AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 17229–17245.
DOI:10.3934/math.2023880
Received: 17 February 2023
Revised: 29 April 2023
Accepted: 11 May 2023
Published: 18 May 2023

Research article

Sturm's Theorem for Min matrices

Efruz Özlem Mersin*

Department of Mathematics, Faculty of Science and Arts, Aksaray University, Aksaray 68100, Turkey

* **Correspondence:** E-mail: efruzmersin@aksaray.edu.tr.

Abstract: In the present paper, we study Min matrix $\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{n}$, where a_s's are the elements of a real sequence $\{a_s\}$. We first obtain a recurrence relation for the characteristic polynomial for matrix $\mathcal{A}_{min}$, and some relations between the coefficients of its characteristic polynomial. Next, we show that the sequence of the characteristic polynomials of the $i \times i\,(i \leq n)$ Min matrices satisfies the Sturm sequence properties according to different required conditions of the sequence $\{a_s\}$. Using Sturm's Theorem, we get some results about the eigenvalues, such as the number of eigenvalues in an interval. Thus, we obtain the number of positive and negative eigenvalues of Min matrix $\mathcal{A}_{min}$. Finally, we give an example to illustrate our results.

Keywords: Sturm's Theorem; Min-Max matrices; characteristic polynomial; eigenvalue
Mathematics Subject Classification: 15A18, 15B99

1. Introduction

Matrices have wide use in a variety of problems in mathematics and many other sciences, such as physics and engineering. Considering that the matrix representation of a particular problem can yield significant results, some concepts, such as eigenvalue, singular value, norm and determinant, are useful for these results. There are some special matrices that attract the attention of researchers; Min and Max matrices are such matrices. Min and Max matrices with minimum and maximum entries were first introduced by Pólya and Szegö [1] as

$$A_{min} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \quad \text{and} \quad A_{max} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}, \tag{1.1}$$

respectively. These matrices are expressed as $A_{min} = [min(i,j)]_{i,j=1}^{n}$ and $A_{max} = [max(i,j)]_{i,j=1}^{n}$. Catalani [2] gave some relations between the principal minors of the matrix A_{min} and the Fibonacci numbers. Bhatia [3] showed that the matrix A_{min} is infinitely divisible, and [4] studied this and related matrices. Eigenvalues and inverse of the matrix $C = [min\{ai-b, aj-b\}]_{i,j=1}^{n}$ were studied by Fonseca [5] for $a > 0$ and $a \neq b$. Moyé [6] studied the covariance matrix of Brownian motion, which appears to be a certain Min matrix. By the motivation of the Moyé's paper, Neudecker et al. [7] posed some problems on the determinant, inverse and positive definiteness of more general type of Min matrices with real number entries, then Chu et al. [8] answered these problems. The general forms of Min and Max matrices given in Eq (1.1) are

$$\mathcal{A}_{min} = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \quad \text{and} \quad \mathcal{A}_{max} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{bmatrix}, \tag{1.2}$$

respectively [7–9]. These Min and Max matrices are expressed as $\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{n}$ and $\mathcal{A}_{max} = \left[a_{max(i,j)}\right]_{i,j=1}^{n}$, where a_s's are the elements of a real sequence $\{a_s\}$. The determinants of the matrices $\mathcal{A}_{min}$ and $\mathcal{A}_{max}$ are [9]

$$\det(\mathcal{A}_{min}) = a_1(a_2-a_1)(a_3-a_2)\ldots(a_n-a_{n-1})$$

and

$$\det(\mathcal{A}_{max}) = (a_1-a_2)(a_2-a_3)\ldots(a_{n-1}-a_n)\,a_n.$$

Bahşi and Solak [10–12] characterized the matrices $A_k = [k+min(i,j)-1]_{i,j=1}^{n}$ and $B_k = [k+max(i,j)-1]_{i,j=1}^{n}$ for $k \in \mathbb{R}$, and studied some of their properties, such as the determinants, inverses and characteristic polynomials. The general form of Min matrix was called a nested symmetric matrix and some of its properties, such as the determinant, inverse, principle minors, LU and QR-decompositions were studied by Stuart [13]. Petroudi and Pirouz [14] defined the exponential form of Min matrix as $A = \left[a^{min(i,j)-1}\right]_{i,j=1}^{n}$, where $a > 1$ is a positive real constant. They investigated some properties of this matrix, such as the determinant, inverse, Hadamard inverse and norm. Petroudi and Pirouz [15–17] examined the matrices $F_{min} = \left[F_{min(i,j)}\right]_{i,j=1}^{n}$, $F_{max} = \left[F_{max(i,j)}\right]_{i,j=1}^{n}$ $F = \left[F_{min(i,j)+1}\right]_{i,j=1}^{n}$ and $e^{\circ F} = \left[e^{F_{min(i,j)+1}}\right]_{i,j=1}^{n}$, where F_n is the nth Fibonacci numbers, for some properties as mentioned above. Some relations between the general forms of Min, Max matrices and meet, join matrices were examined by Mattila and Haukkanen [9]. They used meet and join matrices as a tool to obtain their results. Petroudi and Pirouz [18] studied the particular symmetric matrix $H = \left[H_{min(i,j)}\right]_{i,j=1}^{n}$, where H_n is the nth Harmonic number. The authors investigated its Hadamard exponential matrix, along with some of its properties. They also derived the Euclidean norms and some bounds for the spectral norms of these matrices. Kılıç and Arıkan [19] studied the matrices $\mathcal{A}_{min}$, $\mathcal{A}_{max}$, and their Hadamard inverses as the generalizations of Min and Max matrices. The authors obtained the LU-decompositions, inverse, Cholesky decompositions and LU-decompositions of the inverses of these matrices. The characteristic polynomials, determinants, inverses and Hadamard inverses of Max and Min matrices whose entries consist of the hyper-Fibonacci and hyper-Lucas numbers were examined in [20, 21]. Kızılateş and Terzioğlu [22] defined the matrices r-Min, r-Max and their Hadamard inverses. They investigated some properties of these matrices, such as the determinant, inverse, norm and factorizations.

It is well known that the eigenvalues of a matrix are equivalent to the roots of the matrix's characteristic polynomial. Since the eigenvalues give some important information about matrices, the problem of finding the zeros of a polynomial is important for many sciences. There are some iterative methods, such as Newton's formula [23], and some bounds, such as Cauchy's bound [24], for this need. Also, Descartes' rule of sign [24] and Budan Fourier Theorem [24] give the upper bounds for the number of zeros in an interval. These results do not give the exact number of zeros of a polynomial in an interval. Sturm's Theorem is a very useful tool for just this purpose for any polynomial without multiple zeros. Sturm's Theorem uses the number of sign changes of the consecutive members of the Sturm sequence to get the exact number of zeros in an interval. Sturm's Theorem has been known by means of Sturm's studies [25–27] first appeared in 1829. There are many versions and analogies of the Sturm sequence properties and Sturm's Theorem in the literature [28–31].

Now, we give the Sturm analogy, which we use for this paper.

Definition 1.1. *[28] Let $P_0(x), P_1(x), \ldots, P_n(x)$ be continuous functions on an interval (a, b) (with the possibilities $a = -\infty$, $b = \infty$). If*

(1) $P_0(x)$ has no zeros in (a, b),
(2) The set of zeros of $P_i(x)$ is discrete for $1 \leq i \leq n$,
(3) If $P_i(x_0) = 0$, then $P_{i-1}(x_0)\, P_{i+1}(x_0) < 0$ for $1 \leq i \leq n-1$,
(4) If $P_i(x_0) = 0$, then $P_{i-1}(x_0)\,[P_i(x_0+\varepsilon_2) - P_i(x_0-\varepsilon_1)] < 0$ for $1 \leq i \leq n$ and sufficiently small $\varepsilon_1, \varepsilon_2 > 0$,

then the sequence $P_0(x), P_1(x), \ldots, P_n(x)$ has the Sturm sequence properties.

