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Research article

Action of n-derivations and n-multipliers on ideals of (semi)-prime rings

Shakir Ali[1,*], **Turki M. Alsuraiheed**[2], **Nazia Parveen**[1] **and Vaishali Varshney**[1]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[2] Department of Mathematics, King Saud University, Riyadh, Saudi Arabia

* **Correspondence:** Email: shakir50@rediffmail.com, shakir.ali.mm@amu.ac.in.

Abstract: The present paper aims to investigate the containment of nonzero central ideal in a ring $\mathcal{R}$ when the trace of symmetric n-derivations satisfies some differential identities. Lastly, we prove that in a prime ring $\mathcal{R}$ of suitable torsion restriction, if $\mathfrak{D}, \mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ are two nonzero symmetric n-derivations such that $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0$ holds $\forall\ \vartheta \in \mathcal{W}$, a nonzero left ideal of $\mathcal{R}$ where f and g are the traces of $\mathfrak{D}$ and $\mathcal{G}$, respectively, then either $\mathcal{R}$ is commutative or $\mathcal{G}$ acts as a left n-multiplier. Finally, we characterize symmetric n-derivations in terms of left n-multipliers.

Keywords: semiprime ring; ideal; derivation; symmetric n-derivation; n-multiplier
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1. Introduction

Throughout, $\mathcal{R}$ will be an associative ring with $\mathcal{Z}(\mathcal{R})$ as its center. A ring $\mathcal{R}$ is said to be prime if $\vartheta\mathcal{R}\ell = \{0\}$ implies that either $\vartheta = 0$ or $\ell = 0$ and semiprime if $\vartheta\mathcal{R}\vartheta = \{0\}$ implies that $\vartheta = 0$, where $\vartheta, \ell \in \mathcal{R}$. The symbols $[\vartheta, \ell]$ and $\vartheta \circ \ell$ denote the commutator $\vartheta\ell - \ell\vartheta$ and the anti-commutator $\vartheta\ell + \ell\vartheta$, respectively, for any $\vartheta, \ell \in \mathcal{R}$. A ring $\mathcal{R}$ is said to be n-torsion free if $n\vartheta = 0$ implies that $\vartheta = 0\ \forall\ \vartheta \in \mathcal{R}$. If $\mathcal{R}$ is $n!$-torsion free, then it is m-torsion free for every divisor m of $n!$. An additive mapping $\mathcal{D} : \mathcal{R} \to \mathcal{R}$ is called a derivation if $\mathcal{D}(\vartheta\ell) = \mathcal{D}(\vartheta)\ell + \vartheta\mathcal{D}(\ell)$ holds $\forall\ \vartheta, \ell \in \mathcal{R}$. In order to broaden the scope of derivation, Maksa [12] introduced the notion of symmetric bi-derivations on rings, which Vukman examined in greater detail in [17, 18]. A bi-additive map $\mathfrak{D} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is said to be a bi-derivation if

$$\mathfrak{D}(\vartheta\vartheta', \ell) = \mathfrak{D}(\vartheta, \ell)\vartheta' + \vartheta\mathfrak{D}(\vartheta', \ell),$$

$$\mathfrak{D}(\vartheta, \ell\ell') = \mathfrak{D}(\vartheta, \ell)\ell' + \ell\mathfrak{D}(\vartheta, \ell')$$

hold for any $\vartheta, \vartheta', \ell, \ell' \in \mathcal{R}$. The foregoing conditions are identical if $\mathfrak{D}$ is also a symmetric map, that is, if $\mathfrak{D}(\vartheta, \ell) = \mathfrak{D}(\ell, \vartheta)$ for every $\vartheta, \ell \in \mathcal{R}$. In this case, $\mathfrak{D}$ is referred to as a symmetric bi-derivation on $\mathscr{R}$. Several authors have studied symmetric bi-derivations on rings (see [3, 11, 16] and references therein) and produced highly helpful outcomes.

The study of tri-derivation was initiated in [13], by Öztürk, in which he proved various results. Several results have been obtained by various authors in this direction (see [13, 19] and references therein). In light of the concepts of bi-derivation and tri-derivation, Park [14] introduced the concept of permuting n-derivation as follows:

Definition 1.1. Let $n \geq 2$ be a fixed integer, and $\mathscr{R}^n = \underbrace{\mathscr{R} \times \mathscr{R} \times \cdots \times \mathscr{R}}_{n-\text{times}}$. A map $\mathfrak{D} : \mathscr{R}^n \to \mathscr{R}$ is said to be symmetric (permuting) if

$$\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \mathfrak{D}(\vartheta_{\pi(1)}, \vartheta_{\pi(2)}, \ldots, \vartheta_{\pi(n)})$$

for all permutations $\pi(t) \in S_n$ and $\vartheta_t \in R$, where $t = 1, 2, \ldots, n$.

Definition 1.2. Let $n \geq 2$ be a fixed integer. An n-additive mapping (i.e., additive in each argument) $\mathfrak{D} : \mathscr{R}^n \to \mathscr{R}$ is called an n-derivation on $\mathscr{R}$ if the relations

$$\mathfrak{D}(\vartheta_1\vartheta_1', \vartheta_2, \ldots, \vartheta_n) = \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)\vartheta_1' + \vartheta_1\mathfrak{D}(\vartheta_1', \vartheta_2, \ldots, \vartheta_n),$$

$$\mathfrak{D}(\vartheta_1, \vartheta_2\vartheta_2', \ldots, \vartheta_n) = \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)\vartheta_2' + \vartheta_2\mathfrak{D}(\vartheta_1, \vartheta_2', \ldots, \vartheta_n),$$

$$\vdots$$

$$\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n\vartheta_n') = \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)\vartheta_n' + \vartheta_n\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n')$$

hold for all $\vartheta_t, \vartheta_t' \in \mathscr{R},\ t = 1, 2, \ldots, n$.

If, in addition, $\mathfrak{D}$ is a permuting map, then all the above conditions are equivalent, and in that case $\mathfrak{D}$ is called a permuting n-derivation on $\mathscr{R}$.

Of course, 1-derivation is a derivation, a 2-derivation is a symmetric bi-derivation, and for $n = 3$, $\mathfrak{D}$ is referred to as a permuting 3-derivation (or tri-derivation) on rings (see [17, 19] for details).

A map $\mathscr{d} : \mathscr{R} \to \mathscr{R}$ defined by $\mathscr{d}(\vartheta) = \mathfrak{D}(\vartheta, \vartheta, \ldots, \vartheta)$ is called the trace of $\mathfrak{D}$. If $\mathfrak{D} : \mathscr{R}^n \to \mathscr{R}$ is permuting and n-additive, then the trace $\mathscr{d}$ of $\mathfrak{D}$ satisfies the relation

$$\mathscr{d}(\vartheta + \ell) = \mathscr{d}(\vartheta) + \mathscr{d}(\ell) + \sum_{k=1}^{n-1} {}^nC_k\, h_k(\vartheta; \ell)$$

$\forall\ \vartheta, \ell \in \mathscr{R}$, where ${}^nC_k = \binom{n}{k}$ and

$$h_k(\vartheta; \ell) = \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-k)-\text{times}}, \underbrace{\ell, \ldots, \ell}_{k-\text{times}}).$$

Let S be a nonempty subset of $\mathscr{R}$. A mapping $\mathscr{d} : \mathscr{R} \to \mathscr{R}$ is said to be commuting (respectively, centralizing) on $\mathscr{R}$ if $[\mathscr{d}(\vartheta), \vartheta] = 0$ (respectively, $[\mathscr{d}(\vartheta), \vartheta] \in \mathscr{Z}(\mathscr{R})$) for all $\vartheta \in \mathscr{R}$. The study of commuting and centralizing mappings on a prime ring was initiated by Posner [15], who proved that if a prime ring $\mathscr{R}$ admits a nonzero centralizing derivation, then $\mathscr{R}$ is commutative. Being inspired by this result, Brešar [9, Theorem 4.1] proved this for left ideals. In fact, he proved that if $\mathscr{R}$ is a prime ring, $\mathscr{W}$ is a nonzero left ideal of $\mathscr{R}$, and $\mathscr{d}$ and g are nonzero derivations of $\mathscr{R}$ satisfying $\mathscr{d}(\vartheta)\vartheta - \vartheta g(\vartheta) \in \mathscr{Z}(\mathscr{R})$

$\forall\, \vartheta \in \mathcal{W}$, then $\mathcal{R}$ is commutative. In [2], Argaç gave a partial extension of Brešar's result in the setting of semiprime rings. Motivated by the classical result due to Posner [15], Vukman obtained some results concerning the trace of symmetric bi-derivations in prime rings (see [17, 18] for more details). In [3], Ashraf established similar results for semiprime rings. Further, Ashraf et al. [6, 8] obtained commutativity of rings admitting n-derivations whose traces satisfy certain polynomial conditions. Recently, Ashraf et al. [4] introduced the concepts of permuting n-multipliers and proved that for a fixed integer $n \geq 2$, if $\mathcal{R}$ is a non-commutative $n!$-torsion free prime ring admitting a permuting generalized n-derivation $\mathcal{G}$ with associated n-derivation $\mathcal{D}$ such that the trace ω of $\mathcal{G}$ is commuting on $\mathcal{R}$, then $\mathcal{G}$ is a left n-multiplier on $\mathcal{R}$. Many authors have studied various identities involving traces of bi-derivations and n-derivations and have obtained several interesting results (viz., [3, 4, 6, 11, 16–18] and references therein).

