \circ b$ represents the anticommutator, $ab + ba$. An algebra $\mathcal{A}$ is said to be prime if for any $a, b \in \mathcal{A}$, $a\mathcal{A}b = (0)$ implies $a = 0$ *or* $b = 0$ and $\mathcal{A}$ is semiprime if for any $a \in \mathcal{A}$, $a\mathcal{A}a = (0)$ implies $a = 0$. A linear mapping $\mathcal{D} : \mathcal{A} \longrightarrow \mathcal{A}$ is called a derivation if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ holds for all $a, b \in \mathcal{A}$. In particular, $\mathcal{D}$ defined by $\mathcal{D}(a) = [\lambda, a]$ for all $a \in \mathcal{A}$ is a derivation, called an inner derivation induced by an element $\lambda \in \mathcal{A}$. Let $\mathcal{R}$ be an associative ring and an additive subgroup $\mathcal{U}$ of $\mathcal{R}$ is known to be a Lie ideal of $\mathcal{R}$ if $[u, a] \in \mathcal{U}$, for all $u \in \mathcal{U}$ and $a \in \mathcal{R}$. Let $\mathcal{M}$ be a subspace of $\mathcal{A}$, the linear operator $\mathcal{P} : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a projection of $\mathcal{A}$ on $\mathcal{M}$ if $\mathcal{P}(a) \in \mathcal{M}$ for all $a \in \mathcal{A}$ and

$\mathcal{P}(a) = a$ for all $a \in \mathcal{M}$. Let $\mathcal{M}$ and $\mathcal{N}$ be two subspaces of the Banach algebra $\mathcal{A}$ such that $\mathcal{M} \oplus_{al} \mathcal{N}$ is an algebraic direct sum of $\mathcal{A}$, we provide the two subspaces $\mathcal{M}$ and $\mathcal{N}$ with the induced topology of $\mathcal{A}$ and $\mathcal{M} \times \mathcal{N}$ by the product topology. We say that $\mathcal{M} \oplus_t \mathcal{N}$ is a topological direct sum of $\mathcal{A}$, if the mapping $\psi : \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{A}$, defined by $\psi(a, b) = a + b$ is a homeomorphism, and we say that $\mathcal{M}$ is complemented in $\mathcal{A}$ and $\mathcal{N}$ is its topological complement. In this case, there is a unique continuous projection $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{M}$. Moreover, Sobezyk [12], established that if $\mathcal{A}$ is separable and $\mathcal{M}$ is a subspace of $\mathcal{A}$ isomorphic to C_0 (the subspace of sequences in $\mathbb{C}$ which converge to 0), then $\mathcal{M}$ is complemented in $\mathcal{A}$. In view of [11], every infinite-dimensional Banach space that is not isomorphic to a Hilbert space contains a closed uncomplemented subspace (see also [6] for details).

Many results in literature concerning commutativity of a prime and semiprime Banach algebra are proved, for example, in the year 1990, Yood [15] showed that if $\mathcal{G}$ is a nonvoid open subset of a Banach algebra $\mathcal{A}$, for each $a, b \in \mathcal{G}$, if positive integers $m = m(a, b)$ and $n = n(a, b)$ exist such that $[a^m, b^n] = 0$. Then there is a positive integer r so that $a^r \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. In addition if $\mathcal{A}$ has no nonzero nilpotent ideal, it is sufficient to have $[a^m, b^n] \in \mathcal{Z}(\mathcal{A})$ and $a, b \in \mathcal{G}$, with m, n as above. Then $\mathcal{A}$ is commutative. Yood [13] also proved that if $\mathcal{A}$ is a Banach algebra and $\mathcal{G}_1, \mathcal{G}_2$ are two non-empty open subsets of $\mathcal{A}$ such that for each $a \in \mathcal{G}_1$ and $b \in \mathcal{G}_2$, there is an integer $n = n(a, b) > 1$ where either $(ab)^n = a^n b^n$ or $(ab)^n = b^n a^n$, then $\mathcal{A}$ is commutative (see [4, 7, 8] for recent works). Motivated by Yood's work, Ali and Khan [2] established the commutativity of Banach algebra via derivations. Moreover, they proved that if $\mathcal{A}$ is a unital prime Banach algebra and has a nonzero continuous linear derivation $d : \mathcal{A} \to \mathcal{A}$ such that either $d((ab)^n) - a^n b^n$ or $d((ab)^n) - b^n a^n$ is in the center of $\mathcal{A}$ for an integer $n = n(a, b) > 1$, then $\mathcal{A}$ is commutative (see, also [1, 3, 9, 10] for recent works).

