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Research article

Boundedness of sublinear operators on weighted grand Herz-Morrey spaces

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Abstract: In this paper, we introduce weighted grand Herz-Morrey type spaces and prove the boundedness of sublinear operators and their multilinear commutators on these spaces. The results are still new even in the unweighted setting.

Keywords: sublinear operator; multilinear commutators; grand Herz-Morrey space; variable exponent; weight
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1. Introduction

Motivated by applications to fluid dynamics, variational integrals and partial differential equations with non-standard growth conditions, many research papers focus on the function spaces with variable exponent in harmonic analysis. As a special case of the Musielak-Orlicz spaces, variable exponent Lebesgue spaces $L^{q(\cdot)}$ (together with Sobolev spaces built upon them) were first studied in [1] and some of the properties of the Lebesgue spaces were readily generalized to $L^{q(\cdot)}$. Subsequently, besides the variable Lebesgue and Sobolev spaces, there are diverse spaces with variable exponent were introduced and studied. For example, see [2] for variable exponent Bessel potential spaces, the variable exponent Besov and Triebel-Lizorkin spaces were defined on [3–5], and the variable exponent Morrey spaces and Campanato spaces were also introduced on [6, 7]. The list is not exhausted.

Herz spaces with variable exponent was initially defined by Izuki [8], who used the Haar functions to obtain wavelet characterization of this spaces. Later, Izuki [9] established the boundedness of sublinear operators and their commutators on Herz spaces with variable exponent. As a generalization, Herz-Morrey spaces with variable exponent also were introduced by Izuki in [10]. Indeed, Izuki established the boundedness of vector-valued sublinear operators satisfying a size condition on Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}$. Furthermore, Wang and Xu [11] generalized Izuki's result for the wighted Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q(\cdot)}(\omega)$.

Recently, Sultan et al. [12] introduced weighted grand Herz space $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\omega)$ and they obtained the boundedness of fractional integrals. Inspired by the results of [11, 12], the aim of this paper is to introduce the weighted grand Herz-Morrey spaces and prove the boundedness of sublinear operators and their multilinear commutators on these spaces.

The paper is organized as follows. In Section 2, we will recall some related definitions and auxiliary lemmas, including some basic notions regarding Lebesgue space with variable exponent and grand Lebesgue sequence spaces which are the main ingredients to define weighted grand Herz-Morrey spaces. In Section 3, we introduce the concept of weighted grand Herz-Morrey spaces and investigate the relationship between weighted grand Herz-Morrey spaces and weighted Herz-Morrey spaces. In Section 4, we prove the boundedness of sublinear operators with certain weak size conditions on weighted grand Herz-Morrey spaces. As an application, we obtain the boundedness estimation for some classical sublinear operators on weighted grand Herz-Morrey spaces. Finally, basing on the theories of variable exponent and the generalization of BMO norm, we prove the boundedness for multilinear commutators of sublinear operators on weighted grand Herz-Morrey spaces in Section 5.

Throughout this paper, we denote $\chi_k = \chi_{R_k}$, $R_k = B_k \backslash B_{k-1}$ and $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ for all $k \in \mathbb{Z}$. Constants (often different constant in the same series of inequalities) will mainly be denoted by c or C. $f \lesssim g$ means that $f \leq Cg$ and $f \approx g$ means that $f \lesssim g \lesssim f$.

2. Preliminaries

In this section, we will recall some necessary definitions and auxiliary results.

2.1. Lebesgue space with variable exponent

To begin with, we recall some basic definitions and results on the variable exponent Lebesgue spaces. We can refer to the monographs [13, 14] for more informations. Let $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. $L^{q(\cdot)}(\mathbb{R}^n)$ denotes the spaces of all measurable functions f on $\mathbb{R}^n$ such that

$$I_{q(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{q(x)} \mathrm{d}x < \infty.$$

This is Banach space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : I_{q(\cdot)}(f/\lambda) \leq 1\}.$$

Given an open set $\Omega \subseteq \mathbb{R}^n$, the space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \{f : f \in L^{q(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega\}.$$

For conciseness, we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $q(x)$ on $\mathbb{R}^n$ with range in $[1, \infty)$ such that

$$1 < q_- \leq q(x) \leq q_+ < \infty, \tag{2.1}$$

$$\mathcal{B}(\mathbb{R}^n) := \{q(\cdot) \in \mathcal{P}(\mathbb{R}^n) : M \text{ is bounded on } L^{q(\cdot)}(\mathbb{R}^n)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x \in \mathbb{R}^n, r>0} r^{-n} \int_{B(x,r)} |f(y)| \mathrm{d}y.$$

A real-valued measurable function $g(\cdot):\mathbb{R}^n\to(0,\infty)$ is called globally log-Hölder continuous if there exists a constant $C_{\log}>0$ such that

$$|g(x)-g(y)|\le\frac{C_{\log}}{\log(e+\frac{1}{|x-y|})},\ \forall x,\ y\in\mathbb{R}^n,\tag{2.2}$$

if, for some $g_\infty\in(0,\infty)$ and $C_{\log}>0$, there hold

$$|g(x)-g(0)|\le\frac{C_{\log}}{\log(e+\frac{1}{|x|})},\ \forall x\in\mathbb{R}^n,\tag{2.3}$$

$$|g(x)-g_\infty|\le\frac{C_{\log}}{\log(e+|x|)},\ \forall x\in\mathbb{R}^n,\tag{2.4}$$

then we say $g(\cdot)$ is log-Hölder continuous at the origin (or has a log decay at the origin) and at infinity (or has a log decay at infinity), respectively.