The primary aim of this paper is to prove analogous results related to permuting n-derivations in the setting of prime and semiprime rings. In fact, we investigate the structure of (semi)prime rings and describe the forms of maps (traces of n-derivations) satisfying certain functional identities. More precisely, we prove that: let $n \geq 2$ be a fixed integer, $\mathcal{R}$ be an $n!$-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ with trace $\mathscr{d} : \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ with trace $g : \mathcal{R} \to \mathcal{R}$ satisfying $\mathscr{d}(\vartheta)\ell \pm \vartheta g(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\, \vartheta, \ell \in \mathcal{W}$, then $\mathcal{R}$ has a nonzero central ideal (Theorem 2.5). Further, in the last section, we establish that if $\mathcal{R}$ is an $n!$-torsion free prime ring admitting two symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ and $\mathfrak{G} : \mathcal{R}^n \to \mathcal{R}$ with traces f and g, respectively, satisfying $f(\vartheta)\vartheta + \vartheta g(\vartheta) = 0\ \forall\, \vartheta \in \mathcal{W}$, a left ideal of $\mathcal{R}$, then either $\mathcal{R}$ is commutative or $\mathfrak{G}$ acts as a left n-multiplier on $\mathcal{W}$ (Theorem 3.2). Moreover, we also characterize the traces of q-iterations of n-derivations in prime rings and prove that for a fixed integer $n \geq 2$, if $\mathcal{R}$ is an $n!$-torsion free prime ring and $q \geq 1$, a fixed integer admitting q-iterations of n-derivations $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q : \mathcal{R}^n \to \mathcal{R}$ such that the product of the traces of $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q$, respectively, is zero on a nonzero ideal of $\mathcal{R}$, then either $\mathfrak{D}_1 = 0$ or the rest of $\mathfrak{D}_i's$ act as n-multipliers on $\mathcal{R}$ (Theorem 3.6).

2. Results

In the present section, we state and prove the main results of this article. In order to establish the proofs of our main theorems, we first state a number of well-known results.

Lemma 2.1. *[14] Let n be a fixed positive integer and $\mathcal{R}$ an $n!$-torsion free ring. Suppose that $a_1, a_2, \ldots, a_n \in \mathcal{R}$ satisfy $\lambda a_1 + \lambda^2 a_2 + \cdots + \lambda^n a_n = 0$ (or $\in \mathcal{Z}(\mathcal{R})$) for $\lambda = 1, 2, \ldots, n$. Then, $a_t = 0$ (or $\in \mathcal{Z}(\mathcal{R})$) for $t = 1, 2, \ldots, n$.*

Lemma 2.2. *[10] If $\mathcal{R}$ is a semiprime ring, then the center of a nonzero ideal of $\mathcal{R}$ is contained in the center of $\mathcal{R}$.*

Lemma 2.3. *[16] Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. If $[\mathcal{W}, \mathcal{W}] \subseteq \mathcal{Z}(\mathcal{R})$, then $\mathcal{R}$ contains a nonzero central ideal.*

Lemma 2.4. *[16] Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{W} \circ \mathcal{W} \subseteq \mathcal{Z}(\mathcal{R})$, then $\mathcal{R}$ contains a nonzero central ideal.*

The first main result of this paper is the following theorem:

Theorem 2.5. *Let $n \geq 2$ be a fixed integer, $\mathcal{R}$ be an $n!$-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ with trace $d : \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ with trace $g : \mathcal{R} \to \mathcal{R}$ satisfying $d(\vartheta)\ell \pm \vartheta g(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$, then $\mathcal{R}$ has a nonzero central ideal.*

Proof. It is given that

$$d(\vartheta)\ell \pm \vartheta g(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}. \tag{2.1}$$

Replacing ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, we obtain

$$d(\vartheta)(\ell + m\hbar) \pm \vartheta g(\ell + m\hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Solving further, we get

$$d(\vartheta)\ell + d(\vartheta)m\hbar \pm \vartheta g(\ell) \pm \vartheta g(m\hbar) \pm \vartheta \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R})$$

$\forall\ \vartheta, \ell, \hbar \in \mathcal{W}$. Taking account of the given condition, we find that

$$\vartheta \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W},$$

which implies that

$$m\binom{n}{1}\vartheta h_1(\ell; \hbar) + m^2\binom{n}{2}\vartheta h_2(\ell; \hbar) + \cdots + m^{n-1}\binom{n}{n-1}\vartheta h_{n-1}(\ell; \hbar) \in \mathcal{Z}(\mathcal{R}),$$

where $h_t(\ell; \hbar)$ represents the term in which $\hbar$ appears t- times.

The application of Lemma 2.1 yields

$$n\vartheta\mathcal{G}(\ell, \ldots, \ell, \hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Since $\mathcal{R}$ is $n!$-torsion free, we get

$$\vartheta\mathcal{G}(\ell, \ldots, \ell, \hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Replacing $\hbar$ by ℓ, we find that

$$\vartheta g(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Hence, by the hypothesis, we see that

$$d(\vartheta)\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Now, on commuting with r where $r \in \mathcal{R}$, we get

$$[d(\vartheta)\ell, r] = 0\ \forall\ \vartheta, \ell \in \mathcal{W},\ r \in \mathcal{R},$$

$$\text{or } d(\vartheta)[\ell, r] + [d(\vartheta), r]\ell = 0\ \forall\ \vartheta, \ell \in \mathcal{W},\ r \in \mathcal{R}. \tag{2.2}$$

Replacing ℓ by ℓk where $k \in \mathcal{W}$ in (2.2) and using (2.2), we get

$$\mathcal{d}(\vartheta)\ell[k, r] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}, \ r \in \mathcal{R}.$$

Now, replacing r by $\mathcal{d}(\vartheta)$ in the above equation, we obtain

$$\mathcal{d}(\vartheta)\ell[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}. \tag{2.3}$$

Multiplying by k from left, we get

$$k\mathcal{d}(\vartheta)\ell[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}. \tag{2.4}$$

Taking $k\ell$ in place of ℓ in (2.3), we see that

$$\mathcal{d}(\vartheta)k\ell[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}. \tag{2.5}$$

Subtracting (2.5) from (2.4), we get

$$[k, \mathcal{d}(\vartheta)]\ell[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W},$$

i.e.,

$$[k, \mathcal{d}(\vartheta)]\ell r[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}, \ r \in \mathcal{R},$$

i.e.,

$$[k, \mathcal{d}(\vartheta)]\ell\mathcal{R}[k, \mathcal{d}(\vartheta)]\ell = (0) \ \forall \ \vartheta, \ell, k \in \mathcal{W}.$$

Since $\mathcal{R}$ is a semiprime ring, the last expression gives

$$[k, \mathcal{d}(\vartheta)]\ell = 0 \ \forall \ \vartheta, \ell, k \in \mathcal{W}.$$

Replacing ℓ by $r[k, \mathcal{d}(\vartheta)]$, we get

$$[k, \mathcal{d}(\vartheta)]r[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, k \in \mathcal{W}, \ r \in \mathcal{R}.$$

