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Research article

Existence and uniqueness theorems for pointwise-slant immersions in Sasakian space forms

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Abstract: In this paper we derive the Existence and Uniqueness Theorems for pointwise slant immersions in Sasakian space forms which extend the Existence and Uniqueness Theorems for slant immersions in Sasakian space forms proved by Cabreizo et al in 2001.

Keywords: Sasakian manfold; sectional curvature; pointwise-slant immersion
Mathematics Subject Classification: 53C25, 53C42

1. Introduction

In complex geometry, B.-Y Chen [5, 6] generalized the concept of totally real and holomorphic submanifolds by defining the slant submanifolds; while in contact geometry, A. Lotta [14] extended the notion to almost contact metric manifolds. In addition, Cabreizo et al. established in [3] the existence and uniqueness theorems for slant immersions in Sasakian space forms and obtained similar results of Chen and L. Vrancken [8, 9]. In contact geometry, this subject was studied in many structures, such as cosymplectic space forms [12] and Kenmotsu space forms [15].

The pointwise slant submanifolds were introduced by F. Etayo [11] under the name of quasi-slant submanifolds. Later, Chen and Garay [7] studied pointwise slant submanifolds of almost Hermitian manifolds. After that Park [16] studied this idea in almost contact metric manifolds. In particular, the study of pointwise slant immersions of Sasakian manfolds were presented in [13]. Recently, the pointwise slant submanifolds were studied in different structures on a Riemannian manifold [17,18,20].

As continuation of [1], in this paper we extend the study for existence and uniqueness theorems in contact geometry, especially in Sasakian space forms. First, we review basic formulas and properties for the pointwise slant submanifolds of an almost contact metric manifold which we shall use later. Then, we present the existence and uniqueness theorems. Furthermore, at the end of this paper, we provide some non-trivial examples of general pointwise slant immersions in Euclidean spaces with an almost contact structure.

2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$ dimensional Riemannian manifold, then $\tilde{M}$ is said to be an almost contact metric manifold if it equipped with an almost contact metric structure (ϕ, ξ, η, g) such that ϕ is a tensor field of type $(1,1)$, ξ a structure vector field, η is a 1-form and g is a Riemannian metric on $\tilde{M}$ satisfying the following properties [2]

$$\phi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ be the set of all smooth vector fields on $\tilde{M}$.

In other words, from (2.2), we see that

$$g(\phi X, Y) = -g(X, \phi Y),$$

which means that $g(\phi X, X) = 0$, i.e., $\phi X \perp X$ for each vector field X on $\tilde{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\tilde{M}$ is called a contact metric structure if $d\eta = \Omega$, where Ω is the fundamental 2-form, defined by $\Omega(X, Y) = g(\phi X, Y)$ [2].

If we extend the Riemannian connection $\tilde{\nabla}$ to be a covariant derivative of the tensor field ϕ, then we have the following formula

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

If the structure vector field ξ is Killing with respect to g, the contact metric structure is called a K-contact structure. It is known that a contact metric manifold is K- contact if and only if

$$\tilde{\nabla}_X \xi = -\phi X,$$

for all $X \in \Gamma(T\tilde{M})$.

An almost contact structure on $\tilde{M}$ is said to be normal if,

$$N_\phi + 2d\eta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor of the tensor field ϕ. A normal contact metric manifold is called a Sasakian manifold.

It is easy to show that an almost contact metric manifold is Sasakian if and only if [2]

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

for any $X, Y \in \Gamma(T\tilde{M})$. Furthermore, from (2.4), we find that

$$\tilde{\nabla}_X \xi = -\phi X. \tag{2.5}$$

Every Sasakian manifold is a K-contact metric manifold.

Let $\tilde{M}$ be a $(2m+1)$ dimensional Sasakian manifold, and π be a plane section in the tangent space $T_p\tilde{M}$, then π is said to be a ϕ-section if it is spanned by X and ϕX, such that X be a unit tangent vector field orthogonal to ξ. The sectional curvature $K(\pi)$ of a ϕ-section π is said to be ϕ-sectional curvature. The Sasakian manifold $\tilde{M}$ with the constant ϕ-sectional curvature c is called a Sasakian space form. We denote by $\tilde{M}^{2m+1}(c)$ the complete simply connected Sasakian space form of dimension $(2m+1)$ with the constant ϕ-sectional curvature c [19].

The curvature tensor of $\tilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z =& \frac{c+3}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\} + \frac{c-1}{4}\Big\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(X,\phi Y)\phi Z\Big\},\end{aligned} \tag{2.6}$$

for any $X,Y,Z \in \Gamma(T\tilde{M})$ [2].