In this paper, we will continue the study of the problems on Banach algebras involving projections instead of derivations. The key aim of this work is to discuss the commutativity of prime Banach algebras via its projections. Precisely, we prove that if a prime Banach algebra $\mathcal{A}$ admits a nonzero continuous projection $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$ such that $\mathcal{P}(a^n) = a^n$ for all $a \in \mathcal{G}$, the non-empty open subset of $\mathcal{A}$ and $n \in \mathbb{N}$, then $\mathcal{A}$ is commutative and $\mathcal{P}$ is the identity map on $\mathcal{A}$. Furthermore, apart from proving some other interesting results, we discuss some applications of our study.

2. Results

We recall some well known results which will be helpful in order to prove our results. We begin with the following:

Lemma 2.1. *[5] Let $\mathcal{A}$ be a real or complex Banach algebra and $p(t) = \sum_{k=0}^{n} b_k t^k$ a polynomial in the real variable t with coefficients in $\mathcal{A}$. If for an infinite set of real values of t, $p(t) \in \mathcal{M}$, where $\mathcal{M}$ is a closed linear subspace of $\mathcal{A}$, then every b_k lies in $\mathcal{M}$.*

Lemma 2.2. *[14, Theorem 2] Suppose that there are non-empty open subsets G_1, G_2 of $\mathcal{A}$ (where $\mathcal{A}$ denotes a Banach algebra over the complex field with center $\mathcal{Z}$) such that for each $x \in G_1$ and $y \in G_2$ there are positive integers $n = n(x, y)$, $m = m(x, y)$ depending on x and y, $n > 1, m > 1$, such that either $[x^n, y^m] \in \mathcal{Z}$ or $x^n \cdot y^m \in \mathcal{Z}$. Then $\mathcal{A}$ is commutative if $\mathcal{A}$ is semiprime.*

These lemmas will come handy while proving our results. We shall be proving the following results:

Theorem 2.3. *Let $n > 1$ be a fixed integer. Next, let $\mathcal{A}$ be a real or complex semiprime Banach algebra and $\mathcal{G}$ be a nonvoid open subset of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in*

$\mathcal{Z}(\mathcal{A})$ such that

$$\mathcal{P}(a^n) = a^n \text{ for all } a \in \mathcal{G},$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

Theorem 2.4. *Let $n > 1$ be a fixed integer. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ be two nonvoid open subsets of $\mathcal{A}$ admitting a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$. If for all $a \in \mathcal{G}_1$, there exists $b \in \mathcal{G}_2$ such that*

$$\mathcal{P}(a^n) = b^n,$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

In [2, 14], it was observed that the authors used simple multiplication and Lie product. Motivated by this, in our next result we will be using the two symbols "$\mathcal{T}$" and "$*$" representing either the Lie product "[.,.]", or the Jordan product "$\circ$", or the simple multiplication "$\cdot$" of the algebras.

Theorem 2.5. *Let $n, m > 1$ be fixed integers. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ be two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ such that*

$$\mathcal{P}(a^n \mathcal{T} b^m) = a^n * b^m \text{ for all } (a, b) \in \mathcal{G}_1 \times \mathcal{G}_2,$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

The next theorem provides the necessary and sufficient condition for the commutativity of a real or complex normed algebras.

Theorem 2.6. *The normed algebra $\mathcal{A}$ over $\mathbb{C}$ or $\mathbb{R}$ is commutative if and only if the interior of its center is non-empty.*

In order to prove the above mentioned theorems, we need the following auxiliary result. This result will help to bridge the gap between a complemented subspace and that of a projection map.

Proposition 2.7. *Let $\mathcal{A}$ be a Banach space and $\mathcal{M}$ be a closed subspace of $\mathcal{A}$. $\mathcal{M}$ is complemented if there is a continuous projection $\mathcal{P}$ of $\mathcal{A}$ on $\mathcal{M}$, and its complement is $(\mathcal{I} - \mathcal{P})(\mathcal{A})$, where $\mathcal{I}$ is the identity mapping on $\mathcal{A}$.*