From the semiprimeness of $\mathcal{R}$, we see that

$$[k, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, k \in \mathcal{W}. \tag{2.6}$$

Invoking Lemma 2.2, we have

$$\mathcal{d}(\vartheta) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta \in \mathcal{W}. \tag{2.7}$$

Now, again replacing ϑ by $\vartheta + mw_1$ for $w_1 \in \mathcal{W}$ and $1 \leq m \leq n-1$ in (2.7) and using (2.7), we obtain

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-t)-\text{times}}, \underbrace{mw_1, \ldots, mw_1}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, w_1 \in \mathcal{W},$$

which implies that

$$m\binom{n}{1}h_1(\vartheta; w_1) + m^2\binom{n}{2}h_2(\vartheta; w_1) + \cdots + m^{n-1}\binom{n}{n-1}h_{n-1}(\vartheta; w_1) \in \mathcal{Z}(\mathcal{R})$$

$\forall\, \vartheta, w_1 \in \mathcal{W}$. Invoking Lemma 2.1 and using the fact that $\mathcal{R}$ is $n!$-torsion free, we get

$$\mathfrak{D}(w_1, \vartheta, \ldots, \vartheta) \in \mathcal{Z}(\mathcal{R})\ \forall\, \vartheta, w_1 \in \mathcal{W}. \tag{2.8}$$

Replace ϑ by $\vartheta + mw_2$ for $w_2 \in \mathcal{W}$ and $1 \leq m \leq n-1$ in the above equation to get

$$\mathfrak{D}(w_1, \vartheta + mw_2, \ldots, \vartheta + mw_2) \in \mathcal{Z}(\mathcal{R})\ \forall\, \vartheta, w_1, w_2 \in \mathcal{W},$$

and on further solving and using torsion restriction, we get

$$\mathfrak{D}(w_1, w_2, \vartheta, \ldots, \vartheta) \in \mathcal{Z}(\mathcal{R})\ \forall\, w_1, w_2, \vartheta \in \mathcal{W}.$$

Continuing in the same manner, we get

$$\mathfrak{D}(w_1, w_2, w_3, \ldots, w_n) \in \mathcal{Z}(\mathcal{R})\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}. \tag{2.9}$$

On commuting with r, we get

$$[\mathfrak{D}(w_1, w_2, w_3, \ldots, w_n), r] = 0\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}, r \in \mathcal{R}.$$

After replacing w_1 by $w_1 w_1'$ in the above equation and using torsion restriction of $\mathcal{R}$, we arrive at

$$[w_1, r]\mathfrak{D}(w_1, w_2, \ldots, w_n) = 0\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}, r \in \mathcal{R}.$$

Now taking r to be rr' where $r' \in \mathcal{R}$, we get

$$[w_1, r]r'\mathfrak{D}(w_1, w_2, \ldots, w_n) = 0\ \forall\, r, r' \in \mathcal{R},$$

i.e.,

$$[w_1, r]\mathcal{R}\mathfrak{D}(w_1, w_2, \ldots, w_n) = \{0\}\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}, r \in \mathcal{R}. \tag{2.10}$$

Since $\mathcal{R}$ is a semiprime ring, it must contain a family of prime ideals of $\mathcal{R}$ whose intersection is zero. Let $\mathfrak{P} = \{P_j \mid j \in \Lambda\}$ be the family of all prime ideals such that $\bigcap P_j = \{0\}$. Let P be a typical member of $\mathfrak{P}$. From (2.10), we conclude that for a fixed $w_1 \in \mathcal{W}$,

$$\text{either } [w_1, r] \in P \textit{ or } \mathfrak{D}(w_1, w_2, \ldots, w_n) \in P\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}, r \in \mathcal{R}.$$

Let us set $\mathcal{U} = \{w_1 \in \mathcal{W} \mid [w_1, \mathcal{R}] \subseteq P\}$ and $\mathcal{V} = \{w_1 \in \mathcal{W} \mid \mathfrak{D}(w_1, w_2, \ldots, w_n) \in P\ \forall\, w_2, \ldots, w_n \in \mathcal{W}\}$. Both $\mathcal{U}$ and $\mathcal{V}$ are additive subgroups of $\mathcal{W}$ such that $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$, but a group cannot be the union of two of its proper subgroups. Hence, either $\mathcal{W} = \mathcal{U}$ or $\mathcal{W} = \mathcal{V}$. Let us suppose that $\mathcal{W} \neq \mathcal{U}$. Then, we have $\mathcal{W} = \mathcal{V}$, i.e., $\mathfrak{D}(w_1, w_2, \ldots, w_n) \in P\ \forall\, w_1, w_2, \ldots, w_n \in \mathcal{W}$. Replace w_1 by $w_1 r_1$, i.e., $\mathfrak{D}(w_1 r_1, w_2, \ldots, w_n) \in P$ for any $r_1 \in \mathcal{R}$. On solving, we get $w_1\mathfrak{D}(r_1, w_2, \ldots, w_n) \in P$. Using primeness of P, we get either $w_1 \in P$ or $\mathfrak{D}(r_1, w_2, \ldots, w_n) \in P$ for all $w_1, w_2, \ldots, w_n \in \mathcal{W}$, $r_1 \in \mathcal{R}$. However, $w_1 \in P$ implies that $[w_1, \mathcal{R}] \subseteq P$, which leads to a contradiction. Thus, we have $\mathfrak{D}(r_1, w_2, \ldots, w_n) \in P$ for all $w_2, \ldots, w_n \in \mathcal{W}$, $r_1 \in \mathcal{R}$. Again replace w_2 by $w_2 r_2$, and using the same procedure, we get $\mathfrak{D}(r_1, r_2, w_3, \ldots, w_n) \in P$ for all $w_3, \ldots, w_n \in \mathcal{W}$, $r_1, r_2 \in \mathcal{R}$. Continuing in a similar manner, we arrive at

$$\mathfrak{D}(\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R}) \subseteq P \text{ for any } P \in \mathfrak{P}.$$

Since P was an arbitrary element of $\mathfrak{P}$,

$$\mathfrak{D}(\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R}) \subseteq \bigcap P_j = \{0\},$$

which implies that $\mathfrak{D}(\mathcal{R}, \mathcal{R}, \ldots, \mathcal{R}) = \{0\}$. Hence, we arrive at a contradiction. Therefore, $\mathcal{W} = \mathcal{U}$, i.e., $[w_1, \mathcal{R}] \subseteq P$ for all $w_1 \in \mathcal{W}$ or $[\mathcal{W}, \mathcal{R}] \subseteq \bigcap P_j = \{0\}$. That is, $[\mathcal{W}, \mathcal{R}] = \{0\}$. Therefore, $\mathcal{W}$ is a nonzero central ideal of $\mathcal{R}$. Hence, $\mathcal{R}$ has a nonzero central ideal. □

Theorem 2.6. *For a fixed integer $n \geq 2$, let $\mathcal{R}$ be an $n!$-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ with trace $d : \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ with trace $g : \mathcal{R} \to \mathcal{R}$ satisfying any one of the following conditions:*

(1) $[d(\vartheta), \ell] = \pm\, \vartheta \circ g(\ell)\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(2) $d([\vartheta, \ell]) = [d(\vartheta), \ell] + [d(\ell), \vartheta]\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(3) $d(\vartheta) \circ \ell = \pm\, \vartheta \circ g(\ell)\ \forall\ \vartheta, \ell \in \mathcal{W}$.

Then, $\mathcal{R}$ contains a nonzero central ideal.