Now, let M be a $(n+1)$-dimensional submanifold of an almost contact metric manifold. We denote by TM the tangent bundle of M and by $T^\perp M$ the set of all vectors fields normal to M. For any $X \in \Gamma(TM)$, we put

$$\phi X = TX + FX, \tag{2.7}$$

where TX and FX are the tangential and normal components of ϕX, respectively. Also, for any $U \in \Gamma(T^\perp M)$, we write

$$\phi U = tU + fU, \tag{2.8}$$

where tU and fU are the tangential and normal components of ϕU, respectively.

A submanifold M of an almost contact metric manifold $\tilde{M}$ tangent to the structure vector field ξ is called an invariant if F is identically zero, that is $\phi X \in \Gamma(TM)$, for any $X \in \Gamma(TM)$, while M is called an anti-invariant if T is identically zero, that is $\phi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

Let ∇ indicate the Levi-Civita connection on M, while $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ of M. Then, the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^\perp_X U,$$

for any $X,Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$ such that h is the second fundamental form of M, and A_U is the shape operator corresponding to U, which are related by

$$g(A_U X, Y) = g(h(X,Y),U). \tag{2.9}$$

If we let R be the curvature tensor of M, and $R^\perp$ be the curvature tensor of the normal connection $\nabla^\perp$. Then the equation of Gauss, Ricci and Codazzi are given respectively by [4]

$$\tilde{R}(X,Y;Z,W) = R(X,Y;Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \tag{2.10}$$

$$\tilde{R}(X, Y; U, V) = R^{\perp}(X, Y; U, V) - g([A_U, A_V]X, Y) \tag{2.11}$$

and

$$(\tilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z), \tag{2.12}$$

for all $X, Y, Z, W \in \Gamma(TM)$, and $U, V \in \Gamma(T^{\perp}M)$, where $(\tilde{R}(X, Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X, Y)Z$, and the covariant derivative $\overline{\nabla}h$ is given by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.13}$$

The covariant derivative of T and F, respectively given by

$$(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \tag{2.14}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F(\nabla_X Y). \tag{2.15}$$

3. Pointwise slant submanifolds of an almost contact metric manifold

In this section we recall some results of pointwise slant submanifolds of an almost contact metric manifold $\tilde{M}$.

Definition 3.1. *[16] A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be pointwise slant, if for each point $p \in M$, the Wirtinger angle $\theta(X)$ between ϕX and T_pM is independent of the choice of a non-zero vector $X \in T_pM$. The Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$ which is called the Wirtinger function or slant function of the pointwise slant submanifold.*

We note that a pointwise slant submanifold of $\tilde{M}$ is called slant, if its Wirtinger function θ is globally constant and also, it is called a proper pointwise slant if it is neither invariant nor anti-invariant nor θ is constant on M [5, 6].

We recall the following result for a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$ [20].

Theorem 3.1. *A submanifold M tangent to the structure vector field ξ is a pointwise θ-slant submanifold of an almost contact metric manifold $\tilde{M}$ if and only if*

$$T^2 X = \cos^2\theta(-X + \eta(X)\xi), \tag{3.1}$$

for the slant function θ defined on M.

The following relations are the consequence of Eq (3.1)

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{3.2}$$

$$g(FX, FY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$.

Another useful relation for pointwise slant submanifolds of $\tilde{M}$ comes from (2.1) and (3.1) given in [20] as follows:

$$(a) tFX = \sin^2\theta(-X + \eta(X)\xi), \qquad (3.4)$$
$$(b) fFX = -FTX,$$

for all $X \in \Gamma(TM)$.

Lemma 3.1. *[13] Let M be a pointwise θ-slant submanifold in Sasakian manifold $\tilde{M}$. Then, we have*

$$(\nabla_X T)Y = A_{FY}X + th(X, Y) + g(X, Y)\xi - \eta(Y)X, \qquad (3.5)$$

$$(\nabla_X F)Y = fh(X, Y) - h(X, TY), \qquad (3.6)$$

for any $X, Y \in \Gamma(TM)$.

Throughout this paper, we assume that the structure vector field ξ is tangent to M. Thus, if we denote by $\mathcal{D}$ the orthogonal distribution to ξ in TM, then we can take the orthogonal direct decomposition $TM = \mathcal{D} \oplus \langle\xi\rangle$.