Theorem 1.1. *[28] Suppose that the sequence $P_0(x), P_1(x), \ldots, P_n(x)$ has the Sturm sequence properties on (a, b). Let $\alpha < \beta$ be any numbers in (a, b). Then $P_n(x)$ has exactly $c(\beta) - c(\alpha)$ distinct zeros in interval (α, β), where $c(\alpha)$ denotes the number of changes in sign of consecutive members of the sequence $P_0(\alpha)$, $P_1(\alpha)$, $\ldots$, $P_n(\alpha)$.*

Theorem 1.2. *[28] If $P_0(x), P_1(x), \ldots, P_n(x)$ has the Sturm sequence properties, then the zeros of $P_i(x)$ and $P_{i-1}(x)$ are interlaced for $1 \leq i \leq n$.*

Sturm's Theorem was applied to symmetric tridiagonal matrices by Greenberg [28] to solve some nonlinear eigenvalue problems. Mersin and Bahşi [32] applied Sturm's Theorem to the generalized Frank matrices, and examined their eigenvalues by using the Sturm sequence properties.

In the present paper, we examine the matrix $\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{n}$ given in the Eq (1.2), considering different required conditions for the sequence $\{a_s\}$, such as positive, and either strictly increasing or strictly decreasing. We seek to answer the following questions: What is the recurrence relation for the characteristic polynomial of the matrix $\mathcal{A}_{min}$? Are there any relations between the coefficients of the characteristic polynomials of this matrix? Does the sequence of the characteristic polynomials of the $i \times i\,(i \leq n)$ Min matrices has the Sturm sequence properties? Can we determine the number of the eigenvalues of Min matrix $\mathcal{A}_{min}$ in an interval?

2. Main results

Theorem 2.1. *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{n}$ for any real sequence $\{a_s\}$. Then,*

$$P_n(\lambda) = (a_n - a_{n-1} - 2\lambda)\, P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda), \tag{2.1}$$

with the initial conditions $P_0(\lambda) = 1$ and $P_1(\lambda) = a_1 - \lambda$.

Proof. The characteristic polynomial of the matrix $\mathcal{A}_{min}$ is

$$P_n(\lambda) = \det(\mathcal{A}_{min} - \lambda I) = \begin{vmatrix} a_1-\lambda & a_1 & a_1 & \cdots & a_1 & a_1 \\ a_1 & a_2-\lambda & a_2 & \cdots & a_2 & a_2 \\ a_1 & a_2 & a_3-\lambda & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1}-\lambda & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n-\lambda \end{vmatrix}.$$

Subtracting ith column from the $(i+1)$th column and then ith row from the $(i+1)$th row for $i = n-1, n-2, \ldots, 1$, respectively, we get

$$P_n(\lambda) = \begin{vmatrix} a_1-\lambda & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & a_2-a_1-2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & a_3-a_2-2\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}-a_{n-2}-2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & a_n-a_{n-1}-2\lambda \end{vmatrix}.$$

Thus, we have

$$P_n(\lambda) = (a_n - a_{n-1} - 2\lambda)\, P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda),$$

with initials $P_0(\lambda) = 1$ and $P_1(\lambda) = a_1 - \lambda$. □

Theorem 2.2. *Let $P_n(\lambda) = \lambda^n + \gamma_{n-1}^{(n)}\lambda^{n-1} + \ldots + \gamma_1^{(n)}\lambda + \gamma_0^{(n)}$ be as in Theorem 2.1. Then,*

$$\gamma_0^{(n)} = (-a_n + a_{n-1})\,\gamma_0^{(n-1)} = (-1)^n \det(\mathcal{A}_{min}),$$

$$\gamma_1^{(n)} = (-a_n + a_{n-1})\,\gamma_1^{(n-1)} + 2\gamma_0^{(n-1)},$$

$$\gamma_i^{(n)} = (-a_n + a_{n-1})\,\gamma_i^{(n-1)} + 2\gamma_{i-1}^{(n-1)} - \gamma_{i-2}^{(n-2)}, \quad 2 \le i \le n-2,$$

and

$$\gamma_{n-1}^{(n)} = -a_n + a_{n-1} + 2\gamma_{n-2}^{(n-1)} - \gamma_{n-3}^{(n-2)} = -tr(\mathcal{A}_{min}).$$

Proof. The following recurrence relation

$$P_n(\lambda) = (2\lambda - a_n + a_{n-1})\, P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda), \tag{2.2}$$

is equivalent to the recurrence relation in (2.1). Then, considering Eq (2.2) and the coefficients of $P_n(\lambda)$, $P_{n-1}(\lambda)$ and $P_{n-2}(\lambda)$, we have

$$\begin{aligned}&\lambda^n+\gamma_{n-1}^{(n)}\lambda^{n-1}+\ldots+\gamma_1^{(n)}\lambda+\gamma_0^{(n)}\\=&(2\lambda-a_n+a_{n-1})(\lambda^{n-1}+\gamma_{n-2}^{(n-1)}\lambda^{n-2}+\ldots+\gamma_1^{(n-1)}\lambda+\gamma_0^{(n-1)})-\lambda^2(\lambda^{n-2}+\gamma_{n-3}^{(n-2)}\lambda^{n-3}+\ldots+\gamma_1^{(n-2)}\lambda+\gamma_0^{(n-2)}).\end{aligned}$$

Thus, desired formulas are obtained by comparison of the coefficients. Also,

$$\begin{aligned}\gamma_0^{(n)}&=(-a_n+a_{n-1})\,\gamma_0^{(n-1)}\\&=(-a_n+a_{n-1})\,(-a_{n-1}+a_{n-2})\,\gamma_0^{(n-2)}\\&\vdots\\&=(-a_n+a_{n-1})\,(-a_{n-1}+a_{n-2})\,(-a_{n-2}+a_{n-3})\ldots(-a_2+a_1)\,\gamma_0^{(1)}\\&=(-a_n+a_{n-1})\,(-a_{n-1}+a_{n-2})\,(-a_{n-2}+a_{n-3})\ldots(-a_2+a_1)\,(-a_1)\\&=(-1)^n\det(\mathcal{A}_{min}).\end{aligned}$$

To prove the equality

$$\gamma_{n-1}^{(n)}=-a_n+a_{n-1}+2\gamma_{n-2}^{(n-1)}-\gamma_{n-3}^{(n-2)}=-tr(\mathcal{A}_{min}),$$

we must show that

$$2\gamma_{n-2}^{(n-1)}-\gamma_{n-3}^{(n-2)}=-a_1-a_2-\ldots-a_{n-2}-2a_{n-1} \tag{2.3}$$

is valid for $n\geq 3$. We use the induction method on n. Since

$$\begin{aligned}2\gamma_1^{(2)}-\gamma_0^{(1)}&=2\left((-a_2+a_1)\,\gamma_1^{(1)}+2\gamma_0^{(1)}\right)-\gamma_0^{(1)}\\&=2\,(-a_2+a_1)+3\gamma_0^{(1)}\\&=2\,(-a_2+a_1)+3\,(-a_1+a_0)\,\gamma_0^{(0)}\\&=-a_1-2a_2,\end{aligned}$$

the result is true for $n=3$. Assume that the result is true for $n=k$. That is, the equality

$$2\gamma_{k-2}^{(k-1)}-\gamma_{k-3}^{(k-2)}=-a_1-a_2-\ldots-a_{k-2}-2a_{k-1} \tag{2.4}$$

is true. Considering Eq (2.4), we get

$$\begin{aligned}2\gamma_{k-1}^{(k)}-\gamma_{k-2}^{(k-1)}&=2\left((-a_k+a_{k-1})\,\gamma_{k-1}^{(k-1)}+2\gamma_{k-2}^{(k-1)}-\gamma_{k-3}^{(k-2)}\right)-\gamma_{k-2}^{(k-1)}\\&=2\,(-a_k+a_{k-1})+3\gamma_{k-2}^{(k-1)}-2\gamma_{k-3}^{(k-2)}+\gamma_{k-2}^{(k-1)}-\gamma_{k-2}^{(k-1)}\\&=2\,(-a_k+a_{k-1})+2\left(2\gamma_{k-2}^{(k-1)}-\gamma_{k-3}^{(k-2)}\right)-\gamma_{k-2}^{(k-1)}\\&=2\,(-a_k+a_{k-1})+2\,(-a_1-a_2-\ldots-a_{k-2}-2a_{k-1})-\gamma_{k-2}^{(k-1)},\end{aligned}$$

