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Decay estimates for Schrödinger systems with time-dependent potentials in 2D

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Abstract: We consider the Cauchy problem for systems of nonlinear Schrödinger equations with time-dependent potentials in 2D. Under assumptions about mass resonances and potentials, we prove the global existence of the nonlinear Schrödinger systems with small initial data. In particular, by analyzing the operator Δ and time-dependent potentials V_j separately, we show that the small global solutions satisfy time decay estimates of order $O((t\log t)^{-1})$ when $p = 2$, and the small global solutions satisfy time decay estimates of order $O(t^{-1})$ when $p > 2$.

Keywords: systems of nonlinear Schrödinger equations; time-dependent potentials; time decay estimates; mass resonances
Mathematics Subject Classification: 35B40, 35Q55

1. Introduction

We consider the following Cauchy problem for the nonlinear Schrödinger system in space dimension two

$$\begin{cases} i\partial_t v_j + \frac{1}{2m_j}\Delta_{V_j} v_j = G_j(v_1, v_2), & t > 0, x \in \mathbb{R}^2, \\ v_j(0, x) = \phi_j(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\Delta_{V_j} = \Delta - V_j(t, x)$, $\Delta = \sum_{j=1}^2 \partial^2/\partial x_j^2$, $V_j(t, x)$ is a prescribed $\mathbb{R}$-valued function on $[0, \infty) \times \mathbb{R}^2$, $G_j(v_1, v_2) = K_j(v_1, v_2) + F_j(v_1, v_2)$, $K_j(v_1, v_2) = \lambda_j |v_j|^{p-1} v_j$, $F_1(v_1, v_2) = \overline{v_1} v_2$, $F_2(v_1, v_2) = v_1^2$, $\lambda_j \in \mathbb{C}\backslash\{0\}$, $p \geq 2$, m_j is a mass of a particle, $\phi_j(x)$ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}^2$, and $j = 1, 2$. In this paper, our aim is to prove the time decay estimates of solutions to (1.1)

$$\|v\|_{L^\infty(\mathbb{R}^2)} := \sum_{j=1}^{2} \|v_j\|_{L^\infty(\mathbb{R}^2)} \leqslant \frac{C_1}{1+t}; \quad if \quad p > 2$$

and

$$\|v\|_{L^\infty(\mathbb{R}^2)} := \sum_{j=1}^{2} \|v_j\|_{L^\infty(\mathbb{R}^2)} \le \frac{C_2}{(1+t)\log(2+t)}; \quad if \quad p = 2$$

for all $t \ge 0$, where $C_1, C_2 > 0$, when the small initial data $\phi_j(x)$ belongs to $\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)$, $2m_1 = m_2$, $\Im\lambda_j < 0$ and $V_j(t,x)$ satisfies

$$\|V_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)} \le 2\max\{m_1, m_2\}c_1(1+t)^{-\beta}$$

for all $t \ge 0$, where $c_1 > 0, 0 < \alpha < 1 < \beta < 2$, $j = 1, 2$ and

$$\|U_{\frac{1}{m_j}}(-t)V_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \le 2\max\{m_1, m_2\}c_2(1+t)^{-\theta}$$

for all $t \ge 0$, where $U_\sigma(t) = \mathcal{F}^{-1}E(t)^\sigma\mathcal{F}$, $E(t) = e^{-\frac{i}{2}t|\xi|^2}, \sigma \ne 0, \mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and its inverse transform operators, $c_2 > 0, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$, and $j = 1, 2$.

The Cauchy problem for nonlinear Schrödinger equations with time-dependent potentials

$$\begin{cases} i\partial_t v + \frac{1}{2}\Delta_V v = f(v), & t > 0, x \in \mathbb{R}^n, \\ v(0,x) = \phi(x), & x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

appears in physics, where $\Delta_V = \Delta - V(t,x)$, $V(t,x)$ is a prescribed $\mathbb{R}$-valued function on $[0,\infty)\times\mathbb{R}^n$, $f : \mathbb{C} \to \mathbb{C}$. If the nonlinear term $f(v) = \lambda|v|^{q-1}v, q > 1$ and $\lambda \in \mathbb{R}$, the Cauchy problem (1.2) was considered from the mathematical point of view in [1] and [2]. In [1], the global existence of solutions to (1.2) with $f(v) = \lambda|v|^{q-1}v$ was studied when the initial data $\phi(x) \in H^{1,0}(\mathbb{R}^n) \cap H^{0,1}(\mathbb{R}^n)$ and the external potential $V(t,x)$ satisfy some assumptions. The time decay estimates of solutions to (1.2) with $f(v) = \lambda|v|^{q-1}v$ were considered if the initial data $\phi(x) \in H^{1,0}(\mathbb{R}^n) \cap H^{0,1}(\mathbb{R}^n)$, $\lambda > 0$, and the time-dependent potential $V(t,x)$ meets some conditions. When the time-dependent potential $V(t,x) = \sigma(t)|x|^2/2$ and the nonlinear term $f(v) = \nu F_L(v)v + \mu F_S(v)v$ in (1.2), where $F_L : \mathbb{C} \to \mathbb{R}, F_S : \mathbb{C} \to \mathbb{R}$, and $\nu, \mu \in \mathbb{R}$, the asymptotic behavior and time decay estimates of solutions to (1.2) for small initial data satisfying $\phi(x) \in H^{\gamma,0}(\mathbb{R}^2) \cap H^{0,\gamma}(\mathbb{R}^2)$ with $\gamma > n/2$ were investigated in [3]. In [4], a sharp time decay estimate for the global in time solution to (1.2) with cubic nonlinear term and the potential $V(x)$ which satisfies $< \cdot >^s V \in W^{1,1}(\mathbb{R})$ was obtained in 1D. The cubic nonlinear Schrödinger equation with potential in 1D also has been studied in [5] and [6]. If the time-dependent potential $V(t,x) \equiv 0$, then the Cauchy problem (1.2) becomes

$$\begin{cases} i\partial_t v + \frac{1}{2}\Delta v = f(v), & t > 0, x \in \mathbb{R}^n, \\ v(0,x) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

When $f(v) = \lambda|v|^{q-1}v + \kappa|v|^{\eta-1}v, \lambda, \kappa \in \mathbb{R}$ and $q - 1 = 2/n < \kappa - 1$, (1.3) was investigated in [7]. It is known that $\eta = 2/n$ is the critical exponent if the scattering problem for (1.3) with $f(v) = \kappa|v|^{\eta-1}v$ and $\kappa > 0$ is considered (see [8] and [9]). If the small initial data $\phi(x)$ belongs to $H^{\gamma,0}(\mathbb{R}^n) \cap H^{0,\gamma}(\mathbb{R}^n)$ with $n/2 < \gamma \le q = 1 + 2/n$, the existence of modified scattering states for (1.3) was studied, and the sharp time decay estimate of solutions was proved in [7]. When $n = 1, f(v) = \lambda|v|^{q-1}v$ and $\lambda = \lambda_1 + i\lambda_2, \lambda_j \in \mathbb{R}, \lambda_2 < 0, |\lambda_2| > \frac{q-1}{2\sqrt{q}}|\lambda_1|$ and $1 < q \le 3$, (1.3) with initial data $\phi(x) \in H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$ was studied in [10]. The time decay estimates and large time asymptotics of the solution for arbitrarily large initial data were presented if $q = 3$ or $q < 3$ and q is close to 3. In [11], the same equation was considered. The asymptotic behavior of solutions of the Cauchy problem for initial data $\phi(x) \in H^{0,\gamma}(\mathbb{R})$ with $1/2 < \gamma \leq 1$ was studied. In [12], the scattering for solutions to (1.3) was shown in the case of $f(v) = \lambda |v|^{q-1} v, 1 + \frac{4}{n+2\gamma} < q < 1 + \frac{4}{n}, 0 < \gamma \leq \min\{\frac{n}{2}, 1\}$ and $\lambda = \lambda_1 + i\lambda_2, \lambda_j \in \mathbb{R}, \lambda_2 < 0, |\lambda_2| > \frac{q-1}{2\sqrt{q}} |\lambda_1|$. If $f(v) = \sum_{j\neq 0} \lambda_j |v|^{\sigma_j - j} v^j, \lambda_j \in \mathbb{C}, \sigma_j > 3$ and $n = 1$ in (1.3), the existence of the scattering operator was considered in [13]. If the potential $V(t, x) \equiv V(x)$ and $f(v) = g(|v|^2)v$, then from (1.2) we have

$$\begin{cases} i\partial_t v + \frac{1}{2}\Delta_V v = g(|v|^2)v, \quad t > 0, x \in \mathbb{R}^n, \\ v(0, x) = \phi(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $\Delta_V = \Delta - V(x), V(x)$ is a prescribed $\mathbb{R}$-valued function on $\mathbb{R}^n, g : \mathbb{C} \to \mathbb{R}$. There is some research on the asymptotic behavior of solutions to (1.4) (see [14–17], and references cited therein). In [15], when $V(x)$ is a real-valued measurable function defined in $\mathbb{R}^2$, $f(v) = \lambda |v|^{2\sigma} v, \sigma > \frac{1}{2}, \lambda \in \mathbb{C}\backslash\{0\}$ and $\Im\lambda \leq 0$, the scattering problem for (1.4) was studied, and by using the equivalence between the operators $(-\Delta_V)^{\frac{s}{2}}$ and $(-\Delta)^{\frac{s}{2}}$ in L^2 norm sense for $0 \leq s < 1$, the time decay estimates of the solution were obained as $O(t^{-1})$ in $L^\infty(\mathbb{R}^2)$ as $t \to +\infty$. In the case of $\sigma > \frac{1}{2}$, the solution of a system of the equations in 2D has the same time decay rate under some assumptions. In [18] the solution of the nonlinear Schrödinger systems with quadratic nonlinearities in two space dimensions decays like $O(t^{-1}(\log t)^{-1})$ in $L^\infty(\mathbb{R}^2)$ as $t \to +\infty$, when $V(x) \equiv 0$. In [19], numerical method is considerd by using Fourier spectral method to solve the multidimensional nonlinear fractional-in-space Schrödinger equation involving the fractional Laplacian operator and the numerical method is effective for long time simulation of integer-order Schrödinger equation.

