Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 17197–17207.
DOI: 10.3934/math.2023878
Received: 15 March 2023
Revised: 07 April 2023
Accepted: 26 April 2023
Published: 17 May 2023

Research article

Existence of solution for an impulsive differential system with improved boundary value conditions

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Abstract: In this paper, we study the existence of solution for a class of impulsive integro-differential equations. Different from traditional periodic and anti-periodic boundary value problems, a more general boundary condition introduced in this new system. First, we obtain some new comparison principles. Then, we obtain the expression of the solution for a class of linearized systems. Finally, the existence of extremal solutions for the new boundary value system are obtained by using the monotone iterative technique. The theoretical results obtained have wider applications in practical fields.

Keywords: boundary value problem; integro-differential equation; extremal solutions; monotone iterative technique
Mathematics Subject Classification: 34B37, 34B05

1. Introduction

In this paper, we consider the existence of solution for the follow first-order impulsive integro-differential system with an improved boundary value condition [1]:

$$\begin{cases} h'(\tau) = f(\tau, h(\tau), [\Gamma h](\tau), [\delta h](\tau)), \quad \tau \in I' = I - \{\tau_1, \tau_2, \ldots, \tau_c\}, \\ \triangle h(\tau_i) = I_k(h(\tau_i)), \quad i = 1, 2, \cdots, c, \\ h(0) = kh(T), \end{cases} \tag{1.1}$$

where $I = [0, T], f \in C(I \times R^3), I_k \in C(R, R), 0 = \tau_0 < \tau_1 < \cdots \tau_c < \tau_{c+1} = T, \triangle h(\tau_k) = h(\tau_k^+) - h(\tau_k^-)$, $h(\tau_k^-)$ and $h(\tau_k^+)$ represent the left and right limits of $h(\tau)$ at $\tau = \tau_k$,

$$[\Gamma h](\tau) = \int_0^\tau \varpi(\tau, s)h(s)ds, \qquad [\delta h](\tau) = \int_0^T \varsigma(\tau, s)h(s)ds,$$

$\varpi \in C(D, R^+)$, $D = \{(\tau, s) \in I \times I : \tau \geq s\}$, $\varsigma \in C(I \times I, R^+)$, $R^+ = [0, \infty)$, $k \in R$. Different from traditional periodic and anti-periodic boundary value problems, the boundary conditions introduced in this system are more general.

Impulsive differential equation is a kind of differential equation used to describe the discontinuous development process with jump points (impulses). It has a very wide application background, such as biology, cybernetics, physics and mechanics [2–6]. The integro-differential equation is a kind of equation with unknown functions under both integral and differential signs, this equation will be encountered in physical problems such as diffusion and radiation [7–9]. At present, the boundary value problem of impulsive integro-differential equations has received more and more attention [10–12]. For example, by using conformable fractional derivative, Asawasamrit et al. discuss the existence of solution for impulsive fractional integro-differential equations with periodic boundary value conditions [13]. By using the theorem on fixed points, Mardanov et al. obtain the existence and uniqueness of the solutions for nonlocal boundary problems of nonlinear integro-differential equations [14]. Wang et al. study the uniform and quadratical convergence for monotone sequences of impulsive integro-differential equations with integral boundary conditions [15]. By employing topological degree theory, the existence results for integro-differential systems with robin boundary conditions are established [16]. Existence and stability results of solutions for impulsive integro-differential equations with antiperiodic boundary conditions can refer to [17, 18]. Moreover, the monotone iterative technique combined with upper and lower solutions is a common method to discuss the existence of solution for impulsive integro-differential equations [19–21].

We note that types of boundary conditions described in system (1.1) are rarely discussed by now. It is a linear boundary condition, which can describe the relationship between boundary values more generally. In Section 2, we build some new comparison results for system (1.1). In Section 3, we give the expression of the solution and prove its uniqueness for a class of linearization problems. Finally, we prove main results in Section 4.