for $n=k+1$. Since

$$\begin{aligned}\gamma_{k-2}^{(k-1)}&=(-a_{k-1}+a_{k-2})\,\gamma_{k-2}^{(k-2)}+2\gamma_{k-3}^{(k-2)}-\gamma_{k-4}^{(k-3)}\\&=(-a_{k-1}+a_{k-2})+(-a_1-a_2-\ldots-a_{k-3}-2a_{k-2}),\end{aligned}$$

we have

$$\begin{aligned}&2\gamma_{k-1}^{(k)}-\gamma_{k-2}^{(k-1)}\\=\;&2\,(-a_{k-1}+a_{k-2})+2\,(-a_1-a_2-\ldots-a_{k-2}-2a_{k-1})-(-a_{k-1}+a_{k-2})-(-a_1-a_2-\ldots-a_{k-3}-2a_{k-2})\\=\;&-a_1-a_2-\ldots-a_{k-1}-2a_k.\end{aligned}$$

This completes the proof of Eq (2.3). Hence, we get

$$\begin{aligned}\gamma_{n-1}^{(n)} &= -a_n + a_{n-1} + 2\gamma_{n-2}^{(n-1)} - \gamma_{n-3}^{(n-2)} \\ &= -a_n + a_{n-1} + (-a_1 - a_2 - \ldots - a_{n-2} - 2a_{n-1}) \\ &= -a_1 - a_2 - \ldots - a_n \\ &= -tr\left(\mathcal{A}_{min}\right).\end{aligned}$$

□

Remark 2.1. *We encounter the term a_0 for $n = 1$ in the proof of Theorem 2.2. We should specify that the reader should take $a_0 = 0$ when required.*

Now, we demonstrate that the sequence of the characteristic polynomials of the $i \times i\,(i \leq n)$ Min matrices $\mathcal{A}_{min}$

$$P_0(\lambda) = 1,\ P_1(\lambda),\ P_2(\lambda), \ldots,\ P_{n-1}(\lambda),\ P_n(\lambda)$$

has the Sturm sequence properties according to different required conditions for $\{a_s\}$ such as positive, and either strictly increasing or strictly decreasing. First, we give some Lemmas to use for this purpose.

Lemma 2.1. *If the real sequence $\{a_s\}$ is positive, and either strictly increasing or strictly decreasing, then*

(i) Zero is not a root of $P_i(\lambda)$ for $1 \leq i \leq n$ (or equivalently zero is not an eigenvalue of the $i \times i$ matrix $\mathcal{A}_{min}$ for $1 \leq i \leq n$),
(ii) Two consecutive terms $P_i(\lambda), P_{i+1}(\lambda)$ do not have a common zero for $1 \leq i \leq n-1$.

Proof. Let the real sequence $\{a_s\}$ be positive, and either strictly increasing or strictly decreasing. Then,

(i) From the recurrence relation (2.1) and the equality $P_1(0) = a_1$, we get

$$\begin{aligned}P_i(0) &= (a_i - a_{i-1}) P_{i-1}(0) \\ &= (a_i - a_{i-1})(a_{i-1} - a_{i-2}) P_{i-2}(0) \\ &\vdots \\ &= (a_i - a_{i-1})(a_{i-1} - a_{i-2}) \ldots (a_2 - a_1) a_1 \\ &\neq 0\end{aligned}$$

for $1 \leq i \leq n$.

(ii) Suppose that $P_{i+1}(\lambda_0) = P_i(\lambda_0) = 0$ for some i with $1 \leq i \leq n-1$, then considering the recurrence relation (2.1), we get

$$P_{i-1}(\lambda_0) = P_{i-2}(\lambda_0) = \ldots = P_0(\lambda_0) = 0.$$

Since this result contradicts $P_0(\lambda) = 1$, two consecutive terms $P_i(\lambda), P_{i+1}(\lambda)$ can not have a common zero for $1 \leq i \leq n-1$.

□

Lemma 2.2. *Suppose that the real sequence $\{a_s\}$ is positive, and either strictly increasing or strictly decreasing.*

(i) If the sequence $\{a_s\}$ is strictly increasing, and $J \subset (0, \infty)$ is an interval that contains no zeros of $P_{i-1}(\lambda)$ for $1 \le i \le n$, then $\dfrac{P_i(\lambda)}{P_{i-1}(\lambda)}$ is strictly decreasing on interval J.

(ii) If the sequence $\{a_s\}$ is strictly decreasing, $I_1 \subset (-\infty, 0)$ and $I_2 \subset (0, \infty)$ are any intervals that contain no zeros of $P_{i-1}(\lambda)$ for $1 \le i \le n$, then $\dfrac{P_i(\lambda)}{P_{i-1}(\lambda)}$ is strictly increasing on interval I_1, and strictly decreasing on I_2.

Proof. We use the induction method on i for the proofs.

(i) Since

$$\frac{P_1(\lambda)}{P_0(\lambda)} = \frac{a_1 - \lambda}{1} = a_1 - \lambda$$

is strictly decreasing on interval $(0, \infty)$, the result is true for $i = 1$. Let the result be true for $i \le k$, and $K \subset (0, \infty)$ be an interval that contains no zeros of $P_k(\lambda)$ and $P_{k-1}(\lambda)$. Then, from the recurrence relation (2.1), we have

$$\frac{P_{k+1}(\lambda)}{P_k(\lambda)} = (a_{k+1} - a_k - 2\lambda) - \lambda^2 \frac{P_{k-1}(\lambda)}{P_k(\lambda)}$$

for $k + 1 \le n$. It is clear that $a_{k+1} - a_k - 2\lambda$ is strictly decreasing on interval $(0, \infty)$. Also, considering the assumption (for $i \le k$), we can say that $-\lambda^2 \dfrac{P_{k-1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on K. Then, we have $\dfrac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on K.

If $P_{k-1}(y) = 0$ and $(x, z) \subset (0, \infty)$ is an interval that contains y, but no zeros of $P_k(\lambda)$, then $\dfrac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on intervals (x, y) and (y, z). From the continuity, we have $\dfrac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on interval (x, z).

(ii) Since

$$\frac{P_1(\lambda)}{P_0(\lambda)} = \frac{a_1 - \lambda}{1} = a_1 - \lambda$$

is strictly increasing on interval $(-\infty, 0)$, and strictly decreasing on interval $(0, \infty)$, the result is true for $i = 1$. Let the result be true for $i \le k$, and $K_1 \subset (-\infty, 0)$, $K_2 \subset (0, \infty)$ be two intervals that have no zeros of $P_k(\lambda)$ and $P_{k-1}(\lambda)$. Then for $k + 1 \le n$, considering the recurrence relation (2.1), we have

$$\frac{P_{k+1}(\lambda)}{P_k(\lambda)} = (a_{k+1} - a_k - 2\lambda) - \lambda^2 \frac{P_{k-1}(\lambda)}{P_k(\lambda)}.$$

$a_{k+1} - a_k - 2\lambda$ is strictly increasing on interval $(-\infty, 0)$, and strictly decreasing on $(0, \infty)$. From the assumption (for $i \le k$), $-\lambda^2 \dfrac{P_{k-1}(\lambda)}{P_k(\lambda)}$ is strictly increasing on K_1, and strictly decreasing on K_2. Then, we have $\dfrac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly increasing on K_1, and strictly decreasing on K_2.
Suppose that $P_{k-1}(y_1) = 0$, and $(x_1, z_1) \subset (-\infty, 0)$ is an interval that contains y_1, but no zeros of $P_k(\lambda)$. Then, $\dfrac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly increasing on intervals (x_1, y_1) and (y_1, z_1). Thus, considering the

continuity, we have $\frac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly increasing on interval (x_1, z_1). Similarly, if $P_{k-1}(y_2) = 0$ and $(x_2, z_2) \subset (0, \infty)$ is an interval that contains y_2, but no zeros of $P_k(\lambda)$, then $\frac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on intervals (x_2, y_2) and (y_2, z_2). From the continuity, we have $\frac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing on interval (x_2, z_2).