Proof. (i) It is given that

$$[d(\vartheta), \ell] = \pm\, \vartheta \circ g(\ell)\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Now, replace ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, and we get

$$[d(\vartheta), \ell] + [d(\vartheta), m\hbar] = \pm\, \vartheta \circ g(\ell)\ \pm\ \vartheta \circ g(m\hbar)\ \pm\ \vartheta \circ \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}})$$

$\forall\ \vartheta, \ell, \hbar \in \mathcal{W}$. By using the hypothesis, we get

$$\vartheta \circ \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}) = 0\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W},$$

which implies that

$$mP_1(\vartheta, \ell, \hbar) + m^2 P_2(\vartheta, \ell, \hbar) + \cdots + m^{n-1} P_{n-1}(\vartheta, \ell, \hbar) = 0\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W},$$

where

$$P_t(\vartheta, \ell, \hbar) = \vartheta \circ {}^nC_t \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{\hbar, \ldots, \hbar}_{t-\text{times}})$$

denotes the sum of terms in which $\hbar$ appears t-times. The application of Lemma 2.1 yields that

$$n\{\vartheta \circ \mathcal{G}(\ell, \ldots, \ell, \hbar)\} = 0\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Using the torsion free restriction in $\mathcal{R}$, we find that

$$\vartheta \circ \mathcal{G}(\ell, \ldots, \ell, \hbar) = 0\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

After replacing $\hbar$ by ℓ, we get

$$\vartheta \circ g(\ell) = 0 \;\forall\; \vartheta, \ell \in \mathcal{W}.$$

Again using the hypothesis, we get

$$[\mathscr{d}(\vartheta), \ell] = 0 \;\forall\; \vartheta, \ell \in \mathcal{W}$$

which is the same as (2.6). Hence, proceeding in the same pattern as we have done so far, we conclude that $\mathcal{R}$ contains a nonzero central ideal.

(ii) It is given that

$$\mathscr{d}([\vartheta, \ell]) = [\mathscr{d}(\vartheta), \ell] + [\mathscr{d}(\ell), \vartheta] \;\forall\; \vartheta, \ell \in \mathcal{W}.$$

On replacing ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get

$$\mathscr{d}([\vartheta, \ell] + [\vartheta, m\hbar]) = [\mathscr{d}(\vartheta), \ell] + [\mathscr{d}(\vartheta), m\hbar] + [\mathscr{d}(\ell) + \mathscr{d}(m\hbar) + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}), \vartheta]$$

$\forall\; \vartheta, \ell, \hbar \in \mathcal{W}$. On simplifying, we get

$$\mathscr{d}([\vartheta, \ell]) + \mathscr{d}([\vartheta, m\hbar]) + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{[\vartheta, \ell], \ldots, [\vartheta, \ell]}_{(n-t)-\text{times}}, \underbrace{[\vartheta, m\hbar], \ldots, [\vartheta, m\hbar]}_{t-\text{times}})$$

$$= [\mathscr{d}(\vartheta), \ell] + [\mathscr{d}(\vartheta), m\hbar] + [\mathscr{d}(\ell), \vartheta] + [\mathscr{d}(m\hbar), \vartheta] + [\sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}), \vartheta]$$

$\forall\; \vartheta, \ell, \hbar \in \mathcal{W}$. Using the hypothesis, we get

$$\sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{[\vartheta, \ell], \ldots, [\vartheta, \ell]}_{(n-t)-\text{times}}, \underbrace{[\vartheta, m\hbar], \ldots, [\vartheta, m\hbar]}_{t-\text{times}}) = [\sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}), \vartheta]$$

$\forall\; \vartheta, \ell, \hbar \in \mathcal{W}$. This leads us to the following:

$$mP_1(\vartheta, \ell, \hbar) + m^2 P_2(\vartheta, \ell, \hbar) + \cdots + m^{n-1} P_{n-1}(\vartheta, \ell, \hbar) = 0 \;\forall\; \vartheta, \ell, \hbar \in \mathcal{W}$$

where

$$P_t(\vartheta, \ell, \hbar) = {}^nC_t \mathfrak{D}(\underbrace{[\vartheta, \ell], \ldots, [\vartheta, \ell]}_{(n-t)-\text{times}}, \underbrace{[\vartheta, \hbar], \ldots, [\vartheta, \hbar]}_{t-\text{times}}) - [{}^nC_t \mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{\hbar, \ldots, \hbar}_{t-\text{times}}), \vartheta]$$

denotes the sum of terms in which $\hbar$ appears t- times.

Taking account of Lemma 2.1 and the torsion free restriction in $\mathcal{R}$, we get

$$\mathfrak{D}([\vartheta, \ell], \ldots, [\vartheta, \ell], [\vartheta, \hbar]) = [\mathfrak{D}(\ell, \ldots, \ell, \hbar), \vartheta] \;\forall\; \vartheta, \ell, \hbar \in \mathcal{W}.$$

Replacing $\hbar$ by ℓ, we get

$$\mathscr{d}([\vartheta, \ell]) = [\mathscr{d}(\ell), \vartheta] \;\forall\; \vartheta, \ell \in \mathcal{W}.$$

Using the hypothesis once again, we obtain

$$[\mathcal{d}(\vartheta), \ell] = 0 \ \forall \ \vartheta, \ell \in \mathcal{W},$$

which is the same as (2.6). Hence, the result follows by using the same argument as discussed in Theorem 2.5.

(*iii*) It is given that

$$\mathcal{d}(\vartheta) \circ \ell = \pm \ \vartheta \circ g(\ell) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

On replacing ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get

$$\mathcal{d}(\vartheta) \circ (\ell + m\hbar) = \pm \ \vartheta \circ g(\ell + m\hbar) \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

On simplifying, we get

$$\mathcal{d}(\vartheta) \circ \ell + \mathcal{d}(\vartheta) \circ m\hbar = \pm\vartheta \circ g(\ell) \pm \vartheta \circ g(m\hbar) \pm \vartheta \circ \sum_{t=1}^{n-1} {}^{n}C_{t} \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}})$$

$\forall \ \vartheta, \ell, \hbar \in \mathcal{W}$. On using the given condition, we find that

$$\vartheta \circ \sum_{t=1}^{n-1} {}^{n}C_{t} \mathcal{G}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}) = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Application of Lemma 2.1 gives

$$n(\vartheta \circ \mathcal{G}(\ell, \ldots, \ell, \hbar)) = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Since $\mathcal{R}$ is $n!$-torsion free, we have

$$\vartheta \circ \mathcal{G}(\ell, \ldots, \ell, \hbar) = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

On replacing $\hbar$ by ℓ, we get

$$\vartheta \circ g(\ell) = 0 \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

Using the hypothesis one more time, we see that

$$\mathcal{d}(\vartheta) \circ \ell = 0 \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

Replacing ℓ by $\ell\hbar$ where $\hbar \in \mathcal{W}$, we find that

$$\ell[\hbar, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Replacing ℓ by $[\hbar, \mathcal{d}(\vartheta)]r$ in the above equation, we have

$$[\hbar, \mathcal{d}(\vartheta)]r[\hbar, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Since $\mathcal{R}$ is a semiprime ring, we get

$$[\hbar, \mathcal{d}(\vartheta)] = 0 \ \forall \ \vartheta, \hbar \in \mathcal{W},$$

which is the same as (2.6). Hence, proceeding in the same way, we conclude that $\mathcal{R}$ contains a nonzero central ideal. □

In [7], Ashraf et al. proved that $\mathcal{R}$ is commutative if it satisfies any one of the following conditions: (i) $F(\vartheta\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R})$, (ii) $F(\vartheta\ell) \pm \ell\vartheta \in \mathcal{Z}(\mathcal{R})$, and (iii) $F(\vartheta)F(\ell) - \vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{R}$, where F is a generalized derivation on $\mathcal{R}$. In our next result, we extend Theorems 2.1 and 2.3 of [7] for the traces of permuting n-derivations on semiprime rings.

Theorem 2.7. *Let $n \geq 2$ be a fixed integer and let $\mathcal{R}$ be an $n!$-torsion free semiprime ring and $\mathcal{W}$ be an nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits a symmetric n-derivation $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ with trace $\delta : \mathcal{R} \to \mathcal{R}$ such that any one of the following conditions hold:*

(1) $\delta(\vartheta\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(2) $\delta(\vartheta\ell) \pm \ell\vartheta \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(3) $\delta(\vartheta\ell) \pm [\vartheta, \ell] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(4) $\delta(\vartheta\ell) \pm \vartheta \circ \ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$.

Then, $\mathcal{R}$ contains a nonzero central ideal.