Proof. For any $a \in \mathcal{A}$, we can write $a = \mathcal{P}(a) + (a - \mathcal{P}(a))$, since $\mathcal{P}(a) \in \mathcal{M}$ and $a - \mathcal{P}(a) \in (\mathcal{I} - \mathcal{P})(\mathcal{A})$, we observe that $\mathcal{A} = \mathcal{M} + (\mathcal{I} - \mathcal{P})(\mathcal{A})$. Now, we will show that $\mathcal{M} \cap (\mathcal{I} - \mathcal{P})(\mathcal{A}) = \{0\}$. Let $a \in \mathcal{M} \cap (\mathcal{I} - \mathcal{P})(\mathcal{A})$. Then $a \in (\mathcal{I} - \mathcal{P})(\mathcal{A})$, and hence there is a $x \in \mathcal{A}$ such that $a = (\mathcal{I} - \mathcal{P})(x) = x - \mathcal{P}(x)$. Since $a \in \mathcal{M}$ and $\mathcal{P}(x) \in \mathcal{M}$, we conclude that $x \in \mathcal{M}$ and $\mathcal{P}(x) = x$, so we obtain $a = 0$. Therefore, $\mathcal{M} \oplus_{al} (\mathcal{I} - \mathcal{P})(\mathcal{A})$ is an algebraic direct sum of the Banach space $\mathcal{A}$. It remains for us to show that $(\mathcal{I} - \mathcal{P})(\mathcal{A})$ is closed in $\mathcal{A}$. For this, consider a sequence $(b_n)_{n \in \mathbb{N}} \subset (\mathcal{I} - \mathcal{P})(\mathcal{A})$ that converges to $b \in \mathcal{A}$ as $n \to \infty$. Since $\mathcal{M} \oplus_{al} (\mathcal{I} - \mathcal{P})(\mathcal{A})$ is an algebraic direct sum of $\mathcal{A}$, then there is $c_1 \in \mathcal{M}$ and $c_2 \in (\mathcal{I} - \mathcal{P})(\mathcal{A})$ such that $b = c_1 + c_2$. Thus, we obtain

$$\mathcal{P}(b) = c_1. \tag{2.1}$$

On the other hand, we have $\mathcal{P}(b_n) = 0$ for all $n \in \mathbb{N}$, as $\mathcal{P}$ is a continuous linear operator then $\mathcal{P}(\lim b_n) = 0$. This implies

$$c_1 = \mathcal{P}(b) = 0. \tag{2.2}$$

In view of (2.1) and (2.2), we obtain $b = c_2 \in (\mathcal{I} - \mathcal{P})(\mathcal{A})$, and hence we conclude that $(\mathcal{I} - \mathcal{P})(\mathcal{A})$ is a closed subspace. Consequently, $\mathcal{M} \oplus_t (\mathcal{I} - \mathcal{P})(\mathcal{A})$ is a topological direct sum of the Banach space $\mathcal{A}$. This proves the proposition. □

3. Proofs of the theorems

Proof of Theorem 2.3. Let $\mathcal{A}$ be a semiprime Banach algebra and $\mathcal{P}$ be a continuous projection satisfying

$$\mathcal{P}(a^n) = a^n \text{ for all } a \in \mathcal{G}, \tag{3.1}$$

for a fixed $n \in \mathbb{N}$. Then, clearly $\mathcal{P}$ is not zero. Thus, $\mathcal{Z}(\mathcal{A})$ forms a closed subspace of $\mathcal{A}$ and $\mathcal{P}$ is a continuous projection onto $\mathcal{Z}(\mathcal{A})$, and hence in view of Proposition 2.7, we conclude that $\mathcal{Z}(\mathcal{A})$ is complemented in $\mathcal{A}$ and a topological direct sum of $\mathcal{A}$, that is,

$$\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus_t (\mathcal{I} - \mathcal{P})(\mathcal{A}). \tag{3.2}$$

Now, let $a_0 \in \mathcal{G}$ and $a \in \mathcal{A}$, then $a_0 + ka \in \mathcal{G}$ for any sufficiently small real k, so $\mathcal{P}((a_0 + ka)^n) - (a_0 + ka)^n = 0$. We can write

$$(a_0 + ka)^n = A_0 + A_1 k + A_2 k^2 + \cdots + A_n k^n.$$

We take $p(k) = \mathcal{P}((a_0 + ka)^n) - (a_0 + ka)^n$, since $\mathcal{P}$ is a linear operator, we can write

$$p(k) = (\mathcal{P}(A_0) - A_0) + (\mathcal{P}(A_1) - A_1)k + (\mathcal{P}(A_2) - A_2)k^2 + \cdots + (\mathcal{P}(A_n) - A_n)k^n.$$

The coefficient of k^n in the above polynomial is $\mathcal{P}(A_n) - A_n$. Since $A_n = a^n$, then this coefficient becomes $\mathcal{P}(a^n) - a^n$. Using the Lemma 2.1, we obtain $\mathcal{P}(a^n) - a^n = 0$ and thus $\mathcal{P}(a^n) = a^n$ for all $a \in \mathcal{A}$. Since $\mathcal{P}(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, we conclude that $a^n \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. This implies $a^n b^n \in \mathcal{Z}(\mathcal{A})$ for all $(a, b) \in \mathcal{A} \times \mathcal{A}$. Hence the required result follows from the Lemma 2.2, i.e., $\mathcal{A}$ is commutative. From Eq (3.2), we obtain $(\mathcal{I} - \mathcal{P})(\mathcal{A}) = 0$ and hence $\mathcal{P} = \mathcal{I}$, which completes the proof of theorem.
As an immediate consequence of Theorem 2.3, we have the following results.