□

Theorem 2.3. *Suppose that the real sequence $\{a_s\}$ is positive, either strictly increasing or strictly decreasing and the sequence*

$$P_0(\lambda) = 1,\ P_1(\lambda),\ P_2(\lambda), \ldots, P_{n-1}(\lambda),\ P_n(\lambda) \tag{2.5}$$

consists of the characteristic polynomials of the $i \times i\,(i \le n)$ matrices $\mathcal{A}_{min}$.

(i) If the sequence $\{a_s\}$ is strictly increasing, then the sequence given in Eq (2.5) has the Sturm sequence properties on interval $(0, \infty)$,

(ii) If the sequence $\{a_s\}$ is strictly decreasing, then the sequence given in Eq (2.5) has the Sturm sequence properties on interval $(-\infty, \infty)$.

Proof. (i) Let the sequence $\{a_s\}$ be strictly increasing. We must show that four conditions (1)–(4) in Definition 1.1 are satisfied by the sequence of the characteristic polynomials of the $i \times i\,(i \le n)$ matrices $\mathcal{A}_{min}$.

(1) It is clear that $P_0(\lambda) = 1$ has no zeros.

(2) $P_1(\lambda) = a_1 - \lambda$ has only one zero as $\lambda_0 = a_1$. Thus, (2) is true for $i = 1$. Suppose that (2) is true for $i \le k$, then the set of zeros of $P_k(\lambda)$ is discrete. Considering the recurrence relation (2.1), we have

$$P_{k+1}(\lambda) = P_k(\lambda)\left[(a_{k+1} - a_k - 2\lambda) - \lambda^2 \frac{P_{k-1}(\lambda)}{P_k(\lambda)}\right].$$

By using Lemma 2.1(ii), $P_{k+1}(\lambda)$ and $P_k(\lambda)$ have no common zero, and by using Lemma 2.2(i), $\frac{P_{k+1}(\lambda)}{P_k(\lambda)}$ is strictly decreasing between any two consecutive zeros of $P_k(\lambda)$. Hence, $P_{k+1}(\lambda)$ has at most one zero between any two consecutive zeros of $P_k(\lambda)$. That is, (2) is true for $k + 1 \le n$.

(3) Considering the recurrence relation (2.1) we have

$$P_{i+1}(\lambda) = (a_{i+1} - a_i - 2\lambda)\ P_i(\lambda) - \lambda^2 P_{i-1}(\lambda),$$

for $1 \le i \le n-1$. If $P_i(\lambda) = 0$, then we get $P_{i+1}(\lambda) = -\lambda^2 P_{i-1}(\lambda)$ for $1 \le i \le n-1$. Since $\lambda^2 > 0$, the inequality $P_{i+1}(\lambda)\ P_{i-1}(\lambda) < 0$ is true for $\lambda \in (0, \infty)$.

(4) Suppose that $P_i(\lambda_0) = 0$ for $1 \le i \le n$, and $[\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_2]$ is an interval that contains no zeros of $P_{i-1}(\lambda)$ for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$. Then, the sign of $P_{i-1}(\lambda)$ does not change. By using Lemma 2.2 (i) $\frac{P_i(\lambda)}{P_{i-1}(\lambda)}$ is strictly decreasing on interval $J \subset (0, \infty)$. Thus, the sign

of $\dfrac{P_i(\lambda)}{P_{i-1}(\lambda)}$ (or equivalently the sign of $P_{i-1}(\lambda)\ P_i(\lambda)$) is (+) and (−) in intervals $[\lambda_0 - \varepsilon_1,\ \lambda_0)$ and $(\lambda_0,\ \lambda_0 + \varepsilon_2]$, respectively. That is,

$$P_{i-1}(\lambda_0 - \varepsilon_1)\ P_i(\lambda_0 - \varepsilon_1) > 0 > P_{i-1}(\lambda_0 + \varepsilon_2)\ P_i(\lambda_0 + \varepsilon_2)\,.$$

Since the sign of $P_{i-1}(\lambda)$ does not change in interval $[\lambda_0 - \varepsilon_1,\ \lambda_0 + \varepsilon_2]$, we have

$$P_{i-1}(\lambda_0)\ P_i(\lambda_0 - \varepsilon_1) > 0 > P_{i-1}(\lambda_0)\ P_i(\lambda_0 + \varepsilon_2)$$

and

$$P_{i-1}(\lambda_0)\ [P_i(\lambda_0 + \varepsilon_2) - P_i(\lambda_0 - \varepsilon_1)] < 0.$$

This completes the proof.

(ii) The proof is similar to the proof of (i).

□

Theorem 2.4. *Suppose that the real sequence $\{a_s\}$ is positive, and either strictly increasing or strictly decreasing.*

(i) If $\{a_s\}$ is strictly increasing, then all eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ are distinct and positive,
(ii) If $\{a_s\}$ is strictly decreasing, then one of the eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ is positive, and the remaining $n-1$ eigenvalues are distinct and negative.

Proof. We must compute the numbers of the eigenvalues in intervals $(0, \infty)$ and $(-\infty, 0)$ for the proofs. Assume that λ^l and λ^u are the minimum and maximum zeros of $P_i(\lambda)$ for $1 \leq i \leq n$, respectively. By Theorems 1.1 and 2.3, the number of distinct zeros of $P_i(\lambda)$ in interval $(x,\ y)$ is equal to $c_i(y) - c_i(x)$, where $c_i(\alpha)$ is the number of sign changes of the sequence

$$P_0(\alpha)\,,\ P_1(\alpha)\,,\ P_2(\alpha)\,,\ldots,\ P_{i-1}(\alpha)\,,\ P_i(\alpha)\,,$$

for $1 \leq i \leq n$. Because 0 is not a zero of $P_i(\lambda)$, the sign of $P_i(\lambda)$ does not change in interval $[0,\ \lambda^l)$. Then, the sign of $P_i(x)$ is equal to sign of $P_i(0)$ for $x \in (0,\ \lambda^l)$. Thus, we get $c_i(x) = c_i(0)$. The sign of $P_i(\lambda)$ does not change in interval (λ^u, ∞). Then, we have $c_i(y) = c_i(\infty)$ for $y \in (\lambda^u, \infty)$. Since, the degree of $P_i(\lambda)$ is i, the form of $P_i(\lambda)$ is

$$P_i(\lambda) = (-1)^i\, \lambda^i + \ldots \quad . \tag{2.6}$$

Then, the sign of $P_i(\infty)$ is $(-1)^i$. Hence, we have $c_i(\infty) = i$. Since $c_i(y) - c_i(x) = c_i(\infty) - c_i(0)$ for $x \in (0,\ \lambda^l)$ and $y \in (\lambda^u, \infty)$, we must also calculate $c_i(0)$, to evaluate the number of eigenvalues in interval $(0, \infty)$. Considering a_s is a positive real number, we have

(i) For the strictly increasing sequence $\{a_s\}$, it is clear that

$$P_i(0) = (a_i - a_{i-1})\ (a_{i-1} - a_{i-2}) \ldots (a_2 - a_1)\ a_1 > 0.$$

Then, we have $c_i(0) = 0$. Thus, the number of distinct zeros of $P_i(\lambda)$ in interval $(x,\ y)$ for $x \in (0,\ \lambda^l)$ and $y \in (\lambda^u, \infty)$ is

$$c_i(y) - c_i(x) = c_i(\infty) - c_i(0) = i.$$

Because the number of zeros of $P_i(\lambda)$ is i, we can say that all the zeros of $P_i(\lambda)$ are in interval $(x,\ y)$ for $x \in (0,\ \lambda^l)$ and $y_1 \in (\lambda^u, \infty)$. Thus, all the zeros of $P_i(\lambda)$ are distinct and positive for $1 \leq i \leq n$. In other words, all eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ are distinct and positive.