The set $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot)$ satisfying $q_->0$ and $q_+<\infty$. By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ can be denoted the class of exponents $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, which satisfies conditions (2.3) and (2.4), respectively. $\mathcal{P}^{\log}(\mathbb{R}^n)$ is the set of functions $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying conditions (2.3) and (2.4), with $q_\infty:=\lim_{|x|\to\infty}q(x)$. In particular, we note that if $q(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ with $1<q_-\le q_+<\infty$, then the Hardy-Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, namely, $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, see [14–16].

Lemma 2.1. [1] (Generalized Hölder's inequality) Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $f\in L^{q(\cdot)}(\mathbb{R}^n)$ and $g\in L^{q'(\cdot)}(\mathbb{R}^n)$, the generalized Hölder's inequality holds in the form

$$\int_{\mathbb{R}^n}|f(x)g(x)|\mathrm{d}x\le r_p\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},\tag{2.5}$$

where $r_p=1+1/q_--1/q_+$.

2.2. *Weighted Lebesgue spaces with variable exponent*

Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and ω be a nonnegative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{q(\cdot)}(\omega)$ is the set of all complex-valued measurable functions f such that $f\omega\in L^{q(\cdot)}$. The space $L^{q(\cdot)}(\omega)$ is a Banach space equipped with the norm

$$\|f\|_{L^{q(\cdot)}(\omega)}:=\|f\omega\|_{L^{q(\cdot)}}.$$

Lemma 2.2. [11] Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. A positive measurable function $\omega\in A_{q(\cdot)}$, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that

$$\frac{1}{B}\|\omega\chi_B\|_{L^{q(\cdot)}}\|\omega^{-1}\chi_B\|_{L^{q'(\cdot)}}\le C.$$

The variable Muckenhoupt $A_{q(\cdot)}$ was introduced by Cruz-Uribe et al. in [17]. It is easy to see that if $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $\omega\in A_{q(\cdot)}$, then $\omega^{-1}\in A_{q'(\cdot)}$.

Lemma 2.3. [18] Let $q(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\omega\in A_{q(\cdot)}$. Then there exist constants $\delta_1,\delta_2\in(0,1)$ and $C>0$, such that for all balls in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\omega)}}{\|\chi_B\|_{L^{q(\cdot)}(\omega)}}\le C\left(\frac{|S|}{|B|}\right)^{\delta_1},$$

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\omega^{-1})}}{\|\chi_B\|_{L^{q'(\cdot)}(\omega^{-1})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

A locally integrable function b is called a BMO function, if it satisfies

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B|}\int_B |b(y)-b_B|\mathrm{d}y < \infty,$$

where B is a ball-centered at x and radius of r, $b_B = \frac{1}{|B|}\int_B b(t)\mathrm{d}t$.

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ with j different elements. For any $\sigma \in C_j^m$, the complementary sequence $\sigma^c = \{1,\cdots,m\}\backslash\sigma$. For $\boldsymbol{b} = (b_1,\cdots,b_m)$ and $\sigma = \{\sigma(1),\cdots,\sigma(j)\} \in C_j^m$ with $1 \le j \le m$, we denote

$$\begin{aligned}
&\boldsymbol{b}_\sigma := (b_{\sigma(1)},\cdots,b_{\sigma(j)}),\\
&[b(x)-b(y)]_\sigma := [b_{\sigma(1)}(x)-b_{\sigma(1)}(y)]\cdots[b_{\sigma(j)}(x)-b_{\sigma(j)}(y)],\\
&[(b)_B-b(y)]_\sigma := [(b_{\sigma(1)})_B-b_{\sigma(1)}(y)]\cdots[(b_{\sigma(j)})_B-b_{\sigma(j)}(y)],\\
&\|\boldsymbol{b}_\sigma\|_* = \|b_{\sigma(1)}\|_*\cdots\|b_{\sigma(j)}\|_* \ \text{ for } b_{\sigma(i)} \in \mathrm{BMO}(\mathbb{R}^n).
\end{aligned}$$

In particular,

$$\|\boldsymbol{b}\|_* = \|b_1\|_*\cdots\|b_m\|_*.$$

Lemma 2.4. Let $\omega \in A_{q(\cdot)}, q(\cdot) \in \mathscr{B}(\mathbb{R}^n),\ b_i \in \mathrm{BMO}(\mathbb{R}^n),\ i = 1,2,...,m,\ k > j(k,\ j \in \mathbb{N})$. Then we have