Corollary 3.1. *Let $n > 1$ be a fixed integer. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}$ be a nonvoid open subset of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ such that*

$$\mathcal{P}(a^n) = a^n \text{ for all } a \in \mathcal{G},$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

Corollary 3.2. *Let $\mathcal{J}$ be a part dense in a prime Banach algebra $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ and there is an $n \in \mathbb{N}$ such that*

$$\mathcal{P}(a^n) = a^n \text{ for all } a \in \mathcal{J},$$

then $\mathcal{A}$ is commutative and $\mathcal{P} = \mathcal{I}$, the identity mapping on $\mathcal{A}$.

Proof. Let $a \in \mathcal{A}$, there exists a sequence $(a_k)_{k \in \mathbb{N}}$ in $\mathcal{J}$ converging to a. Since $(a_k)_{k \in \mathbb{N}} \subset \mathcal{J}$, so for each $k \in \mathbb{N}$

$$\mathcal{P}((a_k)^n) - (a_k)^n = 0.$$

By the continuity of $\mathcal{P}$, we conclude that $\mathcal{P}(a^n) - a^n = 0$. Consequently, there exists $n \in \mathbb{N}$ such that

$$\mathcal{P}(a^n) - a^n = 0 \text{ for all } a \in \mathcal{A}.$$

Application of Theorem 2.3, yields the required result. □

Proof of Theorem 2.4. Let us consider two sets,

$$G_n = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \mathcal{P}(a^n) \neq b^n\} \text{ and } H_n = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \mathcal{P}(a^n) = b^n\},$$

for some $n \in \mathbb{N}$. Observe that $(\cap G_n) \cap (\mathcal{G}_1 \times \mathcal{G}_2) = \emptyset$. Indeed, if there exists $a \in \mathcal{G}_1$ and $b \in \mathcal{G}_2$ such that $(a,b) \in G_n$ for all $n \in \mathbb{N}$, then $\mathcal{P}(a^n) \neq b^n$ for all $n \in \mathbb{N}$, which is absurd by the hypothesis of the theorem.
Now we claim that each G_n is open in $\mathcal{A} \times \mathcal{A}$. We show that H_n, the complement of G_n, is closed. Consider a sequence $(a_j, b_j)_{j\in\mathbb{N}} \subset H_n$ converging to $(a,b) \in \mathcal{A} \times \mathcal{A}$. Since $(a_j, b_j)_{j\in\mathbb{N}} \subset H_n$, so

$$\mathcal{P}((a_j)^n) = (b_j)^n \text{ for all } j \in \mathbb{N}.$$

As $\mathcal{P}$ is continuous, we deduce that $\mathcal{P}(a^n) = b^n$. Therefore, $(a,b) \in H_n$, making H_n closed and G_n open. By Baire category theorem, we imply that the intersection of all G_n's is dense if each G_n is dense, contradicting the fact that $(\cap G_n) \cap (\mathcal{G}_1 \times \mathcal{G}_2) = \emptyset$. Hence there exists $p \in \mathbb{N}$ such that G_p is not dense in $\mathcal{A}$ and a nonvoid open subset $G \times G'$ in H_p, such that

$$\mathcal{P}(a^p) = b^p \text{ for all } (a,b) \in G \times G'.$$

Now, let $(a_0, b_0) \in G \times G'$ and $(a,b) \in \mathcal{A} \times \mathcal{A}$. Then $a_0 + ka \in G$ and $b_0 + kb \in G'$ for any sufficiently small real k, making $\mathcal{P}((a_0 + ka)^p) - (b_0 + kb)^p = 0$. We have,

$$(a_0 + ka)^p = A_{p,0}(a_0, a) + A_{p-1,1}(a_0, a)k + A_{p-2,2}(a_0, a)k^2 + \cdots + A_{0,p}(a_0, a)k^p,$$

and

$$(b_0 + kb)^p = B_{p,0}(b_0, b) + B_{p-1,1}(b_0, b)k + B_{p-2,2}(b_0, b)k^2 + \cdots + B_{0,p}(b_0, b)k^p.$$

