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Research article

Looking at Okuda's artwork through GeoGebra: A Citizen Science experience

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Abstract: In this paper, we describe an experience to test the predominant presence of Delaunay triangulations in the artwork of Okuda, a quite famous, young, contemporary Spanish artist. We addressed this task involving, as a citizen science activity in a STEAM (Science, Technology, Engineering, Art, Mathematics) education context, several hundreds of students (of different kinds: secondary education, university undergraduates, in particular, following teacher training degrees). Each student was asked to select an Okuda archive and, with the concourse of a dynamic geometry program provided with some computational geometry commands, to measure the ratio of coincident triangles in Delaunay's and artist's triangulations, over an ample region of the chosen artwork. The results show a very large percentage of coincidence ratios. We conclude with some reflections about how to interpret this fact, and about the potential role of future, enhanced, dynamic geometry systems to automatically address similar issues, concerning mathematical properties of figures from the real world.

Keywords: Delaunay triangulation; GeoGebra; Citizen Science; STEAM education; automated geometer; Okuda; mathematics and arts
Mathematics Subject Classification: 00A66, 68U05, 97U70

1. Introduction

Okuda San Miguel (a.k.a. Okuda) is a young, internationally reputed, artist from Spain. See [1, 2] for information about his biography, curriculum, worldwide presence and for a large set of archives containing images of Okuda's artwork, from 2007 to 2023. A quick glimpse at this collection already shows the persistent, manifold and relevant presence of triangles in its design, making it a suitable context for mathematical thinking.

A more detailed observation of many of Okuda's works, have led us to formulate a mathematical conjecture about some particular geometric properties that seem to be often present in Okuda's triangulations. See [3, 4], where this conjecture is just intuitively formulated and checked in a handful of cases. We do not know of any previous or on-going work in this direction (Okuda vs. Delaunay). Thus, we have considered it could be interesting –both from the artistic, as well as from the scientific and cultural context– to test the truth of this conjecture, in a large number of instances. This is the goal of our paper.

To achieve this objective we have designed a sort of joint STEAM (Science, Technology, Engineering, Arts and Mathematics) education / "citizen science" (c.f [5, 6] for some references regarding these two well–known concepts) research project, involving hundreds of students of various subjects, schools, and educational grades. After delivering a short (and light, in many cases, due to the mathematical level of the students) lecture on the involved concepts and tools, we asked the students to select some Okuda archive of their choice, from those available through the internet web of the artist [2], and to launch in their computers, laptops, etc. the dynamic mathematics program GeoGebra (`www.geogebra.org`).

Then, the students had to place the chosen image over a GeoGebra screen and to start the following process: first, to manually introduce in GeoGebra a collection of points coincident with the vertices of the triangles made by the artist on a (as large as possible) area of the figure. Next, to run in GeoGebra the Delaunay diagram (see [7, 8]) command of the selected set of points. Finally, to count manually and to report about the percentage of Okuda triangles in this region that coincide with those of the Delaunay triangulation for the same set of vertices.

The description of this experience, and the reflection on its output and conclusions, are the goals of this paper. Thus next section presents a short introduction to the concept of Delaunay triangulation, as well as to some key properties of such triangulations that we will need for the discussion of the obtained results. Then we will make a summary description of some of the basic elements of our experience (Citizen Science, STEAM education, GeoGebra), that will be described in detail in Section 3. Section 4 presents the obtained results and their analysis. Finally, Section 5 reflects on the experience from a broader perspective, related to some potential lines of future development of dynamic mathematics tools.

2. Delaunay triangulation

In general, in the Euclidean space $\mathbb{R}^n$, a triangulation of a set of points $\mathcal{P} = \{p_1, p_2, p_3, \dots\}$ is a simplicial complex that covers the convex hull of $\mathcal{P}$, and whose vertices belong to $\mathcal{P}$. In the plane $\mathbb{R}^2$, a triangulation consist of triangles, points $\mathcal{P}$, and edges. See Figure 1 for an illustration. There are many important and well-known triangulations, here we are going to address Delaunay triangulations which were introduced by Boris Delaunay in [7].

