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Research article

Some notes on the tangent bundle with a Ricci quarter-symmetric metric connection

Yanlin Li[1,*], **Aydin Gezer**[2] **and Erkan Karakaş**[2]

[1] School of Mathematics, Key Laboratory of Cryptography of Zhejiang Province, Hangzhou Normal University, Hangzhou 311121, China
[2] Faculty of Science, Department of Mathematics, Ataturk University, Erzurum 25240, Turkey.

* **Correspondence:** Email: liyl@hznu.edu.cn.

Abstract: Let (M, g) be an n-dimensional (pseudo-)Riemannian manifold and TM be its tangent bundle TM equipped with the complete lift metric ${}^{C}g$. First, we define a Ricci quarter-symmetric metric connection $\overline{\nabla}$ on the tangent bundle TM equipped with the complete lift metric ${}^{C}g$. Second, we compute all forms of the curvature tensors of $\overline{\nabla}$ and study their properties. We also define the mean connection of $\overline{\nabla}$. Ricci and gradient Ricci solitons are important topics studied extensively lately. Necessary and sufficient conditions for the tangent bundle TM to become a Ricci soliton and a gradient Ricci soliton concerning $\overline{\nabla}$ are presented. Finally, we search conditions for the tangent bundle TM to be locally conformally flat with respect to $\overline{\nabla}$.

Keywords: complete lift metric; Ricci quarter-symmetric metric connection; Ricci soliton; tangent bundle; vector field
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1. Introduction

The notion of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten in [1]. Using Hayden's idea [2] of a metric connection with torsion, Yano [34] searched properties of a semi-symmetric metric connection on a Riemannian manifold. He proved that a Riemannian manifold endowed with the semi-symmetric metric connection has a vanishing curvature tensor, if and only if, the Riemannian manifold is conformally flat. Later, Golab [11] defined and studied quarter-symmetric connections on differentiable manifolds with linear connections. With this in hand, Yano and Imai [35] gave the most general form of quarter-symmetric metric connections on Riemannian, Hermitian and Kaehlerian manifolds and studied its applications. If the torsion tensor T of a connection is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y, \tag{1.1}$$

then the linear connection is said to be a quarter-symmetric connection. In here, u is a non-zero 1-form, ϕ is a $(1,1)-$tensor, and X, Y are vector fields. In particular, if $\phi = id$, then the quarter-symmetric connection reduces to the semi-symmetric connection. Thus, the notion of a quarter-symmetric connection can be viewed as a generalization of the idea of a semi-symmetric connection. Here, it is obvious that a quarter-symmetric metric connection is a Hayden connection in the form of a torsion tensor (1.1).

Also, if we take the ϕ tensor as a $(1,1)$ type Ricci tensor defined by

$$g(\phi X, Y) = R(X,Y),$$

then the quarter-symmetric connection is called a Ricci quarter-symmetric connection. If a Ricci quarter-symmetric connection ∇ on a Riemannian manifold satisfies the condition

$$(\nabla_X g)(Y,Z) = 0,$$

then ∇ is said to be a Ricci quarter-symmetric metric connection (briefly RQSMC) for all vector fields X, Y, Z on M. Kamilya and De presented the concept of a RQSMC on a Riemannian manifold and found necessary and sufficient conditions for the symmetry of the Ricci tensor of a RQSMC [12]. Also, they studied an Einstein manifold admitting a Ricci quarter-symmetric metric connection whose torsion tensor is defined by means of the Ricci tensor of a Riemannian metric.

Ricci solitons became popular after Grigori Perelman applied Ricci solitons to solve the long-standing Poincare conjecture posed in 1904. The notion of Ricci soliton appeared after Hamilton introduced the Ricci flow in 1982. Let us start with M being a Riemannian manifold with a Riemannian metric g. A Ricci flow satisfies the following equation

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)),$$

where t is the time and Ric denotes the Ricci tensor of M. Ricci solitons correspond to self-similar solutions of Ricci flow, and they model the formation of singularities in the Ricci flow. A smooth vector field V on a Riemannian manifold (M,g) is said to define a Ricci soliton if it satisfies

$$\frac{1}{2} L_V g + Ric = \lambda g,$$

where $L_V g$ is the Lie derivative of the Riemannian metric g with respect to V and λ is a constant. We shall denote a Ricci soliton by triple (g, V, λ). A Ricci soliton is called shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Also, a Ricci soliton is called a gradient Ricci soliton if its potential vector field V is the gradient of some smooth function f on M.

In this paper, first, we shall define a RQSMC on the tangent bundle equipped with complete lift metric over a pseudo-Riemannian manifold. Second, we find all kinds of curvature tensors and study some properties of them. We investigate mean connections of the RQSMC. After that, we investigate some conditions for a vector field $\widetilde{V}$ on TM, such that it becomes $({}^C g, \widetilde{V}, \lambda)$ a Ricci soliton and gradient Ricci soliton. Finally, we study conditions for the tangent bundle TM to be locally conformally flat with respect to the RQSMC.

2. Preliminaries

For all the details about this section, we refer to [36]. Let M be an $n-$dimensional differentiable manifold of class C^∞ and TM its tangent bundle. The natural projection defined by

$$\begin{array}{rcl}\pi & : & TM \to M \\ \widetilde{P} & \to & \pi(\widetilde{P}) = P\end{array}$$

determines the correspondence of $(\widetilde{P} \to P)$ for any point $P \in M$. The set $\pi^{-1}(P) = \widetilde{P} \in T_PM$ is called fibre on $P \in M$. Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighborhood and (x^h), $h = 1, ..., n$ are the coordinate functions. Let $(y^h) = (x^{\overline{h}})$, $\overline{h} = n + 1, ..., 2n$ be the Cartesian coordinates in each tangent space T_PM at $P \in M$ with respect to natural basis $\left\{\frac{\partial}{\partial x^h} |_P\right\}$, where P is an arbitrary point in U with local coordinates (x^h). Then, we can introduce local coordinates (x^h, y^h) on the open set $\pi^{-1}(U) \subset TM$. The coordinate system of $(x^h, y^h) = (x^h, x^{\overline{h}})$ is called induced coordinates on $\pi^{-1}(U)$ from (U, x^h). In the paper, we use Einstein's convention on repeated indices.