Proof. (i) It is given that

$$\delta(\vartheta\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Replace ℓ by $\ell + mk$ for $k \in \mathcal{W}$ and $1 \leq m \leq n-1$, and we get

$$\delta(\vartheta(\ell + mk)) \pm \vartheta(\ell + mk) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, k \in \mathcal{W}.$$

That is,

$$\delta(\vartheta\ell) + \delta(\vartheta mk) + \sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{\vartheta mk, \ldots, \vartheta mk}_{t-\text{times}}) \pm \vartheta\ell \pm \vartheta mk \in \mathcal{Z}(\mathcal{R})$$

$\forall\ \vartheta, \ell, k \in \mathcal{W}$. On using the given condition, we see that

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{m\vartheta k, \ldots, m\vartheta k}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, k \in \mathcal{W}.$$

Now, use Lemma 2.1 and the fact that $\mathcal{R}$ is $n!$- torsion free to get

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta k) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, k \in \mathcal{W}.$$

Replace k by ℓ to get

$$\delta(\vartheta\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Again using the hypothesis, we get

$$\vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}. \tag{2.11}$$

Commuting with $r \in \mathcal{R}$, we obtain

$$[\vartheta\ell, r] = 0\ \forall\ \vartheta, \ell \in \mathcal{W}, r \in \mathcal{R},$$

and so

$$\vartheta[\ell, r] + [\vartheta, r]\ell = 0 \ \forall \ \vartheta, \ell \in \mathcal{W}, r \in \mathcal{R}. \tag{2.12}$$

Replacing ℓ by $\ell\hbar$ in (2.12) and using (2.12), we see that

$$\vartheta\ell[\hbar, r] = 0 \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}, r \in \mathcal{R}.$$

On replacing ϑ by $[\hbar, r]$, we get

$$[\hbar, r]\ell[\hbar, r] = 0 \ \forall \ \ell, \hbar \in \mathcal{W}, r \in \mathcal{R}.$$

That is,

$$[\hbar, r]\ell\mathcal{R}[\hbar, r]\ell = (0) \ \forall \ \ell, \hbar \in \mathcal{W}.$$

Since $\mathcal{R}$ is a semiprime ring, we have

$$[\hbar, r]\ell = 0 \ \forall \ \ell, \hbar \in \mathcal{W}, r \in \mathcal{R}.$$

Taking ℓ to be $t[\hbar, r]$, $t \in \mathcal{R}$, we see that

$$[\hbar, r]t[\hbar, r] = 0.$$

By the semiprimeness of $\mathcal{R}$, we get $\mathcal{W} \subseteq \mathcal{Z}(\mathcal{R})$. Thus, $\mathcal{R}$ contains a nonzero central ideal.

(ii) Use similar arguments as used in (i) to get the required result.

(iii) It is given that

$$\mathscr{d}(\vartheta\ell) \pm [\vartheta, \ell] \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

Replace ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, and we get

$$\mathscr{d}(\vartheta(\ell + m\hbar)) \ \pm \ [\vartheta, \ell + m\hbar] \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

That is,

$$\mathscr{d}(\vartheta\ell) + \mathscr{d}(\vartheta m\hbar) + \sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{\vartheta m\hbar, \ldots, \vartheta m\hbar}_{t-\text{times}}) \ \pm \ [\vartheta, \ell] \ \pm \ [\vartheta, m\hbar] \in \mathcal{Z}(\mathcal{R})$$

$\forall \ \vartheta, \ell, \hbar \in \mathcal{W}$. Using the hypothesis, we see that

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{m\vartheta\hbar, \ldots, m\vartheta\hbar}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Invoking Lemma 2.1 and using the torsion free restriction of $\mathcal{R}$, we get

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\hbar) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Replace k by ℓ, and we obtain

$$\mathscr{d}(\vartheta\ell) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

On using the hypothesis, we see that

$$[\vartheta, \ell] \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

That is, $[\mathcal{W}, \mathcal{W}] \subset \mathcal{Z}(\mathcal{R})$. Hence, by Lemma 2.3, $\mathcal{R}$ contains a nonzero central ideal.

(iv) It is given that

$$\mathscr{d}(\vartheta\ell) \pm \vartheta \circ \ell \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

Taking $\ell + mk$ in the place of ℓ for $k \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get

$$\mathscr{d}(\vartheta(\ell + mk)) \pm \vartheta \circ (\ell + mk) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, k \in \mathcal{W}.$$

That is,

$$\mathscr{d}(\vartheta\ell) + \mathscr{d}(\vartheta mk) + \sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{\vartheta mk, \ldots, \vartheta mk}_{t-\text{times}}) \ \pm \ \vartheta \circ \ell \ \pm \ \vartheta \circ mk \in \mathcal{Z}(\mathcal{R})$$

for all $\vartheta, \ell, k \in \mathcal{W}$. With the help of the given condition, we see that

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{m\vartheta k, \ldots, m\vartheta k}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, k \in \mathcal{W}.$$

Now, using Lemma 2.1 and the fact that $\mathcal{R}$ is $n!$-torsion free, we obtain

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta k) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell, k \in \mathcal{W}.$$

Replace k by ℓ, and we get

$$\mathscr{d}(\vartheta\ell) \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

Making use of the hypothesis, we see that

$$\vartheta \circ \ell \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{W}.$$

That is, $\mathcal{W} \circ \mathcal{W} \in \mathcal{Z}(\mathcal{R})$. Hence, by using Lemma 2.4, $\mathcal{R}$ contains a nonzero central ideal. The proof is complete. □

Based on the preceding findings, we obtain the following known result:

Corollary 2.8. *[8] For any fixed integer $n \geq 2$, let $\mathcal{R}$ be an $n!$-torsion free semiprime ring. If $\mathcal{R}$ admits a nonzero permuting n-derivation $\Delta : \mathcal{R}^n \to \mathcal{R}$ with trace $\mathscr{d} : \mathcal{R} \to \mathcal{R}$ satisfying any one of the conditions*

(1) $\mathscr{d}(\vartheta\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{R}$,

(2) $\mathscr{d}(\vartheta\ell) \pm \ell\vartheta \in \mathcal{Z}(\mathcal{R}) \ \forall \ \vartheta, \ell \in \mathcal{R}$,

then $\mathcal{R}$ is commutative.

Theorem 2.9. *For a fixed integer $n \geq 2$, let $\mathcal{R}$ be an $n!$-torsion free semiprime ring and $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ with $d : \mathcal{R} \to \mathcal{R}$ and $g : \mathcal{R} \to \mathcal{R}$ as traces of $\mathfrak{D}$ and $\mathcal{G}$ satisfying any one of the following conditions:*

(1) $g(\vartheta\ell) + d(\vartheta)d(\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(2) $g(\vartheta\ell) + d(\vartheta)d(\ell) \pm \ell\vartheta \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(3) $g([\vartheta, \ell]) + [d(\vartheta), d(\ell)] \pm [\vartheta, \ell] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$,

(4) $g(\vartheta \circ \ell) + d(\vartheta) \circ d(\ell) \pm \vartheta \circ \ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}$.

Then, $\mathcal{R}$ contains a nonzero central ideal.

Proof. (i) It is given that

$$g(\vartheta\ell) + d(\vartheta)d(\ell) \pm \vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Replacing ℓ by $\ell + m\hbar$ for $\hbar \in \mathcal{W}$ and $1 \leq m \leq n-1$, we arrive at

$$g(\vartheta\ell) + g(\vartheta m\hbar) + \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{\vartheta m\hbar, \ldots, \vartheta m\hbar}_{t-\text{times}})+$$
$$d(\vartheta)\Big(d(\ell) + d(m\hbar) + \sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}})\Big)$$
$$\pm\ \vartheta\ell\ \pm\ \vartheta m\hbar \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Using the given condition, we get

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\vartheta\ell, \ldots, \vartheta\ell}_{(n-t)-\text{times}}, \underbrace{\vartheta m\hbar, \ldots, \vartheta m\hbar}_{t-\text{times}}) + d(\vartheta)\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell, \ldots, \ell}_{(n-t)-\text{times}}, \underbrace{m\hbar, \ldots, m\hbar}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R})$$

$\forall\ \vartheta, \ell, \hbar \in \mathcal{W}$. Using Lemma 2.1, we see that

$$n\mathcal{G}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\hbar) + nd(\vartheta)\mathfrak{D}(\ell, \ldots, \ell, \hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Since $\mathcal{R}$ is $n!$-torsion free, we get

$$\mathcal{G}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\hbar) + d(\vartheta)\mathfrak{D}(\ell, \ldots, \ell, \hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell, \hbar \in \mathcal{W}.$$

Writing ℓ in place of $\hbar$, we get

$$g(\vartheta\ell) + d(\vartheta)d(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

Using the hypothesis, we obtain that

$$\vartheta\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta, \ell \in \mathcal{W}.$$

On using the same arguments as after (2.11), we get the required result.

