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Research article

Decay rate of the solutions to the Lord Shulman thermoelastic Timoshenko model

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Abstract: In this work, we deal with a one-dimensional Cauchy problem in Timoshenko system with thermal effect and damping term. The heat conduction is given by the theory of Lord-Shulman. We prove that the dissipation induced by the coupling of the Timoshenko system with the heat conduction of Lord-Shulman's theory alone is strong enough to stabilize the system, but with slow decay rate. To show our result, we transform our system into a first order system and, applying the energy method in the Fourier space, we establish some pointwise estimates of the Fourier image of the solution. Using those pointwise estimates, we prove the decay estimates of the solution and show that those decay estimates are very slow.

Keywords: partial differential equations; mathematical operators; decay rate; Lord-Shulman; thermoelasticity; Fourier transform
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1. Introduction

Lord Shulman's thermoelasticity has also garnered a lot of interest among scientists in the past few years and there is a broad amount of contributions to explaining this theory. This theory is based on the study of a set of four hyperbolic equations with heat dissipation. In this instance, the heat conduction is also equivalent and hyperbolic in contrast to the one produced for Fourier's law. For more information and explanation about this theory and other theories, see [1, 2]. Green & Naghdi [3, 4] created a thermoelasticity model that incorporates thermal displacement gradient and temperature gradient among the constitutive variables and proposed a heat conduction law.

There are numerous results from the coupling of the Fourier law of heat conduction and various systems, which has been discussed by researchers. For example, the Timoshenko system has been studied in [5, 6], the Bresse system combined with the Cattaneo law of heat conduction (Bresse-Cattaneo) in [7], the Bresse system (Bresse Fourier) has been studied in [8–10] and MGT problem in [11]. The following papers are recommended to the reader for more information [12, 13].

Initially, the basic evolution equations for the one-dimensional Timoshenko thermo-elasticity theories with microtemperature and temperature [14–18] provided by

$$\begin{aligned} \rho\hbar_{tt} &= T_x, \\ I_\rho\Im_{tt} &= H_x + G, \\ \rho T_0\zeta_t &= q_x, \\ \rho E_t &= P^*_x + q - Q. \end{aligned} \tag{1.1}$$

Here $E, H, T, q, \zeta, Q, P^*, T_0, G$ denote the first moment of energy, the equilibrated stress, the stress, the heat flux vector, the entropy, the mean heat flux, the first heat flux moment, the reference temperature and the equilibrated body force, $K, \rho, I_\rho, I\&E$ represent the shear modulus, the density, the polar moment of inertia of a cross section, the moment of inertia of a cross section and Young's modulus of elasticity respectively. In order to make calculations simple, let $T_0 = \rho = I_\rho = K = 1$ and $EI = a^2 > 0$.

In this work, the natural counterpart to the Lord-Shulman theory's microtemperatures is taken into consideration [1,2]. In this case, we must modify the constitutive equations to take the following form.

$$\begin{aligned} T &= T_1 + T_2 \quad & P^* &= -k_2\varpi_x, \\ H &= H_1 + H_2 \quad & \rho\zeta &= \gamma_0\hbar_x + \gamma_1\Im + \beta_1(\kappa\theta_t + \theta), \\ G &= G_1 + G_2 \quad & Q &= (k_1 - k_3)\varpi + (k - k_1)\theta_x, \\ q &= k\theta_x + k_1\varpi \quad & \rho E &= -\beta_2(\kappa\varpi_t + \varpi) - \gamma_2\Im_x, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} T_1 = G_1 &= K(\hbar_x - \Im) \quad & T_2 &= -\gamma_0(\kappa\theta_t + \theta) \\ H_1 &= EI\Im_x \quad & G_2 &= \gamma_1(\kappa\theta_t + \theta) - \mu_1\Im_t, \\ & & H_2 &= -\gamma_2(\kappa\varpi_t + \varpi). \end{aligned} \tag{1.3}$$

Here, the functions $\hbar$ and $\Im$ denote the elastic material displacement, ϖ is the microtemperature vector and the function θ is the temperature difference, $\kappa > 0$ is the relaxation parameter. $\beta_1, \beta_2 > 0$, the coefficients γ_1, k, γ_0 denote, the coupling between the volume fraction and the temperature, the thermal conductivity, the coupling between the displacement and the temperature respectively.

As coupling is considered the coefficients $k_1, k_2, k_3, \gamma_2, \mu_1 > 0$ and satisfy the inequalities

$$k_1^2 < kk_3. \tag{1.4}$$

The main objective of this study is the thermal effects, let the heat capacity $\beta_1 = \beta > 0$, and for more excitement in posing the problem, we do not assume the microtemperature effect, and $\beta_2 = k_1 = k_2 = k_3 = \gamma_2 = 0$.