Let $X = X^h \frac{\partial}{\partial x^h}$ be the local expression in U of a vector field X on M. Given a (torsion-free) linear connection ∇ on M, the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X are respectively given by

$${}^VX = X^h \partial_{\overline{h}},$$

and

$${}^HX = X^h \partial_h - y^s \Gamma^h_{sk} X^k \partial_{\overline{h}}$$

with respect to the induced coordinates, where $\partial_h = \frac{\partial}{\partial x^h}, \partial_{\overline{h}} = \frac{\partial}{\partial y^h}$ and Γ^h_{jk} are the coefficients of the connection ∇. Through these lifts and the connection ∇, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM a frame field which consists of the following $2n$ linearly independent vector fields

$$\begin{array}{rcl}E_j & = & \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \\ E_{\overline{j}} & = & \partial_{\overline{j}}.\end{array}$$

We are calling it as the adapted frame and it will be written as $\left\{E_\beta\right\} = \left\{E_j, E_{\overline{j}}\right\}$ [36]. With respect to adapted frame $\left\{E_\beta\right\}$, the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X is expressed by [36]

$$\begin{array}{rcl}{}^HX & = & X^j E_j, \\ {}^VX & = & X^j E_{\overline{j}}.\end{array}$$

3. Ricci quarter symmetric metric connection on the tangent bundle with the complete lift metric

3.1. The Ricci quarter symmetric metric connection

A linear connection ∇ on an $n-$dimensional differentiable manifold M is said to be a Ricci quarter-symmetric connection if its torsion tensor T satisfies

$$T(X, Y) = \phi(Y)LX - \phi(X)LY,$$

where ϕ is a non-zero 1−form, L is the $(1,1)$ Ricci tensor defined by

$$g(LX, Y) = R(X, Y)$$

and R is the Ricci tensor of M [12]. The tensor T denotes the torsion tensor of ∇, that is,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all vector fields X, Y on M. On a (pseudo-)Riemannian manifold (M, g), a linear connection ∇ is called a metric connection if

$$\nabla g = 0.$$

A linear connection ∇ is said to be a RQSMC if it is both Ricci quarter-symmetric and metric connection. If ∇ is the Levi-Civita connection of M then a RQSMC is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \phi(Y)LX - T(X, Y)\rho,$$

where $\phi(X) = g(X, \rho)$.

Let M be an n−dimensional pseudo-Riemannian manifold with a pseudo-Riemannian metric g and let TM be its tangent bundle. The complete lift metric ${}^C g$ on TM is defined as follows:

$$\begin{aligned} {}^C g\left({}^H X, {}^H Y\right) &= 0, \\ {}^C g\left({}^H X, {}^V Y\right) &= {}^C g\left({}^V X, {}^H Y\right) = g(X, Y), \\ {}^C g\left({}^V X, {}^V Y\right) &= 0, \end{aligned}$$

for all vector fields X and Y on M [36]. ${}^C g$ is a pseudo-Riemannian metric on TM. The covariant and contravariant components of the complete lift metric ${}^C g$ on TM are respectively given in the adapted local frame by

$${}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and

$${}^C g^{\alpha\beta} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & 0 \end{pmatrix}.$$

For the Levi-Civita connection ${}^C\nabla$ of the complete lift metric ${}^C g$, we have the following proposition.

Proposition 1. *The Levi-Civita connection ${}^C\nabla$ of $(TM, {}^C g)$ is given by*

$$\begin{cases} {}^C\nabla_{E_i} E_j = \Gamma_{ij}^k E_k + y^s R_{sij}^{\ \ \ k} E_{\overline{k}}, \\ {}^C\nabla_{E_i} E_{\overline{j}} = \Gamma_{ij}^k E_{\overline{k}}, \\ {}^C\nabla_{E_{\overline{i}}} E_j = 0, \ {}^C\nabla_{E_{\overline{i}}} E_{\overline{j}} = 0, \end{cases} \tag{3.1}$$

with respect to the adapted frame $\left\{E_\beta\right\}$, where Γ_{ij}^h and $R_{sij}^{\ \ \ k}$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor field R of the pseudo-Riemannian metric g on M (see [36]).

Now, we are interested in a RQSMC $\overline{\nabla}$ on $(TM,^C g)$. We denote the components of the RQSMC $\overline{\nabla}$ by $\overline{\Gamma}$. A RQSMC $\overline{\nabla}$ satisfies

$$\overline{\nabla}_\alpha(^C g_{\beta\gamma}) = 0 \text{ and } \overline{\Gamma}^\gamma_{\alpha\beta} - \overline{\Gamma}^\gamma_{\beta\alpha} - [E_\alpha, E_\beta] = \overline{T}^\gamma_{\alpha\beta}, \tag{3.2}$$

where $\overline{T}^\gamma_{\alpha\beta}$ are the components of the torsion tensor of $\overline{\nabla}$. When the Eq (3.2) is solved with respect to $\overline{\Gamma}^\gamma_{\alpha\beta}$, we find the following solution [2]:

$$\overline{\Gamma}^\gamma_{\alpha\beta} = {}^C\Gamma^\gamma_{\alpha\beta} + U^\gamma_{\alpha\beta}, \tag{3.3}$$

where ${}^C\Gamma^\gamma_{\alpha\beta}$ are the components of the Levi-Civita connection of ${}^C g$,

$$U_{\alpha\beta\gamma} = \frac{1}{2}(\overline{T}_{\alpha\beta\gamma} + \overline{T}_{\gamma\alpha\beta} + \overline{T}_{\gamma\beta\alpha}) \tag{3.4}$$

and

$$U_{\alpha\beta\gamma} = U^\epsilon_{\alpha\beta}\,{}^C g_{\epsilon\gamma},\ \overline{T}_{\alpha\beta\gamma} = T^\epsilon_{\alpha\beta}\,{}^C g_{\epsilon\gamma}.$$

We put

$$\overline{T}_{ij}{}^{\bar{k}} = y_j R_i{}^k - y_i R_j{}^k \tag{3.5}$$

and all other $\overline{T}^\gamma_{\alpha\beta}$ not related to $\overline{T}_{ij}{}^{\bar{k}}$ are assumed to be zero, where $y_i = y^s g_{si}$. By using (3.4) and (3.5), we get the only non-zero component of $U^\gamma_{\alpha\beta}$ as follows

$$U_{ij}{}^{\bar{h}} = y_j R_i{}^h - y^h R_{ij}$$

with respect to the adapted frame. From (3.3) and (3.1), we have components of the RQSMC $\overline{\nabla}$ with respect to ${}^C g$ as follows:

$$\begin{array}{ll}
(i)\ \overline{\Gamma}_{ij}{}^{k} = \Gamma_{ij}{}^{k}, & (v)\ \overline{\Gamma}_{ij}{}^{\bar{k}} = y^s R_{sij}{}^{k} + y_j R_i{}^k - y^k R_{ij}, \\
(ii)\ \overline{\Gamma}_{\bar{i}j}{}^{k} = 0, & (vi)\ \overline{\Gamma}_{\bar{i}j}{}^{\bar{k}} = 0, \\
(iii)\ \overline{\Gamma}_{i\bar{j}}{}^{k} = 0, & (vii)\ \overline{\Gamma}_{i\bar{j}}{}^{\bar{k}} = \Gamma_{ij}{}^{k}, \\
(iv)\ \overline{\Gamma}_{\bar{i}\bar{j}}{}^{k} = 0, & (viii)\ \overline{\Gamma}_{\bar{i}\bar{j}}{}^{\bar{k}} = 0,
\end{array} \tag{3.6}$$

which gives the following proposition.