2. Some new comparative principles

To give the solution of system (1.1), we define follow spaces [1, 15, 19, 20]: $PC(I) = \{h : I \to R : h\big|_{(\tau_k,\tau_{k+1}]} \in C((\tau_k, \tau_{k+1}], R), k = 0, 1, \cdots, c$ and there exist $h(\tau_k^+)$ and $h(\tau_k^-)$ with $h(\tau_k^-) = h(\tau_k)$, $k = 1, 2, \cdots, c\}$; $PC'(I) = \{h \in PC(I); h\big|_{(\tau_k,\tau_{k+1}]} \in C'((\tau_k, \tau_{k+1}], R), k = 0, 1, \cdots, c$ and $h'(0^+), h'(T^-), h'(\tau_k^+)$ and $h'(\tau_k^-)$ exist for $k = 1, 2, \cdots, c\}$. Obviously, $PC(I)$ and $PC'(I)$ are Banach spaces with norms

$$||h||_{pc} = sup\{|h(\tau)|; \tau \in I\}, \qquad ||h||_{pc'} = ||h||_{pc} + ||h'||_{pc}.$$

Then, a function $h \in PC'(I)$ is a solution of (1.1) if it satisfies (1.1).

Now, we give and prove some new comparative principles for system (1.1).

Lemma 2.1. Suppose that $\xi > 0$, $\iota_1, \iota_2 \geq 0$, $k < e^{\xi T}$ and $\zeta_k < 1$, $k = 1, 2, \cdots, c$ such that

$$\begin{cases} h'(\tau) + \xi h(\tau) + \iota_1[\Gamma h](\tau) + \iota_2[\delta h](\tau) \leq 0, & \tau \in I', \\ \triangle h(\tau_k) \leq -\zeta_k h(\tau_k), & k = 1, 2, \cdots, c, \\ h(0) \leq kh(T), \end{cases} \tag{2.1}$$

or

$$\begin{cases} h'(\tau) + \xi h(\tau) + \iota_1[\Gamma h](\tau) + \iota_2[\delta h](\tau) + a_h(\tau) \leq 0, & \tau \in I', \\ \triangle h(\tau_k) \leq -\zeta_k h(\tau_k) - \varrho_{hk}, & k = 1, 2, \cdots, c, \\ h(0) > kh(T), \end{cases} \tag{2.2}$$

where $a_h(\tau) = \kappa'(\tau) + \xi\kappa(\tau) + \iota_1[\Gamma\kappa](\tau) + \iota_2[\delta\kappa](\tau)$, $\varrho_{hk} = \Delta\kappa(\tau_k) + \zeta_k\kappa(\tau_k)$ for some $\kappa \in PC'(I)$ with $\kappa \geq 0$, $k\kappa(T) - \kappa(0) > h(0) - kh(T) > 0$.

And,

$$\int_0^T \delta(s)ds \leq \prod_{j=1}^{c}(1 - \overline{\zeta_j}), \tag{2.3}$$

with $\overline{\zeta_j}$ =max$\{\zeta_k, 0\}$, $k = 1, 2, \cdots, c$ and

$$\delta(\tau) = \iota_1 \int_0^\tau \varpi(\tau, s)e^{\xi(\tau-s)} \prod_{s<\tau_k<T}(1 - \zeta_k)ds + \iota_2 \int_0^T \varsigma(\tau, s)e^{\xi(\tau-s)} \prod_{s<\tau_k<T}(1 - \zeta_k)ds.$$

Then, $h \leq 0$ on I.

Proof. Let $\Omega_k = 1 - \zeta_k$, $k = 1, 2, \cdots, c$. If $h(0) \leq kh(T)$, set $\beta(\tau) = \left(\prod_{s<\tau_k<T} \Omega_k^{-1}\right)h(\tau)e^{\xi\tau}$. Then,

$$\begin{cases} \beta'(\tau) \leq -\left(\prod_{s<\tau_k<T} \Omega_k^{-1}\right)\Big(\iota_1 \int_0^t K(\tau, s)e^{\xi(\tau-s)}\beta(s) \prod_{s<\tau_k<T}(\Omega_k)ds \\ \qquad +\iota_2 \int_0^T \varsigma(\tau, s)e^{\xi(\tau-s)}\beta(s) \prod_{s<\tau_k<T}(\Omega_k)ds\Big), \qquad \tau \in I', \\ \beta(\tau_k^+) \leq \Omega_k v(\tau_k), \qquad k = 1, 2, \cdots, c, \\ \beta(0) \leq kv(T)\left(\prod_{s<\tau_k<T} \Omega_k^{-1}\right)e^{-\xi T}. \end{cases}$$

It is clear that h and β have the same sign.