From (2.5) and (2.7) with Gauss formula, we get that

$$\nabla_X \xi = -TX \text{ and } h(X, \xi) = -FX. \qquad (3.7)$$

Now, for each $X \in \Gamma(TM)$ with $\theta \neq 0$, we put

$$X^* = (\csc\theta)FX. \qquad (3.8)$$

Let β be a symmetric bilinear TM-valued form on M defined by

$$\beta(X, Y) = th(X, Y), \qquad (3.9)$$

for any $X, Y \in \Gamma(TM)$. From (3.4) and (3.7), the above equation takes the form

$$\beta(X, \xi) = (\sin^2\theta)(X - \eta(X)\xi). \qquad (3.10)$$

Using (2.7) and (3.8) in (3.9), we obtain

$$\phi\beta(X, Y) = T\beta(X, Y) + (\sin\theta)\beta^*(X, Y). \qquad (3.11)$$

Then by (2.8) and (3.9), we find

$$\phi h(X, Y) = \beta(X, Y) + \gamma^*(X, Y),$$

where γ be a symmetric bilinear $\mathcal{D}$-valued form on M defined by $\gamma^*(X, Y) = fh(X, Y)$. Applying the almost contact structure ϕ on the above equation, then using (2.1), (2.8) and (3.11), we get

$$-h(X, Y) = T\beta(X, Y) + (\sin\theta)\beta^*(X, Y) + t\gamma^*(X, Y) + f\gamma^*(X, Y),$$

as $\eta(h(X,Y))=0$. Equating the tangential and the normal components of the above relation, we obtain

$$T\beta(X,Y)=-t\gamma^*(X,Y),\quad -h(X,Y)=(\sin\theta)\beta^*(X,Y)+f\gamma^*(X,Y).$$

Using (3.1), (3.4) and (3.8), we conclude that

$$\gamma(X,Y)=(\csc\theta)T\beta(X,Y).$$

Also,

$$h(X,Y)=-(\csc\theta)\beta^*(X,Y), \tag{3.12}$$

Using (2.7) and (3.8), the above equation takes the form

$$h(X,Y)=(\csc^2\theta)(T\beta(X,Y)-\phi\beta(X,Y)). \tag{3.13}$$

Taking the inner product of (3.5) with $Z\in\Gamma(TM)$ and using (2.2), (2.7)–(2.9) and (3.9), we derive

$$g((\nabla_X T)Y,Z)=g(\beta(X,Y),Z)-g(\beta(X,Z),Y)+g(X,Y)\eta(Z)-g(X,Z)\eta(Y).$$

For $(n+1)$-dimensional pointwise θ-slant submanifold M of $\tilde{M}^{2m+1}(c)$, we derive the equation of Gauss and Codazzi of M in $\tilde{M}^{2m+1}(c)$ as follows:

From (2.6) and (2.7), we have

$$\begin{aligned}\tilde{R}(X,Y;Z,W)=&\frac{c+3}{4}\Big\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\Big\}\\&+\frac{c-1}{4}\Big\{\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\&+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)\\&+g(TX,W)g(TY,Z)-g(TX,Z)g(TY,W)\\&+2g(X,TY)g(TZ,W)\Big\}.\end{aligned}$$

From (2.10), (3.2) and (3.13), the above equation takes the form

$$\begin{aligned}R(X,Y;Z,W)=&\csc^2\theta\Big\{g(\beta(X,W),\beta(Y,Z))-g(\beta(X,Z),\beta(Y,W))\Big\}\\&+\frac{c+3}{4}\Big\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\Big\}\\&+\frac{c-1}{4}\Big\{\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\&+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)\\&+g(TX,W)g(TY,Z)-g(TX,Z)g(TY,W)\\&+2g(X,TY)g(TZ,W)\Big\},\end{aligned}$$

which gives us the Gauss equation of M in $\tilde{M}^{2m+1}(c)$.

Next, for the Codazzi equation if we take the normal parts of (2.6), we obtain

$$(\tilde{R}(X,Y)Z)^{\perp}=\frac{c-1}{4}\Big\{g(TY,Z)FX-g(TX,Z)FY+2g(X,TY)FZ\Big\}. \tag{3.14}$$

Furthermore, it follows from (3.8) and (3.12) that

$$\nabla^{\perp}_X(h(Y,Z)) = \nabla^{\perp}_X(-(\csc^2\theta)F\beta(Y,Z)),$$

which yields

$$\nabla^{\perp}_X(h(Y,Z)) = -(\csc^2\theta)\nabla^{\perp}_X F\beta(Y,Z) + 2(\csc^2\theta\cot\theta)X(\theta)F\beta(Y,Z).$$