Proposition 2. *The RQSMC $\overline{\nabla}$ of $(TM,^C g)$ is given by*

$$\begin{cases}
\overline{\nabla}_{E_i} E_j = \Gamma^k_{ij} E_k + \{y^s R_{sij}{}^k + y_j R_i{}^k - y^k R_{ij}\} E_{\bar{k}}, \\
\overline{\nabla}_{E_i} E_{\bar{j}} = \Gamma^k_{ij} E_{\bar{k}}, \\
\overline{\nabla}_{E_{\bar{i}}} E_j = 0,\ \overline{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0,
\end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij} and $R_{hji}{}^s$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor field R of the pseudo-Riemannian metric g on M.

Given a pseudo-Riemannian metric g on a differentiable manifold M, another well-known classical pseudo-Riemannian metric on TM is the metric $I + II$ defined by

$$\begin{aligned}
\widetilde{g}\left(X^H, Y^H\right) &= g(X, Y), \\
\widetilde{g}\left(X^H, Y^V\right) &= \widetilde{g}\left(X^V, Y^H\right) = g(X, Y), \\
\widetilde{g}\left(X^V, Y^V\right) &= 0,
\end{aligned}$$

for all vector fields X, Y on M [36]. The metric $I + II$ has the components

$$\widetilde{g}_{\alpha\beta} = \begin{pmatrix} g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

with respect to the adapted frame. Let us consider the covariant derivation of the metric $I + II$ with respect to the RQSMC $\overline{\nabla}$. One checks that

$$\begin{aligned}
\overline{\nabla}_k \overline{g}_{ij} &= E_k \widetilde{g}_{ij} - \overline{\Gamma}^h_{ki} \widetilde{g}_{hj} - \overline{\Gamma}^{\bar{h}}_{ki} \widetilde{g}_{\bar{h}j} - \overline{\Gamma}^h_{kj} \widetilde{g}_{ih} - \overline{\Gamma}^{\bar{h}}_{kj} \widetilde{g}_{i\bar{h}} \\
&= (\partial_k - y^s \Gamma^h_{sk} \partial_{\bar{k}}) g_{ij} - \Gamma^h_{ki} g_{hj} - (y^s R_{ski}{}^h + y_i R^h_k \\
&\quad - y^h R_{ki}) g_{hj} - \Gamma^h_{kj} g_{ih} - (y^s R_{skj}{}^h + y_j R^h_k - y^h R_{kj}) g_{ih} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - y^s R_{skij} - y_i R_{kj} + y_j R_{ki} - \Gamma^h_{kj} g_{ih} \\
&\quad - y^s R_{skji} - y_j R_{ik} + y_i R_{kj} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ih} \\
&= \nabla_k g_{ij} = 0, \\
\overline{\nabla}_k \widetilde{g}_{i\bar{j}} &= E_k \widetilde{g}_{i\bar{j}} - \overline{\Gamma}^h_{ki} \overline{g}_{h\bar{j}} - \overline{\Gamma}^{\bar{h}}_{ki} \underbrace{\widetilde{g}_{\bar{h}\bar{j}}}_{0} - \underbrace{\overline{\Gamma}^h_{k\bar{j}}}_{0} \widetilde{g}_{ih} - \overline{\Gamma}^{\bar{h}}_{k\bar{j}} \widetilde{g}_{i\bar{h}} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ih} = \nabla_k g_{ij} = 0, \\
\overline{\nabla}_k \overline{g}_{\bar{i}j} &= E_k \overline{g}_{\bar{i}j} - \underbrace{\overline{\Gamma}^h_{k\bar{i}}}_{0} \overline{g}_{hj} - \overline{\Gamma}^{\bar{h}}_{k\bar{i}} \overline{g}_{\bar{h}j} - \overline{\Gamma}^h_{kj} \overline{g}_{\bar{i}h} - \overline{\Gamma}^{\bar{h}}_{kj} \underbrace{\overline{g}_{\bar{i}\bar{h}}}_{0} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \overline{\Gamma}^h_{kj} g_{ih} = \nabla_k g_{ij} = 0,
\end{aligned}$$

all others are automatically zero. Hence, we can state following result.

Proposition 3. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^C g$ or the metric $I + II$. The RQSMC $\overline{\nabla}$ with respect to the complete lift metric ${}^C g$ is also a RQSMC with respect to the metric $I + II$.*

3.2. The curvature tensors

The curvature tensor $\overline{R}$ of the RQSMC $\overline{\nabla}$ of $(TM, {}^C g)$ is obtained from the well-known formula

$$\overline{R}\left(\widetilde{X}, \widetilde{Y}\right)\widetilde{Z} = \overline{\nabla}_{\widetilde{X}} \overline{\nabla}_{\widetilde{Y}} \widetilde{Z} - \overline{\nabla}_{\widetilde{Y}} \overline{\nabla}_{\widetilde{X}} \widetilde{Z} - \overline{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on TM. From Proposition 2, we get the following.

Proposition 4. *The curvature tensor $\overline{R}$ of the RQSMC $\overline{\nabla}$ of $(TM,^C g)$ is given as follows*

$$\begin{aligned} \overline{R}(E_i, E_j)E_k &= R_{ijk}{}^l E_l + \{y^s \nabla_s R_{ijk}{}^l\} E_{\bar{l}}, \\ \overline{R}(E_i, E_j)E_{\bar{k}} &= R_{ijk}{}^l E_{\bar{l}}, \\ \overline{R}(E_i, E_{\bar{j}})E_k &= \{R_{ijk}{}^l + R_{ik}\delta^l_j - g_{jk}R_i{}^l\} E_{\bar{l}}, \\ \overline{R}(E_{\bar{i}}, E_j)E_k &= \{R_{ijk}{}^l + g_{ik}R_j{}^l - R_{jk}\delta^l_i\} E_{\bar{l}}, \\ \overline{R}(E_{\bar{i}}, E_{\bar{j}})E_k &= 0, \ \overline{R}(E_{\bar{i}}, E_j)E_{\bar{k}} = 0, \\ \overline{R}(E_i, E_{\bar{j}})E_{\bar{k}} &= 0, \ \overline{R}(E_{\bar{i}}, E_{\bar{j}})E_{\bar{k}} = 0, \end{aligned} \tag{3.7}$$

with respect to the adapted frame $\left\{E_\beta\right\}$.

Since the Levi-Civita connections of the complete lift metric $^C g$ and the metric $I + II$ coincide, their Riemannian curvature tensors coincide [36]. The Riemannian curvature tensor $\widehat{R}$ of the Levi-Civita connection of the complete lift metric $^C g$ (or the metric $I + II$) is given by

$$\begin{aligned} \widehat{R}(E_i, E_j)E_k &= R_{ijk}{}^l E_l + \{y^s \nabla_s R_{ijk}{}^l\} E_{\bar{l}}, \\ \widehat{R}(E_i, E_j)E_{\bar{k}} &= R_{ijk}{}^l E_{\bar{l}}, \\ \widehat{R}(E_i, E_{\bar{j}})E_k &= R_{ijk}{}^l E_{\bar{l}}, \\ \widehat{R}(E_{\bar{i}}, E_j)E_k &= R_{ijk}{}^l E_{\bar{l}}, \\ \widehat{R}(E_{\bar{i}}, E_{\bar{j}})E_k &= 0, \ \widehat{R}(E_{\bar{i}}, E_j)E_{\bar{k}} = 0, \\ \widehat{R}(E_i, E_{\bar{j}})E_{\bar{k}} &= 0, \ \widehat{R}(E_{\bar{i}}, E_{\bar{j}})E_{\bar{k}} = 0. \end{aligned}$$

On comparing the curvature tensors of the Levi-Civita connection of the complete lift metric $^C g$ (or the metric $I + II$) and the RQSMC, we have the result below.