(ii) For the strictly decreasing sequence $\{a_s\}$, the sign of

$$P_i(0) = (a_i - a_{i-1})(a_{i-1} - a_{i-2}) \ldots (a_2 - a_1)\, a_1$$

is $(-1)^{i-1}$. Hence, we have $c_i(0) = i - 1$. Thus, the number of zeros of $P_i(\lambda)$ in interval (x, y) for $x \in (0, \lambda^l)$ and $y \in (\lambda^u, \infty)$ is

$$c_i(y) - c_i(x) = c_i(\infty) - c_i(0) = i - (i - 1) = 1.$$

That is, $P_i(\lambda)$ has one eigenvalue in interval $(0, \infty)$.
Now, we show that the $n \times n$ matrix $\mathcal{A}_{min}$ has $n - 1$ distinct eigenvalues in interval $(-\infty, 0)$. The sign of $P_i(\lambda)$ does not change in interval $(-\infty, \lambda^l)$ and $c_i(x) = c_i(-\infty)$ for $x \in (-\infty, \lambda^l)$. Since the number of distinct zeros of $P_i(\lambda)$ in interval (x, y) is $c_i(y) - c_i(x) = c_i(0) - c_i(-\infty)$ for $x \in (-\infty, \lambda^l)$ and $y \in (\lambda^u, 0)$, we must compute the value $c_i(-\infty)$. Considering Eq (2.6), it is clear that

$$P_i(-\infty) > 0.$$

Then, the number of sign change of $P_i(-\infty)$ is zero. Thus, $c_i(-\infty) = 0$, and we have

$$c_i(y) - c_i(x) = c_i(0) - c_i(-\infty) = (i - 1) - 0 = i - 1.$$

Since the number of zeros of $P_i(\lambda)$ is i, and one of the zeros is positive, remaining $i - 1$ zeros of $P_i(\lambda)$ are in interval (x, y) for $x \in (-\infty, \lambda^l)$ and $y \in (\lambda^u, 0)$. Hence, $P_n(\lambda)$ has $n - 1$ eigenvalues in interval $(-\infty, 0)$. This shows that one of the eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ is positive, and the remaining $n - 1$ eigenvalues are distinct and negative.

□

Remark 2.2. *We note that, we use the notations $P_i(\infty)$ and $c_i(\infty)$ in the proof of Theorem 2.4, rather than $\lim_{\lambda\to\infty} P_i(\lambda)$ and $\lim_{\lambda\to\infty} c_i(\lambda)$, respectively.*

Theorem 2.5. *If the real sequence $\{a_s\}$ is positive, and either strictly increasing or strictly decreasing, then the eigenvalues of $i \times i$ and $(i - 1) \times (i - 1)$ matrices $\mathcal{A}_{min}$ are interlaced for $2 \le i \le n$. That is,*

$$\lambda_1^{(i)} > \lambda_1^{(i-1)} > \lambda_2^{(i)} > \lambda_2^{(i-1)} > \ldots > \lambda_{i-1}^{(i-1)} > \lambda_i^{(i)},$$

where $\lambda_s^{(i)}$'s are the eigenvalues of the $i \times i$ matrices $\mathcal{A}_{min}$ for $s = 1, 2, \ldots, i$.

Proof. Theorems 1.2 and 2.3 give the desired result immediately. □

Remark 2.3. *Our results work even if the real sequence $\{a_s\}$ is negative and strictly decreasing (or strictly increasing). If $b_s = -a_s$, then b_s is a positive real number, the sequence $\{b_s\}$ is strictly increasing (or strictly decreasing). Since $\mathcal{A}_{min} = -\mathcal{B}_{min}$, all eigenvalues of $\mathcal{B}_{min}$ have opposite sign with all eigenvalues of $\mathcal{A}_{min}$. For example, for the negative, either strictly increasing or strictly decreasing real sequence $\{a_s\}$:*

(i) If $\{a_s\}$ is strictly decreasing, then all eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ are distinct and negative,
(ii) If $\{a_s\}$ is strictly increasing, then one of the eigenvalues of the $n \times n$ matrix $\mathcal{A}_{min}$ is negative, and the remaining $n - 1$ eigenvalues are distinct and positive.

3. An example

In this section, we illustrate our results with the following example.

Consider the real sequence $\{a_s\}$ with the elements $a_s = 2s - 1$. Then, the 5×5 Min matrix corresponding to this sequence is

$$\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 7 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{bmatrix},$$

and its Hadamard inverse is

$$\mathcal{A}_{min}^{\circ(-1)} = \left[\frac{1}{a_{min(i,j)}}\right]_{i,j=1}^{5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{7} \\ 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \end{bmatrix}.$$

Theorem 2.1 yields the characteristic polynomials of the $i \times i \, (2 \leq i \leq 5)$ matrices $\mathcal{A}_{min}$ and $\mathcal{A}_{min}^{\circ(-1)}$ as

$$P_i(\lambda) = 2(1-\lambda) P_{i-1}(\lambda) - \lambda^2 P_{i-2}(\lambda)$$

and

$$Q_i(\mu) = -2\left(\frac{1}{a_i a_{i-1}} + \mu\right) Q_{i-1}(\mu) - \mu^2 Q_{i-2}(\mu).$$

Thus,

$$\begin{aligned}
P_0(\lambda) &= 1, \\
P_1(\lambda) &= 1 - \lambda, \\
P_2(\lambda) &= 2(1-\lambda)(1-\lambda) - \lambda^2 = \lambda^2 - 4\lambda + 2, \\
P_3(\lambda) &= 2(1-\lambda)\left(\lambda^2 - 4\lambda + 2\right) - \lambda^2(1-\lambda) = -\lambda^3 + 9\lambda^2 - 12\lambda + 4, \\
P_4(\lambda) &= 2(1-\lambda)\left(-\lambda^3 + 9\lambda^2 - 12\lambda + 4\right) - \lambda^2\left(\lambda^2 - 4\lambda + 2\right) \\
&= \lambda^4 - 16\lambda^3 + 40\lambda^2 - 32\lambda + 8, \\
P_5(\lambda) &= 2(1-\lambda)\left(\lambda^4 - 16\lambda^3 + 40\lambda^2 - 32\lambda + 8\right) - \lambda^2\left(-\lambda^3 + 9\lambda^2 - 12\lambda + 4\right) \\
&= -\lambda^5 + 25\lambda^4 - 100\lambda^3 + 140\lambda^2 - 80\lambda + 16,
\end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
Q_0(\mu) &= 1,\\
Q_1(\mu) &= 1-\mu,\\
Q_2(\mu) &= -2\left(\frac{1}{3}+\mu\right)(1-\mu)-\mu^2=\mu^2-\frac{4}{3}\mu-\frac{2}{3},\\
Q_3(\mu) &= -2\left(\frac{1}{15}+\mu\right)\left(\mu^2-\frac{4}{3}\mu-\frac{2}{3}\right)-\mu^2(1-\mu)=-\mu^3+\frac{23}{15}\mu^2+\frac{68}{45}\mu+\frac{4}{45},\\
Q_4(\mu) &= -2\left(\frac{1}{35}+\mu\right)\left(-\mu^3+\frac{23}{15}\mu^2+\frac{68}{45}\mu+\frac{4}{45}\right)-\mu^2\left(\mu^2-\frac{4}{3}\mu-\frac{2}{3}\right)\\
&= \mu^4-\frac{176}{105}\mu^3-\frac{3848}{1575}\mu^2-\frac{416}{1575}\mu-\frac{8}{1575},\\
Q_5(\mu) &= -2\left(\frac{1}{63}+\mu\right)\left(\mu^4-\frac{176}{105}\mu^3-\frac{3848}{1575}\mu^2-\frac{416}{1575}\mu-\frac{8}{1575}\right)-\mu^2\left(-\mu^3+\frac{23}{15}\mu^2+\frac{68}{45}\mu+\frac{4}{45}\right)\\
&= -\mu^5+\frac{563}{315}\mu^4+\frac{113396}{33075}\mu^3+\frac{51292}{99225}\mu^2+\frac{368}{19845}\mu+\frac{16}{99225}.
\end{aligned}
\tag{3.2}
$$

If we compute $P_{i\le 5}(\lambda)$ and $Q_{i\le 5}(\mu)$ using $det(\mathcal{A}_{min}-\lambda I)$ and $det\left(\mathcal{A}_{min}^{\circ(-1)}-\mu I\right)$ for $i\le 5$ respectively, we obtain the same results as above.