Now, by substituting (1.2) and (1.3) into (1.1), we obtain that:

$$\begin{cases} \hbar_{tt} - (\hbar_x - \Im)_x + \gamma_0(\kappa\theta_t + \theta)_x = 0, \\ \Im_{tt} - a^2\Im_{xx} - (\hbar_x - \Im) - \gamma_1(\kappa\theta_t + \theta) + \mu_1\Im_t = 0, \\ \beta(\kappa\theta_t + \theta)_t + \gamma_0\hbar_{tx} + \gamma_1\Im_t - k\theta_{xx} = 0, \end{cases} \tag{1.5}$$

where

$$(x, t) \in \mathbb{R} \times \mathbb{R}_+,$$

with initial conditions

$$(\hbar, \hbar_t, \Im, \Im_t, \theta, \theta_t)(x, 0) = (\hbar_0, \hbar_1, \Im_0, \Im_1, \theta_0, \theta_1), \quad x \in \mathbb{R}. \tag{1.6}$$

We categorise this paper as follows: We utilise our preliminary findings in the remainder of this section to help us understand our major decay conclusion. The Lyapunov functional is built and the estimate for the Fourier image is discovered in the following section utilising the energy approach in Fourier space. The final portion is devoted to the conclusion.

This, to our knowledge, is one of the first studies to look at this issue in the Fourier space. We require the following Lemma to support our main finding.

Lemma 1.1. *For any $k, \alpha \geq 0, c > 0$, there exist a constant $C > 0$ in such a way that $\forall \ t \geq 0$ the below stated estimate hold*

$$\int_{|\wp|\leq 1} |\wp|^k e^{-c|\wp|^\alpha t} d\wp \leq C(1 + t)^{-(k+n)/\alpha}, \quad \wp \in \mathbb{R}^n. \tag{1.7}$$

2. Energy method and decay estimates

This section provide the decay estimate of the Fourier image of the solution for problems (1.5) and (1.6). This approach enables us to give the decay rate of the solution in the energy space by utilising Plancherel's theorem along with some integral estimates, such as Lemma 1.1. We create the appropriate Lyapunov functionals and apply the energy method in Fourier space to this problem. We conclude by demonstrating our main finding.

2.1. The energy method in the Fourier space

Let us introduce the new variables in order to construct the Lyapunov functional in the Fourier space

$$f = \hbar_x - \Im, \quad j = \hbar_t, \quad b = a\Im_x, \quad m = \Im_t, \quad \zeta = \kappa\theta_t + \theta, \quad \varpi = \theta_x. \tag{2.1}$$

Then, the system (1.5) can also be written as

$$\begin{cases} f_t - j_x + m = 0 \\ j_t - f_x + \gamma_0\zeta_x = 0 \\ b_t - ay_x = 0 \\ m_t - az_x - f - \gamma_1\zeta + \mu_1 m = 0 \\ \beta\zeta_t + \gamma_0 j_x + \gamma_1 m - k\varpi_x = 0 \\ \kappa\varpi_t - \zeta_x + \varpi = 0, \end{cases} \tag{2.2}$$

where initial condition are

$$(f, j, b, m, \zeta, \varpi)(x, 0) = (f_0, j_0, b_0, m_0, \zeta_0, \varpi_0), x \in \mathbb{R}, \tag{2.3}$$

with

$$f_0 = (\hbar_{0,x} - \Im_0),\ j_0 = \hbar_1,\ b_0 = a\Im_{0,x},\ m_0 = \Im_1,\ \zeta_0 = \kappa\theta_1 - \theta_0,\ \varpi_0 = \theta_{0,x}.$$

Hence, the problems (2.2) and (2.3) can be written as

$$\begin{cases} F_t + \mathcal{A}F_x + \mathcal{L}F = 0, \\ F(x,0) = F_0(x), \end{cases} \tag{2.4}$$

with $F = (f, j, b, m, \zeta, \varpi)^T$, $F_0 = (f_0, j_0, b_0, m_0, \zeta_0, \varpi_0)$ and

$$\mathcal{A}F = \begin{pmatrix} -j \\ -f + \gamma_0\zeta \\ -ay \\ -az \\ \frac{1}{\beta}(\gamma_0 j - k\varpi) \\ -\frac{1}{\kappa}\zeta \end{pmatrix}, \quad \mathcal{L}F = \begin{pmatrix} m \\ 0 \\ 0 \\ -f - \gamma_1\zeta + \mu_1 m \\ \frac{1}{\beta}(\gamma_1 m) \\ \frac{1}{\kappa}\varpi \end{pmatrix}.$$