Then by (2.15) and (3.6), the above equation takes the form

$$\nabla^{\perp}_X(h(Y,Z)) = -(\csc^2\theta)\Big[fh(X,\beta(Y,Z)) - h(X,T\beta(Y,Z)) + F(\nabla_X\beta(Y,Z)) - 2(\cot\theta)X(\theta)F\beta(Y,Z)\Big].$$

On the other hand, it also from (3.8) and (3.12) that

$$h(\nabla_X Y,Z) = -(\csc^2\theta)F\beta(\nabla_X Y,Z).$$

Similarly,

$$h(Y,\nabla_X Z) = -(\csc^2\theta)F\beta(Y,\nabla_X Z).$$

Substituting these relations into (2.13), we obtain

$$(\overline{\nabla}_X h)(Y,Z) = -(\csc^2\theta)\Big[fh(X,\beta(Y,Z)) - h(X,T\beta(Y,Z)) + F((\nabla_X\beta)(Y,Z)) - 2(\cot\theta)X(\theta)F\beta(Y,Z)\Big].$$

By (3.4), (3.8) and (3.12), we can write

$$\begin{aligned}(\overline{\nabla}_X h)(Y,Z) = &-(\csc^2\theta)\Big[(\csc^2\theta)FT\beta(X,\beta(Y,Z))\\ &+(\csc^2\theta)F\beta(X,T\beta(Y,Z)) + F((\nabla_X\beta)(Y,Z))\\ &-2(\cot\theta)X(\theta)F\beta(Y,Z)\Big].\end{aligned} \tag{3.15}$$

Similarly, we have

$$\begin{aligned}(\overline{\nabla}_Y h)(X,Z) = &-(\csc^2\theta)\Big[(\csc^2\theta)FT\beta(Y,\beta(X,Z))\\ &+(\csc^2\theta)F\beta(Y,T\beta(X,Z)) + F((\nabla_Y\beta)(X,Z))\\ &-2(\cot\theta)Y(\theta)F\beta(X,Z)\Big].\end{aligned} \tag{3.16}$$

Finally, applying (3.14)–(3.16) in Codazzi equation, we get

$$\begin{aligned}&(\nabla_X\beta)(Y,Z) - g(\beta(Y,Z),TX)\xi - 2(\cot\theta)X(\theta)\beta(Y,Z)\\ &\quad+(\csc^2\theta)\{T\beta(X,\beta(Y,Z)) + \beta(X,T\beta(Y,Z))\}\\ &\quad+\frac{c-1}{4}(\sin^2\theta)\{g(X,TY)(Z-\eta(Z)\xi) + g(X,TZ)(Y-\eta(Y)\xi))\}\\ &=(\nabla_Y\beta)(X,Z) - g(\beta(X,Z),TY)\xi - 2(\cot\theta)Y(\theta)\beta(X,Z)\\ &\quad+(\csc^2\theta)\{T\beta(Y,\beta(X,Z)) + \beta(Y,T\beta(X,Z))\}\\ &\quad+\frac{c-1}{4}(\sin^2\theta)\{\,g(Y,TX)(Z-\eta(Z)\xi) + g(Y,TZ)(X-\eta(X)\xi)\}.\end{aligned}$$

4. Existence and uniqueness theorems

In this section we present the detailed proofs of the existence and uniqueness theorems for pointwise slant immersions into a Sasakian space form.

Theorem 4.1. *(Existence Theorem) Let M be the connected Riemannian manifold of dimension $(n+1)$ equipped with metric tensor g. Suppose that c is a constant and there exists a smooth function θ on M satisfying $0 < \theta \leq \frac{\pi}{2}$, an endomorphism T of the tangent bundle TM, a unit global vector field ξ and a symmetric bilinear TM-valued form β on M such that the following conditions are satisfied:*

$$T(\xi) = 0, \quad g(\beta(X, Y), \xi) = 0, \quad \nabla_X \xi = -TX, \tag{4.1}$$

$$T^2 X = (\cos^2 \theta)(-X + \eta(X)\xi), \tag{4.2}$$

$$g(TX, Y) = -g(X, TY), \tag{4.3}$$

$$\beta(X, \xi) = (\sin^2 \theta)(X - \eta(X)\xi), \tag{4.4}$$

$$g((\nabla_X T)Y, Z) = g(\beta(X, Y), Z) - g(\beta(X, Z), Y) + g(X, Y)\eta(Z) - g(X, Z)\eta(Y), \tag{4.5}$$