We put $p(k) = \mathcal{P}((a_0 + ka)^p) - (b_0 + kb)^p$, since $\mathcal{P}$ is a projection, we can write $p(k) = \mathcal{P}(A_{p,0}(a_0, a)) - B_{p,0}(b_0, b) + \mathcal{P}(A_{p-1,1}(a_0, a)) - B_{p-1,1}(b_0, b)k + \mathcal{P}(A_{p-2,2}(a_0, a)) - B_{p-2,2}(b_0, b)k^2 + \cdots + \mathcal{P}(A_{0,p}(a_0, a)) - B_{0,p}(b_0, b)k^p$. The coefficient of k^p in the above polynomial is just $\mathcal{P}(a^p) - b^p$, according to the Lemma 2.1, we obtain $\mathcal{P}(a^p) - b^p = 0$ and hence, $\mathcal{P}(a^p) = b^p$ for all $(a,b) \in \mathcal{A} \times \mathcal{A}$. In particular, for $a = b$ we have $\mathcal{P}(a^p) = a^p$ for all $a \in \mathcal{A}$. In view of Theorem 2.3, we conclude $\mathcal{A}$ is commutative and $\mathcal{P}$ is the identity mapping of $\mathcal{A}$.

Proof of Theorem 2.5. We know that $\mathcal{Z}(\mathcal{A})$ forms a closed subspace of $\mathcal{A}$ and $\mathcal{P}$ is a given continuous projection onto $\mathcal{Z}(\mathcal{A})$, by Proposition 2.7, $\mathcal{Z}(\mathcal{A})$ is complemented in $\mathcal{A}$. That is,

$$\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus_t (\mathcal{I} - \mathcal{P})(\mathcal{A}), \tag{3.3}$$

a topological direct sum of $\mathcal{Z}(\mathcal{A})$ and its complement. For any pair $n, m \in \mathbb{N}$, we define two sets:

$$G_{n,m} = \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \mathcal{P}(a^n \mathcal{T} b^m) \neq a^n * b^m\}$$

and

$$H_{n,m} = \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid \mathcal{P}(a^n \mathcal{T} b^m) = a^n * b^m\}.$$

We observe that $(\cap G_{n,m}) \cap (\mathcal{G}_1 \times \mathcal{G}_2) = \emptyset$. If not, there exists some $(a, b) \in \mathcal{G}_1 \times \mathcal{G}_2$ such that $(a, b) \in G_{n,m}$ for all $n, m \in \mathbb{N}$, then $\mathcal{P}(a^n \mathcal{T} b^m) \neq a^n * b^m$ for all $n, m \in \mathbb{N}$ which contradicts the hypothesis of the theorem.

Now we claim that each $G_{n,m}$ is open in $\mathcal{A} \times \mathcal{A}$. We show that $H_{n,m}$, the complement of $G_{n,m}$, is closed. For this, consider a sequence $((a_j, b_j))_{j \in \mathbb{N}} \subset H_{n,m}$ converging to $(a, b) \in \mathcal{A} \times \mathcal{A}$. Since $((a_j, b_j))_{j \in \mathbb{N}} \subset H_{n,m}$, we have

$$\mathcal{P}((a_j)^n \mathcal{T} (b_j)^m) = (a_j)^n * (b_j)^m \text{ for all } j \in \mathbb{N}.$$

We conclude that $\mathcal{P}(a^n \mathcal{T} b^m) = a^n * b^m$, as $\mathcal{P}$ is continuous. Therefore, $(a, b) \in H_{n,m}$ and $H_{n,m}$ is closed (i.e., $G_{n,m}$ is open). If every $G_{n,m}$ is dense, we know that their intersection is also dense (by Baire category theorem), which contradicts with $(\cap G_{n,m}) \cap (\mathcal{G}_1 \times \mathcal{G}_2) = \emptyset$. Hence there exists $p, q \in \mathbb{N}$ such that $G_{p,q}$ is not dense and a nonvoid open subset $G \times G'$ in $H_{p,q}$, such that

$$\mathcal{P}(a^p \mathcal{T} b^q) = a^p * b^q \text{ for all } (a, b) \in G \times G'.$$

Since $\mathcal{P}(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, we conclude that $a^p * b^q \in \mathcal{Z}(\mathcal{A})$ for all $(a, b) \in G \times G'$, hence the result follows from the Lemma 2.2.

The following are the immediate consequences of Theorem 2.5.

Remark 3.3. Let $n, m > 1$ be fixed integers. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ be two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ such that

$$\mathcal{P}([a^n, b^m]) = [a^n, b^m] \text{ for all } (a, b) \in \mathcal{G}_1 \times \mathcal{G}_2,$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

Remark 3.4. Let $n, m > 1$ be fixed integers. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ be two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ such that

$$\mathcal{P}(a^n \cdot b^m) = [a^n, b^m] \text{ for all } (a, b) \in \mathcal{G}_1 \times \mathcal{G}_2,$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

Remark 3.5. Let $n, m > 1$ be fixed integers. Next, let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ be two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection $\mathcal{P}$ whose image lies in $\mathcal{Z}(\mathcal{A})$ such that

$$\mathcal{P}([a^n, b^m]) = a^n \circ b^m \text{ for all } (a, b) \in \mathcal{G}_1 \times \mathcal{G}_2,$$

then $\mathcal{P}$ is the identity mapping on $\mathcal{A}$ and $\mathcal{A}$ is commutative.