(*ii*) Following the same steps as in (*i*), we discover that $\mathcal{R}$ contains a nonzero central ideal.

(*iii*) It is given that

$$g([\vartheta,\ell]) + [d(\vartheta), d(\ell)] \pm [\vartheta,\ell] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

Replacing ℓ by $\ell + mk$ for $k \in \mathcal{W}$ and $1 \le m \le n-1$, we conclude that

$$g([\vartheta,\ell]) + g([\vartheta,mk]) + \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{[\vartheta,\ell],\ldots,[\vartheta,\ell]}_{(n-t)-\text{times}},\underbrace{[\vartheta,mk],\ldots,[\vartheta,mk]}_{t-\text{times}})+$$
$$[d(\vartheta),d(\ell)] + [d(\vartheta),d(mk)] + [d(\vartheta),\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell,\ldots,\ell}_{(n-t)-\text{times}},\underbrace{mk,\ldots,mk}_{t-\text{times}})]$$
$$\pm [\vartheta,\ell] \pm [\vartheta,mk] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell,k \in \mathcal{W}.$$

On using the hypothesis, we get

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{[\vartheta,\ell],\ldots,[\vartheta,\ell]}_{(n-t)-\text{times}},\underbrace{[\vartheta,mk],\ldots,[\vartheta,mk]}_{t-\text{times}}) + [d(\vartheta),\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell,\ldots,\ell}_{(n-t)-\text{times}},\underbrace{mk,\ldots,mk}_{t-\text{times}})] \in \mathcal{Z}(\mathcal{R})$$

$\forall\ \vartheta,\ell,k \in \mathcal{W}$. Using Lemma 2.1 and the fact that $\mathcal{R}$ is $n!-$torsion free, we have

$$\mathcal{G}([\vartheta,\ell],\ldots,[\vartheta,\ell],[\vartheta,k]) + [d(\vartheta),\mathfrak{D}(\ell,\ldots,\ell,k)] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell,k \in \mathcal{W}.$$

Writing ℓ in place of k, we obtain

$$g([\vartheta,\ell]) + [d(\vartheta), d(\ell)] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

Using the hypothesis, we obtain that

$$[\vartheta,\ell] \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

By Lemma 2.3, we conclude that $\mathcal{R}$ contains a nonzero central ideal.
(*iv*) It is given that

$$g(\vartheta\circ\ell) + d(\vartheta)\circ d(\ell) \pm \vartheta\circ\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

Replacing ℓ by $\ell + mk$ for $k \in \mathcal{W}$ and $1 \le m \le n-1$, we arrive at

$$g(\vartheta\circ\ell) + g(\vartheta\circ mk) + \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\vartheta\circ\ell,\ldots,\vartheta\circ\ell}_{(n-t)-\text{times}},\underbrace{\vartheta\circ mk,\ldots,\vartheta\circ mk}_{t-\text{times}})+$$
$$d(\vartheta)\circ d(\ell) + d(\vartheta)\circ d(mk) + d(\vartheta)\circ\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell,\ldots,\ell}_{(n-t)-\text{times}},\underbrace{mk,\ldots,mk}_{t-\text{times}})$$
$$\pm \vartheta\circ\ell \pm \vartheta\circ mk \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell,k \in \mathcal{W}.$$

On using the hypothesis, we get

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\vartheta\circ\ell,\ldots,\vartheta\circ\ell}_{(n-t)-\text{times}},\underbrace{\vartheta\circ m\hbar,\ldots,\vartheta\circ m\hbar}_{t-\text{times}}) + \mathscr{d}(\vartheta)\circ\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}(\underbrace{\ell,\ldots,\ell}_{(n-t)-\text{times}},\underbrace{m\hbar,\ldots,m\hbar}_{t-\text{times}}) \in \mathcal{Z}(\mathcal{R})$$

$\forall\ \vartheta,\ell,\hbar \in \mathcal{W}$. Using Lemma 2.1 and using the fact that $\mathcal{R}$ is $n!-$torsion free, we get

$$\mathcal{G}(\vartheta\circ\ell,\ldots,\vartheta\circ\ell,\vartheta\circ\hbar) + \mathscr{d}(\vartheta)\circ\mathfrak{D}(\ell,\ldots,\ell,\hbar) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell,\hbar \in \mathcal{W}.$$

Write ℓ in place of $\hbar$ to get

$$g(\vartheta\circ\ell) + \mathscr{d}(\vartheta)\circ\mathscr{d}(\ell) \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

Using the hypothesis, we obtain that

$$\vartheta\circ\ell \in \mathcal{Z}(\mathcal{R})\ \forall\ \vartheta,\ell \in \mathcal{W}.$$

We conclude by Lemma 2.4 that $\mathcal{R}$ contains a nonzero central ideal. The proof is complete. □

3. Permuting n-multipliers

This section deals with the study of permuting n-multipliers. The idea of a permuting n-multiplier was initially suggested by Ashraf et al. in [4], and they proved some interesting results. In the present section, we examine the action of symmetric n-derivations satisfying the functional identity $f(i)i + ig(i) = 0\ \forall\ i \in \mathcal{W}$, a nonzero left ideal of $\mathcal{R}$ where f and g are the traces of symmetric n-derivations $\mathfrak{D}$ and $\mathcal{G}$, respectively. We begin with the following:

Definition 3.1. A permuting n-additive map $\Lambda : \mathcal{R}^n \to \mathcal{R}$ is called a permuting left n-multiplier (resp., permuting right n-multiplier) if

$$\Lambda(i_1, i_2, \ldots, i_t i'_t, \ldots, i_n) = \Lambda(i_1, i_2, \ldots, i_t, \ldots, i_n)i'_t$$

$$(\text{resp., } \Lambda(i_1, i_2, \ldots, i_t i'_t, \ldots, i_n) = i_t\Lambda(i_1, i_2, \ldots, i'_t, \ldots, i_n))$$

holds $\forall\ i_t, i'_t \in \mathcal{R},\ t = 1, 2, \ldots, n$. If Λ is both a permuting left n-multiplier and a permuting right n-multiplier, it is referred to as a permuting n-multiplier. For related results, see [4, 5].

According to Brešar's proof in [9, Theorem 4.1], if $\mathcal{R}$ is a prime ring, $\mathcal{W}$ is a nonzero left ideal of $\mathcal{R}$, and $\mathscr{d}$ and g are nonzero derivations of $\mathcal{R}$ satisfying $\mathscr{d}(i)i - ig(i) \in \mathcal{Z}(\mathcal{R})\ \forall\ i \in \mathcal{W}$, then $\mathcal{R}$ is commutative. We expand the previous result by demonstrating the following theorem for the trace of n-derivation of $\mathcal{R}$.

Theorem 3.2. *Let $\mathcal{R}$ be an $n!$-torsion free prime ring and $\mathcal{W}$ be a nonzero left ideal of $\mathcal{R}$. Suppose that $\mathcal{R}$ admits two symmetric n-derivations $\mathfrak{D} : \mathcal{R}^n \to \mathcal{R}$ and $\mathcal{G} : \mathcal{R}^n \to \mathcal{R}$ with f and g as traces of $\mathfrak{D}$ and $\mathcal{G}$, respectively. If $f(i)i + ig(i) = 0\ \forall\ i \in \mathcal{W}$, then either $\mathcal{R}$ is commutative or $\mathcal{G}$ acts as a left n-multiplier on $\mathcal{W}$. Furthermore, in the last case, either $\mathfrak{D} = 0$ or $\mathcal{W}[\mathcal{W}, \mathcal{W}] = 0$.*