Utilizing the Fourier transform on (2.4), we obtain that:

$$\begin{cases} \widehat{F}_t + i\wp\mathcal{A}\widehat{F} + \mathcal{L}\widehat{F} = 0, \\ \widehat{F}(\wp, 0) = \widehat{F}_0(\wp), \end{cases} \tag{2.5}$$

where $\widehat{F}(\wp, t) = (\widehat{f}, \widehat{j}, \widehat{b}, \widehat{m}, \widehat{\zeta}, \widehat{\varpi})^T(\wp, t)$. The quation $(2.5)_1$ can be written as

$$\begin{cases} \widehat{f}_t - i\wp\widehat{j} + \widehat{m} = 0 \\ \widehat{j}_t - i\wp\widehat{f} + i\wp\gamma_0\widehat{\zeta} = 0 \\ \widehat{b}_t - ai\wp\widehat{m} = 0 \\ \widehat{m}_t - ai\wp\widehat{b} - \widehat{f} - \gamma_1\widehat{\zeta} + \mu_1\widehat{m} = 0 \\ \beta\widehat{\zeta}_t + i\wp\gamma_0\widehat{j} + \gamma_1\widehat{m} - i\wp k\widehat{\varpi} = 0 \\ \kappa\widehat{\varpi}_t - i\wp\widehat{\zeta} + \widehat{\varpi} = 0. \end{cases} \tag{2.6}$$

Lemma 2.1. *Let $\widehat{F}(\wp, t)$ be a solution of (2.5). Then the energy functional $\widehat{E}(\wp, t)$, given by*

$$\widehat{E}(\wp, t) = \frac{1}{2}\Big\{|\widehat{f}|^2 + |\widehat{j}|^2 + |\widehat{b}|^2 + |\widehat{m}|^2 + \beta|\widehat{\zeta}|^2 + k\kappa|\widehat{\varpi}|^2\Big\}, \tag{2.7}$$

satisfies

$$\frac{d\widehat{E}(\wp, t)}{dt} = -\mu_1|\widehat{m}|^2 - k|\widehat{\varpi}|^2 \leq 0. \tag{2.8}$$

Proof. Firstly, multiplying $(2.6)_{1,2,3,4}$ by $\overline{\widehat{f}}, \overline{\widehat{j}}, \overline{\widehat{b}}$ and $\overline{\widehat{m}}$ respectively, and multiplying $(2.6)_{5,6}$ by $\overline{\widehat{\zeta}}, k\overline{\widehat{\varpi}}$, adding these equalities and taking the real part, we obtain that

$$\frac{1}{2}\frac{d}{dt}\Big[|\widehat{f}|^2 + |\widehat{j}|^2 + |\widehat{b}|^2 + |\widehat{m}|^2 + \beta|\widehat{\zeta}|^2 + k\kappa|\widehat{\varpi}|^2\Big] + \mu_1|\widehat{m}|^2 + k|\widehat{\varpi}|^2 = 0, \tag{2.9}$$

we find

$$\frac{d\widehat{E}(\wp, t)}{dt} = -\mu_1|\widehat{m}|^2 - k|\widehat{\varpi}|^2.$$

Hence, we obtain (2.7) and (2.8) ($\widehat{E}$ is non-increasing function). □

Now, to obtain the main result, we need the following lemmas.

Lemma 2.2. *The functional*

$$\mathcal{D}_1(\wp, t) := \Re e\Big\{i\wp\Big(\widehat{f}\overline{\widehat{j}} + \widehat{m}\overline{\widehat{b}}\Big)\Big\}, \tag{2.10}$$

satisfies,

$$\begin{aligned} \frac{d\mathcal{D}_1(\wp, t)}{dt} \leq & -\frac{1}{2}\wp^2|\widehat{j}|^2 - \frac{a}{2}\wp^2|\widehat{b}|^2 + c(1+\wp^2)|\widehat{f}|^2 \\ & + c(1+\wp^2)|\widehat{m}|^2 + c\wp^2|\widehat{\zeta}|^2. \end{aligned} \tag{2.11}$$

Proof. Differentiating $\mathcal{D}_1$ and using (2.6), we get

$$\begin{aligned} \frac{d\mathcal{D}_1(\wp, t)}{dt} &= \Re e\Big\{i\wp\widehat{f}_t\overline{\widehat{j}} - i\wp\widehat{j}_t\overline{\widehat{f}} + i\wp\widehat{m}_t\overline{\widehat{b}} - i\wp\widehat{b}_t\overline{\widehat{m}}\Big\} \\ &= -\wp^2|\widehat{j}|^2 - a\wp^2|\widehat{b}|^2 + \wp^2|\widehat{f}|^2 + a\wp^2|\widehat{m}|^2 - \Re e\Big\{i\wp\widehat{m}\overline{\widehat{j}}\Big\} + \Re e\Big\{i\wp\widehat{f}\overline{\widehat{b}}\Big\} \\ &\quad + \Re e\Big\{i\gamma_1\wp\widehat{\zeta}\overline{\widehat{b}}\Big\} - \Re e\Big\{i\mu_1\wp\widehat{m}\overline{\widehat{b}}\Big\} - \Re e\Big\{\gamma_0\wp^2\widehat{\zeta}\overline{\widehat{f}}\Big\}. \end{aligned} \tag{2.12}$$