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\omega)}}\left\|\prod_{i=1}^m (b_i-(b_i)_B)\chi_B\right\|_{L^{q(\cdot)}(\omega)} \approx \prod_{i=1}^m \|b_i\|_*,$$

and

$$\left\|\prod_{i=1}^m (b_i-(b_i)_{B_j})\chi_{B_k}\right\|_{L^{q(\cdot)}(\omega)} \lesssim (k-j)^m \prod_{i=1}^m \|b_i\|_* \|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}.$$

Lemma 2.4 is a generalization of the well-known properties for BMO spaces (see [19]) and it's also a generalized version of Izuki's and Wang's results in [9, 20].

2.3. Weighted Herz-Morrey spaces with variable exponent

In this subsection, we recall definition of the weighted Herz-Morrey spaces with variable exponent.

Definition 2.1. [11] Let $0 \le \lambda < \infty,\ 0 < p < \infty,\ q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous weighted Herz-Morrey space $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\omega)$ with variable exponent is defined by

$$M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\omega) = \left\{f \in L_{\mathrm{loc}}^{q(\cdot)}(\mathbb{R}^n\backslash\{0\},\omega) : \|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\omega)} < \infty\right\},$$

where

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\omega)} := \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\omega)}^p\right)^{\frac{1}{p}}.$$

(ii) The non-homogeneous weighted Herz-Morrey space $MK^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega)$ with variable exponent is defined by

$$MK^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n, \omega) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega)} < \infty \right\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega)} := \sup_{k_0 \in \mathbb{N}_0} 2^{-k_0\lambda} \left(\sum_{k=0}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\omega)} \right)^{\frac{1}{p}}.$$

2.4. Grand Lebesgue sequence space

In this subsection, we introduce grand Lebesgue sequence space. $\mathbb{X}$ stands for one of the sets $\mathbb{Z}^n$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$.

Definition 2.2. [21] Let $1 \le p < \infty$ and $\theta > 0$. The grand Lebesgue sequence space $l^{p),\theta}$ is defined by the norm

$$\|\boldsymbol{x}\|_{l^{p),\theta}(\mathbb{X})} := \sup_{\varepsilon>0} \left(\varepsilon^\theta \sum_{k\in\mathbb{X}} |x_k|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|\boldsymbol{x}\|_{l^{p(1+\varepsilon)}(\mathbb{X})},$$

where $\boldsymbol{x} = \{x_k\}_{k\in\mathbb{X}}$.

Note that the following nesting properties hold:

$$l^{p(1-\varepsilon)} \hookrightarrow l^p \hookrightarrow l^{p),\theta_1} \hookrightarrow l^{p),\theta_2} \hookrightarrow l^{p(1+\delta)}, \tag{2.6}$$

for $0 < \varepsilon < \frac{1}{p}$, $\delta > 0$ and $0 < \theta_1 \le \theta_2$.

3. Weighted grand Herz-Morrey spaces with variable exponent

In this section, we give the definition of weighted grand Herz-Morrey spaces in a natuaral way from Definition 2.2.

Definition 3.1. Let $0 \le \lambda < \infty$, $k \in \mathbb{Z}$, $1 \le p < \infty$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $\theta > 0$. We define the homogeneous weighted grand Herz-Morrey space by

$$M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}, \omega) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} &: = \sup_{\varepsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &= \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p(1+\varepsilon),q(\cdot)}(\omega)}. \end{aligned}$$

The non-homogeneous weighted grand Herz-Morrey space by

$$MK^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n, \omega) : \|f\|_{MK^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} < \infty \right\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} := \sup_{\varepsilon>0}\sup_{k_0\in\mathbb{N}_0} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0}\|2^{k\alpha(\cdot)}f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$= \sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{p(1+\varepsilon)}}\|f\|_{MK^{\alpha(\cdot),\lambda,\theta}_{p(1+\varepsilon),q(\cdot)}(\omega)}.$$

Remark 3.1. From Definition 3.1, it is not hard to see that if $\alpha(\cdot) = c$ and $\lambda = 0$, $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega) = \dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\omega)$ is weighted grand Herz space with variable exponent in [12]. If $\alpha(\cdot), q(\cdot)$ are constants and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega) = \dot{K}^{\alpha,p),\theta}_{q}(\omega)$ in [22]. When $\theta = 0$, the weighed grand Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$ is weighed Herz-Morrey space in [11]. In the meantime, there is a analog for the non-homogeneous case.

In the following theorem, we prove that homogeneous weighed Herz-Morrey space is contained in homogeneous weighed grand Herz-Morrey space.

Theorem 3.1. For $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\lambda \in [0,\infty)$, $k \in \mathbb{Z}$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$ and $1 \le p < \infty$. We have $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega) \subset M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$, $\theta > 0$.