Proof. By hypothesis, we have
$$f(i)i + ig(i) = 0 \ \forall \ i \in \mathcal{W}.$$
Replacing i by $i + m\ell$ for $\ell \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get
$$f(i + m\ell)(i + m\ell) + (i + m\ell)g(i + m\ell) = 0 \ \forall \ i, \ell \in \mathcal{W}.$$
On using the definition of f and g, we see that
$$\Big(f(i) + f(m\ell) + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{i, \ldots, i}_{(n-t)-\text{times}}, \underbrace{m\ell, \ldots, m\ell}_{t-\text{times}})\Big)(i + m\ell) + (i + m\ell)\Big(g(i) + g(m\ell) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{i, \ldots, i}_{(n-t)-\text{times}}, \underbrace{m\ell, \ldots, m\ell}_{t-\text{times}})\Big) = 0 \ \forall \ i, \ell \in \mathcal{W}.$$
On using the given condition, we get
$$f(i)m\ell + f(m\ell)i + ig(m\ell) + m\ell g(i) + \Big(\sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{i, \ldots, i}_{(n-t)-\text{times}}, \underbrace{m\ell, \ldots, m\ell}_{t-\text{times}})\Big)(i + m\ell) + (i + m\ell)\Big(\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{i, \ldots, i}_{(n-t)-\text{times}}, \underbrace{m\ell, \ldots, m\ell}_{t-\text{times}})\Big) = 0$$
$\forall \ i, \ell \in \mathcal{W}$. On using Lemma 2.1, we get
$$f(\ell)i + n\mathfrak{D}(i, \ell, \ldots, \ell)\ell + ig(\ell) + n\ell\mathcal{G}(i, \ell, \ldots, \ell) = 0 \tag{3.1}$$
$\forall \ i, \ell \in \mathcal{W}$. Replace i by ik to obtain
$$f(\ell)ik + ni\mathfrak{D}(k, \ell, \ldots, \ell)\ell + n\mathfrak{D}(i, \ell, \ldots, \ell)k\ell + ikg(\ell) + n\ell i\mathcal{G}(k, \ell, \ldots, \ell) + n\ell\mathcal{G}(i, \ell, \ldots, \ell)k = 0 \ \forall \ i, \ell, k \in \mathcal{W}. \tag{3.2}$$
On comparing (3.1) and (3.2), we get
$$-ig(\ell)k - n\mathfrak{D}(i, \ell \ldots, \ell)\ell k + ni\mathfrak{D}(k, \ell, \ldots, \ell)\ell + n\mathfrak{D}(i, \ell, \ldots, \ell)k\ell + ikg(\ell) + n\ell i\mathcal{G}(k, \ell, \ldots, \ell) = 0 \ \forall \ i, \ell, k \in \mathcal{W}.$$
This implies that
$$i[k, g(\ell)] + n\mathfrak{D}(i, \ell, \ldots, \ell)[k, \ell] + ni\mathfrak{D}(k, \ell, \ldots, \ell)\ell + n\ell i\mathcal{G}(k, \ell, \ldots, \ell) = 0 \tag{3.3}$$
$\forall \ i, \ell, k \in \mathcal{W}$. Substitute ri for i in (3.3) to get
$$ri[k, g(\ell)] + nr\mathfrak{D}(i, \ell, \ldots, \ell)[k, \ell] + n\mathfrak{D}(r, \ell, \ldots, \ell)i[k, \ell] + nri\mathfrak{D}(k, \ell, \ldots, \ell)\ell + n\ell ri\mathcal{G}(k, \ell, \ldots, \ell) = 0 \tag{3.4}$$

$\forall\ i, \ell, k \in \mathcal{W}, r \in \mathcal{R}$. Compare (3.4) and (3.3).

$$n\mathfrak{D}(r, \ell, \ldots, \ell)i[k, \ell] + n\ell ri\mathcal{G}(k, \ell, \ldots, \ell) - nr\ell i\mathcal{G}(k, \ell, \ldots, \ell) = 0 \quad \forall\ i, \ell, k, \in \mathcal{W}, r \in \mathcal{R}.$$

Since $\mathcal{R}$ is $n!-$torsion free, we obtain

$$\mathfrak{D}(r, \ell, \ldots, \ell)i[k, \ell] + [\ell, r]i\mathcal{G}(k, \ell, \ldots, \ell) = 0 \tag{3.5}$$

$\forall\ i, \ell, k \in \mathcal{W}, r \in \mathcal{R}$. Replacing ℓ by k in (3.5), we see that

$$[k, r]ig(k) = 0\ \forall\ i, k \in \mathcal{W}, r \in \mathcal{R}.$$

Substituting ri for i, we get

$$[k, r]rig(k) = 0\ \forall\ i, k \in \mathcal{W}.$$

Since $\mathcal{R}$ is a prime ring, it yields that either $[k, r] = 0$ or $ig(k) = 0$. If $[k, r] = 0\ \forall\ k \in \mathcal{W}$ and $r \in \mathcal{R}$, then replacing k by sk, we get $[s, r]k = 0\ \forall\ k \in \mathcal{W}, r, s \in \mathcal{R}$. Again, replace k by rk such that $[s, r]rk = 0\ \forall\ k \in \mathcal{W}, r, s \in \mathcal{R}$. Since $\mathcal{R}$ is a prime ring, we conclude that $\mathcal{R}$ is commutative. Next, if $ig(k) = 0\ \forall\ i, k \in \mathcal{W}$, then replacing k by $k + m\ell$, we get

$$ig(k + m\ell) = 0\ \forall\ i, \ell, k \in \mathcal{W}.$$

That is,

$$ig(k) + ig(m\ell) + i\sum_{t=1}^{n-1} {}^{n}C_t\mathcal{G}(\underbrace{k, \ldots, k}_{(n-t)-\text{times}}, \underbrace{m\ell, \ldots, m\ell}_{t-\text{times}}) = 0\ \forall\ i, \ell, k \in \mathcal{W}.$$

By using Lemma 2.1 and the fact that $\mathcal{R}$ is $n!-$torsion free, we get

$$i\mathcal{G}(k, \ell \ldots, \ell) = 0\ \forall\ i, \ell, k \in \mathcal{W}.$$

This implies that

$$\mathcal{G}(ik, \ell, \ldots, \ell) = \mathcal{G}(i, \ell, \ldots, \ell)k.$$

Hence, $\mathcal{G}$ acts as a left n-multiplier. Since $i\mathcal{G}(k, \ell \ldots, \ell) = 0\ \forall\ i, \ell, k \in \mathcal{W}$, using (3.5), we arrive at

$$\mathfrak{D}(r, \ell, \ldots, \ell)i[k, \ell] = 0\ \forall\ i, \ell, k, \in \mathcal{W}, r \in \mathcal{R}.$$

Replace r by sr to get

$$\mathfrak{D}(s, \ell, \ldots, \ell)\mathcal{R}i[k, \ell] = 0\ \forall\ i, \ell, k \in \mathcal{W}.$$

Primeness of $\mathcal{R}$ yields that either $\mathfrak{D}(s, \ell, \ldots, \ell) = 0$ or $i[k, \ell] = 0\ \forall\ i, \ell, k \in \mathcal{W}, s \in \mathcal{R}$. If $\mathfrak{D} \neq 0$, the latter results in $\mathcal{W}[\mathcal{W}, \mathcal{W}] = 0$. □

Following the same vein, we can also demonstrate the following:

Theorem 3.3. *Let $\mathcal{R}$ be an $n!-$ torsion free prime ring and $\mathcal{W}$ be a nonzero right ideal of $\mathcal{R}$. Assume that $\mathfrak{D}$ and $\mathcal{G}$ are two symmetric n-derivations of $\mathcal{R}$ with trace f and g, respectively. If $f(i)i + ig(i) = 0$ $\forall\ i \in \mathcal{W}$, then either $\mathcal{R}$ is commutative or $\mathfrak{D}$ acts as a left n-multiplier on $\mathcal{W}$. Furthermore, in the last case, either $\mathcal{G} = 0$ or $\mathcal{W}[\mathcal{W}, \mathcal{W}] = 0$.*

In view of the above result, we obtain the following known results:

Corollary 3.4. *[1] Let $\mathcal{R}$ be a prime ring of characteristic not two, $\mathcal{W}$ be a nonzero left ideal of $\mathcal{R}$ and Δ_1, Δ_2 be symmetric bi-derivations of $\mathcal{R}$ with traces δ_1 and δ_2, respectively. If $\Delta_1(i,i)i + i\Delta_2(i,i) = 0\ \forall\ i \in \mathcal{W}$, then either $\mathcal{R}$ is commutative or Δ_2 acts as a left bi-multiplier on $\mathcal{W}$. Moreover, in the last case either $\Delta_1 = 0$ or $\mathcal{W}[\mathcal{W},\mathcal{W}] = 0$.*