There are the following relations between the coefficients of the characteristic polynomials given in Eq (3.1) considering the form of $P_n(\lambda)=\lambda^n+\gamma_{n-1}^{(n)}\lambda^{n-1}+\ldots+\gamma_1^{(n)}\lambda+\gamma_0^{(n)}$ as mentioned in Theorem 2.2, we have the coefficients as

$$
\begin{aligned}
\gamma_0^{(1)} &= (-a_1+a_0)\gamma_0^{(0)}=(-1+0)1=-1,\\
\gamma_0^{(2)} &= (-a_2+a_1)\gamma_0^{(1)}=(-3+1)(-1)=2,\\
\gamma_0^{(3)} &= (-a_3+a_2)\gamma_0^{(2)}=(-5+3)(2)=-4,\\
\gamma_0^{(4)} &= (-a_4+a_3)\gamma_0^{(3)}=(-7+5)(-4)=8,\\
\gamma_0^{(5)} &= (-a_5+a_4)\gamma_0^{(4)}=(-9+7)(8)=-16,\\
\gamma_1^{(2)} &= (-a_2+a_1)+2\gamma_0^{(1)}=(-3+1)+2\gamma_0^{(1)}=-4,\\
\gamma_1^{(3)} &= (-a_3+a_2)\gamma_1^{(2)}+2\gamma_0^{(2)}=(-5+3)(-4)+2(2)=12,\\
\gamma_1^{(4)} &= (-a_4+a_3)\gamma_1^{(3)}+2\gamma_0^{(3)}=(-7+5)(12)+2(-4)=-32,\\
\gamma_1^{(5)} &= (-a_5+a_4)\gamma_1^{(4)}+2\gamma_0^{(4)}=(-9+7)(-32)+2(8)=80,\\
\gamma_2^{(3)} &= (-a_3+a_2)+2\gamma_1^{(2)}-\gamma_0^{(1)}=(-3+1)+2(-4)-(-1)=-9,\\
\gamma_2^{(4)} &= (-a_4+a_3)\gamma_2^{(3)}+2\gamma_1^{(3)}-\gamma_0^{(2)}=(-5+3)(-9)+2(12)-(2)=40,\\
\gamma_2^{(5)} &= (-a_5+a_4)\gamma_2^{(4)}+2\gamma_1^{(4)}-\gamma_0^{(3)}=(-7+5)(40)+2(-32)-(-4)=-140,\\
\gamma_3^{(4)} &= (-a_4+a_3)+2\gamma_2^{(3)}-\gamma_1^{(2)}=(-7+5)+2(-9)-(-4)=-16,\\
\gamma_3^{(5)} &= (-a_5+a_4)\gamma_3^{(4)}+2\gamma_2^{(4)}-\gamma_1^{(3)}=(-7+5)(-16)+2(40)-(12)=100,\\
\gamma_4^{(5)} &= (-a_5+a_4)+2\gamma_3^{(4)}-\gamma_2^{(3)}=(-9+7)+2(-16)-(-9)=-25.
\end{aligned}
$$

Considering the values

$$det(\mathcal{A}_{min})=1(3-1)(5-3)(7-5)(9-7)=16,$$

$$tr(\mathcal{A}_{min})=1+3+5+7+9=25,$$

we observe that the equalities given in Theorem 2.2 are provided for the 5×5 matrix $\mathcal{A}_{min}$. For example

$$\gamma_0^{(5)} = -16\,(-1)^5\, det\,(\mathcal{A}_{min})$$

and

$$\gamma_4^{(5)} = -25 = -tr\,(\mathcal{A}_{min})\,.$$

The relations for the coefficients of the characteristic polynomials of the matrix $\mathcal{A}_{min}^{\circ(-1)}$ given in Eq (3.2) can be obtained similarly.

The roots of $P_{i\leq 5}\,(\lambda)$ and $Q_{i\leq 5}\,(\mu)$ (or the eigenvalues of $i \times i\,(i \leq 5)$ matrices $\mathcal{A}_{min}$ and $\mathcal{A}_{min}^{\circ(-1)}$, respectively) are

$$\begin{gathered}
\lambda_1^{(1)} = 1,\\
\lambda_1^{(2)} = 3.41, \quad \lambda_2^{(2)} = 0.59,\\
\lambda_1^{(3)} = 7.46, \quad \lambda_2^{(3)} = 1, \quad \lambda_3^{(3)} = 0.54,\\
\lambda_1^{(4)} = 13.1, \quad \lambda_2^{(4)} = 1.62, \quad \lambda_3^{(4)} = 0.723, \quad \lambda_4^{(4)} = 0.520,\\
\lambda_1^{(5)} = 20.4, \quad \lambda_2^{(5)} = 2.42, \quad \lambda_3^{(5)} = 1, \quad \lambda_4^{(5)} = 0.630, \quad \lambda_5^{(5)} = 0.512,
\end{gathered}$$

and

$$\begin{gathered}
\mu_1^{(1)} = 1,\\
\mu_1^{(2)} = 1.72, \quad \mu_2^{(2)} = -0.383,\\
\mu_1^{(3)} = 2.229, \quad \mu_2^{(3)} = -0.063, \quad \mu_3^{(3)} = -0.63,\\
\mu_1^{(4)} = 2.64, \quad \mu_2^{(4)} = -0.025, \quad \mu_3^{(4)} = -0.091, \quad \mu_4^{(4)} = -0.847,\\
\mu_1^{(5)} = 2.991, \quad \mu_2^{(5)} = -0.013, \quad \mu_3^{(5)} = -0.034, \quad \mu_4^{(5)} = -0.114, \quad \mu_5^{(5)} = -1.042,
\end{gathered}$$

where $\lambda_s^{(i)}$ and $\mu_s^{(i)}$ denote the roots of $P_{i\leq 5}\,(\lambda)$ and $Q_{i\leq 5}\,(\mu)$, respectively for $s = 1, 2, \ldots, i$. Hence, we have

- $P_{i\leq 5}\,(\lambda)$ and $Q_{i\leq 5}\,(\mu)$ do not vanish for $\lambda = \mu = 0$.
- $P_{i\leq 4}\,(\lambda)$ and $P_{i+1}\,(\lambda)$ (or $Q_{i\leq 4}\,(\mu)$ and $Q_{i+1}\,(\mu)$) have not a common zero.
- The sets of zeros of both $P_{i\leq 5}\,(\lambda)$ and $Q_{i\leq 5}\,(\mu)$ are discrete.
- If $P_{i\leq 4}\,(\lambda) = 0$, then $P_{i-1}\,(\lambda)\,P_{i+1}\,(\lambda) < 0$. For example, since $P_2\,(1) = -1$, $P_3\,(1) = 0$, $P_4\,(1) = 1$, we have $P_2\,(1)\,P_4\,(1) = -1 < 0$. Similarly, if $Q_{i\leq 4}\,(\mu) = 0$, then $Q_{i-1}\,(\lambda)\,Q_{i+1}\,(\lambda) < 0$. For example, since $Q_2\,(-0.063) = -0.579$, $Q_3\,(-0.063) = 0$, $Q_4\,(-0.063) = 0.002$, we have $Q_2\,(-0.063)\,Q_4\,(-0.063) = -0.001 < 0$.
- If $P_i\,(\lambda_0) = 0$ and $Q_i\,(\mu_0) = 0$, then for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$,