Proof. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega)$, from (2.6), we can obtained that

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} = \sup_{\varepsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\|2^{k\alpha(\cdot)}f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\|\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}(\omega)}\|_{l^{p),\theta}}$$
$$\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\|\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}(\omega)}\|_{l^{p}}$$
$$= C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega)}.$$

4. Boundedness of sublinear operators

In this section, we show that sublinear operators are bounded on homogeneous weighted grand Herz-Morrey spaces. To this end, we need the following lemma.

Lemma 4.1. Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, ω be a weight, $\lambda \in [0,\infty)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $k \in \mathbb{Z}$ and $1 \le p < \infty$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} \approx \max\Big\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\Big)^{\frac{1}{p(1+\varepsilon)}},$$
$$\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}$$
$$+\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\Big)^{\frac{1}{p(1+\varepsilon)}}\Big\}.$$

The proof of this lemma is essentially similar to the proof of Proposition 3.8 in [23] of $\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ space. Indeed, when $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$ and $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, there exist positive constants C_1, C_2 such that if $k \le 0$ and $x \in R_k$, then $C_1 2^{k\alpha(0)} \le 2^{k\alpha(x)} \le C_2 2^{k\alpha(0)}$; if $k > 0$ and $x \in R_k$, then $C_1 2^{k\alpha_\infty} \le 2^{k\alpha(x)} \le C_2 2^{k\alpha_\infty}$. Thus, we obtain Lemma 4.1, which is also true for non-homogeneous weighted grand Herz-Morrey space.

Theorem 4.2. Let $1 < p < \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\omega \in A_{q(\cdot)}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ such that $-n\delta_1 < \alpha(0), \alpha_\infty < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ be the constants in Lemma 2.2. Suppose that sublinear operator T satisfies the size conditions

$$|Tf(x)| \lesssim \|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{4.1}$$

when supp$f \subseteq R_k$ and $|x| \ge 2^{k+1}$ with $k \in \mathbb{Z}$ and

$$|Tf(x)| \lesssim 2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{4.2}$$

when supp$f \subseteq R_k$ and $|x| \le 2^{k-2}$ with $k \in \mathbb{Z}$. If T is bounded on $L^{q(\cdot)}(\omega)$, then T is bounded on $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$.

Proof. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$. We decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x).$$

From Lemma 4.1, we have

$$\begin{aligned}
\|Tf\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} \approx & \max\Big\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\|T(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\Big)^{\frac{1}{p(1+\varepsilon)}},\\
& \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\|T(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\\
& + \varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\|T(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\Big)^{\frac{1}{p(1+\varepsilon)}}\Big\}\\
& =: \max\{\mathrm{E}, \mathrm{F}+\mathrm{G}\},
\end{aligned}$$

where

$$\mathrm{E} = \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty} T(f\chi_l)\right\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$\mathrm{F} = \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty} T(f\chi_l)\right\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$G = \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty} T(f\chi_l)\right\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}.$$

Since the estimate of F is essentially similar to the estimate of E, it suffices to show that E and G are bounded on homogeneous weighted grand Herz-Morrey space. It is easy to see that

$$\mathrm{E} \lesssim \sum_{\mathrm{i}=1}^{3}\mathrm{E_i}, \qquad \mathrm{G} \lesssim \sum_{\mathrm{i}=1}^{3}\mathrm{G_i},$$

where

$$E_1 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$E_2 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$E_3 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$G_1 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$G_2 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

$$G_3 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}.$$

Firstly, we consider E_1. For a.e. $x \in R_k$ with $k \in \mathbb{Z}$ and $l \le k-2$, from size condition of T and generalized Hölder's inequality, it follows that

$$\begin{aligned} |T(f\chi_l)(x)| &\lesssim 2^{-kn}\int_{R_l} |f(y)|\mathrm{d}y \\ &\lesssim 2^{-kn}\|f\chi_l\omega\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\omega^{-1}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\lesssim 2^{-kn}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}. \end{aligned} \tag{4.3}$$

By Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} &\lesssim 2^{-kn}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}\|\chi_k\|_{L^{q(\cdot)}(\omega)} \\ &\lesssim 2^{-kn}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}|B_k|\|\chi_k\|^{-1}_{L^{q'(\cdot)}(\omega^{-1})} \\ &\lesssim 2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}. \end{aligned} \tag{4.4}$$

From (4.4), it follows that

$$\begin{aligned} E_1 &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{(l-k)n\delta_2}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} 2^{\alpha(0)l}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{v(l-k)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}, \end{aligned}$$

where $v := n\delta_2 - \alpha(0) > 0$. And then, by Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)} < 2^{-p}$, we obtain that

$$\begin{aligned}
\mathrm{E}_1 &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} 2^{\alpha(0)lp(1+\varepsilon)} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} 2^{vp(1+\varepsilon)(l-k)/2} \right) \right. \\
&\quad \left. \times \left(\sum_{l=-\infty}^{k-2} 2^{v(l-k)(p(1+\varepsilon))'/2} \right)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} 2^{\alpha(0)lp(1+\varepsilon)} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} 2^{v(l-k)p(1+\varepsilon)/2} \right) \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \sum_{k=l+2}^{k_0} 2^{v(l-k)p(1+\varepsilon)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \sum_{k=l+2}^{k_0} 2^{v(l-k)p/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.
\end{aligned}$$