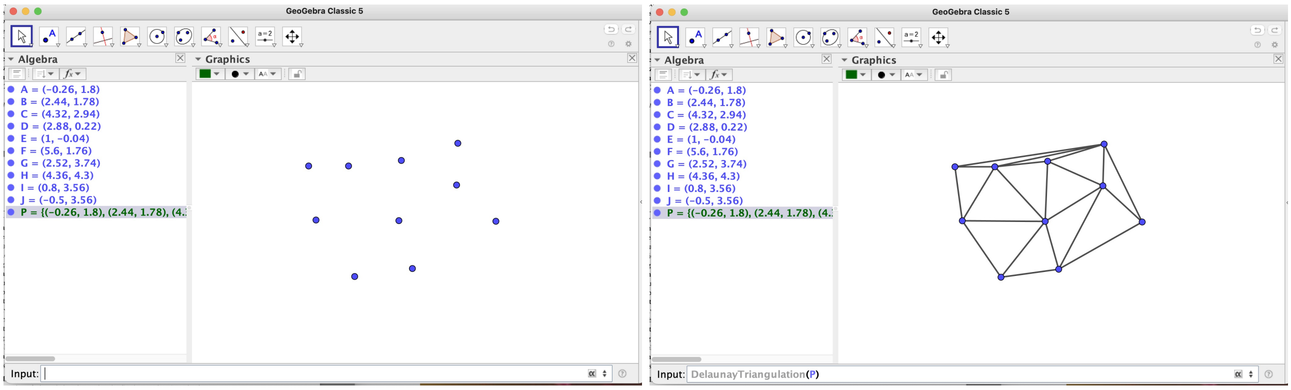

Figure 1. A set $\mathcal{P} = \{A, B, C, D, E, F\}$ of points in the plane (left figure) and the Delaunay triangulation of $\mathcal{P}$ (right figure).

Delaunay triangulation is a type of triangulation that satisfies the following criterion: the circumcircle of every triangle in the triangulation contains no other vertices from the set being triangulated. The Delaunay triangulation has many useful properties, including that it maximizes the minimum angle of all the triangles, minimizes the number of triangles with angles greater than 90 degrees, and minimizes the size difference between adjacent triangles. Moreover, the shortest path between two vertices, along Delaunay edges, is not longer than $1.998 \cdot d$, where d is the Euclidean distance between the two vertices (see [9]). In other words, the Delaunay triangulation results in more "regular" triangles, that are less likely to be skinny or acute.

This kind of triangulation is also closely related to Voronoi diagrams (see, e.g. [8], for some basic concepts in Computational Geometry), i.e., the Delaunay triangulation is the dual graph of the Voronoi diagram, where the vertices of the Voronoi diagram correspond to the triangles in the Delaunay triangulation, and the edges of the Voronoi diagram correspond to the edges of the triangles in the Delaunay triangulation.

The Delaunay triangulation of a set of points in the plane is not always unique, however, it is guaranteed to be **unique** if the set of points are in general position, that is, if no four points in the set are cocircular (i.e., lie on a common circle) with empty interior, and no three points are collinear ([10, Chapter 7.2.], [11, 12]). See [13, 14] for more properties and recent applications, close to our context, of Delaunay diagrams.

Given a set of points $\mathcal{P}$, all the triangulations of $\mathcal{P}$ have the same number of triangles, say $2n-2-k$, and the same number of edges, say $3n-3-k$. Here, we denote by $n = |\mathcal{P}|$ and $k = |\mathrm{CH}(\mathcal{P})|$, where CH stands for the convex hull. A natural question arising from this data is the number of different triangulations that can be constructed from a set of points $\mathcal{P}$, so we can have an idea of the "oddness" of Delaunay triangulation.