We find that the time decay estimates of the solutions of the Cauchy problem for two dimensional critical NLS system with potentials is an unsolved problem. So we consider the following nonlinear Schrödinger system. Let $W_j(t, x) = \frac{1}{2m_j} V_j(t, x)$ for $j = 1, 2$. From (1.1) we have

$$\begin{cases} i\partial_t v_j + \frac{1}{2m_j}\Delta v_j - W_j v_j = G_j(v_1, v_2), \quad t > 0, x \in \mathbb{R}^2, \\ v_j(0, x) = \phi_j(x), \quad x \in \mathbb{R}^2. \end{cases} \tag{1.5}$$

We assume that the masses of particles in the Cauchy problem (1.5) and the time-dependent $\mathbb{R}$-valued potential $W_j(t, x)$ satisfies the following hypotheses.

(H1) $2m_1 = m_2$.

(H2) $\|W_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap \dot{H}^{\beta,0}(\mathbb{R}^2)} \leq c_1(1 + t)^{-\beta}$ for $t \geq 0$, where $c_1 > 0, 0 < \alpha < 1 < \beta < 2, j = 1, 2$.

(H3) $\|U_{\frac{1}{m_j}}(-t)W_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap \dot{H}^{0,\beta}(\mathbb{R}^2)} \leq c_2(1 + t)^{-\theta}$ for all $t \geq 0$, where $U_\sigma(t) = \mathcal{F}^{-1}E(t)^\sigma\mathcal{F}$, $E(t) = e^{-\frac{i}{2}t|\xi|^2}, \sigma \neq 0, \mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and its inverse transform operators, $c_2 > 0, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$, and $j = 1, 2$.

We use the assumption of mass resonance (H1) to deal with the nonlinearity $F_j(v_1, v_2)$ in (1.5) for $j = 1, 2$. The assumptions of the potential $W_j(t, x)$ are (H2) and (H3). The assumption (H2) is used to investigate Lemma 2.3 and Lemma 3.2. The assumption (H3) is applied to the proofs of Lemma 3.2 and Lemma 3.4.

If the mass resonance condition (H1) holds, then the Cauchy problem (1.5) meet the following gauge condition

$$G_j(v_1, v_2) = e^{im_j\theta} G_j(e^{-im_1\theta} v_1, e^{-im_2\theta} v_2), \quad j = 1, 2$$

for any $\theta \in \mathbb{R}$. If $W_j = 0$ for $j = 1, 2$ in (1.5), then the system (1.5) becomes

$$\begin{cases} i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = \lambda_j |v_j| v_j + F_j(v_1, v_2), & t > 0, x \in \mathbb{R}^2, \\ v_j(0, x) = \phi_j(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.6}$$

where $F_1(v_1, v_2) = \overline{v_1} v_2, F_2(v_1, v_2) = v_1^2, m_j$ is a mass of a particle, and $\phi_j(x)$ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}^2$ for $j = 1, 2$. The time decay of small solutions to (1.6) with $\Im\lambda_j \leq 0$ for $j = 1, 2$ was studied in the situation of small initial data $\phi_j \in H^{\gamma,0}(\mathbb{R}^2) \cap H^{0,\gamma}(\mathbb{R}^2)$ with $1 < \gamma < 2$ in [18]. If $\lambda_j = 0$ for $j = 1, 2$, then from (1.6) we have

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \overline{v_1} v_2, & t > 0, x \in \mathbb{R}^2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = v_1^2, & t > 0, x \in \mathbb{R}^2, \\ (v_1(0, x), v_2(0, x)) = (\phi_1(x), \phi_2(x)), & x \in \mathbb{R}^2. \end{cases} \tag{1.7}$$

In [20], global existence and time decay estimates of solutions to (1.7) for small initial data $v_j(0, x) \in H^{2,0}(\mathbb{R}^2) \cap H^{0,2}(\mathbb{R}^n)$ with $j = 1, 2$ were investigated under the mass resonance condition $2m_1 = m_2$. Large-time asymptotic behavior of solutions to the Cauchy problem for nonlinear Schrödinger equations

$$\begin{cases} i\partial_t v_1 + \frac{1}{2}\Delta v_1 = -i|v_2|^2 v_1, & t > 0, x \in \mathbb{R}, \\ i\partial_t v_2 + \frac{1}{2}\Delta v_2 = -i|v_1|^2 v_2, & t > 0, x \in \mathbb{R}, \\ (v_1(x, 0), v_2(x, 0)) = (\phi_1, \phi_2), & x \in \mathbb{R} \end{cases} \tag{1.8}$$

was considered in [21] and [22]. As far as we know, the time decay of the Cauchy problem (1.5) with the time-dependent potentials $V_j(t, x)$ has not been shown, where $j = 1, 2$.

Multiplying the equations of (1.5) by $i\overline{v_j}$ respectively, and taking the real parts of the result, we obtain

$$\begin{cases} \partial_t |v_1|^2 - \frac{1}{m_1}\Re((i\Delta v_1)\overline{v_1}) + 2Re(iW_1|v_1|^2) = -2\Re(i\lambda_1|v_1|^{p+1}) - 2\Re(i\overline{v_1}^2 v_2), \\ \partial_t |v_2|^2 - \frac{1}{m_2}\Re((i\Delta v_2)\overline{v_2}) + 2Re(iW_2|v_2|^2) = -2\Re(i\lambda_1|v_2|^{p+1}) - 2\Re(iv_1^2\overline{v_2}). \end{cases}$$

Since the assumption $W_j(t, x)$ is the $\mathbb{R}$-valued potential, and by integrating in space, we find

$$\begin{cases} \partial_t \|v_1\|^2_{L^2(\mathbb{R}^2)} = 2\Im\lambda_1 \|v_1\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} - 2\int_{\mathbb{R}^2} \Re\left(i\overline{v_1}^2 v_2\right) dx, \\ \partial_t \|v_2\|^2_{L^2(\mathbb{R}^2)} = 2\Im\lambda_2 \|v_2\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} - 2\int_{\mathbb{R}^2} \Re\left(iv_1^2\overline{v_2}\right) dx. \end{cases} \tag{1.9}$$

Under the assumption that $\Im\lambda_j \leq 0$ for $j = 1, 2$, by (1.9) we obtain

$$\partial_t \left(\|v_1\|^2_{L^2(\mathbb{R}^2)} + \|v_2\|^2_{L^2(\mathbb{R}^2)} \right) \leq 0.$$

Therefore, we prove stability in time of solutions in the neighborhood of solutions to a suitable approximate equation. Our main purpose in this paper is to show time decay estimates of solutions to the Cauchy problem (1.5) with small initial data in $\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2), 0 < \alpha < 1 < \beta < 2$. Combining the methods of [7] and [18], we obtain the following result under the assumptions that $\Im\lambda_j < 0$ for $j = 1, 2$, (H1), (H2) and (H3) hold. Our main idea is to consider Δv_j and $W_j v_j$ separately. By the assumptions (H2) and (H3) of the potential W_j, we can analyze the linear term $W_j v_j$ of the system (1.5). The time decay estimates of the solutions to the nonlinear Schrödinger equations are studies in [15] by regarding the $\frac{1}{m_j}\Delta - W_j$ as one whole. Our method differs from the approaches. The potential weakness of this paper is that the potentials W_j satisfy the decaying condition (H2) and (H3). So if not, whether the time decay estimates of the solutions to the system (1.5) can be obtained is an open problem.

Theorem 1.1. *Assume that (1.5) satisfies the mass resonance condition (H1), the time-dependent potential $W_j(t,x)$ satisfies (H2), (H3) and $\Im\lambda_j < 0, j = 1,2$. Then there exist constants $\varepsilon_0 > 0$ and $C_1, C_2 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and*

$$\|\phi\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} = \sum_{j=1}^{2}\|\phi_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \leq \varepsilon,$$

where $0 < \alpha < 1 < \beta < 2$, there exist a unique global solution $v = (v_1, v_2)$ of (1.5) satisfying

$$U_{\frac{1}{m_j}}(-t)v_j \in C([1,\infty), \dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)), \quad j = 1,2$$

and the time decay estimates

$$\|v\|_{L^\infty(\mathbb{R}^2)} = \sum_{j=1}^{2}\|v_j\|_{L^\infty(\mathbb{R}^2)} \leqslant \frac{C_1}{1+t}; \quad if \quad p > 2, \tag{1.10}$$

$$\|v\|_{L^\infty(\mathbb{R}^2)} = \sum_{j=1}^{2}\|v_j\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_2}{(1+t)\log(2+t)}; \quad if \quad p = 2 \tag{1.11}$$

for $t \geq 0$.