Next we show E_2, since the operator T is bounded on $L^{q(\cdot)}(\omega)$, we get

$$\begin{aligned}
\mathrm{E}_2 &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \|f\chi_k\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.
\end{aligned}$$

Now we turn to estimate E_3. For each $k \in \mathbb{Z}$, $l \ge k+2$ and a.e. $x \in R_k$, size condition of T and generalized Hölder's inequality imply that

$$\begin{aligned}
|T(f\chi_l)(x)| &\lesssim 2^{-ln} \int_{R_l} |f(y)| \mathrm{d}y \\
&\lesssim 2^{-ln} \|f\chi_l \omega\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l \omega^{-1}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\lesssim 2^{-ln} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}. \qquad (4.5)
\end{aligned}$$

Splitting E_3 by means of Minkowski's inequality, we deduce

$$\begin{aligned}
\mathrm{E}_3 &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=k+2}^{-1} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=0}^{\infty} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}
\end{aligned}$$

$$=: \mathrm{E}_{31} + \mathrm{E}_{32}.$$

By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}\|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} &\lesssim 2^{-ln}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}\|\chi_k\|_{L^{q(\cdot)}(\omega)} \\ &\lesssim 2^{-ln}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_k\|_{L^{q(\cdot)}(\omega)}|B_l|\|\chi_l\|^{-1}_{L^{q(\cdot)}(\omega)} \\ &\lesssim 2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}. \end{aligned} \tag{4.6}$$

For E_{31}, using (4.6) we get

$$\begin{aligned}\mathrm{E}_{31} &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{k\alpha(0)}2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{l\alpha(0)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{k\alpha(0)-l\alpha(0)+(k-l)n\delta_1}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{-1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{(n\delta_1+\alpha(0))(k-l)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{d(k-l)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},\end{aligned}$$

where $d := n\delta_1 + \alpha(0) > 0$. Then applying Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)} < 2^{-p}$, we obtain that

$$\begin{aligned}\mathrm{E}_{31} &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{dp(1+\varepsilon)(k-l)/2}\right)\right. \\ &\quad\left.\times\left(\sum_{l=k+2}^{-1}2^{d(k-l)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{d(k-l)p(1+\varepsilon)/2}\right)\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\sum_{k=-\infty}^{l-2}2^{d(k-l)p(1+\varepsilon)/2}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\sum_{k=-\infty}^{l-2}2^{d(k-l)p/2}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\end{aligned}$$

$$\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.$$

On the other hand, applying Hölder's inequality, (4.6) and note that $h := n\delta_1 + \alpha_\infty > 0$, for E_{32} we get

$$\begin{aligned}
E_{32} &\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=0}^{\infty} 2^{(k-l)n\delta_1} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k(\alpha(0)+n\delta_1)p(1+\varepsilon)} \left(\sum_{l=0}^{\infty} 2^{-ln\delta_1} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} 2^{-lh} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} \right) \right. \\
&\quad \left. \times \left(\sum_{l=0}^{\infty} 2^{-lh(p(1+\varepsilon))'} \right)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.
\end{aligned}$$

This combine the estimate of E_{31} implies that $E_3 \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}$. Furthermore, we can get $E \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}$.

Next, we consider G_1. By the notion of G_1, splitting G_1 as follows

$$\begin{aligned}
G_1 &\lesssim \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=-\infty}^{-1} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\quad + \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=0}^{k-2} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&=: G_{11} + G_{12}.
\end{aligned}$$

To estimate G_{11}, by (4.4) and the fact that $e := n\delta_2 - \alpha_\infty > 0$, we get

$$\begin{aligned}
G_{11} &\lesssim \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} 2^{(l-k)n\delta_2} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k(\alpha_\infty - n\delta_2)p(1+\varepsilon)} \left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} 2^{ln\delta_2} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q(\cdot)}(\omega)} 2^{ln\delta_2} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}
\end{aligned}$$

$$\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}2^{lv}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}.$$

According to Hölder inequality and noting that $v := n\delta_2 - \alpha(0) > 0$, we further have

$$\begin{aligned}\mathrm{G}_{11} &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)\right.\\ &\quad\left.\times\left(\sum_{l=-\infty}^{-1} 2^{lv(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.\end{aligned}$$

The estimate for G_{12} follows from a similar method to E_1 and using the fact that $e := n\delta_2 - \alpha_\infty > 0, k > 0$. So we cancel the proof of G_{12}.

For G_2, in the view of the boundedness of T on $L^{q(\cdot)}(\omega)$, we obtain that

$$\begin{aligned}\mathrm{G}_2 &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.\end{aligned}$$

We cancel the proof of G_3. Since the estimate for G_3 can be obtained by similar way to E_{31} and note that $h := n\delta_1 + \alpha_\infty > 0$, $k > 0$.