There are only some partial answers to this question depending on the distribution of the set of points ([15]):

- If the points are in convex position (i.e., the $n = |\mathcal{P}|$ points are the vertices of a convex n-gon, the number of triangulations is equal to the Catalan number C_{n-2}, that is,

$$C_{n-2} := \frac{1}{n-1}\binom{2n-4}{n-2} \in \Theta(4^n n^{-3/2}).$$

- Different configurations of points $\mathcal{P}$ in general position give different numbers of triangulations. It is known that there are configurations for which the number of triangulations is of the order 8.5^n.
- A general upper bound for the number of triangulations is of order 30^n.

We conclude that, given a certain "generic" collection of points, and a random triangulation, the probability that it agrees with the corresponding Delaunay triangulation is very low.

3. Educational context

3.1. STEAM education and Citizen Science

There is a general consensus on the crucial importance of scientific education for the development of today's society. The so called Rocard report ([16]) in 2007 stresses the need to provide all citizens with scientific literacy and positive attitudes towards science, equipping them with the necessary competences to live and work in society. These skills will provide future generations with the necessary tools to develop critical thinking and scientific reasoning that will enable them to make informed decisions and be free citizens.

The report proposes the inclusion of Inquiry-Based Science Education (IBSE) approaches in schools as a method of teaching science to increase interest and positive experiences with science at early ages. They are methods based on curiosity and observations, followed by problem solving and experimentation. Using critical thinking and reflection, students are able to make inferences from the data collected, providing boys and girls with the opportunity to develop a wide range of complementary skills such as teamwork, written and oral expression, open-ended problem solving, and other interdisciplinary skills. All this accompanied by the necessary training of teachers in these methods and the development of teacher networks ([16]).

This ideas gave rise to the so-called STEM education (Science, Technology, Engineering and Mathematics), an integrated teaching and learning approach based on interdisciplinarity and applicability of concepts in Science and Mathematics in connection with real contexts with the aim of developing basic competences in these areas. While STEM education traditionally focuses on convergent methods and skills like problem-solving [6], a further step includes Arts into STEAM education, incorporating divergent skills and becoming a more personal and engaging approach for students ([17]). It encourages problem-posing skills, prompts student's own interpretations and visualizations, stimulates the exploration of possibilities, therefore producing alternative vehicles of understanding ([18]). The integration of arts and science in STEAM projects produces a balance between analytic and creative thinking, becoming an opportunity to motivate learners and foster innovative solutions ([19]).

This approach supports the general understanding about the role and benefits of interdisciplinary approaches for teaching and learning mathematics. Indeed, the National Council of Teachers in Mathematics [20] states in the Connections Standard that "school mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics, from other subject areas or disciplines as well as to students “daily lives” ([20, p. 65–66]).

The project we have developed is part of a Citizen Science project. As defined in [5, p.8], Citizen Science refers to the general public engagement in scientific research activities when citizens actively contribute to science either with their intellectual effort or surrounding knowledge or with their tools and resources. We have worked in a Contributive Project, where the project and the hypotheses are posed by the researchers, and the citizens contributed with the recollection of data.

Let us mention that some of our school-experiences have been done in the context of activities programmed for the dissemination of Science, by the Unidad de Cultura Científica of the Universidad de Cantabria `https://web.unican.es/unidades/cultura-cientifica`.

3.2. GeoGebra and artistic analysis

The interrelation between mathematics and arts is well known: old, deep, relevant. See [21] for a recent book collecting contributions in this regard, as described in its title: MACAS = mathematics and its connections to the arts and sciences. On the other hand, GeoGebra is a free, dynamic geometry and computer algebra program, available in computers, laptops, tablets, smartphones, etc., with over 100 million users all over the world, mostly in the educational system. Thus, algorithms implemented in GeoGebra are prone to get a large potential impact.