$$\begin{aligned} R(X, Y; Z, W) = {} & (\csc^2 \theta)\Big\{g(\beta(X, W), \beta(Y, Z)) - g(\beta(X, Z), \beta(Y, W))\Big\} \\ & + \frac{c+3}{4}\Big\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\Big\} \\ & + \frac{c-1}{4}\Big\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) \\ & + g(TX, W)g(TY, Z) - g(TX, Z)g(TY, W) \\ & + 2g(X, TY)g(TZ, W)\Big\}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} & (\nabla_X \beta)(Y, Z) - g(\beta(Y, Z), TX)\xi - 2(\cot \theta)X(\theta)\beta(Y, Z) \\ & \quad + (\csc^2 \theta)\Big\{T\beta(X, \beta(Y, Z)) + \beta(X, T\beta(Y, Z))\Big\} \\ & \quad + \frac{c-1}{4}(\sin^2 \theta)\Big\{g(X, TY)(Z - \eta(Z)\xi) + g(X, TZ)(Y - \eta(Y)\xi)\Big\} \\ & = (\nabla_Y \beta)(X, Z) - g(\beta(X, Z), TY)\xi - 2(\cot \theta)Y(\theta)\beta(X, Z) \\ & \quad + (\csc^2 \theta)\Big\{T\beta(Y, \beta(X, Z)) + \beta(Y, T\beta(X, Z))\Big\} \\ & \quad + \frac{c-1}{4}(\sin^2 \theta)\Big\{ g(Y, TX)(Z - \eta(Z)\xi) + g(Y, TZ)(X - \eta(X)\xi)\Big\}. \end{aligned} \tag{4.7}$$

for $X, Y, Z \in \Gamma(TM)$, where η be the dual 1-form of ξ. Then there exists a pointwise θ-slant isometric immersion of M into a Sasakian space form $\tilde{M}^{2m+1}(c)$ such that the second fundamental form h of M is given by

$$h(X, Y) = (\csc^2 \theta)(T\beta(X, Y) - \phi\beta(X, Y)). \tag{4.8}$$

Proof. Assume that $c,\ \theta,\ \xi,\ T$ and M satisfy the given conditions. Suppose that $TM\oplus\mathcal{D}$ be a Whitney sum. For each $X\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$ we simply denote $(X,0)$ by X, $(0,Z)$ by Z^* and $\hat{\xi}=(\xi,0)$ with ξ.

Let $\hat{g}$ be the product metric on $TM\oplus\mathcal{D}$. So, if we set $\hat{\eta}$ as the dual 1-form of $\hat{\xi}$, then $\hat{\eta}(X,Z)=\eta(X)$, for any $X\in TM$ and $Z\in\mathcal{D}$.

We define the endomorphism $\hat{\phi}$ on $TM\oplus\mathcal{D}$ by

$$\hat{\phi}(X,0)=(TX,(\sin\theta)(X-\eta(X)\xi)),\quad \hat{\phi}(0,Z)=(-(\sin\theta)Z,-TZ), \tag{4.9}$$

for each $X\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$. Then, we find $\hat{\phi}^2(X,0)=-(X,0)+\hat{\eta}(X,0)\hat{\xi}$. Also, $\hat{\phi}^2(0,Z)=-(0,Z)$. Hence, $\hat{\phi}^2(X,Z)=-(X,Z)+\hat{\eta}(X,Z)\hat{\xi}$ for any $X\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$. From (4.2), (4.3) and (4.9) it is directly to check that $(\hat{\phi},\hat{\eta},\hat{\xi},\hat{g})$ is an almost contact metric structure on $TM\oplus\mathcal{D}$.

Now, we can define an endomorphism A on TM, a $(\mathcal{D})^*$-valued symmetric bilinear form h on TM and a metric connection $\nabla^\perp$ of the vector bundle $(\mathcal{D})^*$ over M by

$$A_{Z^*}X=(\csc\theta)\{(\nabla_X T)Z-\beta(X,Z)-g(X,Z)\xi\}, \tag{4.10}$$

$$h(X,Y)=-(\csc\theta)\beta^*(X,Y), \tag{4.11}$$

$$\nabla^\perp_X Z^*=(\nabla_X Z-\eta(\nabla_X Z)\xi)^*-(\cot\theta)X(\theta)Z^*+(\csc^2\theta)\{(T\beta(X,Z))^*+\beta^*(X,TZ)\}, \tag{4.12}$$

for $X,Y\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$.

Denote by $\hat{\nabla}$ the canonical connection on $TM\oplus\mathcal{D}$ induced from Eqs (4.9)–(4.12). Using (4.1), (4.2), (4.4) and (4.9), we get

$$\begin{aligned}&(\hat{\nabla}_{(X,0)}\hat{\phi})(Y,0)=\hat{g}((X,0),(Y,0))\hat{\xi}-\hat{\eta}(Y,0)(X,0),\\&(\hat{\nabla}_{(X,0)}\hat{\phi})(0,Z)=0,\end{aligned}$$

for any $X,Y\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$.