Remark 1.1. *By Lemma 2.1, we have*

$$\begin{aligned}\|W_j\|_{L^\infty(\mathbb{R}^2)} &= \|\mathcal{F}^{-1}\mathcal{F}W_j\|_{L^\infty(\mathbb{R}^2)}\\ &\leq C\|\mathcal{F}W_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}\\ &\leq C\|W_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)}\end{aligned} \tag{1.12}$$

where $0 < \alpha < 1 < \beta < 2$, $j = 1,2$. By the assumption (H2) and (1.12), we have

$$\|W_j\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-\beta} \tag{1.13}$$

for $t \geq 1$, which is used in the proof of Lemma 3.4, where $0 < \alpha < 1 < \beta < 2, j = 1,2$. We also get Lemma 2.3 by the assumption (H2) and some other conditions, which is applied to the proof of Lemma 3.2.

The rest of this paper is organized as follows. In Section 2, we give some notations and basic lemmas. We prove Theorem 1.1 in Section 3 by using the strategy introduced in [18,23].

2. Preliminaries

In this section, we give some estimates as preliminaries. In what follows, we use the same notations both for the vector function spaces and the scalar ones. For any p with $1 \leq p \leq \infty$, $L^p(\mathbb{R}^2)$ denotes the usual Lebesgue space with the norm $\|\phi\|_{L^p(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\phi(x)|^p dx\right)^{\frac{1}{p}}$, if $1 \leq p < \infty$ and $\|\phi\|_{L^\infty(\mathbb{R}^2)} = \text{ess} \cdot \sup_{x\in\mathbb{R}^2} |\phi(x)|$. For any $m, s \in \mathbb{R}$, weighted Sobolev space $H^{m,s}(\mathbb{R}^2)$ is defined by

$$H^{m,s}(\mathbb{R}^2) = \left\{ f = (f_1, f_2) \in L^2(\mathbb{R}^2); \|f\|_{H^{m,s}(\mathbb{R}^2)} = \sum_{j=1}^{2} \left\|f_j\right\|_{H^{m,s}(\mathbb{R}^2)} < \infty \right\},$$

where the Sobolev norm is defined as

$$\|f_j\|_{H^{m,s}(\mathbb{R}^2)} = \left\|(1 + |x|^2)^{\frac{s}{2}}(1 - \Delta)^{\frac{m}{2}} f_j\right\|_{L^2(\mathbb{R}^2)}$$

for $j = 1, 2$. Also we define the homogeneous Sobolev seminorm $\|f_j\|_{\dot{H}^{m,s}(\mathbb{R}^2)}$ as

$$\|f_j\|_{\dot{H}^{m,s}(\mathbb{R}^2)} = \left\||x|^s(-\Delta)^{\frac{m}{2}} f_j\right\|_{L^2(\mathbb{R}^2)}$$

for $j = 1, 2$.

We define the dilation operator by

$$(D_\alpha\phi)(x) = \frac{1}{i\alpha}\phi\left(\frac{x}{\alpha}\right), \quad \text{for } \alpha \neq 0,$$

and

$$E(t) = e^{-\frac{i}{2}t|\xi|^2},\ M(t) = e^{-\frac{i}{2}t|x|^2}, \quad \text{for } t \neq 0.$$

Let $U_\alpha(t) = \mathcal{F}^{-1}E(t)^\alpha\mathcal{F}$ with $\alpha \neq 0$, where the Fourier transform of f is

$$(\mathcal{F}f)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

and the inverse Fourier transform of g is

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi} g(\xi)d\xi.$$

The evolution operator $U_\alpha(t)$ and inverse evolution operator $U_\alpha(-t)$ for $t \neq 0$, are written as

$$(U_\alpha(t)\phi)(x) = M(t)^{-\frac{1}{\alpha}} D_{\alpha t}\left(\mathcal{F}M(t)^{-\frac{1}{\alpha}}\phi\right)(x)$$

and

$$(U_\alpha(-t)\phi)(x) = M(t)^{\frac{1}{\alpha}}\left(\mathcal{F}^{-1}D_{\alpha t}^{-1}M(t)^{\frac{1}{\alpha}}\phi\right)(x),$$

respectively.

The operator $|J_{\frac{1}{m}}|^s(t)$ is given by

$$|J_{\frac{1}{m}}|^s(t) = U_{\frac{1}{m}}(t)|x|^s U_{\frac{1}{m}}(-t), \quad s > 0,$$

which is represented as

$$|J_{\frac{1}{m}}|^s(t) = M^{-m}\left(-\frac{t^2}{m^2}\Delta\right)^{\frac{s}{2}} M^m$$

for $t \neq 0$. Let $[E, F] = EF - FE$. We have the commutator relations

$$\left[i\partial_t + \frac{1}{2m}\Delta, |J_{\frac{1}{m}}|^s(t) \right] \equiv 0$$

for $s > 0$. These formulas are essential tools for studying the asymptotic behavior of solutions to (1.1) (see [24]). And in what follows, we denote several positive constants by the same letter C, which may vary from one line to another.

We start with the following lemma.

Lemma 2.1. *Let* $0 < s_1 < 1 < s_2$. *Then we have*

$$\|f\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{\dot{H}^{0,s_1}(\mathbb{R}^2)\cap\dot{H}^{0,s_2}(\mathbb{R}^2)}. \tag{2.1}$$

By the Cauchy-Schwarz inequality, we have Lemma 2.1. We shall not give the proof here.

We next recall the well known results (see [7]).

Lemma 2.2. *Let* $0 \leq s < 2, \rho \geq 2, m > 0$. *Then we have*

$$\|U_{\frac{1}{m}}(-t)|v|^{\rho-1}v\|_{\dot{H}^{0,s}(\mathbb{R}^2)} \leq C\|v\|^{\rho-1}_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}, \tag{2.2}$$

$$\||v|^{\rho-1}v\|_{\dot{H}^{s,0}(\mathbb{R}^2)} \leq C\|v\|^{\rho-1}_{L^\infty(\mathbb{R}^2)}\|v\|_{\dot{H}^{s,0}(\mathbb{R}^2)}, \tag{2.3}$$

and

$$\|fg\|_{\dot{H}^{s,0}(\mathbb{R}^2)} \leq C\left(\|f\|_{L^\infty(\mathbb{R}^2)}\|g\|_{\dot{H}^{s,0}(\mathbb{R}^2)} + \|f\|_{\dot{H}^{s,0}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R}^2)}\right). \tag{2.4}$$

Using the factorization formula $U_{\frac{1}{m_j}}(-t) = -M^{m_j}\mathcal{F}^{-1}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}$ and the assumption (H2), we have

Lemma 2.3. *Let* $\mathcal{M}^{-1}_{m_j} = \mathcal{F}M^{-m_j}\mathcal{F}^{-1}$, $m_j > 0$ *and* $j = 1, 2$. *If* W_j *satisfies the assumption (H2), then there exists a constant* $C_3 > 0$ *such that*

$$\left\|E^{-\frac{1}{m_j}}\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)} \leq C_3$$

for $t \geq 1$, *where* $0 < \alpha < 1 < \beta < 2$.

Proof. By using the factorization formula $U_{\frac{1}{m_j}}(-t) = -M^{m_j}\mathcal{F}^{-1}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}$, we have

$$\begin{aligned}
&\left\|E^{-\frac{1}{m_j}}\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j(t,x)\right\|_{\dot{H}^{s,0}(\mathbb{R}^2)} \\
&= \left\|-E^{-\frac{1}{m_j}}\mathcal{F}M^{-m_j}M^{m_j}\mathcal{F}^{-1}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}W_j(t,x)\right\|_{\dot{H}^{s,0}(\mathbb{R}^2)} \\
&\leq \frac{t}{m_j}\left\||\xi|^s\mathcal{F}\left(W_j(t,\frac{xt}{m_j})\right)\right\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{m_j}{t}\left\||\xi|^s\left(\mathcal{F}W_j\right)(t,\frac{\xi m_j}{t})\right\|_{L^2(\mathbb{R}^2)} \\
&\leq \frac{t^s}{m_j^s}\|W_j\|_{\dot{H}^{s,0}(\mathbb{R}^2)}
\end{aligned}$$

for $t \geq 1$, where $s = \alpha$ or β. Then we obtain that

$$\|E^{-\frac{1}{m_j}}\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)} \leq \frac{t^\beta}{\min\{m_1^s, m_2^s\}}\|W_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)} \tag{2.5}$$

for $t \geq 1$, where $0 < \alpha < 1 < \beta < 2$. By the assumption (H2) and (2.5), we get

$$\|E^{-\frac{1}{m_j}}\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\|_{\dot{H}^{\alpha,0}(\mathbb{R}^2)\cap\dot{H}^{\beta,0}(\mathbb{R}^2)} \leq C_3$$

for $t \geq 1$, where $C_3 = \frac{c_1}{\min\{m_1^s, m_2^s\}}$, and $0 < \alpha < 1 < \beta < 2$. □