Corollary 1. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric $^C g$. The curvature tensors of the Levi-Civita connection of the complete lift metric $^C g$ (or the metric $I + II$) and the RQSMC if and only if $R_{ik}\delta^l_j - g_{jk}R_i{}^l = 0$.*

Theorem 1. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric $^C g$. The curvature $(0, 4)-$tensor $\overline{R}$ of the RQSMC $\overline{\nabla}$ holds the followings*

i) $\overline{R}_{\alpha\beta\gamma\sigma} + \overline{R}_{\beta\alpha\gamma\sigma} = 0$,

ii) $\overline{R}_{\alpha\beta\gamma\sigma} + \overline{R}_{\alpha\beta\sigma\gamma} = 0$.

Proof. On lowering the upper index of the curvature tensor $\overline{R}$ of the RQSMC $\overline{\nabla}$, the non-zero components of the curvature $(0, 4)-$tensor are obtained as follows

$$\begin{aligned} \overline{R}_{ijkh} &= y^s \nabla_s R_{ijkh}, \\ \overline{R}_{ijk\bar{h}} &= R_{ijkh}, \\ \overline{R}_{ij\bar{k}h} &= R_{ijkh}, \\ \overline{R}_{i\bar{j}kh} &= R_{ijkh} - g_{jk}R_{im} + R_{ik}g_{jm}, \\ \overline{R}_{\bar{i}jkh} &= R_{ijkh} + g_{ik}R_{jm} - R_{jk}g_{im}. \end{aligned} \tag{3.8}$$

i) and *ii)* The results immediately follows from the above relations. □

Let $\overline{K}_{\alpha\beta} = \overline{R}_{\sigma\alpha\beta}{}^{\sigma}$ denote the Ricci tensor of the RQSMC $\overline{\overline{\nabla}}$. Then

$$\begin{aligned} \overline{K}_{jk} &= (3-n)R_{jk}, \\ \overline{K}_{j\bar{k}} &= 0, \\ \overline{K}_{\bar{j}k} &= 0, \\ \overline{K}_{\bar{j}\bar{k}} &= 0, \end{aligned} \tag{3.9}$$

from which the following result follows.

Theorem 2. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. The Ricci tensor of the RQSMC $\overline{\overline{\nabla}}$ is symmetric.*

Theorem 3. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. Then TM is Ricci flat with respect to the RQSMC $\overline{\overline{\nabla}}$ if and only if M is Ricci flat.*

A (pseudo-)Riemannian manifold (M, g) is called Ricci semi-symmetric if the following condition is satisfied [33]

$$R(X, Y).K = 0,$$

where $R(X, Y)$ is a linear operator acting as a derivation on the Ricci curvature tensor K of (M, g).

The curvature operator $\overline{R}(\widetilde{X}, \widetilde{Y})$ is a differential operator on TM for all vector fields $\widetilde{X}$ and $\widetilde{Y}$. Now we operate the curvature operator $\overline{R}(\widetilde{X}, \widetilde{Y})$ to the Ricci curvature tensor $\overline{K}$, that is, for all $\widetilde{Z}, \widetilde{W}$, we consider the condition $(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W}) = 0$. In this case, we shall call TM Ricci semi-symmetric with respect to the Ricci quarter-symmetric metric connection $\overline{\overline{\nabla}}$.

In the adapted frame $\left\{E_{\beta}\right\}$, the tensor $(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})$ is locally expressed as follows

$$(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})_{\alpha\beta\gamma\theta} = \overline{R}_{\alpha\beta\gamma}{}^{\varepsilon}\overline{K}_{\varepsilon\theta} + \overline{R}_{\alpha\beta\theta}{}^{\varepsilon}\overline{K}_{\gamma\varepsilon}. \tag{3.10}$$

Similarly, in local coordinates,

$$((\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})_{ijkl} = \overline{R}_{ijk}{}^{p}\overline{K}_{pl} + \overline{R}_{ijl}{}^{p}\overline{K}_{kp}.$$

Theorem 4. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. Then TM is Ricci semi-symmetric with respect to the RQSMC $\overline{\overline{\nabla}}$ if and only if M is Ricci semi-symmetric and $n \neq 3$.*

Proof. By putting $\alpha = i, \beta = j, \gamma = k, \theta = l$ in (3.10), we find

$$\begin{aligned} & (\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})_{ijkl} \\ = & \ \overline{R}_{ijk}{}^{h}\overline{K}_{hl} + \overline{R}_{ijl}{}^{h}\overline{K}_{kh} \\ = & \ R_{ijk}^{\ \ h}[(3-n)R_{hl}] + R_{ijl}^{\ \ h}[(3-n)R_{kh}] \\ = & \ (3-n)[R_{ijk}^{\ \ h}R_{hl} + R_{ijl}^{\ \ h}R_{kh}] \\ = & \ (3-n)[(R(X, Y)Ric)(Z, W)]_{ijkl}, \end{aligned}$$

all the others being zero. This finishes the proof. □

For the scalar curvature $\overline{r}$ of the RQSMC $\overline{\nabla}$ with respect to ${}^{C}g$, we find

$$\overline{r} = \overline{K}_{\alpha\beta}{}^{C}g^{\alpha\beta} = \overline{K}_{jk}{}^{C}g^{jk} + \overline{K}_{\bar{j}k}{}^{C}g^{\bar{j}k} + \overline{K}_{j\bar{k}}{}^{C}g^{j\bar{k}} + \overline{K}_{\bar{j}\bar{k}}{}^{C}g^{\bar{j}\bar{k}} = 0.$$

Then we can state the following.

Theorem 5. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. The scalar curvature of TM with the RQSMC $\overline{\nabla}$ with respect to ${}^{C}g$ vanishes.*

As an application concerning the torsion tensor $\overline{T}$ of the RQSMC $\overline{\nabla}$, we get the following.

Theorem 6. *Let $\overline{\nabla}$ be a RQSMC on TM. Then for all vector felds $\overline{X}$,$\overline{Y}$ and $\overline{Z}$ on TM*

$$\underset{\overline{X},\overline{Y},\overline{Z}}{\sigma} \overline{T}\left(\overline{T}(\overline{X}, \overline{Y})\overline{Z}\right) = 0,$$

where σ is the cyclic sum by three arguments and $\overline{T}$ is the torsion tensor of the RQSMC $\overline{\nabla}$.