Let $R^\perp$ be the curvature tensor associated with the connection $\nabla^\perp$ on $(\mathcal{D})^*$, which gives by

$$R^\perp(X,Y)Z^*=\nabla^\perp_X\nabla^\perp_Y Z^*-\nabla^\perp_Y\nabla^\perp_X Z^*-\nabla^\perp_{[X,Y]}Z^*,$$

for any $X,Y\in\Gamma(TM)$ and $Z\in\Gamma(\mathcal{D})$. Then by (4.12), we have

$$\begin{aligned}R^\perp(X,Y)Z^*=&\ \nabla^\perp_X[(\nabla_Y Z-\eta(\nabla_Y Z)\xi)^*-(\cot\theta)Y(\theta)Z^*\\&+\csc^2\theta\{(T\beta(Y,Z))^*+\beta^*(Y,TZ)\}]\\&-\nabla^\perp_Y[(\nabla_X Z-\eta(\nabla_X Z)\xi)^*-(\cot\theta)X(\theta)Z^*\\&+\csc^2\theta\{(T\beta(X,Z))^*+\beta^*(X,TZ)\}]\\&-(\nabla_{[X,Y]}Z-\eta(\nabla_{[X,Y]}Z)\xi)^*+(\cot\theta)[X,Y](\theta)Z^*\\&-(\csc^2\theta)\{(T\beta([X,Y],Z))^*+\beta^*([X,Y],TZ)\}.\end{aligned}$$

Using (2.14), (4.1), (4.3), (4.7) and (4.12), we simplify

$$R^\perp(X,Y)Z^*=(\csc^2\theta)[Y(\theta)-X(\theta)]Z^*+(R(X,Y)Z-\eta(R(X,Y)Z)\xi)^*$$

$$
\begin{aligned}
&+\Big\{\frac{c-1}{4}T[g(Y,TZ)X-g(X,TZ)Y-2g(X,TY)Z]\\
&+\frac{c-1}{4}[g(Y,T^2Z)(X-\eta(X)\xi)-g(X,T^2Z)(Y-\eta(Y)\xi)\\
&-2g(X,TY)TZ] \qquad (4.13)\\
&+\csc^2\theta[(\tilde{\nabla}_X T)\beta(Y,Z)-(\nabla_Y T)\beta(X,Z)-\eta(\nabla_X T\beta(Y,Z))\xi\\
&+\eta(\nabla_Y T\beta(X,Z))\xi-\beta(X,(\nabla_Y T)Z)+\beta(Y,\nabla_X T)Z)]\Big\}. \qquad (4.14)
\end{aligned}
$$

Moreover, from (4.1), (4.5), (4.10) and (4.11), we derive

$$
\begin{aligned}
g([A_{Z^*},A_{W^*}]X,Y)=\csc^2\theta\Big\{&g((\nabla_X T)W,(\nabla_Y T)Z)-g((\nabla_X T)Z,(\nabla_Y T)W)\\
&+g((\nabla_X T)Z,\beta(Y,W))+g((\nabla_Y T)W,\beta(X,Z))\\
&-g((\nabla_X T)W,\beta(Y,Z))-g((\nabla_Y T)Z,\beta(X,W)) \qquad (4.15)\\
&+g(\beta(X,W),\beta(Y,Z))-g(\beta(X,Z),\beta(Y,W))\\
&+(1-2\cos^2\theta)(g(X,W)g(Y,Z)-g(X,Z)g(Y,W))\Big\}.
\end{aligned}
$$

Also, using (4.3), we can write

$$g(\beta(Y,Z),TW)+g(T\beta(Y,Z),W)=0.$$

Taking the covariant derivative of the above equation with respect to X and using (4.3), we obtain

$$g(\beta(Y,Z),(\nabla_X T)W)+g((\nabla_X T)\beta(Y,Z),W)=0.$$

Furthermore, from (4.5), we find

$$g((\nabla_X T)Z,(\nabla_Y T)W)=g((\nabla_X T)Z,\beta(Y,W))-g(\beta(Y,(\nabla_X T)Z),W)+\cos^2\theta g(X,Z)g(Y,W).$$