We evaluate the terms in the RHS of (2.12), utilizing Young's inequality, for any $\delta_1, \delta_2 > 0$ we get

$$\begin{aligned} -\Re e\Big\{i\wp\widehat{m}\overline{\widehat{j}}\Big\} &\leq \delta_1\wp^2|\widehat{j}|^2 + c(\delta_1)|\widehat{m}|^2, \\ +\Re e\Big\{i\wp\widehat{f}\overline{\widehat{b}}\Big\} &\leq \delta_2\wp^2|\widehat{b}|^2 + c(\delta_2)|\widehat{f}|^2, \\ +\Re e\Big\{i\gamma_1\wp\widehat{\zeta}\overline{\widehat{b}}\Big\} &\leq \delta_2\wp^2|\widehat{b}|^2 + c(\delta_2)|\widehat{\zeta}|^2, \\ -\Re e\Big\{i\mu_1\wp\widehat{m}\overline{\widehat{b}}\Big\} &\leq \delta_2\wp^2|\widehat{b}|^2 + c(\delta_2)|\widehat{m}|^2, \\ -\Re e\Big\{\gamma_0\wp^2\widehat{\zeta}\overline{\widehat{f}}\Big\} &\leq c\wp^2|\widehat{\zeta}|^2 + c\wp^2|\widehat{f}|^2. \end{aligned} \tag{2.13}$$

Inserting the above estimates (2.13) into (2.12) and by letting $\delta_1 = \frac{1}{2}, \delta_2 = \frac{a}{6}$, we get (2.11). □

Lemma 2.3. *The functional*

$$\mathcal{D}_2(\wp, t) := \Re e\Big\{i\wp\Big(\kappa\beta\widehat{\varpi}\overline{\widehat{\zeta}}\Big)\Big\}, \tag{2.14}$$

holds, for any $\varepsilon_1 > 0$

$$\begin{aligned}\frac{d\mathcal{D}_2(\wp, t)}{dt} \;\leq\;& -\frac{\beta}{2}\wp^2|\widehat{\zeta}|^2 + \varepsilon_1\frac{\wp^4}{(1+\wp^2)^2}|\widehat{j}|^2 + c\wp^2|\widehat{m}|^2\\ &+c(\varepsilon_1)(1+\wp^2)^2|\widehat{\varpi}|^2.\end{aligned} \tag{2.15}$$

Proof. Differentiating $\mathcal{D}_2$ and using (2.6), we get

$$\begin{aligned}\frac{\mathcal{D}_2(\wp, t)}{dt} \;=\;& \Re e\Big\{i\wp\Big(\kappa\beta\widehat{\varpi}_t\overline{\widehat{\zeta}} + \kappa\beta\widehat{\varpi}\overline{\widehat{\zeta}_t}\Big)\Big\}\\ =\;& -\beta\wp^2|\widehat{\zeta}|^2 + k\kappa\wp^2|\widehat{\varpi}|^2 - \Re e\Big\{i\wp\beta\widehat{\varpi}\overline{\widehat{\zeta}}\Big\}\\ &+\Re e\Big\{i\gamma_1\kappa\wp\widehat{m}\overline{\widehat{\varpi}}\Big\} - \Re e\Big\{\gamma_0\kappa\wp^2\overline{\widehat{j}\widehat{\varpi}}\Big\}.\end{aligned} \tag{2.16}$$

Similarly, we evaluate the terms in the RHS of (2.16), by applying Young's inequality, for any $\varepsilon_1, \delta_3 > 0$ we find

$$\begin{aligned}-\Re e\Big\{i\wp\beta\widehat{\varpi}\overline{\widehat{\zeta}}\Big\} \;\leq\;& \delta_3\wp^2|\widehat{\zeta}|^2 + c(\delta_3)|\widehat{\varpi}|^2,\\ +\Re e\Big\{i\gamma_1\kappa\wp\widehat{m}\overline{\widehat{\varpi}}\Big\} \;\leq\;& c\wp^2|\widehat{m}|^2 + c|\widehat{\varpi}|^2,\\ -\Re e\Big\{\gamma_0\kappa\wp^2\overline{\widehat{j}\widehat{\varpi}}\Big\} \;\leq\;& \varepsilon_1\frac{\wp^4}{(1+\wp^2)^2}|\widehat{j}|^2 + c(\varepsilon_1)(1+\wp^2)^2|\widehat{\varpi}|^2.\end{aligned} \tag{2.17}$$

By substituting (2.17) into (2.16) and letting $\delta_3 = \frac{\beta}{2}$, we obtain (2.15). □

Next, we will discuss the the below mentioned lemmas.