  $$P_{i-1}\,(\lambda_0)\,[P_i\,(\lambda_0 + \varepsilon_2) - P_i\,(\lambda_0 - \varepsilon_1)] < 0,$$

  and

  $$Q_{i-1}\,(\mu_0)\,[Q_i\,(\mu_0 + \varepsilon_2) - Q_i\,(\mu_0 - \varepsilon_1)] < 0,$$

  where $1 \leq i \leq 5$. For example, since

  $$P_3\,(1) = 0, \quad P_2\,(1) = -1, \quad P_3\left(1 + \frac{1}{100}\right) = 0.031, \quad P_3\left(1 - \frac{1}{1000}\right) = -0.003,$$

  we have

  $$P_2\,(1)\left[P_3\left(1 + \frac{1}{100}\right) - P_3\left(1 - \frac{1}{1000}\right)\right] = -0.034 < 0,$$

where $\varepsilon_1 = \dfrac{1}{1000}$, $\varepsilon_2 = \dfrac{1}{100}$. Similarly, since

$$Q_4(2.64) = 0, \quad Q_3(2.64) = -3.635, \quad Q_4\left(2.64 + \frac{1}{10000}\right) = 0.01, \quad Q_4\left(2.64 - \frac{1}{100}\right) = -0.248,$$

we have

$$Q_3(2.64)\left[Q_4\left(2.64 + \frac{1}{10000}\right) - Q_4\left(2.64 - \frac{1}{100}\right)\right] = -0.938 < 0,$$

where $\varepsilon_1 = \dfrac{1}{100}$, $\varepsilon_2 = \dfrac{1}{10000}$.

- The sequences $P_0(\lambda)$, $P_1(\lambda)$, $P_2(\lambda)$, $P_3(\lambda)$, $P_4(\lambda)$, $P_5(\lambda)$, and $Q_0(\mu)$, $Q_1(\mu)$, $Q_2(\mu)$, $Q_3(\mu)$, $Q_4(\mu)$, $Q_5(\mu)$ have the Sturm sequence properties.
- All of eigenvalues of the $i \times i\,(i \leq 5)$ matrices $\mathcal{A}_{min}$ (or the zeros of $P_{i\leq 5}(\lambda)$) are distinct and positive. Also one of the eigenvalues of the $i \times i\,(i \leq 5)$ matrices $\mathcal{A}_{min}^{\circ(-1)}$ (or the zeros of $Q_{i\leq 5}(\mu)$) is positive, and the remaining $i-1$ eigenvalues are distinct and negative.
- The eigenvalues of the $i \times i$ and $(i-1)\times(i-1)$ matrices $\mathcal{A}_{min}$ are interlaced for $2 \leq i \leq 5$. For example,

$$\begin{aligned}&\lambda_1^{(5)} = 20.4 > \lambda_1^{(4)} = 13.1 > \lambda_2^{(5)} = 2.42 > \lambda_2^{(4)} = 1.62 > \\ &\lambda_3^{(5)} = 1 > \lambda_3^{(4)} = 0.723 > \lambda_4^{(5)} = 0.630 > \lambda_4^{(4)} = 0.520 > \lambda_5^{(5)} = 0.512.\end{aligned}$$

Similarly, the eigenvalues of the $i \times i$ and $(i-1)\times(i-1)$ matrices $\mathcal{A}_{min}^{\circ(-1)}$ are interlaced for $2 \leq i \leq 5$. For example,

$$\begin{aligned}&\mu_1^{(5)} = 2.991 > \mu_1^{(4)} = 2.64 > \mu_2^{(5)} = -0.013 > \mu_2^{(4)} = -0.025 > \\ &\mu_3^{(5)} = -0.034 > \mu_3^{(4)} = -0.091 > \mu_4^{(5)} = -0.114 > \mu_4^{(4)} = -0.847 > \mu_5^{(5)} = -1.042.\end{aligned}$$

Finally, we compute the number of eigenvalues of the 5×5 matrix $\mathcal{A}_{min}$ in intervals $(0,2)$ and $(2,25)$. So then, we need the number of sign changes of $P_{i\leq 5}(\lambda)$ for $\lambda = 0$, $\lambda = 2$, and $\lambda = 25$. Table 1 serves this need.

Table 1. The number of sign changes of $P_{i\leq 5}(\lambda)$ for $\lambda = 0$, $\lambda = 2$, and $\lambda = 25$.

Characteristic polynomials of the $i \times i\,(i \leq 5)$ matrices $\mathcal{A}_{min}$	Sign of $P_i(\lambda)$ for $\lambda = 0$	Sign of $P_i(\lambda)$ for $\lambda = 2$	Sign of $P_i(\lambda)$ for $\lambda = 25$
$P_0(\lambda) = 1$	+	+	+
$P_1(\lambda) = 1 - \lambda$	+	−	−
$P_2(\lambda) = \lambda^2 - 4\lambda + 2$	+	−	+
$P_3(\lambda) = -\lambda^3 + 9\lambda^2 - 12\lambda + 4$	+	+	−
$P_4(\lambda) = \lambda^4 - 16\lambda^3 + 40\lambda^2 - 32\lambda + 8$	+	−	+
$P_5(\lambda) = -\lambda^5 + 25\lambda^4 - 100\lambda^3 + 140\lambda^2 - 80\lambda + 16$	+	−	−
Number of sign changes	$c_5(0) = 0$	$c_5(2) = 3$	$c_5(25) = 5$

From Table 1, we have $c_5(0) = 0$, $c_5(2) = 3$, and $c_5(25) = 5$, where $c_5(\alpha)$ denotes the number of changes in sign of $P_{i\leq 5}(\alpha)$. Thus, the number of eigenvalues of the 5×5 matrix $\mathcal{A}_{min}$ in interval $(0,2)$ is

$c_5(2) - c_5(0) = 3 - 0 = 3$, and the number of eigenvalues in interval $(2, 25)$ is $c_5(25) - c_5(2) = 5 - 3 = 2$. Really, the eigenvalues of the 5×5 matrix $\mathcal{A}_{min}$ are $20.4, 2.42, 1, 0.630$, and 0.512.

Similarly, we compute the number of eigenvalues of the 5×5 matrix $\mathcal{A}_{min}^{\circ(-1)}$ in intervals $(-2, 0)$ and $(0, 3)$. Table 2 includes the number of sign changes of $Q_{i \le 5}(\mu)$ for $\mu = -2$, $\mu = 0$, and $\mu = 3$.

Table 2. The number of sign changes of $Q_{i \le 5}(\mu)$ for $\mu = -2$, $\mu = 0$, and $\mu = 3$.

Characteristic polynomials of the $i \times i\,(i \le 5)$ matrices $\mathcal{A}_{min}^{\circ(-1)}$	Sign of $Q_i(\mu)$ for $\mu = -2$	Sign of $Q_i(\mu)$ for $\mu = 0$	Sign of $Q_i(\mu)$ for $\mu = 3$
$Q_0(\mu) = 1$	+	+	+
$Q_1(\mu) = 1 - \mu$	+	+	−
$Q_2(\mu) = \mu^2 - \frac{4}{3}\mu - \frac{2}{3}$	+	−	+
$Q_3(\mu) = -\mu^3 + \frac{23}{15}\mu^2 + \frac{68}{45}\mu + \frac{4}{45}$	+	+	−
$Q_4(\mu) = \mu^4 - \frac{176}{105}\mu^3 - \frac{3848}{1575}\mu^2 - \frac{416}{1575}\mu - \frac{8}{1575}$	+	−	+
$Q_5(\mu) = -\mu^5 + \frac{563}{315}\mu^4 + \frac{113396}{33075}\mu^3 + \frac{51292}{99225}\mu^2 + \frac{368}{19845}\mu + \frac{16}{99225}$	+	+	−
Number of sign changes	$c_5(-2) = 0$	$c_5(0) = 4$	$c_5(3) = 5$

According to Table 2, we have $c_5(-2) = 0$, $c_5(0) = 4$, and $c_5(3) = 5$. Thus, the number of eigenvalues of the 5×5 matrix $\mathcal{A}_{min}^{\circ(-1)}$ in interval $(-2, 0)$ is $c_5(0) - c_5(-2) = 4 - 0 = 4$, and the number of eigenvalues in interval $(0, 3)$ is $c_5(3) - c_5(0) = 5 - 4 = 1$. Really, the eigenvalues of the 5×5 matrix $\mathcal{A}_{min}^{\circ(-1)}$ are $2.991, -0.013, -0.034, -0.114$, and -1.042.