Dynamic Geometry systems, and in particular GeoGebra, have been used in connection with Arts as a versatile tool to both highlight mathematical features of artworks and to enhance artistic creativity. Different examples in both directions can be found in the contributions of the Facebook group `https://www.facebook.com/groups/GeoGebraSTEAM/` devoted to GeoGebra Arts & STEAM, moderated by Prof. Zsolt Lavicza, Professor in STEAM Education Research Methods, Johannes Kepler University, Austria. Or in the “Análisis de obras de Arte” of the Spanish “geometriadinamica” group `https://www.geometriadinamica.es`, [22], for some Spanish examples.

Concerning our specific context, related to the detection of computational geometry patterns in artistic works, we can refer to [23], that used the GeoGebra’s dynamic colors to describe Voronoi diagrams of certain focal zones in paintings, and to [3], showing preliminary evidences of Delaunay triangulations in the work of artists such as Curra Rueda (`http://currarueda.com`), and Okuda.

The interrelation GeoGebra/Arts, from the specific perspective we would like to highlight here, goes well beyond the verification of the Okuda/Delaunay conjecture. Indeed, we consider it just as a first step towards the achievement of an ambitious project, involving GeoGebra and Arts, that has been already announced in [24]. In what follows we will summarily describe it.

Let us recall that GeoGebra, through the fork version GeoGebra Discovery (freely available at `https://github.com/kovzol/geogebra/releases`) has already automated reasoning tools, allowing the discovery –without other human intervention than that of asking GeoGebra to start exploring a certain figure– of elementary geometric properties in a geometric construction: perpendicularity, parallelism, co-linearity, co-circularity, etc. (see [25, 26]). Let us remark that the output of such explorations is mathematically sound (i.e. mathematically rigorous, assuming the validity of the construction steps, as it is based on symbolic computation, not on numerical or probabilistic methods). Now, our examination of triangulation properties over Okuda’s artwork, can be considered as a toy example of a future attempt to extend the performance of GeoGebra Discovery to the automated exploration of other, non-elementary, geometric properties, such as those pertaining to the Computational Geometry realm: Voronoi diagrams, Delaunay triangulations, etc. Yet, currently, as we will describe in the next section, such exploration has to be done with collaboration of humans: for instance, in the framework of citizen science activities.

4. Description of the experience

As already mentioned, our immediate goal is to verify the truth, in many instances, of a conjecture about the mathematical properties that seem to be present in Okuda's work: *Okuda's triangulation is Delaunay's triangulation up to artistic and anthropological minor exceptions*.

To achieve this objective, we have designed a STEAM "citizen science" research project, involving students from various subjects, schools and University educational degrees for pre-service teachers. In addition to being part of a citizen science project, students had the opportunity to perceive, in an practical, hands-on, way, a particular case of the relationship between mathematics and art. Moreover, as a consequence of the whole activity, prospective teachers may incorporate this type of citizen science projects in their future professional practice, as a suitable learning tool.

The project started in 2020, and in total 375 Secondary School students and 203 University students have participated in it. The Secondary students had ages between 12 and 15 coming from Castroverde, Torrevelo, Peñalabra and Miguel Bravo Schools, all of them in the surroundings of Santander; University students were pursuing the Primary Education Degree in different level at three universities in Madrid: Universidad Autónoma de Madrid, Universidad Rey Juan Carlos and Centro Universitario La Salle.

To verify the conjecture, the data collected by the participants of this project were: Name of the work, number of Okuda's triangles, number of matching Delaunay triangles, and the coincidence ratio (in percentage).

The steps of the process were the following:

1. Delivering a short lecture on the involved concepts and tools.
2. Students' selection of an Okuda work of their choice.
3. Students' GeoGebra analysis of the work.
4. Students' collection of the data.
5. Revision of the data by the researchers.