Lemma 2.4. *The functional*

$$\mathcal{D}_3(\wp, t) := -\Re e\Big\{\widehat{f}\overline{\widehat{m}} + a\,\widehat{j}\overline{\widehat{b}}\Big\}, \tag{2.18}$$

satisfies:

(1) For $a = 1$. *Then, for any* $\varepsilon_2 > 0$

$$\frac{d\mathcal{D}_3(\wp, t)}{dt} \;\leq\; -\frac{1}{2}|\widehat{f}|^2 + \varepsilon_2\frac{\wp^2}{1+\wp^2}|\widehat{b}|^2 + c|\widehat{m}|^2 + c(\varepsilon_2)(1+\wp^2)|\widehat{\zeta}|^2. \tag{2.19}$$

(2) For $a \neq 1$. *Then, for any* $\varepsilon_2, \varepsilon_3 > 0$

$$\begin{aligned}\frac{d\mathcal{D}_3(\wp, t)}{dt} \;\leq\;& -\frac{1}{2}|\widehat{f}|^2 + \varepsilon_2\frac{\wp^2}{1+\wp^2}|\widehat{b}|^2 + \varepsilon_3\frac{\wp^2}{1+\wp^2}|\widehat{j}|^2\\ &+c(\varepsilon_3)(1+\wp^2)|\widehat{m}|^2 + c(\varepsilon_2)(1+\wp^2)|\widehat{\zeta}|^2.\end{aligned} \tag{2.20}$$

Proof. Firstly, differentiating $\mathcal{D}_3$ and using (2.6), we get

$$\begin{aligned}\frac{d\mathcal{D}_3(\wp,t)}{dt} &= -|\widehat{f}|^2+|\widehat{m}|^2-\Re e\Big\{\gamma_1\widehat{\zeta}\overline{\widehat{f}}\Big\}+\Re e\Big\{\mu_1\widehat{m}\overline{\widehat{f}}\Big\}\\ &\quad +\Re e\Big\{ia\gamma_0\wp\widehat{\zeta}\overline{\widehat{b}}\Big\}-\Re e\Big\{i(1-a^2)\wp\widehat{j}\overline{\widehat{m}}\Big\}. \end{aligned} \tag{2.21}$$

Now we will discuss two cases:
Case1. ($a = 1$).
In this case, by applying the Young's inequality to the terms on the RHS of (2.21), for any $\varepsilon_2, \delta_4 > 0$ we get

$$\begin{aligned} -\Re e\Big\{\gamma_1\widehat{\zeta}\overline{\widehat{f}}\Big\} &\le \delta_4|\widehat{f}|^2+c(\delta_4)|\widehat{\zeta}|^2,\\ \Re e\Big\{\mu_1\widehat{m}\overline{\widehat{f}}\Big\} &\le \delta_4|\widehat{f}|^2+c(\delta_4)|\widehat{m}|^2,\\ \Re e\Big\{ia\gamma_0\wp\widehat{\zeta}\overline{\widehat{b}}\Big\} &\le \varepsilon_2\frac{\wp^2}{1+\wp^2}|\widehat{b}|^2+c(\varepsilon_2)(1+\wp^2)|\widehat{\zeta}|^2. \end{aligned} \tag{2.22}$$

Substituting the estimates (2.22) in (2.21) and by letting $\delta_4 = \frac{1}{4}$, we find (2.19).
Case2. ($a \neq 1$).
In this case, using the Young's inequality to the last terms on the RHS of (2.21), gives for any $\varepsilon_3 > 0$

$$(a^2-1)\Re e\Big\{i\wp\widehat{j}\overline{\widehat{m}}\Big\} \le \varepsilon_3\frac{\wp^2}{1+\wp^2}|\widehat{j}|^2+c(\varepsilon_3)(1+\wp^2)|\widehat{m}|^2. \tag{2.23}$$

Inserting (2.23) and (2.22) in (2.21), we get (2.20). Which completes the proof of Lemma 2.4. □

At this stage, we define the Lyapunov functional for the two cases by

$$\begin{aligned}\mathcal{H}(\wp,t) &:= N(1+\wp^2)^2\widehat{E}(\wp,t)+\frac{\wp^2}{1+\wp^2}\Big\{\frac{1}{1+\wp^2}N_1\mathcal{D}_1(\wp,t)+N_3\mathcal{D}_3(\wp,t)\Big\}\\ &\quad +N_2\mathcal{D}_2(\wp,t), \end{aligned} \tag{2.24}$$

where $N, N_i, i = 1, 2, 3$ are positive constants which will be selected later.

Lemma 2.5. *There exist $\mu_2, \mu_3, \mu_4 > 0$ such that the functional $\mathcal{H}(\wp, t)$ stated by (2.24) satisfies*

$$\begin{cases} \mu_2(1+\wp^2)^2\widehat{E}(\wp,t) \le \mathcal{H}(\wp,t) \le \mu_3(1+\wp^2)^2\widehat{E}(\wp,t),\\ \mathcal{H}'(\wp,t) \le -\mu_4\rho(\wp)\mathcal{H}(\wp,t), \quad \forall t>0, \end{cases} \tag{2.25}$$