Proof of Theorem 2.6. Suppose $\mathcal{A}$ is commutative, then $\mathcal{A} = \mathcal{Z}(\mathcal{A})$ and hence the interior of $\mathcal{Z}(\mathcal{A})$ is the interior of $\mathcal{A}$ itself, $\mathcal{A}$ being open implies $Int(\mathcal{Z}((A)) = Int(\mathcal{A}) = \mathcal{A} \neq 0$.

Now, we prove the other way round. If $Int(\mathcal{Z}(\mathcal{A})) \neq \emptyset$, then there exists $0 \neq a \in Int(\mathcal{Z}(\mathcal{A}))$. Let $c \in \mathcal{A}$, we have $a + kc \in Int(\mathcal{Z}(\mathcal{A}))$ for any sufficiently small nonzero real k, therefore, we have

$$[a + kc, b] = 0 \text{ for all } b \in \mathcal{A},$$

that is,

$$[a, b] + k[c, b] = 0 \text{ for all } b \in \mathcal{A}.$$

Since,

$$[a, b] = 0 \text{ for all } b \in \mathcal{A},$$

we obtain

$$k[c, b] = 0 \text{ for all } b \in \mathcal{A}.$$

This implies that

$$[c, b] = 0 \text{ for all } b \in \mathcal{A}.$$

Hence, $\mathcal{A}$ is commutative. This completes the proof.

Corollary 3.6. *If $\mathcal{Z}(\mathcal{A})$ contains an isolated point of $\mathcal{A}$, then $\mathcal{A}$ is commutative.*

Proof. Let x be an isolated point of $\mathcal{A}$ contained in $\mathcal{Z}(\mathcal{A})$. We have $\{x\} \subset \mathcal{Z}(\mathcal{A})$. This gives $Int(\{x\}) \subset Int(\mathcal{Z}(\mathcal{A}))$. Since x is an isolated point of $\mathcal{A}$, the singleton set $\{x\}$ is open in $\mathcal{A}$, that is, $Int(\{x\}) = \{x\}$. That is, $Int(\mathcal{Z}(\mathcal{A})) \neq \emptyset$. Hence, $\mathcal{A}$ is commutative by Theorem 2.6. □

In particular, we get the following result.

Corollary 3.7. *Let $\mathcal{A}$ be a normed algebra over $\mathbb{R}$ or $\mathbb{C}$. If 0 is an isolated point in $\mathcal{A}$, then $\mathcal{A}$ is commutative.*

Example 3.8. Let $\mathbb{R}$ be the field of real numbers. Next, let us consider

$$\mathcal{A} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 2 \right\}.$$

Clearly, $\mathcal{A}$ is a real prime Banach algebra under the norm defined by $\|A\|_1 = max_j(\sum_{i=1}^{2} |a_{ij}|)$ for all $A = (a_{ij})_{1 \leq i,j \leq 2} \in \mathcal{A}$.

Consider the sets, $\mathcal{S}_1 = \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s > 0 \right\}$ and $\mathcal{S}_2 = \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s \in \mathbb{R}^* \right\}$. Clealy, $\mathcal{S}_1$ and $\mathcal{S}_1$ are not open in $\mathcal{A}$. Take $\mathcal{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{E}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathcal{E}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{E}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we observe that the family $\mathcal{O} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ is a basis of $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A}) = span(\mathcal{E}_1)$, so we can write $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus_t span(\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$. The mapping $\mathcal{P}$ defined from $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$ by $\mathcal{P}(M) = a_1\mathcal{E}_1$ for all $M = \sum_{i=0}^{4} a_i\mathcal{E}_i \in \mathcal{A}$ is a continuous projection of $\mathcal{A}$ on $\mathcal{Z}(\mathcal{A})$. For all $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{S}_1$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in \mathcal{S}_2$ and for all $n, m \in \mathbb{N}$, it is easy to see that

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \text{ and } B^m = \begin{pmatrix} b^m & 0 \\ 0 & b^m \end{pmatrix}.$$

Moreover, we compute

$$A^nB^m = \begin{pmatrix} a^nb^m & 0 \\ 0 & a^nb^m \end{pmatrix}, A^n \circ B^m = \begin{pmatrix} 2a^nb^m & 0 \\ 0 & 2a^nb^m \end{pmatrix} \text{ and } [A^n, B^m] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that

$$A^nB^m = a^nb^m\mathcal{E}_1;\ A^n \circ B^m = 2a^nb^m\mathcal{E}_1;\ [A^n, B^m] = 0\mathcal{E}_1.$$

Thus, we have

(1) $\mathcal{P}(A^n \circ B^m) = A^n \circ B^m$

(2) $\mathcal{P}(A^n B^m) = A^n B^m$

(3) $\mathcal{P}([A^n, B^m]) = [A^n, B^m]$

(4) $\mathcal{P}(A^n) = A^n$.