In Step 1, the researchers started with a brief presentation of the work of the artist Okuda, showing his characteristic use of colored triangles. Then they introduced the notion of a triangulation of a set of points and the main properties of the particular case of the Delaunay triangulation. The researchers raised the question "*is it true that Okuda follows Delaunay?*" and they propose to the students to try to answer it by participating in a Citizen Science project. Next, as a digital tool to collect the data, the researchers show how GeoGebra can construct the Delaunay triangulation from a set of points placed over an image (see Step 3 for more details).

In Step 2, the students selected (at their own choice, without any previous suggestion from the teachers, except that of trying not to repeat the same artwork as the one selected by a schoolmate sitting nearby) an image from the artist's web ([2]) and placed it over a GeoGebra window. Most of the students didn't know the software, so they used GeoGebra Classic 6 in its on-line version (`https://www.geogebra.org/classic`).

Then, in Step 3, students were asked to manually introduce in GeoGebra a collection of points coincident with the vertices of the triangles made by the artist on a (as large as possible) area of the figure. Yet, in order to avoid unfair comparisons, the students were advised to choose an area that was mainly a collection of triangles, i.e. avoiding zones including non-triangular objects, such eyes, ears, etc. Also, it was suggested that the selected region was, as much as possible, convex, since Delaunay triangulates the convex hull of the given set of vertices.

Once they had chosen the collection of points, students ran the command *DelaunayTriangulation(⟨List of Points⟩)* in GeoGebra to obtain the Delaunay diagram of the given set of points. Figure 2 shows an example of the result obtained by an student after following Steps 2–3.

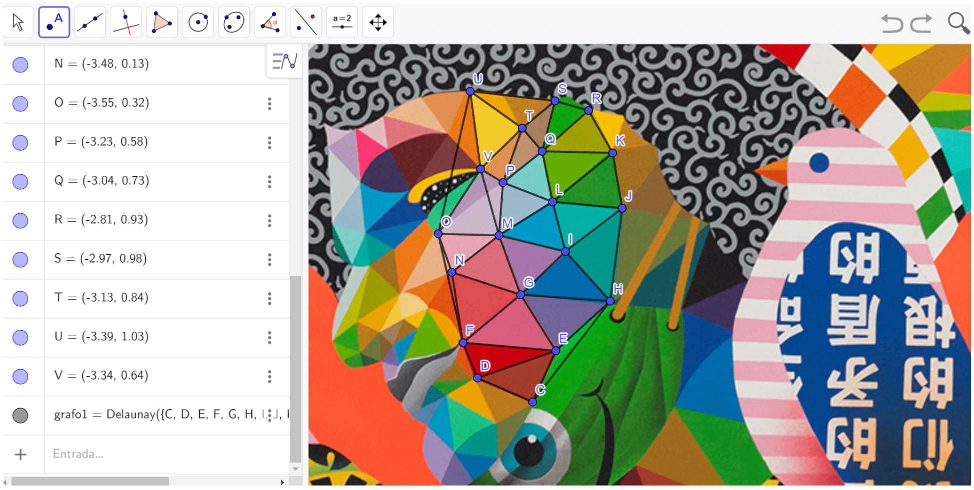

Figure 2. Image of the result obtained by a student with the work *Liberty Statue I* (2021).

In Step 4, students had to calculate the number of triangles in the Delaunay triangulation, the number of coincidences with Okuda's triangles and finally to establish the percentage (ratio of coincidence) of Okuda triangles in this region that coincide with those of the obtained Delaunay triangulation.

Steps 1–4 were carried out in a single session with a duration between 1 and 2 hours depending on the group.

After the sessions, the researchers found several mistakes made by the students that lead to a revision of some of the data (in general), and in particular, of some cases with very low ratios. The main mistakes found were the following:

- Errors in the selection of points, causing misplaced triangles.
- Not inclusion of some of the selected points in the input of the *DelaunayTriangulation* command, producing a coarser triangulation than the one in the artwork.
- Students counted, sometimes, too many triangles, by taking into account triangles outside of a figure in the artwork. For example, considering the triangulation produced in Figure 3 and the border triangles containing the point A, only the three triangles at the bottom of the artwork must be counted.