4. Conclusions

In this paper, we obtained a recurrence relation for the characteristic polynomials of the real symmetric Min matrix $\mathcal{A}_{min} = \left[a_{min(i,j)}\right]_{i,j=1}^{n}$, where a_s's are the elements of a real sequence $\{a_s\}$. We also gave some relations between the coefficients of the characteristic polynomials of this matrix. Additionally, we obtained that the sequence of the characteristic polynomials of the $i \times i\,(i \le n)$ Min matrices satisfies the Sturm sequence properties considering different required conditions for the real sequence $\{a_s\}$. We showed that the eigenvalues of the $i \times i$ and $(i-1) \times (i-1)$ Min matrices are interlaced as a consequence of Sturm's Theorem, where $2 \le i \le n$. It is well known that the eigenvalues of real symmetric matrices are real; we specified how many of the real eigenvalues are positive, and how many are negative of the $n \times n$ matrices $\mathcal{A}_{min}$ with the help of Sturm's Theorem.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. G. Pólya, G. Szegö, *Problems and theorems in analysis II*, Berlin: Springer, 1998. http://dx.doi.org/10.1007/978-3-642-61905-2

2. M. Catalani, A particular matrix and its relationships with Fibonacci numbers, arXiv:math/0209249.
3. R. Bhatia, Infinitely divisible matrices, *Am. Math. Mon.*, **113** (2006), 221–235. http://dx.doi.org/10.2307/27641890
4. R. Bhatia, Min matrices and mean matrices, *Math. Intelligencer*, **33** (2011), 22–28. http://dx.doi.org/10.1007/s00283-010-9194-z
5. C. Da Fonseca, On the eigenvalues of some tridiagonal matrices, *J. Comput. Appl. Math.*, **200** (2007), 283–286. http://dx.doi.org/10.1016/j.cam.2005.08.047
6. L. Moyé, *Statistical monitoring of clinical trials: fundamentals for investigators*, New York: Springer, 2006. http://dx.doi.org/10.1007/0-387-27782-X
7. H. Neudecker, G. Trenkler, S. Liu, Problem section, *Stat. Papers*, **50** (2009), 221–223. http://dx.doi.org/10.1007/s00362-008-0174-8
8. K. Chu, S. Puntanen, G. Styan, Problem section, *Stat. Papers*, **52** (2011), 257–262. http://dx.doi.org/10.1007/s00362-010-0363-0
9. M. Mattila, P. Haukkanen, Studying the various properties of Min and Max matrices-elemantary vs. more advanced methods, *Spec. Matrices*, **4** (2016), 101–109. http://dx.doi.org/10.1515/spma-2016-0010
10. M. Bahşi, S. Solak, A particular matrix and its properties, *Int. J. Math. Sci. Appl.*, **1** (2011), 971–974.
11. S. Solak, M. Bahşi, A particular matrix and its some properties, *Sci. Res. Essays*, **8** (2013), 1–5. http://dx.doi.org/10.5897/SRE11.410
12. M. Bahşi, S. Solak, Some particular matrices and their characteristic polynomials, *Linear Multilinear A.*, **63** (2015), 2071–2078. http://dx.doi.org/10.1080/03081087.2014.940940
13. J. Stuart, Nested matrices and inverse M-matrices, *Czech. Math. J.*, **65** (2015), 537–544. http://dx.doi.org/10.1007/s10587-015-0192-3
14. S. Jafari-Petroudi, B. Pirouz, A particular matrix, its inversion and some norms, *Appl. Comput. Math.*, **4** (2015), 47–52. http://dx.doi.org/10.11648/j.acm.20150402.13
15. S. Jafari-Petroudi, B. Pirouz, A note on Hadamard inverse and Hadamard exponential of a matrix with Fibonacci numbers, *Proceedings of The 7th National Conference on Mathematics*, 2015, 28–29.
16. S. Jafari-Petroudi, M. Pirouz, A note on tribonacci numbers with particular matrices, *Proceedings of The 28th International Conference of The Jangjeon Mathematical Society*, 2015, 71.
17. S. Jafari-Petroudi, M. Pirouz, On the bounds for the spectral norm of particular matrices with Fibonacci and Lucas numbers, *Int. J. Adv. Appl. Math. Mech.*, **3** (2016), 82–90.
18. S. Jafari-Petroudi, M. Pirouz, Toward special symmetric matrices with harmonic numbers, *Proceedings of The 8th National Conference on Mathematics*, 2016, 11–12.
19. E. Kılıç, T. Arıkan, Studying new generalizations of Max-Min matrices with a novel approach, *Turk. J. Math.*, **43** (2019), 2010–2024.

20. T. Solmaz, M. Bahşi, Max and Min matrices with hyper-Fibonacci numbers, *Asian-Eur. J. Math.*, **15** (2022), 2250084. http://dx.doi.org/10.1142/S179355712250084X

21. D. Özgul, M. Bahşi, Min matrices with hyper-Lucas numbers, *J. Sci. Arts*, **4** (2020), 855–864. http://dx.doi.org/10.46939/J.Sci.Arts-20.4-a07

22. C. Kızılateş, N. Terzioğlu, On r-min and r-max matrices, *J. Appl. Math. Comput.*, **68** (2022), 4559–4588. http://dx.doi.org/10.1007/s12190-022-01717-y

23. M. Jain, S. Iyengar, R. Jain, *Numerical methods: problems and solutionsz*, Bangalore: New Age International, 2007.

24. P. Ciarlet, J. Lions, Solution of equations in $\mathbb{R}^n$, In: *Handbook of numerical analysis*, Amsterdam: Elsevier, (1994), 625–778.

25. M. Sturm, Analyse d' un Mémoire sur la résolution des équations numériques, In: *Collected works of Charles François Sturm*, Basel: Birkhäuser, 2009, 323–326. http://dx.doi.org/10.1007/978-3-7643-7990-2_24

26. M. Sturm, Extrait d'un Mémoire sur L'intécration d'un système d'équations différentielles linéaires, présenté à l'Académie des sciences, In: *Collected works of Charles François Sturm*, Basel: Birkhäuser, 2009, 334–342. http://dx.doi.org/10.1007/978-3-7643-7990-2_27

27. P. Sturm, Mémoire sur la résolution des équations numériques, In: *Collected works of Charles François Sturm*, Basel: Birkhäuser, 2009, 345–390. http://dx.doi.org/10.1007/978-3-7643-7990-2_29

28. L. Greenberg, Sturm sequences for nonlinear eigenvalue problems, *SIAM J. Math. Anal.*, **20** (1989), 182–199. http://dx.doi.org/10.1137/0520015

29. E. Isaacson, H. Keller, *Analysis of numerical methods*, 2 Eds., New York: John Wiley-Sons, 1966.

30. J. Stoer, R. Bulirsch, *Introduction to numerical analysis*, New York: Springer-Verlag, 2002. http://dx.doi.org/10.1007/978-0-387-21738-3

31. A. Mostowski, M. Stark, *Introduction to higher algebra*, Pergamon: Elsevier, 1964. http://dx.doi.org/10.1016/C2013-0-10019-0

32. E. Mersin, M. Bahşi, Sturm theorem for the generalized Frank matrix, *Hacet. J. Math. Stat.*, **50** (2021), 1002–1011. http://dx.doi.org/10.15672/hujms.773281

AIMS Press