One might think that the projection map $\mathcal{P} = \mathcal{I}$, but this is not true as $\mathcal{P}(\mathcal{E}_3) = 0_{\mathcal{A}} \neq \mathcal{E}_3$. So the conditions of $\mathcal{G}_1$ and $\mathcal{G}_2$ to be open in Theorem 2.5 is indispensable.

Our next example shows that we cannot replace $\mathbb{R}$ or $\mathbb{C}$ in Theorem 2.5 by the finite field $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_3$.

Example 3.9. Let $\mathcal{A} = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{Z}_3\right\}$. It is easy to check that $\mathcal{A}$ forms a prime Banach algebra under the norm, $||A||_\infty = max_i(\sum_{j=1}^{2} |a_{ij}|)$ for all $A = (a_{ij})_{1 \leq i,j \leq 2} \in \mathcal{A}$, where $|.|$ on $\mathbb{Z}_3$ is defined as,

$$|\bar{0}| = 0,\ |\bar{1}| = 1 \text{ and } |\bar{2}| = 2.$$

Next, let $\mathcal{G} = \left\{\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s \in \mathbb{Z}_3\right\}$ be an open set in $\mathcal{A}$. Consider $A \in \mathcal{G}$, the open ball $B(A, 1) = \{Y \in \mathcal{A}$ such that $||A - Y||_\infty < 1\} = \{A\} \subset \mathcal{G}$, so $\mathcal{G}$ is a nonvoid open subset of $\mathcal{A}$. Take, $\mathcal{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{E}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathcal{E}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{E}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, observe that the family $\mathcal{O} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ forms a basis of $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A}) = span(\mathcal{E}_1)$, so we can write $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus_t span(\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$. The mapping $\mathcal{P}$ defined from $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$ by $\mathcal{P}(M) = a_1\mathcal{E}_1$ for all $M = \sum_{i=0}^{4} a_i\mathcal{E}_i \in \mathcal{A}$ is a nonzero continuous projection of $\mathcal{A}$ on $\mathcal{Z}(\mathcal{A})$. For all $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in \mathcal{G}$ and for all $n, m \in \mathbb{N}$, we have

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \text{ and } B^m = \begin{pmatrix} b^m & 0 \\ 0 & b^m \end{pmatrix}.$$

Thus, we obtain

$$A^n B^m = \begin{pmatrix} a^n b^m & 0 \\ 0 & a^n b^m \end{pmatrix}, A^n \circ B^m = \begin{pmatrix} 2a^n b^m & 0 \\ 0 & 2a^n b^m \end{pmatrix} \text{ and } [A^n, B^m] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we can write,

$$A^n B^m = a^n b^m \mathcal{E}_1;\ A^n \circ B^m = 2a^n b^m \mathcal{E}_1;\ [A^n, B^m] = 0\mathcal{E}_1.$$

Thus, it is easy to see that

(1) $\mathcal{P}(A^n \circ B^m) = A^n \circ B^m$

(2) $\mathcal{P}(A^n B^m) = A^n B^m$

(3) $\mathcal{P}([A^n, B^m]) = [A^n, B^m]$

(4) $\mathcal{P}(A^n) = A^n$.

Observe that $\mathcal{P} \neq \mathcal{I}$ as $\mathcal{P}(\mathcal{E}_2) = 0_{\mathcal{A}} \neq \mathcal{E}_2$. Hence we conclude that $\mathbb{R}$ or $\mathbb{C}$ cannot be replaced by the field $\mathbb{F}_3$ in case of Theorem 2.5.

4. Applications

In this section, we will discuss an application of Theorem 2.5.

Let $\mathbb{C}$ be the field of complex numbers. Next, let us consider

$$\mathcal{A} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{C}, 1 \leq i, j \leq 2 \right\}.$$