5. Results and discussion

The overall final results of the experience can be found in Table 1 obtaining an 81.5% of coincidence (in mean) between Okuda and Delaunay. A total of 123 different Okuda's works were analyzed (23 students did not provide the title), where most of the works where selected by just one or two students (almost 70% of the collection analyzed) but some others were chosen many times.

Table 1. Overall results of the Citizen Science project.

Students	Okuda's works	Coincidence ratio (mean)
378	123	81.5%

The results of the two main groups, Secondary School students and prospective teachers, are shown in Table 2. Generally speaking, students did not select too many points to create the Delaunay diagrams (in total the mean of considered triangles was slightly over 25), except one group of 36 University 4th-year students which constructed triangulations with more than 80 triangles in mean. This group had previous contact with Voronoi's diagrams and were taking a STEAM related course.

Table 2. Results of the project by groups of students.

	Students	Okuda's works	Coincidence ratio (mean)
Secondary Education	175	51	81.9%
Prospective Teachers	203	107	81.2%

For a more detailed analysis, we selected two Okuda's works with high percentage and two with low percentage of coincidences, all with a sufficiently large number of triangles (see Table 3). The coincidence ratio shows the mean of the ratios obtained by the students. We can remark that the high standard deviations in examples with low coincidence ratios are produced by the presence of the students' mistakes mentioned at the end of Section 4.

Table 3. Selection of Okuda's works and their coincidence ratio.

Okuda's work	No. of times selected	Coincidence ratio (mean)	Standard deviation
Magic Deer (2020)	9	94.4%	7.8
English Teacher (2021)	13	94.1%	5.9
Monkey Brother 2 (2021)	7	71%	16.4
Cougar (2016)	12	68%	15.7

In what follows we describe some of the triangulations obtained by the students, showing certain features of coincidences and non-coincidences between Okuda and Delaunay. An example of a high ratio of coincidence between Okuda and Delaunay (94.4%) can be found in Figure 3.

Figure 3. Magic Deer (2020).

In Figure 4, Okuda's triangulations for the quadrilaterals $ABCD$ and $EFGH$ do not agree with Delaunay's as computed by GeoGebra, but since the four points in both quadrilaterals are co-circular, we can assume both triangulations as coincident.

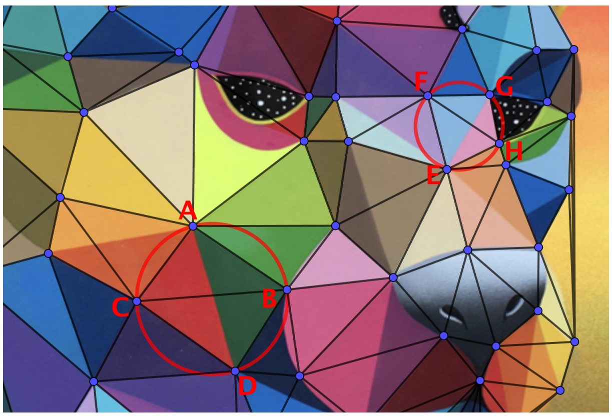

Figure 4. Cougar (2016). Example of a co-circular coincidence.

As examples of non-coincidences, in Figure 5 Okuda's triangulation of $CDFE$ is not Delaunay and the four points are not co-circular. This may be because Okuda might want to highlight the cheek bone lines, going against Delaunay. The same phenomenon happens in Figure 6 with the quadrilaterals $ABCD$ and $EFGH$ at the sides of the mask.