Therefore, combining the estimates for E and G to deduce that

$$\|Tf\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.$$

This finishes the proof of Theorem 4.2.

Corollary 4.1. Let $p, \alpha(\cdot)$ and $q(\cdot)$ as in Theorem 4.2. If a sublinear operator T satisfies the condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}\mathrm{d}y, \quad x \notin \mathrm{supp}f \tag{4.7}$$

for any integrable function f with compact support and T is bounded on $L^{q(\cdot)}(\omega)$, then T is bounded on $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$.

Remark 4.1. We remark (4.7) is satisfied by many operators in harmonic analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and Bochner-Riesz means and so on. In particular, the Hardy-Littlewood maximal function also satisfies the hypotheses of Theorem 4.2. Due to the proof for the non-homogenous case can be treated by the similar method, in this article, our results are valid for the homogenous weighted grand Herz-Morrey space.

5. Boundedness for multilinear commutators of sublinear operators

In this section, we study the boundedness for multilinear commutators of sublinear operators on homogeneous weighted grand Herz-Morrey space.

Let $\boldsymbol{b} = (b_1, b_2, \cdots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, 2, \cdots, m\}$, $m \in \mathbb{N}$, $x \notin \operatorname{supp} f$. The multilinear commutators of sublinear operators $T^{\boldsymbol{b}}$ is defined as

$$T^{\boldsymbol{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(x) - b_j(y)| K(x, y) f(y) \mathrm{d}y,$$

where $K(x, y)$ is the integral kernel of the operator T, see [24].

Theorem 5.1. Let $\boldsymbol{b} = (b_1, b_2, \cdots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, 2, \cdots, m\}$, $m \in \mathbb{N}$. For $1 < p < \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\omega \in A_{q(\cdot)}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, such that $-n\delta_1 < \alpha(0), \alpha_\infty < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ be the constants in Lemma 2.3. Suppose that sublinear operator T satisfying the size conditions (4.1) and (4.2). If $T^{\boldsymbol{b}}$ is bounded on $L^{q(\cdot)}(\omega)$, then $T^{\boldsymbol{b}}$ is bounded on $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$.

Proof. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$, we decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x).$$

From Lemma 4.1, we have

$$\begin{aligned}
\|T^{\boldsymbol{b}} f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} \approx \max\Big\{ &\sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \|T^{\boldsymbol{b}}(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} \Big)^{\frac{1}{p(1+\varepsilon)}}, \\
&\sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \|T^{\boldsymbol{b}}(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} \\
&+ \varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \|T^{\boldsymbol{b}}(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} \Big)^{\frac{1}{p(1+\varepsilon)}} \Big\} \\
&=: \max\{\mathrm{A}, \mathrm{N} + \mathrm{S}\},
\end{aligned}$$

where

$$\mathrm{A} = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\left\| \chi_k \sum_{l=-\infty}^{\infty} T^{\boldsymbol{b}}(f\chi_l) \right\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$\mathrm{N} = \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left(\left\| \chi_k \sum_{l=-\infty}^{\infty} T^{\boldsymbol{b}}(f\chi_l) \right\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$\mathrm{S} = \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \left(\left\| \chi_k \sum_{l=-\infty}^{\infty} T^{\boldsymbol{b}}(f\chi_l) \right\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}.$$

Since the estimate of N is essentially similar to the estimate of A, it suffices to show that A and S are bounded on homogeneous grand weighted Herz-Morrey space. It is easy to see that

$$\mathrm{E} \lesssim \sum_{\mathrm{i}=1}^{3} \mathrm{A_i}, \qquad \mathrm{S} \lesssim \sum_{\mathrm{i}=1}^{3} \mathrm{S_i},$$

where

$$A_1 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$A_2 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=k-1}^{k+1} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$A_3 = \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=k+2}^{\infty} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$S_1 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$S_2 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \left(\sum_{l=k-1}^{k+1} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}},$$

$$S_3 = \sup_{\varepsilon>0} \sup_{k_0> 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)} \left(\sum_{l=k+2}^{\infty} \|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}.$$

Firstly, we consider A_1. For a.e. $x \in R_k$ with $k \in \mathbb{Z}$ and $l \le k-2$, from size condition of T and generalized Hölder's inequality, it follows that

$$\begin{aligned}
|T^{\boldsymbol{b}}(f\chi_l)(x)| &\lesssim 2^{-kn} \int_{R_l} \prod_{j=1}^{m} |b_j(x) - b_j(y)||f(y)|\mathrm{d}y \\
&\lesssim 2^{-kn} \int_{R_l} \prod_{j=1}^{m} |b_j(x) - (b_j)_{B_l} + (b_j)_{B_l} - b_j(y)||f(y)|\mathrm{d}y \\
&\lesssim 2^{-kn} \sum_{j=0}^{m} \sum_{\sigma\in C_j^m} |[b(x) - (b)_{B_l}]_\sigma| \int_{R_l} |[b(y) - (b)_{B_l}]_{\sigma^c}||f(y)|\mathrm{d}y \\
&\lesssim 2^{-kn} \sum_{j=0}^{m} \sum_{\sigma\in C_j^m} |[b(x) - (b)_{B_l}]_\sigma| \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \|[b(y) - (b)_{B_l}]_{\sigma^c}\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}. \qquad (5.1)
\end{aligned}$$