Clearly, $\mathcal{A}$ is prime algebra over $\mathbb{C}$ under the norm defined by $\|A\|_1 = max_j(\sum_{i=1}^{2} |a_{ij}|)$ for all $A = (a_{ij})_{1\leq i,j\leq 2} \in \mathcal{A}$. Consider the sets, $\mathcal{G}_1 = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}$ and $\mathcal{G}_2 = \left\{ \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \mid t \in \mathbb{R} \right\}$. Clearly, $\mathcal{G}_1$ and $\mathcal{G}_2$ are open in $\mathcal{A}$. Take $\mathcal{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{E}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathcal{E}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{E}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we observe that the family $O = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ is a basis of $\mathcal{A}$ and $\mathcal{Z}(\mathcal{A}) = span(\mathcal{E}_1)$, so we can write $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus_t span(\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$. The mapping $\mathcal{P}$ defined from $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$ by $\mathcal{P}(M) = a_1\mathcal{E}_1$ for all $M = \sum_{i=0}^{4} a_i\mathcal{E}_i \in \mathcal{A}$ is a continuous projection of $\mathcal{A}$ on $\mathcal{Z}(\mathcal{A})$. For all $A = \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix} \in \mathcal{G}_1$, $B = \begin{pmatrix} e^{-ib} & 0 \\ 0 & e^{ib} \end{pmatrix} \in \mathcal{G}_2$ and for all $n, m \in \mathbb{N}$, it is easy to see that

$$A^n = \begin{pmatrix} e^{ina} & 0 \\ 0 & e^{-ina} \end{pmatrix} \text{ and } B^m = \begin{pmatrix} e^{-imb} & 0 \\ 0 & e^{imb} \end{pmatrix}.$$

Moreover, we compute

$$[A^n, B^m] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that

$$[A^n, B^m] = 0\mathcal{E}_1.$$

Thus, we have $\mathcal{P}([A^n, B^m]) = [A^n, B^m]$. Observe that $\mathcal{P}(\mathcal{E}_3) = 0_{\mathcal{A}} \neq \mathcal{E}_3$. Hence, it follows from Theorem 2.5, that $\mathcal{A}$ is not a Banach algebra under the defined norm.

5. A direction for further research

Several papers in the literature evidence how the behaviour of some linear mappings is closely connected to the structure of the Banach algebras (cf. [1–3] and [9, 10]). In our main results (Theorems 2.3 and 2.5), we investigate the structure of prime Banach algebras equipped with a continuous projections. Nevertheless, there are various interesting open problems related to our work. In this final section, we will propose a direction for future further research. In view of [2] and Theorems 2.3 and 2.4, the following problems remain unanswered.

Problem 5.1. Let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{P} : \mathcal{A} \longrightarrow \mathcal{Z}(\mathcal{A})$ be a nonzero continuous projection. Suppose that there is an open subset $\mathcal{G}$ of $\mathcal{A}$ such that $\mathcal{P}(a^n) - a^n \in \mathcal{Z}(\mathcal{A})$ for each $a \in \mathcal{G}$ and an integer $n > 1$. Then, what we can say about the structure of $\mathcal{A}$ and $\mathcal{P}$?

Problem 5.2. Let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{P} : \mathcal{A} \longrightarrow \mathcal{Z}(\mathcal{A})$ be a nonzero continuous projection. Suppose that there are open subsets $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{A}$ such that $\mathcal{P}(a^n) - b^n \in \mathcal{Z}(\mathcal{A})$ for each $(a, b) \in \mathcal{G}$ and an integer $n = n(a, b) > 1$. Then, what we can say about the structure of $\mathcal{A}$ and $\mathcal{P}$?

Problem 5.3. Let $\mathcal{A}$ be a real or complex prime Banach algebra and $\mathcal{P} : \mathcal{A} \longrightarrow \mathcal{Z}(\mathcal{A})$ be a nonzero continuous projection. Suppose that there are open subsets $\mathcal{G}_1$, $\mathcal{G}_2$ of $\mathcal{A}$ such that $\mathcal{P}((ab)^n) - a^n b^n \in \mathcal{Z}(\mathcal{A})$ for each $a \in \mathcal{G}_1$ and $b \in \mathcal{G}_2$ and an integer $n = n(x, y) > 1$. Then, what we can say about the structure of $\mathcal{A}$ and $\mathcal{P}$?

Problem 5.4. Let $\mathcal{A}$ be a commutative Banach algebra such that it admits a continuous projection $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{Z}(\mathcal{A})$ satisfying $\mathcal{P}(a)^n = a^n$ for all $a \in \mathcal{G}$, where $\mathcal{G}$ is a non-empty open subset of $\mathcal{A}$. Then, what we can say about the structure of $\mathcal{A}$ and $\mathcal{P}$?

6. Conclusions and discussions

In this paper, we discussed new criteria to study the commutativity of Banach algebras. Particularly, we described the commutativity of prime Banach algebras over the field of real and complex via its projections. In this direction, Yood [13, 14] established the commutativity of Banach algebras using the polynomial identities. Similarly, taking this idea forward [2, 3, 10] were able to accomplished the commutativity of Banach algebras via derivations. It would be interesting to discuss the commutativity of Banach algebras involving more general functional identities via projections (see Open Problems 5.1–5.4).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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