3. Proof of Theorem 1.1

3.1. A priori estimates of solutions

We define the function space X_T as follows

$$\begin{aligned} X_T = \Big\{ & U_{\frac{1}{m_j}}(t)f_j \in \left((C\cap L^\infty)\left([0,T];\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)\right)\right); \\ & \|f\|_{X_T} = \||J_{\frac{1}{m}}|^\alpha f\|_{L^\eta([0,T];L^\zeta(\mathbb{R}^2))} + \||J_{\frac{1}{m}}|^\beta f\|_{L^\eta([0,T];L^\zeta(\mathbb{R}^2))} \\ & +\|U_{\frac{1}{m}}(-t)f\|_{L^\infty\left([0,T];\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)\right)}, \eta = \frac{4}{1-\alpha}, \zeta = \frac{4}{1+\alpha}\Big\}, \end{aligned}$$

where $T > 0$ and $U_{\frac{1}{m}}(t)f = \left(U_{\frac{1}{m_1}}(t)f_1, U_{\frac{1}{m_2}}(t)f_2\right)$. We can obtain the local existence of solutions to (1.1) by the standard contraction mapping principle (see [23]).

Multiplying both sides of (1.5) by $D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)$, $j = 1, 2$, we use the factorization formula $\mathcal{F}U_{\frac{1}{m_j}}(-t) = -\mathcal{M}_{m_j}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}$ with $\mathcal{M}_{m_j} = \mathcal{F}M^{m_j}\mathcal{F}^{-1}$ to get

$$\mathcal{F}U_{\frac{1}{m_j}}(-t)G_j(v_1, v_2) = -\mathcal{M}_{m_j}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}G_j(v_1, v_2).$$

Using the identity operator $I = -D_{\frac{t}{m_j}}D_{\frac{m_j}{t}}$, $j = 1, 2$, we have

$$\begin{aligned} D_{\frac{m_j}{t}}G_j(v_1, v_2) &= D_{\frac{m_j}{t}}G_j\left(-D_{\frac{t}{m_1}}D_{\frac{m_1}{t}}v_1, -D_{\frac{t}{m_2}}D_{\frac{m_2}{t}}v_2\right) \\ &= \frac{t}{im_j}G_j\left(-\frac{im_j}{t}D_{\frac{m_j}{t}}D_{\frac{t}{m_1}}D_{\frac{m_1}{t}}v_1, -\frac{im_j}{t}D_{\frac{m_j}{t}}D_{\frac{t}{m_2}}D_{\frac{m_2}{t}}v_2\right) \\ &= \frac{t}{im_j}G_j\left(-\frac{m_j}{t}D_{\frac{m_j}{m_1}}D_{\frac{m_1}{t}}v_1, -\frac{m_j}{t}D_{\frac{m_j}{m_2}}D_{\frac{m_2}{t}}v_2\right). \end{aligned}$$

We now use the identity $D_aE^{-b}f(t, x) = \frac{1}{ia}e^{\frac{ibt|\xi|^2}{2a^2}}f\left(t, \frac{x}{a}\right) = E^{\frac{b}{a^2}}D_af(t, x)$ for $a \neq 0$ to get

$$D_{\frac{m_j}{m_k}}D_{\frac{m_k}{t}}v_k = D_{\frac{m_j}{m_k}}E^{-\frac{1}{m_k}}E^{\frac{1}{m_k}}D_{\frac{m_k}{t}}v_k = E^{-\frac{m_k}{m_j^2}}D_{\frac{m_j}{m_k}}E^{\frac{1}{m_k}}D_{\frac{m_k}{t}}v_k.$$

Let $\theta_j = e^{-\frac{t|\xi|^2}{2m_j^2}}$, then we have

$$E^{-\frac{m_k}{m_j^2}} = e^{\frac{i}{2}t\frac{m_k}{m_j^2}|\xi|^2} = e^{-im_k\theta_j}.$$

Let $\widetilde{v}_k = E^{\frac{1}{m_k}} D_{\frac{m_k}{t}} v_k, k = 1,2$. By the factorization formula $\mathcal{F}U_{\frac{1}{m_k}}(-t) = -\mathcal{M}_{m_k} E^{\frac{1}{m_k}} D_{\frac{m_k}{t}}$, we have $\widetilde{v}_k = -\mathcal{M}_{m_k}^{-1}\mathcal{F}U_{\frac{1}{m_k}}(-t)v_k$. By the mass resonance condition (H1), we get

$$\begin{aligned}
&\mathcal{F}U_{\frac{1}{m_j}}(-t)G_j(v_1, v_2)\\
&= i\mathcal{M}_{m_j}E^{\frac{1}{m_j}}\frac{t}{m_j}G_j\left(-\frac{m_j}{t}e^{-im_1\theta_j}D_{\frac{m_j}{m_1}}\widetilde{v}_1, -\frac{m_j}{t}e^{-im_2\theta_j}D_{\frac{m_j}{m_2}}\widetilde{v}_2\right)\\
&= i\mathcal{M}_{m_j}E^{\frac{1}{m_j}}\frac{t}{m_j}e^{-im_j\theta_j}G_j\left(-\frac{m_j}{t}D_{\frac{m_j}{m_1}}\widetilde{v}_1, -\frac{m_j}{t}D_{\frac{m_j}{m_2}}\widetilde{v}_2\right)\\
&= i\mathcal{M}_{m_j}\frac{t}{m_j}K_j\left(-\frac{m_j}{t}D_{\frac{m_j}{m_1}}\widetilde{v}_1, -\frac{m_j}{t}D_{\frac{m_j}{m_2}}\widetilde{v}_2\right) + i\mathcal{M}_{m_j}\frac{t}{m_j}F_j\left(-\frac{m_j}{t}D_{\frac{m_j}{m_1}}\widetilde{v}_1, -\frac{m_j}{t}D_{\frac{m_j}{m_2}}\widetilde{v}_2\right)\\
&= i\mathcal{M}_{m_j}\frac{{m_j}^{p-1}}{t^{p-1}}K_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right)\\
&\quad + i\mathcal{M}_{m_j}\frac{m_j}{t}F_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right), j = 1, 2.
\end{aligned}$$

Next we consider $\mathcal{F}U_{\frac{1}{m_j}}(-t)\left(W_jv_j\right)$ similarly, we have

$$D_{\frac{m_j}{t}}W_jv_j = \frac{t}{im_j}\left(\frac{im_j}{t}D_{\frac{m_j}{t}}W_j\right)\left(\frac{im_j}{t}D_{\frac{m_j}{t}}v_j\right).$$

Let $D_{\frac{m_j}{t}}W_j = E^{-\frac{1}{m_j}}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}W_j$, $D_{\frac{m_j}{t}}v_j = E^{-\frac{1}{m_j}}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}v_j$, and $\widetilde{W}_j = E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}W_j, \widetilde{v}_j = E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}v_j$, $j = 1, 2$. By the factorization formula $\mathcal{F}U_{\frac{1}{m_j}}(-t) = -\mathcal{M}_{m_j}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}$, we have $\widetilde{W}_j = -\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j$, $\widetilde{v}_j = -\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j$. By the definition of the operator $E(t)$, we get

$$\begin{aligned}
&\mathcal{F}U_{\frac{1}{m_j}}(-t)W_jv_j\\
&= i\mathcal{M}_{m_j}E^{\frac{1}{m_j}}\frac{t}{m_j}\left(\frac{im_j}{t}E^{-\frac{1}{m_j}}\widetilde{W}_j\right)\left(\frac{im_j}{t}E^{-\frac{1}{m_j}}\widetilde{v}_j\right)\\
&= i\mathcal{M}_{m_j}\frac{t}{m_j}\left(\frac{im_j}{t}E^{-\frac{1}{m_j}}\widetilde{W}_j\right)\left(\frac{im_j}{t}\widetilde{v}_j\right)\\
&= i\mathcal{M}_{m_j}\frac{m_j}{t}\left(-iE^{-\frac{1}{m_j}}\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right)\left(-i\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\right), j = 1, 2.
\end{aligned}$$