By Lemmas 2.2–2.4, we get

$$\begin{aligned}
\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)} &\lesssim \|\boldsymbol{b}\|_* 2^{-kn}(k-l)^m \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})} \|\chi_k\|_{L^{q(\cdot)}(\omega)} \\
&\lesssim \|\boldsymbol{b}\|_* 2^{-kn}(k-l)^m \|f\chi_l\|_{L^{q(\cdot)}(\omega)} \|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})} |B_k| \|\chi_k\|^{-1}_{L^{q'(\cdot)}(\omega^{-1})} \\
&\lesssim \|\boldsymbol{b}\|_* (k-l)^m 2^{(l-k)n\delta_2} \|f\chi_l\|_{L^{q(\cdot)}(\omega)}. \qquad (5.2)
\end{aligned}$$

From (5.2), it follows that

$$A_1 \lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \right.$$

$$\times\left(\sum_{l=-\infty}^{k-2}(k-l)^m 2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\Bigg)^{\frac{1}{p(1+\varepsilon)}}$$

$$\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)l}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(k-l)^m 2^{v(l-k)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$

where $v := n\delta_2-\alpha(0)>0$. And then, by Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)}<2^{-p}$, we obtain that

$$\begin{aligned}
\mathrm{A}_1 &\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{vp(1+\varepsilon)(l-k)/2}\right)\right.\\
&\quad\times\left.\left(\sum_{l=-\infty}^{k-2}(k-l)^{m(p(1+\varepsilon))'}2^{v(l-k)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{v(l-k)p(1+\varepsilon)/2}\right)\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\sum_{k=l+2}^{k_0}2^{v(l-k)p(1+\varepsilon)/2}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\sum_{k=l+2}^{k_0}2^{v(l-k)p/2}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.
\end{aligned}$$

To show A_2, if $T^{\boldsymbol{b}}$ is bounded on $L^{q(\cdot)}(\omega)$, we get

$$\begin{aligned}
\mathrm{A}_2 &\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.
\end{aligned}$$

Now we turn to estimate A_3. For each $k\in\mathbb{Z}$, $l\ge k+2$ and a.e. $x\in R_k$, size condition of T and generalized Hölder's inequality imply that

$$\begin{aligned}
|T^{\boldsymbol{b}}(f\chi_l)(x)| &\lesssim 2^{-ln}\int_{R_l}\prod_{j=1}^m|b_j(x)-b_j(y)||f(y)|\mathrm{d}y\\
&\lesssim 2^{-ln}\int_{R_l}\prod_{j=1}^m|b_j(x)-(b_j)_{B_l}+(b_j)_{B_l}-b_j(y)||f(y)|\mathrm{d}y
\end{aligned}$$

$$\lesssim 2^{-ln}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|[b(x)-(b)_{B_l}]_\sigma|\int_{R_l}|[b(y)-(b)_{B_l}]_{\sigma^c}||f(y)|\mathrm{d}y$$

$$\lesssim 2^{-ln}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|[b(x)-(b)_{B_l}]_\sigma|\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|[b(y)-(b)_{B_l}]_{\sigma^c}\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}. \tag{5.3}$$

Applying Lemmas 2.2–2.4, we get

$$\begin{aligned}\|\chi_k T^{\boldsymbol{b}} f\chi_l\|_{L^{q(\cdot)}(\omega)} &\lesssim \|\boldsymbol{b}\|_*(l-k)^m 2^{-ln}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_l\|_{L^{q'(\cdot)}(\omega^{-1})}\|\chi_k\|_{L^{q(\cdot)}(\omega)}\\ &\lesssim \|\boldsymbol{b}\|_*(l-k)^m 2^{-ln}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\|\chi_k\|_{L^{q(\cdot)}(\omega)}|B_l|\|\chi_l\|^{-1}_{L^{q'(\cdot)}(\omega)}\\ &\lesssim \|\boldsymbol{b}\|_*(l-k)^m 2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}.\end{aligned} \tag{5.4}$$

Splitting A_3 by means of Minkowski's inequality, we deduce

$$\begin{aligned}A_3 &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{-1}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &+\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &=: A_{31}+A_{32}.\end{aligned}$$