where

$$\rho(\wp) = \frac{\wp^4}{(1+\wp^2)^4}. \tag{2.26}$$

Proof. Firstly, for the case $a = 1$, by differentiating (2.24) and using (2.8), (2.11), (2.15) and (2.19), with the fact that $\frac{\wp^2}{1+\wp^2} \le \min\{1, \wp^2\}$ and $\frac{1}{1+\wp^2} \le 1$, we find

$$
\begin{aligned}
\mathcal{H}'(\wp,t) \;\; \le \;\; & -\frac{\wp^4}{(1+\wp^2)^2}\Big\{\Big[\frac{1}{2}N_1-\varepsilon_1 N_2\Big]|\widehat{j}|^2+\Big[\frac{a}{2}N_1-\varepsilon_2 N_3\Big]|\widehat{b}|^2\Big\}\\
& -(1+\wp^2)^2\Big[\mu_1 N-cN_1-cN_2-cN_3\Big]|\widehat{m}|^2\\
& -(1+\wp^2)^2\Big[kN-c(\varepsilon_1)N_2\Big]|\widehat{\varpi}|^2-\frac{\wp^2}{1+\wp^2}\Big[\frac{1}{2}N_3-cN_1\Big]|\widehat{f}|^2\\
& -\wp^2\Big[\frac{\beta}{2}N_2-cN_1-c(\varepsilon_2)N_3\Big]|\widehat{\zeta}|^2. \qquad (2.27)
\end{aligned}
$$

By setting

$$
\varepsilon_1=\frac{N_1}{4N_2}, \quad \varepsilon_2=\frac{aN_1}{4N_3},
$$

we obtain

$$
\begin{aligned}
\mathcal{H}'(\wp,t) \;\; \le \;\; & -\frac{\wp^4}{(1+\wp^2)^2}N_1\Big\{\frac{1}{4}|\widehat{j}|^2+\frac{a}{4}|\widehat{b}|^2\Big\}\\
& -(1+\wp^2)^2\Big[\mu_1 N-cN_1-cN_2-cN_3\Big]|\widehat{m}|^2\\
& -(1+\wp^2)^2\Big[kN-c(N_1,N_2)N_2\Big]|\widehat{\varpi}|^2-\frac{\wp^2}{1+\wp^2}\Big[\frac{1}{2}N_3-cN_1\Big]|\widehat{f}|^2\\
& -\wp^2\Big[\frac{\beta}{2}N_2-cN_1-c(N_1,N_3)N_3\Big]|\widehat{\zeta}|^2. \qquad (2.28)
\end{aligned}
$$

Similarly, for the case $a \neq 1$, by differentiating (2.24) and using (2.8), (2.11), (2.15) and (2.20), with the fact that $\frac{\wp^2}{1+\wp^2} \le \min\{1,\wp^2\}$ and $\frac{1}{1+\wp^2} \le 1$, we find

$$
\begin{aligned}
\mathcal{H}'(\wp,t) \;\; \le \;\; & -\frac{\wp^4}{(1+\wp^2)^2}\Big\{\Big[\frac{1}{2}N_1-\varepsilon_1 N_2-\varepsilon_3 N_3\Big]|\widehat{j}|^2+\Big[\frac{a}{2}N_1-\varepsilon_2 N_3\Big]|\widehat{b}|^2\Big\}\\
& -(1+\wp^2)^2\Big[\mu_1 N-cN_1-cN_2-c(\varepsilon_3)N_3\Big]|\widehat{m}|^2\\
& -(1+\wp^2)^2\Big[kN-c(\varepsilon_1)N_2\Big]|\widehat{\varpi}|^2-\frac{\wp^2}{1+\wp^2}\Big[\frac{1}{2}N_3-cN_1\Big]|\widehat{f}|^2\\
& -\wp^2\Big[\frac{\beta}{2}N_2-cN_1-c(\varepsilon_2)N_3\Big]|\widehat{\zeta}|^2. \qquad (2.29)
\end{aligned}
$$

By setting

$$
\varepsilon_1=\frac{N_1}{8N_2}, \quad \varepsilon_2=\frac{aN_1}{4N_3}, \quad \varepsilon_3=\frac{N_1}{8N_3},
$$

we obtain

$$\begin{aligned}\mathcal{H}'(\wp,t) &\leq -\frac{\wp^4}{(1+\wp^2)^2}N_1\left\{\frac{1}{4}|\widehat{j}|^2+\frac{a}{4}|\widehat{b}|^2\right\}\\ &\quad -(1+\wp^2)^2\Big[\mu_1 N - cN_1 - cN_2 - c(N_1,N_3)N_3\Big]|\widehat{m}|^2\\ &\quad -(1+\wp^2)^2\Big[kN - c(N_1,N_2)N_2\Big]|\widehat{\varpi}|^2 - \frac{\wp^2}{1+\wp^2}\Big[\frac{1}{2}N_3 - cN_1\Big]|\widehat{f}|^2\\ &\quad -\wp^2\Big[\frac{\beta}{2}N_2 - cN_1 - c(N_1,N_3)N_3\Big]|\widehat{\zeta}|^2. \end{aligned} \tag{2.30}$$