Proof. For all vector felds $\overline{X}$,$\overline{Y}$ and $\overline{Z}$ on TM, $\underset{\overline{X},\overline{Y},\overline{Z}}{\sigma} \overline{T}\left(\overline{T}(\overline{X}, \overline{Y})\overline{Z}\right)$ can be written as follows

$$\underset{\overline{X},\overline{Y},\overline{Z}}{\sigma} \overline{T}\left(\overline{T}(\overline{X}, \overline{Y})\overline{Z}\right) = \overline{T}_{\alpha\beta}^{\ \ \epsilon}\overline{T}_{\epsilon\gamma}^{\ \ \sigma} + \overline{T}_{\gamma\alpha}^{\ \ \epsilon}\overline{T}_{\epsilon\beta}^{\ \ \sigma} + \overline{T}_{\beta\gamma}^{\ \ \epsilon}\overline{T}_{\epsilon\alpha}^{\ \ \sigma}$$

in the adapted frame $\{E_\beta\}$. By using (3.5), standard calculations directly give the result. □

A $(0, 2)$ generalized tensor Z is defined by

$$Z(X, Y) = Ric(X, Y) + \phi g(X, Y)$$

for all vectors X and Y on M. Analogous to this definition, a tensor $\overline{Z}$ may be locally defined on TM as follows

$$\overline{Z}_{\alpha\beta} = \overline{R}_{\alpha\beta} + \phi\overline{g}_{\alpha\beta}.$$

Here $\overline{R}_{\alpha\beta}$ denote the components of the Ricci tensor of the RQSMC $\overline{\nabla}$ and $\overline{g}_{\alpha\beta}$ denote the components of the complete lift metric ${}^{C}g$. Putting the values of $\overline{R}_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ in the above equation, we have the non-zero components

$$\begin{aligned} \overline{Z}_{ij} &= (3 - n)R_{ij}, \\ \overline{Z}_{i\bar{j}} &= \overline{Z}_{\bar{i}j} = \phi g_{ij}. \end{aligned} \tag{3.11}$$

Hence, we have the following result.

Theorem 7. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle. The tensor $\overline{Z}$ of the RQSMC $\overline{\nabla}$ is symmetric.*

Theorem 8. *Let ∇ be the Levi-Civita connection on a Riemannian manifold (M, g) and TM be the tangent bundle. TM is $\overline{Z}$ semi-symmetric with respect to the RQSMC $\overline{\nabla}$ if and only if the Riemannian manifold (M, g) is Ricci semi-symmetric with respect to ∇ and $n \neq 3$.*

Proof. The tensor $\overline{R}(\overline{X},\overline{Y}).\overline{Z}$ has the components

$$(\overline{R}(\overline{X},\overline{Y}).\overline{Z})_{\alpha\beta\gamma\sigma}=\overline{R}_{\alpha\beta\gamma}^{\ \ \ \epsilon}\overline{Z}_{\epsilon\sigma}+\overline{R}_{\alpha\beta\sigma}^{\ \ \ \epsilon}\overline{Z}_{\gamma\epsilon}$$

with respect to the adapted frame $\{E_\beta\}$. By using (3.7) and (3.11) on the above equation we find the only non-zero component

$$(\overline{R}(\overline{X},\overline{Y}).\overline{Z})_{ijkm}=(3-n)(R(X,Y)Ric)_{ijkm}.$$

This completes the proof. □

Next, we are interested in the mean connection of the RQSMC $\overline{\nabla}$ on $(TM,^C g)$. We denote the components of the mean connection $\widetilde{\nabla}$ by $\widetilde{\Gamma}$. From (3.5) and (3.6), by using $\widetilde{\Gamma}^{\gamma}_{\alpha\beta}=\overline{\Gamma}^{\gamma}_{\alpha\beta}-\frac{1}{2}\overline{T}^{\gamma}_{\alpha\beta}$ we have components of the mean connection with respect to RQSMC $\overline{\nabla}$ as follows:

$$\begin{array}{ll}(i)\ \overline{\Gamma}_{ij}^{\ \ k}=\Gamma_{ij}^{\ \ k} & (v)\ \overline{\Gamma}_{ij}^{\ \ \bar{k}}=y^sR_{sij}^{\ \ \ k}-y^kR_{ij}+\frac{1}{2}(y_jR_i^{\ k}+y_iR_j^{\ k})\\ (ii)\ \overline{\Gamma}_{\bar{i}j}^{\ \ k}=0 & (vi)\ \overline{\Gamma}_{\bar{i}j}^{\ \ \bar{k}}=0\\ (iii)\ \overline{\Gamma}_{i\bar{j}}^{\ \ k}=0 & (vii)\ \overline{\Gamma}_{i\bar{j}}^{\ \ \bar{k}}=\Gamma_{ij}^{\ \ k}\\ (iv)\ \overline{\Gamma}_{\bar{i}\bar{j}}^{\ \ k}=0 & (viii)\ \overline{\Gamma}_{\bar{i}\bar{j}}^{\ \ \bar{k}}=0.\end{array}$$

Hence we get the following proposition.

Proposition 5. *The mean connection of the RQSMC $\overline{\nabla}$ of $(TM,^C g)$ is given by*

$$\begin{cases}\widetilde{\nabla}_{E_i}E_j=\Gamma^k_{ij}E_k+\{y^sR_{sij}^{\ \ \ k}-y^kR_{ij}+\frac{1}{2}(y_jR_i^{\ k}+y_iR_j^{\ k})\}E_{\bar{k}},\\ \widetilde{\nabla}_{E_i}E_{\bar{j}}=\Gamma^k_{ij}E_{\bar{k}},\\ \widetilde{\nabla}_{E_{\bar{i}}}E_j=0,\ \widetilde{\nabla}_{E_{\bar{i}}}E_{\bar{j}}=0,\end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij} and $R_{hji}^{\ \ \ s}$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor R of the pseudo-Riemannian metric g on M.

From Proposition 5, we get the following.

Proposition 6. *The curvature tensor $\widetilde{R}$ of the mean connection $\widetilde{\nabla}$ of $(TM,^C g)$ is given as follows:*

$$\begin{aligned}\widetilde{R}(E_i,E_j)E_k &= R_{ijk}^{\ \ \ l}E_l+\{y^s\nabla_sR_{ijk}^{\ \ \ l}\}E_{\bar{l}},\\ \widetilde{R}(E_i,E_j)E_{\bar{k}} &= R_{ijk}^{\ \ \ l}E_{\bar{l}},\\ \widetilde{R}(E_i,E_{\bar{j}})E_k &= \{R_{ijk}^{\ \ \ l}+\delta^l_jR_{ik}-\frac{1}{2}(g_{jk}R_i^{\ l}+g_{ji}R_k^{\ l})\}E_{\bar{l}},\\ \widetilde{R}(E_{\bar{i}},E_j)E_k &= \{R_{ijk}^{\ \ \ l}-\delta^l_iR_{jk}+\frac{1}{2}(g_{ik}R_j^{\ l}+g_{ij}R_k^{\ l})\}E_{\bar{l}},\\ \widetilde{R}(E_{\bar{i}},E_{\bar{j}})E_k &= 0,\ \widetilde{R}(E_{\bar{i}},E_j)E_{\bar{k}}=0,\\ \widetilde{R}(E_i,E_{\bar{j}})E_{\bar{k}} &= 0,\ \widetilde{R}(E_{\bar{i}},E_{\bar{j}})E_{\bar{k}}=0,\end{aligned}$$

with respect to the adapted frame $\left\{E_\beta\right\}$.