We set

$$\begin{aligned}
R_{1j} = i\left(\mathcal{M}_{m_j} - I\right)&\frac{{m_j}^{p-1}}{t^{p-1}}K_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right)\\
&+ i\left(\mathcal{M}_{m_j} - I\right)\frac{m_j}{t}F_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right)\\
&+ i\left(\mathcal{M}_{m_j} - I\right)\frac{m_j}{t}\left(-iE^{-\frac{1}{m_j}}\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right)\left(-i\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\right),
\end{aligned}$$

$$
\begin{aligned}
R_{2j} = & \, i\frac{{m_j}^{p-1}}{t^{p-1}} K_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right) \\
& - i\frac{{m_j}^{p-1}}{t^{p-1}} K_j\left(-D_{\frac{m_j}{m_1}}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right) \\
& + i\frac{m_j}{t} F_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{M}_{m_2}^{-1}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right) \\
& - i\frac{m_j}{t} F_j\left(-D_{\frac{m_j}{m_1}}\mathcal{F}U_{\frac{1}{m_1}}(-t)v_1, -D_{\frac{m_j}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2\right) \\
& + i\frac{m_j}{t}\left(-iE^{-\frac{1}{m_j}}\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right)\left(-i\mathcal{M}_{m_j}^{-1}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\right) \\
& - i\frac{m_j}{t}\left(-iE^{-\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right)\left(-i\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\right)
\end{aligned}
$$

and $u_j = D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j, S_j = D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j$, then we have

$$
\begin{aligned}
& \left(\mathcal{F}U_{\frac{1}{m_j}}(-t)G_j(v_1,v_2)\right)(t,\xi) + \left(\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j v_j\right)(t,\xi) \\
& = i\frac{{m_j}^{p-1}}{t^{p-1}} K_j\left(-D_{\frac{m_j}{m_1}}D^{-1}_{\frac{1}{m_1}}u_1, -D_{\frac{m_j}{m_2}}D^{-1}_{\frac{1}{m_2}}u_2\right)(t,\xi) \\
& \quad + i\frac{m_j}{t} F_j\left(-D_{\frac{m_j}{m_1}}D^{-1}_{\frac{1}{m_1}}u_1, -D_{\frac{m_j}{m_2}}D^{-1}_{\frac{1}{m_2}}u_2\right)(t,\xi) \\
& \quad + i\frac{m_j}{t}\left(-iE^{-\frac{1}{m_j}}D^{-1}_{\frac{1}{m_j}}S_j\right)\left(-iD^{-1}_{\frac{1}{m_j}}u_j\right)(t,\xi) + \sum_{l=1}^{2} R_{lj} \\
& = i\frac{{m_j}^{p-1}}{t^{p-1}} K_j\left(-\frac{1}{m_j}u_1, -\frac{1}{m_j}u_2\right)\left(t, \frac{\xi}{m_j}\right) \\
& \quad + i\frac{m_j}{t} F_j\left(-\frac{1}{m_j}u_1, -\frac{1}{m_j}u_2\right)\left(t, \frac{\xi}{m_j}\right) \\
& \quad + i\frac{m_j}{t}\left(\frac{1}{m_j}E^{-\frac{1}{m_j}}S_j\left(t,\frac{\xi}{m_j}\right)\right)\left(\frac{1}{m_j}u_j\left(t,\frac{\xi}{m_j}\right)\right) + \sum_{l=1}^{2} R_{lj}.
\end{aligned}
$$

Multiplying both sides of the above identity by $D_{\frac{1}{m_j}}$, we have

$$
\begin{aligned}
& D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)G_j(v_1,v_2)\left(t,\frac{\xi}{m_j}\right) + D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\left(t,\frac{\xi}{m_j}\right)v_j\left(t,\frac{\xi}{m_j}\right) \\
& = \frac{1}{t^{p-1}}K_j(u_1,u_2)(t,\xi) + \frac{1}{t}F_j(u_1,u_2)(t,\xi) + \frac{1}{t}\left(\left(E^{-m_j}S_j\right)u_j\right)(t,\xi) + D_{\frac{1}{m_j}}\sum_{l=1}^{2} R_{lj}.
\end{aligned}
$$

Therefore, from (1.5) we have

$$
i\partial_t u_j = \frac{\lambda_j}{t^{p-1}}|u_j|^{p-1}u_j + \frac{1}{t}F_j(u_1,u_2) + \frac{1}{t}\left(E^{-m_j}S_j\right)u_j + D_{\frac{1}{m_j}}\sum_{l=1}^{2} R_{lj} \tag{3.1}
$$

for $j = 1, 2$. Multiplying both sides of (3.1) by $\overline{u_j}$, taking the imaginary parts, we obtain

$$\partial_t \sum_{j=1}^{2} |u_j|^2 = \frac{2}{t^{p-1}} \Im \sum_{j=1}^{2} (\lambda_j |u_j|^{p+1}) + 2\Im \left(\sum_{j=1}^{2} \left(\frac{1}{t} F_j(u_1, u_2) \right) \overline{u_j} \right) + 2\Im \left(\sum_{j=1}^{2} \left(\frac{1}{t} \left(E^{-m_j} S_j \right) u_j \right) \overline{u_j} \right) + 2\Im \left(\sum_{j=1}^{2} \left(D_{\frac{1}{m_j}} \sum_{l=1}^{2} R_{lj} \right) \overline{u_j} \right).$$

By the definitions of the nonlinear terms $F_1(u_1, u_2) = \overline{u_1} u_2$, $F_2(u_1, u_2) = u_1^2$, and $\Im \lambda_j < 0$ for $j = 1, 2$, we obtain

$$\partial_t \sum_{j=1}^{2} |u_j|^2 \le 2\Im \left(\sum_{j=1}^{2} \left(\frac{1}{t} \left(E^{-m_j} S_j \right) u_j \right) \overline{u_j} \right) + 2\Im \left(\sum_{j=1}^{2} \left(D_{\frac{1}{m_j}} \sum_{l=1}^{2} R_{lj} \right) \overline{u_j} \right).$$

We integrate the inequality above in time and use $u_j = D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t) v_j$ to obtain that

$$\begin{aligned} \|\mathcal{F} U_{\frac{1}{m}}(-t) v\|_{L^\infty(\mathbb{R}^2)} &\le C \|\mathcal{F} U_{\frac{1}{m}}(-1) v(1, x)\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + C \int_1^t \tau^{-1} \sum_{j=1}^{2} \| \left(E^{-m_j} S_j \right) u_j \|_{L^\infty(\mathbb{R}^2)} d\tau \\ &\quad + C \int_1^t \sum_{j=1}^{2} \sum_{l=1}^{2} \|R_{lj}\|_{L^\infty(\mathbb{R}^2)} d\tau. \end{aligned} \tag{3.2}$$

Next we estimate $\sum_{j=1}^{2} \| \left(E^{-m_j} S_j \right) u_j \|_{L^\infty(\mathbb{R}^2)}$ and $\sum_{j=1}^{2} \sum_{l=1}^{2} \|R_{lj}\|_{L^\infty(\mathbb{R}^2)}$.

Lemma 3.1. *We have*

$$\| (E^{-m} S) u \|_{L^\infty(\mathbb{R}^2)} := \sum_{j=1}^{2} \| \left(E^{-m_j} S_j \right) u_j \|_{L^\infty(\mathbb{R}^2)} \le C t^{-\theta} \|U_{\frac{1}{m}}(-t) v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)}$$

for $t \ge 1$, where $0 < \alpha < 1 < 1 + 2\mu < \beta < 2$, $0 < \theta < \mu$, $\|U_{\frac{1}{m}}(-t) v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)} := \sum_{j=1}^{2} \|U_{\frac{1}{m_j}}(-t) v_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)}$.

Proof. By the definitions of S_j and u_j, Lemma 2.1 and the assumption (H3), we obtain

$$\begin{aligned} &\| \left(E^{-m_j} D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t) W_j \right) \left(D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t) v_j \right) \|_{L^\infty(\mathbb{R}^2)} \\ &\le \|\mathcal{F} U_{\frac{1}{m_j}}(-t) W_j\|_{L^\infty(\mathbb{R}^2)} \|\mathcal{F} U_{\frac{1}{m_j}}(-t) v_j\|_{L^\infty(\mathbb{R}^2)} \\ &\le C \|U_{\frac{1}{m}}(-t) W\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)} \|U_{\frac{1}{m}}(-t) v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)} \\ &\le C t^{-\theta} \|U_{\frac{1}{m}}(-t) v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)}, \end{aligned}$$

where $0 < \alpha < 1 < 1 + 2\mu < \beta < 2$, $0 < \theta < \mu$. Thus we have the desired result. □

Lemma 3.2. *We have*

$$\sum_{j=1}^{2}\sum_{l=1}^{2}\|R_{lj}\|_{L^\infty(\mathbb{R}^2)} \le Ct^{1-p-\mu}\|U_{\frac{1}{m}}(-t)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + Ct^{-1-\theta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}$$

for $t \ge 1$, where $p \ge 2, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$, and

$$\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} = \sum_{j=1}^{2}\|U_{\frac{1}{m_j}}(-t)v_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}.$$