Next, for the two cases (2.28) and (2.30), we fixed N_1 and choosing N_3 large enough such that

$$\frac{1}{2}N_3 - cN_1 > 0,$$

then we select N_2 large enough such as

$$\frac{\beta}{2}N_2 - cN_1 - c(N_1,N_3)N_3 > 0.$$

Hence, for the two cases we arrive at

$$\begin{aligned}\mathcal{H}'(\wp,t) &\leq -\frac{\wp^4}{(1+\wp^2)^2}\Big(\alpha_0|\widehat{j}|^2+\alpha_1|\widehat{b}|^2\Big) - (1+\wp^2)^2\Big[\mu_1 N - c\Big]|\widehat{m}|^2\\ &\quad -\frac{\wp^2}{1+\wp^2}\alpha_2|\widehat{f}|^2 - \wp^2\alpha_3|\widehat{\zeta}|^2 - (1+\wp^2)^2\Big[kN-c\Big]|\widehat{\varpi}|^2. \end{aligned} \tag{2.31}$$

Additionally, we have

$$\begin{aligned}\Big|\mathcal{H}(\wp,t) - N(1+\wp^2)^2\widehat{E}(\wp,t)\Big| &= N_1\frac{\wp^2}{(1+\wp^2)^2}\Big|\mathcal{D}_1(\wp,t)\Big| + N_3\frac{\wp^2}{1+\wp^2}\Big|\mathcal{D}_3(\wp,t)\Big|\\ &\quad +N_2\Big|\mathcal{D}_2(\wp,t)\Big|. \end{aligned}$$

Utilizing Young's inequality and the fact that $\frac{\wp^2}{1+\wp^2} \leq \min\{1,\wp^2\}$ and $\frac{1}{1+\wp^2} \leq 1$, we find

$$\Big|\mathcal{H}(\wp,t) - N(1+\wp^2)^2\widehat{E}(\wp,t)\Big| \leq c(1+\wp^2)^2\widehat{E}(\wp,t).$$

Therefore, we get

$$(N-c)(1+\wp^2)^2\widehat{E}(\wp,t) \leq \mathcal{H}(\wp,t) \leq (N+c)(1+\wp^2)^2\widehat{E}(\wp,t). \tag{2.32}$$

Now, we pick N large enough such as

$$N-c>0, \;\; \mu_1 N - c > 0, \;\; kN - c > 0,$$

and exploiting (2.7), estimates (2.31) and (2.32), respectively, there exists a positive constant $\alpha > 0$, $\forall$ $t > 0$ and $\forall$ $\wp \in \mathbb{R}$, such that

$$\mu_2(1+\wp^2)^2\widehat{E}(\wp,t) \leq \mathcal{H}(\wp,t) \leq \mu_3(1+\wp^2)^2\widehat{E}(\wp,t), \tag{2.33}$$

and

$$\mathcal{H}'(\wp,t) \leq -\alpha\frac{\wp^4}{(1+\wp^2)^2}\Big(|\widehat{j}|^2+|\widehat{b}|^2+|\widehat{\varpi}|^2+|\widehat{f}|^2+|\widehat{m}|^2+|\widehat{\zeta}|^2\Big), \tag{2.34}$$

then

$$\mathcal{H}'(\wp,t) \leq -\lambda_1\rho(\wp)\widehat{E}(\wp,t), \quad \forall t \geq 0. \tag{2.35}$$

Consequently, for some positive constant $\mu_4 = \frac{\lambda_1}{\mu_3} > 0$, we get

$$\mathcal{H}'(\wp,t) \leq -\mu_4\rho(\wp)\mathcal{H}(\wp,t), \quad \forall t \geq 0, \tag{2.36}$$

where $\rho(\wp) = \frac{\wp^4}{(1+\wp^2)^4}$, for some $\lambda_1, \mu_i > 0, i = 2,3,4$. The proof of the first result (2.25) is completed. □

The pointwise estimates of the functional $\widehat{E}(\wp,t)$ is stated by the following Proposition.

Proposition 2.1. *For any $t \geq 0$ and $\wp \in \mathbb{R}$, positive constants $d_1 > 0$ exist in such a way that the energy functional given by (2.7) satisfies*

$$\widehat{E}(\wp,t) \leq d_1\widehat{E}(\wp,0)e^{-\mu_4\rho(\wp)t}, \tag{2.37}$$

where $\rho(\wp) = \dfrac{\wp^4}{(1+\wp^2)^4}$.