Let $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ denotes the Ricci tensor of the mean connection with respect to the RQSMC $\overline{\nabla}$. Then

$$\widetilde{R}_{jk} = (3-n)R_{jk},$$
$$\widetilde{R}_{\bar{j}k} = 0,$$
$$\widetilde{R}_{j\bar{k}} = 0,$$
$$\widetilde{R}_{\bar{j}\bar{k}} = 0,$$

from which the following result follows.

Theorem 9. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. The Ricci tensor of the mean connection and Ricci tensor of the RQSMC coincide.*

3.3. Ricci soliton structures on $(TM, {}^{C}g)$

A Ricci soliton is defined by a smooth vector field V on a Riemannian manifold (M, g) such that

$$\frac{1}{2}L_V g + Ric = \lambda g \tag{3.12}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g with respect to V, Ric is the Ricci tensor of (M, g) and λ is a constant. The vector field V is called the potential vector field of the Ricci soliton.

Now, we give some conditions for a vector field $\widetilde{V}$ on TM , such that $({}^{C}g, \widetilde{V}, \lambda)$ becomes a Ricci soliton with respect to the RQSMC $\overline{\nabla}$. Let $\widetilde{V}$ be a fibre-preserving vector field on TM with components $\left(v^h, v^{\bar{h}}\right)$ with respect to the adapted frame $\left\{E_\beta\right\}$, that is, v^h depend only on the variables $\left(x^h\right)$. From (3.12), $({}^{C}g, \widetilde{V}, \lambda)$ is a Ricci soliton on TM if and only if the following equations are satisfied

$$\frac{1}{2}L_{\widetilde{V}}\,{}^{C}g(X^v, X^h) + \overline{K}(X^v, X^h) = \lambda^{C}g(X^v, X^h), \tag{3.13}$$

$$\frac{1}{2}L_{\widetilde{V}}\,{}^{C}g(X^h, X^v) + \overline{K}(X^h, X^v) = \lambda^{C}g(X^h, X^v), \tag{3.14}$$

$$\frac{1}{2}L_{\widetilde{V}}\,{}^{C}g(X^h, X^h) + \overline{K}(X^h, X^h) = \lambda^{C}g(X^h, X^h), \tag{3.15}$$

for any vector fields X and Y on M. Here $\overline{K}$ denotes the Ricci tensor of the RQSMC $\overline{\nabla}$. With respect to the adapted frame $\left\{E_\beta\right\}$, a vector field $\widetilde{V}$ on $(TM, {}^{C}g)$ is said to define a Ricci soliton if there exists a real constant λ such that

$$\frac{1}{2}L_{\widetilde{V}}\widetilde{g}_{\alpha\beta} + \overline{K}_{\alpha\beta} = \lambda\widetilde{g}_{\alpha\beta}.$$

Putting $(\alpha, \beta) = (\bar{i}, j)$, $(i, \bar{j})$ and (i, j), from the above equation, it can be written the following system by using (3.9)

$$\begin{array}{ll} i) & \left(E_{\bar{i}}v^{\bar{h}}\right)g_{hj} + \left(\nabla_j v^h\right)g_{hi} = 2\lambda g_{ij}, \\ ii) & \left(\nabla_i v^h\right)g_{hj} + \left(E_{\bar{j}}v^{\bar{h}}\right)g_{hi} = 2\lambda g_{ij}, \\ iii) & \begin{array}{l} \left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^{h} + y_a R_i^h - y^h R_{ia}\right)v^a + \Gamma_{ia}^{h} v^{\bar{a}}\right]g_{hj} \\ +\left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^{h} + y_a R_j^h - y^h R_{ja}\right)v^a + \Gamma_{ja}^{h} v^{\bar{a}}\right]g_{hi} \\ +2(3-n)R_{jk} \end{array} = 0. \end{array} \tag{3.16}$$

Next, we will give a series of propositions. They will use the proof of the main theorem, which will be given at the end of this section.

Proposition 7. *The scalar function* λ *on* TM *depends only on the variables* $\left(x^h\right)$ *with respect to the induced coordinates* $\left(x^h, y^h\right)$.

Proof. Applying $E_{\bar{k}}$ to the both sides of the equation *(i)* in (3.16), we have

$$g_{hj}E_{\bar{k}}E_{\bar{i}}v^{\bar{h}} = 2E_{\bar{k}}(\lambda)\, g_{ij}$$

from which we get

$$E_{\bar{k}}(\lambda)\, g_{ij} = E_{\bar{i}}(\lambda)\, g_{kj},$$

it follows that

$$(n-1)E_{\bar{k}}(\lambda) = 0.$$

This shows that the scalar function λ on TM depends only on the variables $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, y^h\right)$. Thus we can regard λ as a function on M and in the following we write ρ instead of λ. □

From (3.16) and Proposition 7, $E_{\bar{i}}\left(v^{\bar{h}}\right)$ depends only the variables $\left(x^h\right)$, thus we can put

$$v^{\bar{h}} = y^a A_a^h + B^h, \tag{3.17}$$

where A_a^h and B^h are certain functions which depend only on the variable $\left(x^h\right)$. Furthermore, we can easily show that A_a^h and B^h are the components of a $(1,1)-$tensor field and a contravariant vector field on M, respectively.

Proposition 8. *If we put*

$$B = B^h \frac{\partial}{\partial x^h},$$

then we get $L_B g_{ij} = 2(n-3)R_{ij}$ *on* M .

Proof. Substituting (3.17) and (3.9) into the equation *(iii)* in (3.16) we have

$$\nabla_i B_j + \nabla_j B_i + 2(3-n)R_{ij} = 0 \tag{3.18}$$

and

$$v^a(R_{siaj} + R_{sjai} + g_{sa}R_{ij} - g_{sj}R_{ia} + g_{sa}R_{ji} - g_{si}R_{ja}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0 \tag{3.19}$$

where $B_i = g_{im}B^m$ and $A_{sj} = g_{hj}A_s^h$. Hence by (3.18), it follows

$$L_B g_{ij} = 2(n-3)R_{ij}.$$

□

Substituting (3.17) into the equation *(i)* in (3.16), we have

$$\begin{aligned} E_{\bar{i}}\left(v^{\bar{h}}\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} &= 2\rho g_{ij} \\ &\Rightarrow \partial_{\bar{i}}\left(y^s A_s^h + B^h\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\rho g_{ij} \\ &\Rightarrow A_i^h g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\rho g_{ij} \\ &\Rightarrow g_{hj} A_i^h = 2\rho g_{ij} - g_{hi}\left(\nabla_j v^h\right). \end{aligned} \tag{3.20}$$