Proof. Let $h_{k,j} = -D_{\frac{m_j}{m_k}}\mathcal{M}^{-1}_{m_k}\mathcal{F}U_{\frac{1}{m_k}}(-t)v_k$. By the definition of R_{1j}, the Cauchy-Schwarz inequality, Lemmas 2.1–2.3 and the assumption (H3), we have

$$\begin{aligned}
&\|R_{1j}\|_{L^\infty(\mathbb{R}^2)}\\
&\le Ct^{1-p-\mu}\||x|^{2\mu}\mathcal{F}^{-1}K_j(h_{1,j},h_{2,j})\|_{L^1(\mathbb{R}^2)} + Ct^{-1-\mu}\||x|^{2\mu}\mathcal{F}^{-1}F_j(h_{1,j},h_{2,j})\|_{L^1(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\left\||x|^{2\mu}\mathcal{F}^{-1}\left(E^{-\frac{1}{m_j}}(\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j)(\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j)\right)\right\|_{L^1(\mathbb{R}^2)}\\
&\le Ct^{1-p-\mu}\|K_j(h_{1,j},h_{2,j})\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\|F_j(h_{1,j},h_{2,j})\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\left\|\left(E^{-\frac{1}{m_j}}\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)W_j\right)\left(\mathcal{M}^{-1}_{m_j}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\right)\right\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\le Ct^{1-p-\mu}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|^{p-1}_{L^\infty(\mathbb{R}^2)}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)W\|_{L^\infty(\mathbb{R}^2)}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)}\|E^{-\frac{1}{m}}\mathcal{M}^{-1}_{m}\mathcal{F}U_{\frac{1}{m}}(-t)W\|_{\dot{H}^{s_1+2\mu,0}(\mathbb{R}^2)\cap\dot{H}^{s_2+2\mu,0}(\mathbb{R}^2)}\\
&\le Ct^{1-p-\mu}\|U_{\frac{1}{m}}(-t)v\|^{p-1}_{\dot{H}^{0,s_1}(\mathbb{R}^2)\cap\dot{H}^{0,s_2}(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s_1+2\mu}(\mathbb{R}^2)\cap\dot{H}^{0,s_2+2\mu}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s_1}(\mathbb{R}^2)\cap\dot{H}^{0,s_2}(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s_1+2\mu}(\mathbb{R}^2)\cap\dot{H}^{0,s_2+2\mu}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\mu-\theta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s_1}(\mathbb{R}^2)\cap\dot{H}^{0,s_2}(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s_1+2\mu}(\mathbb{R}^2)\cap\dot{H}^{0,s_2+2\mu}(\mathbb{R}^2)}\\
&\le Ct^{1-p-\mu}\|U_{\frac{1}{m}}(-t)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}\\
&\quad + Ct^{-1-\theta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}
\end{aligned}$$

for $t \ge 1$, where $p \ge 2, 0 < s_1 < 1 < s_2, s_1 = \alpha, s_2 + 2\mu = \beta < 2, 0 < \theta < \mu$.

We next consider the estimate $\|R_{2j}\|_{L^\infty(\mathbb{R}^2)}$. By the definition of R_{2j}, Lemma 2.1 and the assumption (H3), we get

$$\begin{aligned}
\|R_{2j}\|_{L^\infty(\mathbb{R}^2)} \le\ & Ct^{1-p}\|\mathcal{F}(M^{-m}-I)U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} \\
&\quad \times \left(\|\mathcal{F}U_{\frac{1}{m}}(-t)v\|^{p-1}_{L^\infty(\mathbb{R}^2)} + \|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|^{p-1}_{L^\infty(\mathbb{R}^2)}\right) \\
&+ Ct^{-1}\|\mathcal{F}(M^{-m}-I)U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} \\
&\quad \times \left(\|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} + \|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)}\right) \\
&+ Ct^{-1}\left(\|\mathcal{F}(M^{-m})U_{\frac{1}{m}}(-t)W\|_{L^\infty(\mathbb{R}^2)} + \|\mathcal{F}U_{\frac{1}{m}}(-t)W\|_{L^\infty(\mathbb{R}^2)}\right) \\
&\quad \times \left(\|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} + \|\mathcal{F}M^{-m}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)}\right) \\
\le\ & Ct^{1-p-\mu}\||x|^{2\mu}U_{\frac{1}{m}}(-t)v\|_{L^1(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|^{p-1}_{L^1(\mathbb{R}^2)} \\
&+ Ct^{-1-\mu}\||x|^{2\mu}U_{\frac{1}{m}}(-t)v\|_{L^1(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|_{L^1(\mathbb{R}^2)} \\
&+ Ct^{-1}\|U_{\frac{1}{m}}(-t)W\|_{L^1(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-t)v\|_{L^1(\mathbb{R}^2)} \\
\le\ & Ct^{1-p-\mu}\|U_{\frac{1}{m}}(-t)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\
&+ Ct^{-1-\theta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)},
\end{aligned}$$

where $p \ge 2, 0 < \mu < 1, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$.
Therefore we obtain

$$\begin{aligned}
\sum_{j=1}^{2}\sum_{l=1}^{2}\|R_{lj}\|_{L^\infty(\mathbb{R}^2)} \le\ & Ct^{1-p-\mu}\|U_{\frac{1}{m}}(-t)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\
&+ Ct^{-1-\theta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)},
\end{aligned}$$

where $p \ge 2, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$. □

By Lemmas 3.1 and 3.2, we have from (3.2)

$$\begin{aligned}
\|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} \le\ & C\|\mathcal{F}U_{\frac{1}{m}}(-1)v(1,x)\|_{L^\infty(\mathbb{R}^2)} + C\int_1^t \tau^{1-p-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\theta}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau.
\end{aligned}$$

By the Lemma 2.1 and the existence of local solutions of (1.5), we have

$$\begin{aligned}
\|\mathcal{F}U_{\frac{1}{m}}(-1)v(1,x)\|_{L^\infty(\mathbb{R}^2)} &\le \|U_{\frac{1}{m}}(-1)v(1,x)\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\
&\le \|U_{\frac{1}{m}}(-\cdot)v\|_{L^\infty([0,T],\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2))} \\
&\le C\varepsilon,
\end{aligned}$$

where $T > 1$. Then we obtain

$$\begin{aligned}\|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon &+ C\int_1^t \tau^{1-p-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\ &+ C\int_1^t \tau^{-1-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\ &+ C\int_1^t \tau^{-1-\theta}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau,\end{aligned} \tag{3.3}$$

where $0 < \alpha < 1 < 1 + 2\mu < \beta < 2$, $0 < \theta < \mu$.

Lemma 3.3. *Let $f \in \dot{H}^{0,\alpha}(\mathbb{R}^2) \cap \dot{H}^{0,\beta}(\mathbb{R}^2)$. Then we get*

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^2)} \le & Ct^{-1}\|\mathcal{F}U_{\frac{1}{m}}(-t)f\|_{L^\infty(\mathbb{R}^2)} \\ &+ Ct^{-1-\mu}\|U_{\frac{1}{m}}(-t)f\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}\end{aligned}$$

for $t \ge 1$, where $0 < \alpha < 1 < 1 + 2\mu < \beta$.

Similar to the proof of Lemma 3.2, we obtain Lemma 3.3. We omit the proof of Lemma 3.3 here. By Lemmas 2.2 and 3.3, the assumptions (H2) and (H3), we have the following lemma.

Lemma 3.4. *We have*

$$\begin{aligned}&\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\ &\le C\varepsilon + C\int_1^t \tau^{1-p}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|^{p-1}_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\ &\quad + C\int_1^t \tau^{(-1-\mu)(p-1)}\|U_{\frac{1}{m}}(-\tau)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\ &\quad + C\int_1^t \tau^{-1}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\ &\quad + C\int_1^t \tau^{-1-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau + C\int_1^t \tau^{-1-\theta}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau\end{aligned}$$

for any $t \in [1, T]$, where $p \ge 2, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$.

Proof. Let us consider the integral equation of (1.5) which is written as

$$v_j(t) = U_{\frac{1}{m_j}}(t)U_{\frac{1}{m_j}}(-1)v_j(1) - i\int_1^t U_{\frac{1}{m_j}}(t-\tau)G_j(v_1, v_2) + U_{\frac{1}{m_j}}(t-\tau)\left(W_jv_j\right)d\tau. \tag{3.4}$$

Multiplying both sides of (3.4) by $|J_{\frac{1}{m_j}}|^s(t) = U_{\frac{1}{m_j}}(t)|x|^s U_{\frac{1}{m_j}}(-t)$, $s = \alpha$ or β, by Lemma 2.2 we have

$$\begin{aligned}
\|U_{\frac{1}{m_j}}(-t)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)} \le{}& \|U_{\frac{1}{m_j}}(-1)v_j(1)\|_{\dot{H}^{0,s}(\mathbb{R}^2)} + \sum_{j=1}^{2}\int_1^t \|U_{\frac{1}{m_j}}(-\tau)G_j(v_1,v_2)\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\sum_{j=1}^{2}\int_1^t \|U_{\frac{1}{m_j}}(-\tau)(W_j v_j)\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
\le{}& \|U_{\frac{1}{m_j}}(-1)v_j(1)\|_{\dot{H}^{0,s}(\mathbb{R}^2)} + C\int_1^t \|v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \|v\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \|W\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \|v\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)W\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau
\end{aligned}$$

for $j = 1, 2$.