Corollary 3.5. *[1] Let $\mathcal{R}$ be a prime ring of characteristic not two, $\mathcal{W}$ be a nonzero right ideal of $\mathcal{R}$ and Δ_1, Δ_2 be symmetric bi-derivations of $\mathcal{R}$ with traces δ_1 and δ_2, respectively. If $\Delta_1(i,i)i+i\Delta_2(i,i) = 0\ \forall\ i \in \mathcal{W}$, then either $\mathcal{R}$ is commutative or Δ_1 acts as a left bi-multiplier on $\mathcal{W}$. Moreover, in the last case either $\Delta_2 = 0$ or $\mathcal{W}[\mathcal{W},\mathcal{W}] = 0$.*

The next result is the generalization of Vukman's result [18]. Indeed, Vukman showed that if $\mathcal{R}$ is a prime ring of characteristic different from two and three, and there exist symmetric bi-derivations $\mathfrak{D}_1 : \mathcal{R}\times\mathcal{R} \to \mathcal{R}$ and $\mathfrak{D}_2 : \mathcal{R}\times\mathcal{R} \to \mathcal{R}$, such that $f_1(a)f_2(a) = 0,\ \forall\ a \in \mathcal{R}$ holds, where f_1 and f_2 are the traces of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively, then either $\mathfrak{D}_1 = 0$ or $\mathfrak{D}_2 = 0$. We extend this theorem for q-iterations of n-derivations.

Theorem 3.6. *Let $\mathcal{R}$ be an $n!$-torsion free prime ring, $\mathcal{W}$ be a nonzero ideal of $\mathcal{R}$ and $q \geq 1$, be a fixed integer. Consider $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q : \mathcal{R}^n \to \mathcal{R}$ to be n-derivations on $\mathcal{R}$ such that $\delta_1(i_1)\delta_2(i_2)\cdots\delta_q(i_q) = 0\ \forall\ i_1, i_2, \ldots, i_q \in \mathcal{W}$ where $\delta_i's$, are traces of $\mathfrak{D}_i's$, respectively. Then, one of the following holds:*

(1) $\delta_1(i_1) = 0\ \forall\ i_1 \in \mathcal{W}$;

(2) All $\mathfrak{D}_p$ act as left n-multipliers on $\mathcal{R}$ for $p = 2, 3, \ldots, q$.

Proof. We will prove it through induction. If we put $q = 1$ in our hypothesis, then it is obvious that $\delta_1(i_1) = 0\ \forall\ i_1 \in \mathcal{W}$. Now, consider $q = 2$, and by the hypothesis, we have

$$\delta_1(i_1)\delta_2(i_2) = 0\ \forall\ i_1, i_2 \in \mathcal{W}. \tag{3.6}$$

Replacing i_2 by $i_2 + m\ell_2$ for $\ell_2 \in \mathcal{W}$ and $1 \leq m \leq n-1$, we get

$$\delta_1(i_1)\delta_2(i_2 + m\ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathcal{W}.$$

On simplifying, we get

$$\delta_1(i_1)\delta_2(i_2) + \delta_1(i_1)\delta_2(m\ell_2) + \delta_1(i_1)\sum_{t=1}^{n-1} {}^nC_t\mathfrak{D}_2(\underbrace{i_2,\ldots,i_2}_{(n-t)-\text{times}},\underbrace{m\ell_2,\ldots,m\ell_2}_{t-\text{times}}) = 0 \tag{3.7}$$

$\forall\ i_1, i_2, \ell_2 \in \mathcal{W}$. Compare (3.6) and (3.7) and use Lemma 2.1 to get

$$n\delta_1(i_1)\mathfrak{D}_2(i_2,\ldots,i_2,\ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathcal{W}.$$

Since $\mathcal{R}$ is $n!$-torsion free, we obtain

$$\delta_1(i_1)\mathfrak{D}_2(i_2,\ldots,i_2,\ell_2) = 0\ \forall\ i_1, i_2, \ell_2 \in \mathcal{W}. \tag{3.8}$$

Replacing ℓ_2 by $\ell_2 r$ in (3.8), we obtain

$$\mathscr{d}_1(i_1)\ell_2\mathfrak{D}_2(i_2,\ldots,i_2,r) = 0\ \forall\ i_1, i_1, \ell_2 \in \mathscr{W}, r \in \mathscr{R},$$

i.e.,

$$\mathscr{d}_1(i_1)\ell_2\mathscr{R}\mathfrak{D}_2(i_2,\ldots,i_2,r) = (0)\ \forall\ i_1, i_2, \ell_2 \in \mathscr{W}.$$

Since $\mathscr{R}$ is a prime ring, we can find either $\mathscr{d}_1(i_1)\ell_2 = 0$ or $\mathfrak{D}_2(i_2,\ldots,i_2,r) = 0$. Consider the first case, $\mathscr{d}_1(i_1)\ell_2 = 0$. Again, $\mathscr{R}$ is a prime ring, and we get $\mathscr{d}_1(i_1) = 0$. Now, consider the latter case, $\mathfrak{D}_2(i_2,\ldots,i_2,r) = 0\ \forall\ i_2 \in \mathscr{W},\ r \in \mathscr{R}$. A straightforward modification shows that $\mathfrak{D}_2(i_2,\ldots,i_2,w_1 r) = \mathfrak{D}_2(i_2,\ldots,i_2,w_1)r\ \forall\ w_1 \in \mathscr{W}, r \in \mathscr{R}$. Hence, $\mathfrak{D}_2$ acts as a left n-multiplier as desired.

Next, suppose that it is true for $n = q-1$, and we shall prove it for $n = q$. Let us assume the hypothesis:

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_q(i_q) = 0\ \forall\ i_1, i_2,\ldots,i_q \in \mathscr{W}. \tag{3.9}$$

Replacing i_q by $i_q + m\ell_q$ for $\ell_q \in \mathscr{W}$ and $1 \le m \le n-1$ in (3.9) and taking account of Lemma 2.1, we get

$$n\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\mathfrak{D}_q(i_q,\ldots,i_q,\ell_q) = 0$$

$\forall\ i_1, i_2,\ldots,i_q,\ell_q \in \mathscr{W}$. Since $\mathscr{R}$ is $n!$-torsion free, we see that

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\mathfrak{D}_q(i_q,\ldots,i_q,\ell_q) = 0. \tag{3.10}$$

Substituting $\ell_q u$ for ℓ_q in (3.10) and using (3.10), we arrive at

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\ell_q\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0,$$

i.e.,

$$\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1})\ell_q\mathscr{R}\mathfrak{D}_q(i_q,\ldots,i_q,u) = (0)$$

$\forall\ i_1, i_2,\ldots,i_q,\ell_q \in \mathscr{W}, u \in \mathscr{R}$. Primeness of $\mathscr{R}$ gives that either $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1}) = 0$ or $\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0\ \forall\ i_1, i_2,\ldots,i_q \in \mathscr{W},\ u \in \mathscr{R}$. If $\mathscr{d}_1(i_1)\mathscr{d}_2(i_2)\cdots\mathscr{d}_{q-1}(i_{q-1}) = 0$, then we are done by the former case. If $\mathfrak{D}_q(i_q,\ldots,i_q,u) = 0\ \forall\ i_q \in \mathscr{W},\ u \in \mathscr{R}$, then we can easily compute that $\mathfrak{D}_q(i_q,\ldots,i_q,w_{q-1}u) = \mathfrak{D}_q(i_q,\ldots,i_q,w_{q-1})u\ \forall\ i_q, w_{q-1} \in \mathscr{W},\ u \in \mathscr{R}$. Hence, $\mathfrak{D}_q$ acts as a left n-multiplier on $\mathscr{R}$ as desired. The theorem's proof is completed with this conclusion. □

4. Conclusions and further remarks

In this article, we discussed some results concerning the containment of a nonzero central ideal in a ring $\mathscr{R}$ satisfying certain functional identities involving the traces $\mathscr{d}$ and g of symmetric n-derivations $\mathcal{D}$ and $\mathcal{G}$, respectively. Besides proving some results concerning the traces of permuting n-derivations, some results related to permuting n-multipliers are also discussed in the last section. In fact, we characterized symmetric n-derivations of prime rings in terms of left n-multipliers. In future, it would be interesting to study these functional identities in the setting of generalized permuting n-derivations and its related maps in rings with involution.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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