Let ∇ be a linear connection on M. A vector field V on M is said to be a projective vector field if there exists a 1-form θ such that

$$(L_V\nabla)(X, Y) = \theta(X)Y + \theta(Y)X$$

for any vector fields X and Y on M. In this case θ is called the associated 1-form of V. It can locally be expressed in the following form

$$L_V\Gamma_{ij}^h = \theta_i\delta_j^h + \theta_j\delta_i^h.$$

Proposition 9. *The vector field V with components $\left(v^h\right)$ is a projective vector field (infinitesmal projective transformation) on M with respect to the Levi-Civita connection ∇, if*

$$2\delta_a^h R_{ij} - R_{ia}\delta_j^h - R_{ja}\delta_i^h = 0.$$

Proof. Applying the covariant derivative ∇_k to the both sides of (3.20), we obtain

$$\begin{aligned} g_{hj}\nabla_k A_i^h &= \nabla_k\left[2\rho g_{ij} - g_{hi}\left(\nabla_j v^h\right)\right] \\ &= 2\left(\nabla_k\rho\right) g_{ij} - g_{hi}\nabla_k\nabla_j v^h \\ &= 2\rho_k g_{ij} - g_{hi}\left(L_V\Gamma_{kj}^{\ h} - R_{akj}^{\ \ \ h} v^a\right) \\ \nabla_k A_{ij} &= 2\rho_k g_{ij} - L_V\Gamma_{kj}^{\ h} g_{hi} - R_{akij} v^a. \end{aligned} \tag{3.21}$$

Substituting (3.21) into (3.19), we have

$$v^a(R_{siaj} + R_{sjai} + g_{sa}R_{ij} - g_{sj}R_{ia} + g_{sa}R_{ji} - g_{si}R_{ja}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0$$

$$\Rightarrow \begin{array}{c} v^a(R_{siaj} + R_{sjai} + g_{sa}R_{ij} - g_{sj}R_{ia} + g_{sa}R_{ji} - g_{si}R_{ja}) \\ +2\rho_i g_{sj} - L_V\Gamma_{ij}^{\ h} g_{hs} - R_{aisj}v^a + 2\rho_j g_{si} - L_V\Gamma_{ji}^{\ h} g_{hs} - R_{ajsi}v^a \end{array} = 0$$

$$\Rightarrow v^a\left(g_{sa}R_{ij} - g_{sj}R_{ia} + g_{sa}R_{ji} - g_{si}R_{ja}\right) + 2\left(\rho_i g_{sj} + \rho_j g_{si}\right) = 2L_V\Gamma_{ij}^{\ h} g_{hs}$$

$$\Rightarrow L_V\Gamma_{ij}^{\ h} = \rho_i\delta_j^h + \rho_j\delta_i^h + \frac{1}{2}v^a\left(2\delta_a^h R_{ij} - R_{ia}\delta_j^h - R_{ja}\delta_i^h\right).$$

where $\rho_i = \nabla_i\rho$. Hence, V is a projective vector field on M with respect to the Levi-Civita connection ∇. □

Theorem 10. *Let $\widetilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ be a vector field on $(TM, {}^C g)$ with respect to the adapted frame $\left\{E_\beta\right\}$. Then $({}^C g, \widetilde{V}, \lambda)$ is a Ricci soliton on TM if and only if the following conditions are satisfied:*

i) λ on TM depends only the variables (x^h).

ii) The vector field V with the components (v^h) is an infinitesimal projective transformation on M.

iii) $v^{\bar{h}} = y^a A_a^h + B^h$.

iv) $A_{ij} = 2pg_{ij} - \nabla_j v_i$. v) $L_B g_{ij} = 2(n-3)R_{ij}$.

Proof. The Propositions 7–9 and the facts that have already been shown complete the proof of Theorem. □

3.4. Gradient Ricci soliton structures on $(TM,^C g)$

A Ricci soliton (g, V, λ) is called a gradient Ricci soliton if $V = \nabla f$. Here the smooth function f is called the potential function and the Eq (3.12) assumes the form:

$$Hess f + Ric = \lambda g, \tag{3.22}$$

where ∇f is the gradient of f and $Hess$ denotes the Hessian. We denote as usually the Hessian (with respect to the connection ∇) of any function f on M, by

$$(Hess_{\nabla} f)(X, Y) = XY f - (\nabla_X Y) f,$$

for any vector fields X and Y on M.

Lemma 1. *Let f be a smooth function on a Riemannian manifold (M, g). Then, the Hessian of its vertical lift is expressed by with respect to the RQSMC $\overline{\nabla}$ on $(TM,^C g)$:*

$$\begin{aligned}
Hess_{\overline{\nabla}}\,{}^V f({}^H X,{}^H Y) &= {}^H X\,{}^H Y\,{}^V f - (\overline{\nabla}_{{}^H X}\,{}^H Y)^V f.\\
Hess_{\overline{\nabla}}\,{}^V f(E_i, E_j) &= E_i\,E_j{}^V f - (\overline{\nabla}_{E_i} E_j)^V f\\
&= (\partial_i - y^s\Gamma^h_{si}\partial_{\overline{h}})(\partial_j - y^m\Gamma^l_{mj}\partial_{\overline{l}}) f\\
&\quad -[\Gamma^h_{ij}E_h + (y^s R_{sij}{}^h + y_j R^h_i - y^h R_{ij})E_{\overline{h}}]^V f\\
&= (\partial_i - y^s\Gamma^h_{si}\partial_{\overline{h}})(\partial_j f) - \Gamma^h_{ij}(\partial_h - y^s\Gamma^m_{sh}\partial_{\overline{m}}) f\\
&= \partial_i\partial_j f - \Gamma^h_{ij}\partial_h f\\
&= \nabla_i\nabla_j f. \qquad (3.23)\\
Hess_{\overline{\nabla}}\,{}^V f({}^V X,{}^V Y) &= {}^V X\,{}^V Y\,{}^V f - (\overline{\nabla}_{{}^V X}\,{}^V Y)^V f.\\
Hess_{\overline{\nabla}}\,{}^V f(E_{\overline{i}}, E_{\overline{j}}) &= E_{\overline{i}}\,E_{\overline{j}}{}^V f - (\overline{\nabla}_{E_{\overline{i}}} E_{\overline{j}})^V f\\
&= 0. \qquad (3.24)\\
Hess_{\overline{\nabla}}\,{}^V f({}^H X,{}^V Y) &= {}^H X\,{}^V Y\,{}^V f - (\overline{\nabla}_{{}^H X}\,{}^V Y)^V f.\\
Hess_{\overline{\nabla}}\,{}^V f(E_i, E_{\overline{j}}) &= E_i\,E_{\overline{j}}{}^V f - (\overline{\nabla}_{E_i} E_{\overline{j}})^V f\\
&= -(\Gamma^h_{i\overline{j}}E_h + \Gamma^{\overline{h}}_{i\overline{j}}E_{\overline{h}})^V f\\
&= 0. \qquad (3.25)\\
Hess_{\overline{\nabla}}\,{}^V f({}^V X,{}^H Y) &= {}^V X\,{}^H Y\,{}^V f - (\overline{\nabla}_{{}^V X}\,{}^H Y)^V f.\\
Hess_{\overline{\nabla}}\,{}^V f(E_{\overline{i}}, E_j) &= E_{\overline{i}}\,E_j{}^V f - (\overline{\nabla}_{E_{\overline{i}}} E_j)^V f\\
&= \partial_{\overline{i}}(\partial_j - y^s\Gamma^h_{sj}\partial_{\overline{h}})^V f\\
&= \partial_{\overline{i}}\partial_j f\\
&= 0. \qquad (3.26)
\end{aligned}$$

Now, we focus the gradient Ricci soliton.