Substituting the pervious relations in (4.13) and (4.15) with a direct computation, we arrive at

$$
\begin{aligned}
&g(R^\perp(X,Y)Z^*,W^*)-g([A_{Z^*},A_{W^*}]X,Y)\\
&\quad=\frac{c-1}{4}\Big[(\sin^2\theta)\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}-2g(X,TY)g(TZ,W)\Big]\\
&\quad+(\csc^2\theta)[Y(\theta)-X(\theta)]\,g(Z,W).
\end{aligned}
$$

Notice that the above equation together with (2.6), (4.2) and (4.3) implies that $(M,A,\nabla^\perp)$ satisfies the Ricci equation of a $(n+1)$-dimensional pointwise θ-slant submanifold of the Sasakian space form $\tilde{M}^{2m+1}(c)$, while (4.6) and (4.7) mean that (M,h) satisfies the equations of Gauss and Codazzi, respectively. Therefore, we have a vector bundle $TM\oplus\mathcal{D}$ over M equipped with product metric g, the second fundamental form h, the shape operator A, and the connections $\nabla^\perp$ and $\hat{\nabla}$ which satisfy the structure equations of a $(n+1)$-dimensional pointwise θ-slant submanifold of $\tilde{M}^{2m+1}(c)$. Consequently, by applying Theorem 1 of [10] we conclude that there exists a pintwise θ-slant isometric immersion from M into $\tilde{M}^{2m+1}(c)$ whose second fundamental form is given by $h(X,Y)=\csc^2\theta(T\beta(X,Y)-\phi\beta(X,Y))$. □

The next result provides the sufficient conditions to have the uniqueness property for pointwise slant immersions.

Theorem 4.2. *(Uniqueness Theorem) Let $\tilde{M}^{2m+1}(c)$ be a Sasakian space form and M be a connected Riemannian manifold of dimension $(n + 1)$. Let $x^1, x^2 : M \to M^{2m+1}(c)$ be two pointwise θ-slant isometric immersions such that $0 < \theta \leq \frac{\pi}{2}$. Suppose that h_1 and h_2 be the second fundamental forms of x^1 and x^2, respectively. Assume that there exists a vector field $\hat{\xi}$ on M satisfies $x^i_{*p}(\hat{\xi}_p) = \xi_{x^i(p)}$, for any $p \in M$ and any $i = 1, 2$. Suppose that*

$$g(h_1(X, Y), \phi x^1_* Z) = g(h_2(X, Y), \phi x^2_* Z), \tag{4.16}$$

for all $X, Y, Z \in \Gamma(TM)$. In addition, if we consider that at least one of the following conditions is satisfied:

(i) $\theta = \frac{\pi}{2}$,
(ii) *there exists a point p in M such that $T_1 = T_2$,*
(iii) $c \neq 0$,

then $T_1 = T_2$ and there exists an isometry α of $\tilde{M}^{2m+1}(c)$ such that $x^1 = \alpha(x^2)$.

Proof. The proof of this Theorem is similar of the Uniqueness Theorem in the complex space forms (see [1, 8]) by taking $\hat{\xi}$ in the orthonormal frame tangent to M. □

5. Some examples of pointwise slant immersions

In this section we give some examples of general pointwise slant immersions in an almost contact metric manifolds.

Example 5.1. *Consider the Euclidean 5-space $\mathbb{R}^5$ with the cartesian coordinates (x_1, x_2, y_1, y_2, t) and the almost contact structure*

$$\phi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \;\; \phi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \;\; \phi\Big(\frac{\partial}{\partial t}\Big) = 0, \;\; 1 \leq i, j \leq 2,$$

such that $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and g be the standard Euclidean metric on $\mathbb{R}^5$. It is easy to show $\mathbb{R}^5$ is an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g). Let M be a submanifold of $\mathbb{R}^5$ given by the immersion ψ as follows:

$$\psi(u,\ v,\ t) = (2v,\ \cos u,\ \sin u,\ 0,\ t),$$

where u, v are non vanishing real valued functions on M. Then the tangent space of M is spanned by the following vectors

$$X_1 = -\sin u\,\frac{\partial}{\partial x_2} + \cos u\,\frac{\partial}{\partial y_1}, \;\; X_2 = 2\frac{\partial}{\partial x_1}, \;\; X_3 = \frac{\partial}{\partial t}.$$

Then,

$$\phi X_1 = \sin u\,\frac{\partial}{\partial y_2} + \cos u\,\frac{\partial}{\partial x_1}, \;\; \phi X_2 = -2\frac{\partial}{\partial y_1}, \;\; \phi X_3 = 0.$$

Hence, the slant angle is given by

$$\cos\theta = \frac{g(X_2, \phi X_1)}{\|X_2\|\|\phi X_1\|} = \frac{2\cos u}{2} = \cos u.$$

So, $\theta = u$ is a slant function and from it ψ is a pointwise slant immersion with pointwise slant distribution $\mathcal{D}^\theta = \mathrm{Span}\{X_1,\ X_2\}$ and $TM = \mathcal{D}^\theta \oplus \langle\xi\rangle$.