For A_{31}, according to (5.4), we have

$$\begin{aligned}A_{31} &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}(l-k)^m\right.\\ &\left.\times\left(\sum_{l=k+2}^{-1}2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{\alpha(0)l}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(l-k)^m 2^{d(k-l)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},\end{aligned}$$

where $d := n\delta_1+\alpha(0) > 0$. Then applying Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)} < 2^{-p}$, we obtain that

$$\begin{aligned}A_{31} &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{dp(1+\varepsilon)(k-l)/2}\right)\right.\\ &\left.\times\left(\sum_{l=k+1}^{-1}(l-k)^{m(p(1+\varepsilon))'}2^{d(k-l)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}2^{d(k-l)p(1+\varepsilon)/2}\right)\right)^{\frac{1}{p(1+\varepsilon)}}\end{aligned}$$

$$\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \sum_{k=-\infty}^{l-2} 2^{d(k-l)p(1+\varepsilon)/2}\right)^{\frac{1}{p(1+\varepsilon)}}$$

$$\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \sum_{k=-\infty}^{l-2} 2^{d(k-l)p/2}\right)^{\frac{1}{p(1+\varepsilon)}}$$

$$\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$

$$\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}_{p),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

On the other hand, applying Hölder's inequality, (5.4), and note that $h := n\delta_1 + \alpha_\infty > 0$, for A_{32} we get

$$\begin{aligned}
A_{32} &\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\right.\\
&\quad\times\left.\left(\sum_{l=0}^{\infty}(l-k)^m 2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k(\alpha(0)+n\delta_1)p(1+\varepsilon)}\right.\\
&\quad\times\left.\left(\sum_{l=0}^{\infty}(l-k)^m 2^{-ln\delta_1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(l-k)^m 2^{-lh}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_* \sup_{\varepsilon>0} \sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)\right.\\
&\quad\times\left.\left(\sum_{l=0}^{\infty}(l-k)^{m(p(1+\varepsilon))'} 2^{-lh(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}_{p),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.
\end{aligned}$$

This combine with the estimate of A_{31} to obtained that $A_3 \lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}_{p),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}$. Furthermore, we can get $A \lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}_{p),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}$.

Next, we consider S_1. By the notion of S_1, splitting S_1 as follows

$$S_1 \lesssim \sup_{\varepsilon>0} \sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$

$$+\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=0}^{k-2}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$=: \mathrm{S}_{11}+\mathrm{S}_{12}.$$

To show S_{11}, using (5.2) and note that $e := n\delta_2-\alpha_\infty>0$, we have

$$\begin{aligned}\mathrm{S}_{11}&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\right.\\&\quad\left.\times\left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(k-l)^m2^{(l-k)n\delta_2}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0}2^{k(\alpha_\infty-n\delta_2)p(1+\varepsilon)}\right.\\&\quad\left.\times\left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(k-l)^m2^{ln\delta_2}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(k-l)^m2^{ln\delta_2}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{l\alpha(0)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}(k-l)^m2^{lv}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}.\end{aligned}$$

We can obtained S_{11} further caculation by Hölder inequality and use the fact that $v := n\delta_2-\alpha(0)>0$, we get

$$\begin{aligned}\mathrm{S}_{11}&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)\right.\\&\quad\left.\times\left(\sum_{l=-\infty}^{-1}(k-l)^{m(p(1+\varepsilon))'}2^{lv(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}\\&\lesssim\|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.\end{aligned}$$

The estimate for S_{12} follows from a similar method to A_1 and note that $e := n\delta_2-\alpha_\infty>0, k>0$. So we cancel the proof of S_{12}.

For S_2, in the view of the boundedness of $T^{\boldsymbol{b}}$ on $L^{q(\cdot)}(\omega)$, we obtain that

$$\begin{aligned}\mathrm{S}_2&\lesssim\|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\&\lesssim\|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)}.\end{aligned}$$

We cancel the proof of S_3. Since the estimate for S_3 can be obtained by similar way to A_{31} and using the fact that $h := n\delta_1 + \alpha_\infty > 0$, $k > 0$.

Therefore, combining the estimates for A and S to deduce that

$$\|T^{\boldsymbol{b}} f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)} \lesssim \|\boldsymbol{b}\|_* \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)},$$

which ends the proof.

Corollary. Let $p, \alpha(\cdot)$ and $q(\cdot)$ as in Theorem 5.1. If a sublinear operator T satisfies the condition (4.7), for any integrable function f with compact support and $T^{\boldsymbol{b}}$ is bounded on $L^{q(\cdot)}(\omega)$, then $T^{\boldsymbol{b}}$ is bounded on $MK^{\alpha(\cdot),\lambda,\theta}_{p),q(\cdot)}(\omega)$.

6. Conclusions

In this article, we introduced the concept of weighted grand Herz-Morrey spaces and investigate the relationship between weighted grand Herz-Morrey spaces and weighted Herz-Morrey spaces. In addition, we proved the boundedness of sublinear operators with certain weak size conditions on weighted grand Herz-Morrey spaces. As an application, we obtained the boundedness estimation for multilinear commutators of sublinear operators on weighted grand Herz-Morrey spaces. These results are new even in unweighted setting.

Use of AI tools declaration

The authors declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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