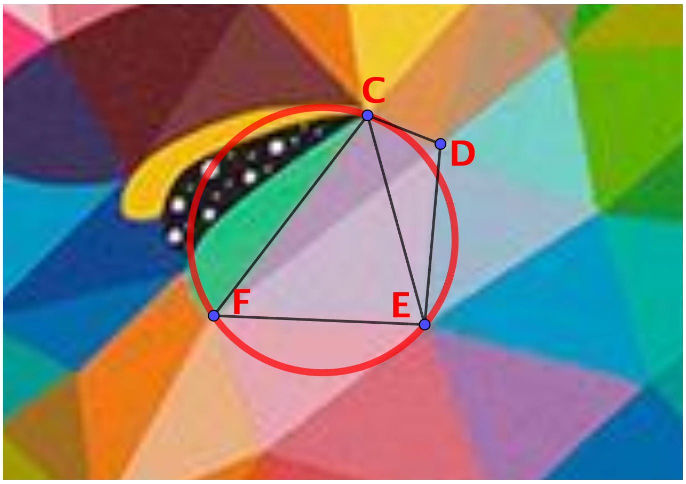

Figure 5. Liberty Statue I (2021). Example of a non-coincidence by anthropological reasons (sharpening the eye line).

Figure 6 shows an example of an Okuda's work that can be explained as the union of two Delaunay triangulations: the top of the skull (with a fan-shape triangulation) and the middle part (maxilla) of the skull. With this union, the ratio of coincidence in this case is 92.6%.

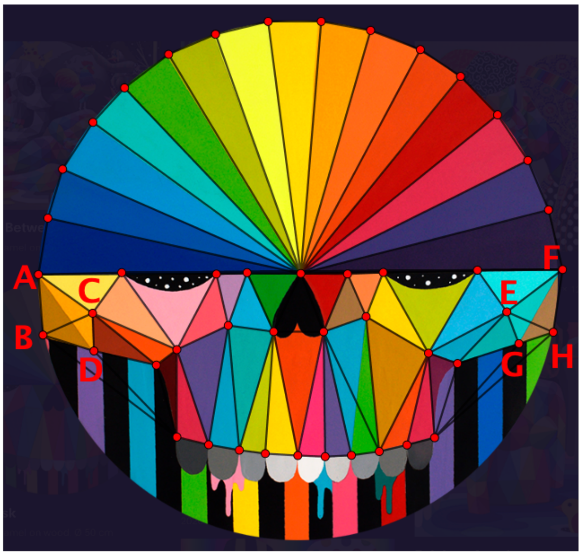

Figure 6. Skull Mask (2020). Okuda as the union of two Delaunay triangulations.

Last, we include here an example where Okuda triangulates a convex polygon with no co-circular points. In this case, Delaunay triangulation is unique, and, since there are no anthropological objections, the ratio of coincidence obtained by a student who triangulated the complete figure is 100% (Figure 7).

Figure 7. The Moon Loving The Lake II (2021). Okuda matches Delaunay.

6. Conclusions

A first conclusion is that our initial conjecture (relevant coincidence of Delaunay's triangulations and of the characteristic triangulations in many of Okuda's artworks) can be considered as confirmed. This is the main, and novel, contribution of our research. The minor exceptions of non-coincidence are mostly the cases of artworks containing four co-circular vertices, or the inclusion, within the convex hull of the examined set of vertices, of *non-linear objects* from the figure (i.e. ears, eyes, etc.).

Indeed, as remarked in Section 2, the lack of uniqueness for Delaunay triangulations when there are four co-circular points, turns arbitrary the choice of a concrete triangulation in this case, and GeoGebra is forced to display just one of the two possible ones. Thus, in the event of "co-circular quadrilaterals" within the vertices of an Okuda artwork, any of the displayed triangulations of the corresponding quadrilateral should be considered as coincident with Delaunay's. In fact, it happens that slightly dragging in GeoGebra the vertices of such quadrilateral, yields the interchange of the two possible Delaunay triangulations, making finally one of them coincident with the artist's choice. We think that if students would have taken into account this exceptional situation, they would have reported an —even if already very high— increased percentage of coincidences.