Theorem 11. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^Cg$. For any smooth function f on M, the triple $({}^Cg, {}^HV, {}^Vf)$ is a gradient Ricci soliton on TM with respect to the RQSMC $\overline{\nabla}$ if and only if the Ricci tensor is equal to its Hessian metric obtained by means of the Levi-Civita connection and $n \neq 3$.*

Proof. Taking into accounts (3.23)–(3.26) and (3.9) in the Eq (3.22), it can be writen

$$\nabla_i \nabla_j f = (n-3)R_{ij}.$$

This completes the proof. □

3.5. Locally conformally flatness

In this section, we investigate the local conformal flatness property of $(TM, {}^Cg)$ with respect to the RQSMC $\overline{\nabla}$.

Theorem 12. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^Cg$. Then TM is locally conformally flat with respect to the Ricci quarter-symmetric metric connection $\overline{\nabla}$ if and only if M is locally flat.*

Proof. Here, we prove the only necessary conditions of the theorem because the sufficient condition directly follows. Let $\overline{\nabla}$ be the RQSMC on the tangent bundle $(TM, {}^Cg)$. The tangent bundle $(TM, {}^Cg)$ is locally conformally flat with respect to the RQSMC $\overline{\nabla}$ if and only if the components of the curvature $(0,4)-$tensor $\overline{R}$ of TM satisfy the following relation:

$$\begin{aligned} \overline{R}_{\alpha\gamma\beta\mu} &= -\frac{\overline{r}}{2(2n-1)(n-1)} \left\{ {}^Cg_{\alpha\beta} {}^Cg_{\gamma\mu} - {}^Cg_{\alpha\mu} {}^Cg_{\gamma\beta} \right\} \\ &\quad + \frac{1}{2(n-1)} \left({}^Cg_{\gamma\mu} \overline{K}_{\alpha\beta} - {}^Cg_{\alpha\mu} \overline{K}_{\gamma\beta} + {}^Cg_{\alpha\beta} \overline{K}_{\gamma\mu} - {}^Cg_{\gamma\beta} \overline{K}_{\alpha\mu} \right). \end{aligned}$$

From (3.9) we find

$$\overline{R}_{\bar{i}\bar{j}kh} = \frac{3-n}{2(n-1)} \left(g_{jh}R_{ik} - g_{jk}R_{ih} \right). \tag{3.27}$$

$$\overline{R}_{\bar{i}jkh} = \frac{3-n}{2(n-1)} \left(g_{ik}R_{jh} - g_{ih}R_{jk} \right). \tag{3.28}$$

$$\overline{R}_{i\bar{j}\bar{k}h} = \frac{3-n}{2(n-1)} \left(g_{ik}R_{jh} - g_{jk}R_{ih} \right). \tag{3.29}$$

$$\overline{R}_{ij\bar{k}\bar{h}} = \frac{3-n}{2(n-1)} \left(g_{jh}R_{ik} - g_{ih}R_{jk} \right). \tag{3.30}$$

On the other hand, by using $\overline{R}_{\alpha\gamma\beta\sigma} = \overline{g}_{\sigma\epsilon} \overline{R}_{\alpha\gamma\beta}^{\ \ \ \ \epsilon}$ we find

$$\overline{R}_{\bar{i}\bar{j}kh} = R_{ijkh} - g_{jk}R_{ih} + g_{jh}R_{ik}. \tag{3.31}$$

$$\overline{R}_{\bar{i}jkh} = R_{ijkh} + g_{ik}R_{jh} - g_{ih}R_{jk}. \tag{3.32}$$

$$\overline{R}_{i\bar{j}\bar{k}h} = R_{ijkh} \tag{3.33}$$

$$\overline{R}_{ij\bar{k}\bar{h}} = R_{ijkh}. \tag{3.34}$$

In this case, by means of (3.27) and (3.31), we get

$$\frac{5-3n}{2(n-1)}\left(g_{jh}R_{ik}-g_{jk}R_{ih}\right)=R_{ijkh}.$$

Similarly by means of (3.28) and (3.32), we get

$$\frac{5-3n}{2(n-1)}\left(g_{ik}R_{jh}-g_{hi}R_{jk}\right)=R_{ijkh} \tag{3.35}$$

and by means of (3.29) and (3.33), we get

$$\frac{3-n}{2(n-1)}\left(g_{ik}R_{jh}-g_{jk}R_{ih}\right)=R_{ijkh}$$

and by means of (3.30) and (3.34), we get

$$\frac{3-n}{2(n-1)}\left(g_{jh}R_{ik}-g_{ih}R_{jk}\right)=R_{ijkh}.$$

Changing (3.35) by g^{ih}, we obtain

$$(\frac{3n-7}{2})R_{jk}=0. \tag{3.36}$$

Thus, by (3.36), we obtain $R_{jk}=0$, then it follows from (3.35) $R_{ijkh}=0$. This completes the proof. □

4. Conclusions

In this paper, we consider a tangent bundle over a Riemannian manifold (M,g) admitting a Ricci quarter-symmetric metric connection with torsion tensor $T(X,Y)=\phi(Y)LX-\phi(X)LY$, where ϕ is a non-zero 1−form, L is the $(1,1)$ Ricci tensor defined by $g(LX,Y)=R(X,Y)$ and R is the Ricci tensor of (M,g). We obtain the form of the Ricci quarter-symmetric metric connection by using the Levi-Civita connection of the complete lift metric ${}^{C}g$. We show that the Ricci quarter-symmetric metric connection with respect to the complete lift metric ${}^{C}g$ is also a Ricci quarter-symmetric metric connection with respect to the metric $I+II$ which is another well-known classical pseudo-Riemannian metric on the tangent bundle. We compute the all forms of the curvature tensor of this connection and present its curvature properties. Also, we give the conditions for the tangent bundle to be semi-symmetric and $\overline{Z}$ semi-symmetric with respect to the Ricci quarter-symmetric metric connection. Ricci flow was introduced by Hamilton in 1982. It turned out to be a very powerful tool in Riemannian geometry and is now intensively studied. Important objects of this study are solitons. Ricci solitons generate self-similar solutions to the Ricci flow; in fact, a great deal of their relevance lies in their occurrence as models for the asymptotic profile of singularities developed under the Ricci flow. We present the necessary and sufficient conditions for the tangent bundle to become a Ricci soliton and a gradient Ricci soliton with respect to the Ricci quarter-symmetric metric connection. Finally, we close this paper with the locally conformally flatness property of the tangent bundle with respect to this connection. Furthermore, the research on singularity theory and submanifold theory, etc. as evidenced by recent papers [3–32], provides a promising foundation for advancing the field with the results and theorems. Future research can build upon the ideas presented in these papers [3–32] to push the boundaries of our understanding even further.

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Conflict of interest

The authors declare no conflicts of interest.

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