By Lemma 3.3, (1.13) in Remark 1.1, the assumption (H3), we obtain

$$\begin{aligned}
\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)} \le{}& C\varepsilon + C\int_1^t \tau^{1-p}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{(-1-\mu)(p-1)}\|U_{\frac{1}{m}}(-\tau)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-\beta}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\theta}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\theta-\mu}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
\le{}& C\varepsilon + C\int_1^t \tau^{1-p}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{(-1-\mu)(p-1)}\|U_{\frac{1}{m}}(-\tau)v\|^p_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1}\|\mathcal{F}U_{\frac{1}{m}}(-\tau)v\|_{L^\infty(\mathbb{R}^2)}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,s}(\mathbb{R}^2)}d\tau \\
&+ C\int_1^t \tau^{-1-\mu}\|U_{\frac{1}{m}}(-\tau)v\|^2_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau
\end{aligned}$$

$$+ C\int_1^t \tau^{-1-\theta}\|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)}d\tau,$$

where $s = \alpha$ or β, $p \geq 2, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2$, $0 < \theta < \mu$. This completes the proof of the lemma. □

Lemma 3.5. *There exists a small $\delta > 0$ such that*

$$t^{-\delta}\|U_{\frac{1}{m}}(-t)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} + \|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)} < \varepsilon^{\frac{1}{2}}$$

for any $t \in [1, T]$, where $p \geq 2$, $\varepsilon^{\frac{1}{2}} < \delta < \min\{\theta, \frac{\mu}{p}\}$, $0 < \alpha < 1 < 1 + 2\mu < \beta < 2$, $0 < \theta < \mu$.

Proof. Let

$$H(t) = \|U_{\frac{1}{m}}(-\tau)v\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)},$$
$$K(t) = \|\mathcal{F}U_{\frac{1}{m}}(-t)v\|_{L^\infty(\mathbb{R}^2)},$$
$$\widetilde{H}(t) = t^{-\delta}H(t).$$

By (3.3), Lemma 3.4, we get

$$K(t) \leq C\varepsilon + C\int_1^t \tau^{1-p-\mu}H(\tau)^p d\tau + C\int_1^t \tau^{-1-\mu}H(\tau)^2 d\tau + C\int_1^t \tau^{-1-\theta}H(\tau)d\tau,$$
$$H(t) \leq C\varepsilon + C\int_1^t \tau^{1-p}K(\tau)^{p-1}H(\tau) + \tau^{(-1-\mu)(p-1)}H(\tau)^p d\tau$$
$$+ C\int_1^t \tau^{-1}K(\tau)H(\tau) + \tau^{-1-\mu}H(\tau)^2 d\tau + C\int_1^t \tau^{-1-\theta}H(\tau)d\tau.$$

Then we have

$$\begin{aligned} K(t) \leq C\varepsilon + C\int_1^t \tau^{1-p-\mu+p\delta}\widetilde{H}(\tau)^p d\tau + C\int_1^t \tau^{-1-\mu+2\delta}\widetilde{H}(\tau)^2 d\tau \\ + C\int_1^t \tau^{-1-\theta+\delta}\widetilde{H}(\tau)d\tau, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \frac{d}{dt}H(t) \leq Ct^{1-p}K(t)^{p-1}H(t) + Ct^{(-1-\mu)(p-1)}H(t)^p \\ + Ct^{-1}K(t)H(t) + Ct^{-1-\mu}H(t)^2 + Ct^{-1-\theta}H(t). \end{aligned} \tag{3.6}$$

Since

$$\frac{d}{dt}\widetilde{H}(t) = t^{-\delta}\frac{d}{dt}H(t) - \delta t^{-\delta-1}H(t),$$

then from (3.6) we obtain

$$\frac{d}{dt}\widetilde{H}(t) + \delta t^{-1}\widetilde{H}(t) \leq Ct^{1-p}K(t)^{p-1}\widetilde{H}(t) + Ct^{(-1-\mu+\delta)(p-1)}\widetilde{H}(t)^p$$
$$+ Ct^{-1}K(t)\widetilde{H}(t) + Ct^{-1-\mu+\delta}\widetilde{H}(t)^2 + Ct^{-1-\theta}\widetilde{H}(t).$$

Thus we get

$$\begin{aligned}\widetilde{H}(t) + \delta \int_1^t \tau^{-1}\widetilde{H}(\tau)d\tau &\le C\varepsilon + C\int_1^t \tau^{1-p}K(\tau)^{p-1}\widetilde{H}(\tau)d\tau \\ &+ C\int_1^t C\tau^{(-1-\mu+\delta)(p-1)}\widetilde{H}(\tau)^p d\tau + C\int_1^t \tau^{-1}K(\tau)\widetilde{H}(\tau)d\tau \\ &+ C\int_1^t \tau^{-1-\mu+\delta}\widetilde{H}(\tau)^2 d\tau + C\int_1^t \tau^{-1-\theta}\widetilde{H}(\tau)d\tau.\end{aligned} \tag{3.7}$$

If we assume that there exists a time $t \in [1, T]$ such that $K(t) + \widetilde{H}(t) \le \varepsilon^{\frac{1}{2}}$, then by (3.7), we have

$$\begin{aligned}\widetilde{H}(t) + \delta \int_1^t \tau^{-1}\widetilde{H}(\tau)d\tau &\le C\left(\varepsilon + \varepsilon^{\frac{p}{2}}\right) + C\varepsilon^{\frac{1}{2}}\int_1^t \tau^{-1}\widetilde{H}(\tau)d\tau + C\int_1^t \tau^{-1-\theta}\widetilde{H}(\tau)d\tau \\ &\le C\varepsilon + C\varepsilon^{\frac{1}{2}}\int_1^t \tau^{-1}\widetilde{H}(\tau)d\tau + C\int_1^t \tau^{-1-\theta}\widetilde{H}(\tau)d\tau,\end{aligned}$$

where $\varepsilon^{\frac{1}{2}} < \delta < \mu, p \ge 2$. By Gronwall's inequality, we have

$$\begin{aligned}\widetilde{H}(t) &\le C\varepsilon \cdot e^{C\int_1^t \tau^{-1-\theta}d\tau} \\ &\le C_4\varepsilon,\end{aligned}$$

where $C_4 = Ce^{C\int_1^t \tau^{-1-\theta}d\tau}$. Therefore if we choose $\varepsilon > 0$ small enough, from (3.5) we get

$$\begin{aligned}K(t) &\le C\varepsilon + C\varepsilon^p \int_1^t \tau^{1-p-\mu+p\delta}d\tau + C\varepsilon^2\int_1^t \tau^{-1-\mu+2\delta}d\tau \\ &\quad + C\varepsilon\int_1^t \tau^{-1-\theta+\delta}d\tau \\ &\le C\left(\varepsilon + \varepsilon^p + \varepsilon^2\right) \\ &\le C\varepsilon,\end{aligned}$$

where $p \ge 2, 0 < \delta < \min\{\frac{\mu}{p}, \theta\}$, $0 < \theta < \mu$.
Thus we have

$$\widetilde{H}(t) + K(t) \le C\varepsilon < \varepsilon^{\frac{1}{2}}.$$

This contradicts the assumption that there exists a time $t \in [1, T]$ such that $K(t) + \widetilde{H}(t) \le \varepsilon^{\frac{1}{2}}$. This completes the proof of the lemma. □

3.2. Time decay estimates of the global solutions

By Lemma 3.5, we have global existence of solutions to the Cauchy problem (1.5). By Lemmas 3.3 and 3.5, we have

$$\begin{aligned}\|v_j\|_{L^\infty(\mathbb{R}^2)} &\le Ct^{-1}\|\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\|_{L^\infty(\mathbb{R}^2)} + Ct^{-1-\mu}\|U_{\frac{1}{m_j}}(-t)v_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\ &\le Ct^{-1}\varepsilon^{\frac{1}{2}} + Ct^{-1-\mu+\delta}\varepsilon^{\frac{1}{2}} \\ &\le Ct^{-1}\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Therefore we get the time decay estimate (1.10)

$$\|v\|_{L^\infty(\mathbb{R}^2)} = \sum_{j=1}^{2} \|v_j\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-1}\varepsilon^{\frac{1}{2}}$$

for $t \ge 1$.

We are now in the position of proving the delicate decay estimates of solutions to (1.5). From (3.1), we have

$$\begin{aligned}\partial_t \sum_{j=1}^{2} |u_j|^2 =& \frac{2}{t^{p-1}} \Im \sum_{j=1}^{2} (\lambda_j |u_j|^{p+1}) + 2\Im\left(\sum_{j=1}^{2}\left(\frac{1}{t}\left(E^{-m_j}S_j\right)u_j\right)\overline{u_j}\right)\\ &+ 2\Im\left(\sum_{j=1}^{2}\left(D_{\frac{1}{m_j}}\sum_{l=1}^{2} R_{lj}\right)\overline{u_j}\right),\end{aligned} \tag{3.8}$$

where $u_j = D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j$ for $j = 1, 2$. Let $|u| = \left(\sum_{j=1}^{2}|u_j|^2\right)^{\frac{1}{2}}$. We can get positive constants λ_*, λ^* such that

$$-\lambda_*|u|^{p+1} \le \sum_{j=1}^{2} \Im\lambda_j |u_j|^{p+1} \le -\lambda^*|u|^{p+1}. \tag{3.9}$$

By Lemmas 3.1, 3.2, 3.5, (3.8) and (3.9), we obtain

$$\partial_t|u|^2 \le -\frac{2\lambda^*}{t^{p-1}}|u|^{p+1} + Ct^{-1-\theta+\delta}\varepsilon \tag{3.10}$$

for $t \ge 1$, where $\varepsilon^{\frac{1}{2}} < \delta < \min\{\theta, \frac{\mu}{p}, \mu-\theta, \frac{p+\mu-2-\theta}{p-1}\}$, $p \ge 2, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$.