Once the conjecture is confirmed, it might be interesting to find a (intuitive) reason for its validity. Why there is such coincidence? Of course, we are convinced —after consultation with some of his managers— that the artist is not even aware of the existence of some mathematical concept named "Delaunay's triangulations", neither does he use any kind of technological tool, producing these triangulations, in his creativity process.

As mentioned in [3], a possible reason for the validity of this conjecture could be related to the fact that handling an artist tool (a brush, or a spray, for instance), when painting triangles, becomes easier —and the artist is implicitly pushed to do so— when maximizing the minimum displayed angle, and thus it leads, as explained in Section 2, to output Delaunay's triangulations. Other possible interpretations could highlight the higher artistic value of collections of close-to-regular triangles (as the ones appearing in Delaunay's triangulations) versus those including more acute ones. We think that looking for such explanations pertain more to the artistic realm (technical requirements for the use of some painting instruments, aesthetic consideration of certain sets of triangles), quite far from our expertise. It could be a subject of future analysis for experts in that field.

In this paper we have described in detail both an educational experience in the STEAM/citizen science context and the obtained data, as well as its possible interpretation. We think that the description of the way we have developed our project —involving so many students from different levels, including future teachers— could be also considered as a relevant contribution of our paper, providing mathematics teachers with a novel example of how to merge, in the classroom, Mathematics, Technology and Art.

The use of technology (through GeoGebra) in our research protocol is not just a way to simplify tasks (such as constructing the Delaunay triangulation of a given set of points) that could be also achieved by human hands —although with much effort and many possible errors. Indeed, we would like to remark here the role of this experience as providing a challenging antecedent and benchmark for testing a future advancement of GeoGebra's automated reasoning tools. Namely, developing in GeoGebra instruments for the automated exploration and determination of features in geometric figures.

As described in Subsection 3.1, currently, such exploration is restricted to traditional euclidean geometry properties (perpendicularity, parallelism, co-linearity, etc.) and requires the introduction of the figure by the user. We consider as possible in a near future, to enhance GeoGebra with the automated recognition of geometric elements (lines, points, etc.) from images captured, for instance, through an smartphone that includes GeoGebra as an app, or taken from a web page. Indeed, we might assume that GeoGebra could be able to automatically convert such images into a standard GeoGebra input, without requiring the manual introduction, say, of the vertices in Okuda's work, by the students. This could be done by implementing some techniques for identifying lines, circles, etc. in a given image, based, for instance, on the Hough transform [27].

Once the visual representation of reality has been turned into a precise GeoGebra input, the discovery of the mathematics lying behind this geometric version of reality could be automatically obtained by GeoGebra, e.g. using the already implemented Delaunay algorithm, and then comparing both the input and output triangulations, through a kind of extended "Discover" command, such as the one currently able to explore automatically, without human guide, the presence of elementary geometry properties in a figure [28].

We are talking, thus, very roughly speaking, of extending the current, quite performing, automated discovery exploration, from euclidean to computational geometry: i.e. now GeoGebra finds if two given lines are parallel, in the future, it could find how many triangles belong to the Delaunay diagram of some given set of points.

The final goal that we envision in this modest contribution is not other than that of yielding GeoGebra as a kind of Automatically Augmented Reality tool (see [4, 24] for a detailed description of this proposal), a kind of combination of Artificial Intelligence and Artificial Senses, that could have a extraordinary potential for education and human beings, in general (think, for example, of its contribution to the perception of reality for people with visual problems). Yet, this is just a future dream (but see, e.g. [29–31] for some already presented work in this direction): as we have described in Section 4, currently, the protocol we have followed in our experience (uploading some image to GeoGebra, identifying vertices, introducing the list of such points in the Delaunay command, analyzing coincidences) has to be done with the collaboration of humans, in the framework of citizen science activities. We would like to end our article with a recognition to the value of their contribution, and to thank them for their collaboration.

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Illustrations along the paper are taken from Okuda's web page [2]. We would like to acknowledge his permission.

Conflict of interest

Authors acknowledge no conflict of interest whatsoever.

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