Example 5.2. *Let $\mathbb{R}^7$ be the Euclidean 7-space with the cartesian coordinates (x_i, y_j, t) and the almost contact structure*

$$\phi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i},\ \ \phi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j},\ \ \phi\Big(\frac{\partial}{\partial t}\Big) = 0,\ \ 1 \le i, j \le 3,$$

such that $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and g be the standard Euclidean metric on $\mathbb{R}^7$. Consider a submanifold M of $\mathbb{R}^7$ given by the following immersion:

$$\psi(u,\ v,\ t) = (u\cos v,\ v\cos u,\ u,\ u\sin v,\ v\sin u,\ kv,\ t),$$

for any u, v non vanishing real valued functions and $k \neq 0$ be a real number. Then, the tangent space of M is spanned by the following vectors

$$\begin{aligned}
X_1 &= \cos v\frac{\partial}{\partial x_1} - v\sin u\,\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \sin v\,\frac{\partial}{\partial y_1} + v\cos u\,\frac{\partial}{\partial y_2},\\
X_2 &= -u\sin v\,\frac{\partial}{\partial x_1} + \cos u\,\frac{\partial}{\partial x_2} + u\cos v\,\frac{\partial}{\partial y_1} + \sin u\,\frac{\partial}{\partial y_2} + k\frac{\partial}{\partial y_3},\\
X_3 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Clearly, we obtain

$$\begin{aligned}
\phi X_1 &= -\cos v\frac{\partial}{\partial y_1} + v\sin u\,\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} + \sin v\,\frac{\partial}{\partial x_1} + v\cos u\,\frac{\partial}{\partial x_2},\\
\phi X_2 &= u\sin v\,\frac{\partial}{\partial y_1} - \cos u\,\frac{\partial}{\partial y_2} + u\cos v\,\frac{\partial}{\partial x_1} + \sin u\,\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3},\\
\phi X_3 &= 0.
\end{aligned}$$

Then, we find that the slant angle satisfies $\theta = \cos^{-1}\Big(\frac{u-v+k}{\sqrt{v^2+2}\sqrt{u^2+k^2+1}}\Big)$, since u, v $(u \neq v)$ are non vanishing real valued functions on M, hence the slant angle is none constant. Thus, M is a pointwise slant submanifold of $\mathbb{R}^7$ with the slant function θ and pointwise slant distribution $\mathcal{D}^\theta = \mathrm{Span}\{X_1,\ X_2\}$, where $TM = \mathcal{D}^\theta \oplus \langle\xi\rangle$.

Example 5.3. A submanifold M of $\mathbb{R}^{13}$ given by the following immersion

$$\psi(u,\ v,\ t) = (e^u,\ 2u,\ \sin v,\ u,\ u-v,\ \cos u,\ -e^u,\ v,\ \cos v,\ -v,\ u+v,\ \sin u,\ t),$$

for non vanishing u, v. We set the the tangent space TM of M is spanned by the following vectors

$$X_1 = e^u\frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \sin u\frac{\partial}{\partial x_6} - e^u\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_5} + \cos u\,\frac{\partial}{\partial y_6},$$

$$X_2 = \cos v \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} + \frac{\partial}{\partial y_2} - \sin v \frac{\partial}{\partial y_3} - \frac{\partial}{\partial y_4} + \frac{\partial}{\partial y_5},$$
$$X_3 = \frac{\partial}{\partial t}.$$

Clearly, we have

$$\phi X_1 = -e^u \frac{\partial}{\partial y_1} - 2\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_4} - \frac{\partial}{\partial y_5} + \sin u \frac{\partial}{\partial y_6} - e^u \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} + \cos u \frac{\partial}{\partial x_6},$$
$$\phi X_2 = -\cos v \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_5} + \frac{\partial}{\partial x_2} - \sin v \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},$$
$$\phi X_3 = 0.$$

It is easy to see that M is a pointwise slant submanifold with slant function $\theta = \cos^{-1}\left(\frac{3}{\sqrt{2}\sqrt{5}}(e^{2u}+4)^{\frac{-1}{2}}\right)$ and $\mathcal{D}^\theta = \mathrm{Span}\{X_1, X_2\}$, where $TM = \mathcal{D}^\theta \oplus \langle \xi \rangle$.

Conflict of interest

The authors declare no conflict of interest.

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