Proof. From $(2.25)_2$, we have by integration over $(0,t)$

$$\mathcal{H}(\wp,t) \leq \mathcal{H}(\wp,0)e^{-\mu_4\rho(\wp)t}, \quad \forall t \geq 0. \tag{2.38}$$

Hence, by according of (2.25) and (2.38), we establish (2.37) . □

2.2. Decay estimates

Theorem 2.1. *Let s be a nonnegative integer, and $F_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the solution F of problems (2.2) and (2.3) holds, $\forall$ $t \geq 0$ the following decay estimates*

$$\|\partial_x^k F(t)\|_2 \leq C\|F_0\|_1(1+t)^{-\frac{1}{8}-\frac{k}{4}} + C(1+t)^{-\frac{\ell}{4}}\|\partial_x^{k+\ell}F_0\|_2, \tag{2.39}$$

where ℓ and k are nonnegative integers such that $k+\ell \leq s$ and $C > 0$ is a positive constant.

Proof. From (2.7), we get $|\widehat{F}(\wp,t)|^2 \sim \widehat{E}(\wp,t)$, then by utilizing the Plancherel theorem and exploiting (2.37), we achieve that

$$
\begin{aligned}
\|\partial_x^k F(t)\|_2^2 &= \int_{\mathbb{R}} |\wp|^{2k} |\widehat{F}(\wp,t)|^2 d\wp \\
&\leq c\int_{\mathbb{R}} |\wp|^{2k} e^{-\mu_4 \rho(\wp)t} |\widehat{F}(\wp,0)|^2 d\wp \\
&\leq \underbrace{c\int_{|\wp|\leq 1} |\wp|^{2k} e^{-\mu_4 \rho(\wp)t} |\widehat{F}(\wp,0)|^2 d\wp}_{R_1} \\
&\quad + \underbrace{c\int_{|\wp|\geq 1} |\wp|^{2k} e^{-\mu_4 \rho(\wp)t} |\widehat{F}(\wp,0)|^2 d\wp}_{R_2}. \qquad (2.40)
\end{aligned}
$$

Now, we estimate R_1, R_2, the low-frequency part $|\wp| \leq 1$ and the high-frequency part $|\wp| \geq 1$ respectevly. Firstly, we have $\rho(\wp) \geq \frac{1}{16}\wp^4$, for $|\wp| \leq 1$. Then

$$
\begin{aligned}
R_1 &\leq c\int_{|\wp|\leq 1} |\wp|^{2k} e^{-\frac{\mu_4}{16}|\wp|^4 t} |\widehat{F}(\wp,0)|^2 d\wp \\
&\leq c\sup_{|\wp|\leq 1}\{|\widehat{F}(\wp,0)|^2\} \int_{|\wp|\leq 1} |\wp|^{2k} e^{-\frac{\mu_4}{16}|\wp|^4 t} d\wp, \qquad (2.41)
\end{aligned}
$$

by utilizing Lemma 1.1, we get that

$$
\begin{aligned}
R_1 &\leq c\sup_{|\wp|\leq 1}\{|\widehat{F}(\wp,0)|^2\}(1+t)^{-\frac{k}{2}-\frac{1}{4}} \\
&\leq c\|F_0\|_1^2 (1+t)^{-\frac{k}{2}-\frac{1}{4}}. \qquad (2.42)
\end{aligned}
$$

Secondly, we have $\rho(\wp) \geq \frac{1}{16}\wp^{-4}$, for $|\wp| \geq 1$. Then

$$
R_2 \leq c\int_{|\wp|\geq 1} |\wp|^{2k} e^{-\frac{\mu_4}{16}|\wp|^{-4} t} |\widehat{F}(\wp,0)|^2 d\wp, \quad \forall t \geq 0. \qquad (2.43)
$$

Expoiting the inequality

$$
\sup_{|\wp|\geq 1}\left\{|\wp|^{-2\ell} e^{-c|\wp|^{-2} t}\right\} \leq C(1+t)^{-\ell}, \qquad (2.44)
$$

we get that

$$
\begin{aligned}
R_2 &\leq c\sup_{|\wp|\geq 1}\left\{|\wp|^{-2\ell} e^{-\frac{\mu_4}{16}|\wp|^{-4} t}\right\} \int_{|\wp|\geq 1} |\wp|^{2(k+\ell)} |\widehat{F}(\wp,0)|^2 d\wp \\
&\leq c(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} F(x,0)\|_2^2, \quad \forall t \geq 0. \qquad (2.45)
\end{aligned}
$$

Substituting (2.42) and (2.45) into (2.40), we find (2.39).

□

3. Conclusions

The investigation of the general decay estimate of solutions to a one-dimensional Lord-Shulman Timoshenko system with thermal effect and damping term is the goal of this study, we prove some optimal decay results for the L^2-norm of the solution. More precisely, we prove that the decay rate of the solution is of the form $(1 + t)^{-1/8}$.

To prove our results, we used the energy method in the Fourier space to build some very delicate Lyapunov functionals that give the desired results. In system (1.5) the presence of the mechanical damping $\mu_1 \Im_t$ seems to be necessary for our treatment. It is an interesting problem to prove the same result for $\mu_1 = 0$.

In the upcoming study, we will attempt to utilize the same methodology in the same systems, but with the different types of the memory and the delay terms, we believe that we will obtain similar results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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