Let us consider the case of $p = 2$. Multiplying both sides of (3.10) by $(\log t)^3$, we have

$$\partial_t\left((\log t)^3|u|^2\right) \le \frac{3}{t}(\log t)^2|u|^2 - \frac{2\lambda^*}{t}(\log t)^3|u|^3 + Ct^{-1-\theta+\delta}(\log t)^3\varepsilon$$

for $t \ge 1$. By Young's inequality, we obtain:

$$\frac{3}{t}(\log t)^2|u|^2 \le \frac{2\lambda^*}{t}(\log t)^3|u|^3 + \frac{1}{\lambda^{*2}}\frac{1}{t}$$

for $t \ge 1$. Thus we have

$$\partial_t\left((\log t)^3|u|^2\right) \le \frac{1}{\lambda^{*2}}\frac{1}{t} + Ct^{-1-\theta+\delta}(\log t)^3\varepsilon,$$

for $t \ge 1$, where $\varepsilon^{\frac{1}{2}} < \delta < \min\{\theta, \frac{\mu}{2}, \mu-\theta\}, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$.

Integrating the above inequality in time, we have

$$|u| \le C(\log t)^{-1} \tag{3.11}$$

for $t \ge 2$.

By Lemmas 3.3, 3.5 and (3.11), we have

$$\begin{aligned}\|v_j\|_{L^\infty(\mathbb{R}^2)} &\le Ct^{-1}\|\mathcal{F}U_{\frac{1}{m_j}}(-t)v_j\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + Ct^{-1-\mu}\|U_{\frac{1}{m_j}}(-t)v_j\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} \\ &\le Ct^{-1}(\log t)^{-1} + Ct^{-1-\mu+\delta}\varepsilon^{\frac{1}{2}} \\ &\le Ct^{-1}(\log t)^{-1}\end{aligned} \tag{3.12}$$

for $t \ge 2$, where $\varepsilon^{\frac{1}{2}} < \delta < \min\{\theta, \frac{\mu}{2}, \mu - \theta\}, 0 < \alpha < 1 < 1 + 2\mu < \beta < 2, 0 < \theta < \mu$. Therefore, we get the desired time decay estimate (1.11).

From (3.4) , we have

$$v_j(t) = U_{\frac{1}{m_j}}(t)\left(U_{\frac{1}{m_j}}(-1)\phi_j - i\int_1^t U_{\frac{1}{m_j}}(-\tau)G_j(v_1, v_2) + U_{\frac{1}{m_j}}(-\tau)\left(W_j v_j\right)d\tau\right).$$

Let $v_{j+} = U_{\frac{1}{m_j}}(-1)\phi_j - i\int_1^\infty U_{\frac{1}{m_j}}(-\tau)G_j(v_1, v_2) + U_{\frac{1}{m_j}}(-\tau)\left(W_j v_j\right)d\tau$. Then we have

$$v_j(t) = U_{\frac{1}{m_j}}(t)v_{j+} + i\int_t^\infty U_{\frac{1}{m_j}}(t-\tau)G_j(v_1, v_2) + U_{\frac{1}{m_j}}(t-\tau)\left(W_j v_j\right)d\tau.$$

We can also obtain the scattering $\lim_{t\to+\infty}\|U_{\frac{1}{m_j}}(-t)v(t) - v_+\|_{\dot{H}^{0,\alpha}(\mathbb{R}^2)\cap\dot{H}^{0,\beta}(\mathbb{R}^2)} = 0$ from the time decay estimate (1.10), where $0 < \alpha < 1 < \beta < 2$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

References

1. R. Carles, Nonlinear Schrödinger equation with time dependent potential, *Commun. Math. Sci.*, **9** (2011), 937–964. https://doi.org/10.4310/cms.2011.v9.n4.a1
2. R. Carles, J. D. Silva, Large time behavior in nonlinear Schrödinger equations with time dependent potential, *Commun. Math. Sci.*, **13** (2015), 443–460. http://doi.org/10.4310/CMS.2015.v13.n2.a9

3. M. Kawamoto, R. Muramatsu, Asymptotic behavior of solutions to nonlinear Schrödinger equations with time-dependent harmonic potentials, *J. Evol. Equ.*, **21** (2021), 699–723. https://doi.org/10.1007/s00028-020-00597-8
4. I. P. Naumkin, Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential, *J. Math. Phys.*, **57** (2016), 051501. https://doi.org/10.1063/1.4948743
5. P. Germain, F. Pusateri, F. Rousset, The nonlinear Schrödinger equation with a potential, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **35** (2018), 1477–1530. https://doi.org/10.1016/j.anihpc.2017.12.002
6. S. Masaki, J. Murphy, J. Segata, Modified scattering for the one-dimensional cubic NLS with a repulsive delta potential, *Int. Math. Res. Notices*, **2019** (2019), 7577–7603. https://doi.org/10.1093/imrn/rny011
7. N. Hayashi, P. I. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, *Am. J. Math.*, **120** (1998), 369–389. https://doi.org/10.1353/ajm.1998.0011
8. J. E. Barab, Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation, *J. Math. Phys.*, **25** (1984), 3270–3273. https://doi.org/10.1063/1.526074
9. Y. Tsutsumi, K. Yajima, The asymptotic behavior of nonlinear Schrödinger equations, *Bull. Amer. Math. Soc.*, **11** (1984), 186–188. https://doi.org/10.1090/S0273-0979-1984-15263-7
10. N. Kita, A. Shimomura, Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data, *J. Math. Soc. Japan*, **61** (2009), 39–64. https://doi.org/10.2969/jmsj/06110039
11. G. Hoshino, Dissipative nonlinear Schrödinger equations for large data in one space dimension, *Commun. Pur. Appl. Anal.*, **19** (2020), 967–981. https://doi.org/10.3934/cpaa.2020044
12. G. Hoshino, Scattering for solutions of a dissipative nonlinear Schrödinger equation, *J. Differ. Equations*, **266** (2019), 4997–5011. https://doi.org/10.1016/j.jde.2018.10.016
13. N. Hayashi, P. I. Naumkin, Scattering problem for the supercritical nonlinear Schrödinger equation in 1d, *Funkc. Ekvacioj*, **58** (2015), 451–470. https://doi.org/10.1619/fesi.58.451
14. S. Cuccagna, N. Visciglia, V. Georgiev, Decay and scattering of small solutions of pure power NLS in R with p>3 and with a potential, *Commun. Pure Appl. Math.*, **67** (2014), 957–981. https://doi.org/10.1002/cpa.21465
15. V. Georgiev, C. Li, On the scattering problem for the nonlinear Schrödinger equation with a potential in 2D, *Physica D*, **398** (2019), 208–218. https://doi.org/10.1016/j.physd.2019.03.010
16. V. Georgiev, B. Velichkov, Decay estimates for the supercritical 3-D Schrödinger equation with rapidly decreasing potential, In: *Evolution equations of hyperbolic and Schrödinger type. Progress in mathematics*, Basel: Birkhäuser, **301** (2012), 145–162. https://doi.org/10.1007/978-3-0348-0454-7_8
17. Z. Li, L. Zhao, Decay and scattering of solutions to nonlinear Schrödinger equations with regular potentials for nonlinearities of sharp growth, *J. Math. Study*, **50** (2017), 277–290. https://doi.org/10.4208/jms.v50n3.17.05

18. S. Katayama, C. Li, H. Sunagawa, A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D, *Differ. Integral Equ.*, **27** (2014), 301–312. https://doi.org/10.57262/die/1391091368

19. W. Tang, Y. Wang, Z. Li, Numerical simulation of fractal wave propagation of a multi-dimensional nonlinear fractional-in-space Schrödinger equation, *Phys. Scripta*, **98** (2023), 045205. https://doi.org/10.1088/1402-4896/acbdd0

20. N. Hayashi, C. Li, P. I. Naumkin, On a system of nonlinear Schrödinger equations in 2d, *Differ. Integral Equ.*, **24** (2011), 417–434. https://doi.org/10.57262/die/1356018911

21. C. Li, Y. Nishii, Y. Sagawa, H. Sunagawa, Large time asymptotics for a cubic nonlinear Schrödinger system in one space dimension, *Funkc. Ekvacioj*, **64** (2021), 361–377. https://doi.org/10.1619/fesi.64.361

22. C. Li, Y. Nishii, Y. Sagawa, H. Sunagawa, Large time asymptotics for a cubic nonlinear Schrödinger system in one space dimension, II, *Tokyo J. Math.*, **44** (2021), 411–416. https://doi.org/10.3836/tjm/1502179340

23. C. Li, On a system of quadratic nonlinear Schrödinger equations and scale invariant spaces in 2D, *Differ. Integral Equ.*, **28** (2015), 201–220. https://doi.org/10.57262/die/1423055224

24. N. Hayashi, T. Ozawa, Scattering theory in the weighted $\mathbf{L}^2(\mathbb{R}^n)$ spaces for some Schrödinger equations, *Annales De L Institut Henri Poincare-physique Theorique*, **48** (1988), 17–37. http://eudml.org/doc/76388

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