In the follow, we discuss by three cases:

(i) If $\beta \geq 0$ and $\beta \neq 0$, then $\beta'(\tau) \leq 0$ on I' and $\beta(T) \leq \beta(0)\prod_{i=1}^{m} c_j \leq kv(T)e^{-\xi T}$. If $\beta(T) = 0$, then $\beta(0) = 0$; thus $\beta \leq 0$ on I and so $\beta \equiv 0$, this is a contradiction. If $\beta(T) > 0$, then $ke^{-\xi T} \geq 1$, which is impossible.

(ii) If $\beta(0) > 0$, then $\beta(T) > 0$. Let $o_1 \in (0, T)$ such that $\beta(o_1) < 0$. Suppose that $\beta(\tau^*) = \min_{\tau\in[0,T]}\beta(\tau) = b$, then $b < 0$. We obtain

$$\beta'(\tau) \leq -b\left(\prod_{\tau<\tau_k<T} \Omega_k^{-1}\right)\delta(\tau),$$

so,

$$\begin{aligned} 0 < \beta(T) &\leq \beta(\tau^*) \prod_{\tau^*<\tau_k<T} \Omega_k - b \int_{\tau^*}^T \left(\prod_{s<\tau_k<T} \Omega_k\right)\left(\prod_{s<\tau_k<T} \Omega_k^{-1}\right)\delta(s)ds \\ &\leq b \prod_{\tau^*<\tau_k<T} \Omega_k - b \int_{\tau^*}^T \delta(s)ds. \end{aligned}$$

Then,

$$\int_0^T \delta(s)ds \geq \int_{\tau^*}^T \delta(s)ds > \prod_{\tau^*<\tau_k<T} \Omega_k \geq \prod_{j=1}^{c} \overline{\Omega_j},$$

where $\overline{\Omega_j} = 1 - \overline{\zeta_j}$, $j = 1, 2, \cdots, c$. We know that it is a contradiction from (2.3), so $\beta \geq 0$. However, we can see from case (i) that it is impossible, so $\beta(0) \leq 0$.

(iii) Let $o_1 \in (0, T]$ such that $\beta(o_1) > 0$. Suppose that $\beta(o_2) = \min_{\tau\in[0,o_1]}\beta(\tau) = b$, then $b < 0$. Otherwise, $\beta'(\tau) \le 0$ on $[0, o_1] \bigcap I'$ and β is nonincreasing. So, $\beta(o_1) \le \beta(0) \prod_{0<\tau_k<o_1} \Omega_k \le 0$, which is impossible. Then, we obtain

$$0 < \beta(o_1) \le \beta(o_2) \prod_{o_2<\tau_k<o_1} \Omega_k - b \int_{o_2}^{o_1} \left(\prod_{s<\tau_k<o_1} \Omega_k \right) \left(\prod_{s<\tau_k<T} \Omega_k^{-1} \right) \delta(s)ds,$$

hence,

$$\int_{o_2}^{o_1} \left(\prod_{o_1\le\tau_k<T} \Omega_k^{-1} \right) \delta(s)ds > \prod_{o_2<\tau_k<o_1} \Omega_k.$$

Then,

$$\int_0^T \delta(s)\,ds \ge \int_{o_2}^{o_1} \delta(s)\,ds > \prod_{o_2<\tau_k<T} \Omega_k \ge \prod_{j=1}^{c} \overline{\Omega_j},$$

it contradicts with (2.3).

When $h(0) > kh(T)$, we assume that $F(\tau) = h(\tau) + \kappa(\tau)$. Then,

$$\begin{cases} F'(\tau) + \xi F(\tau) + \iota_1[\Gamma F](\tau) + \iota_2[\delta F](\tau) \le 0, & \tau \in I', \\ \Delta F(\tau_k) \le -\zeta_k h(\tau_k) - \varrho_{hk} + \Delta\kappa(\tau_k) = -\zeta_k F(\tau_k), & k = 1, 2, \cdots, c, \\ F(0) \le kF(T). \end{cases}$$

Obviously, we have $F \le 0$ on I and so $h \le 0$ on I from the above proof. This completes the proof.

Remark 2.1. The proof and description of Lemma 2.1 are similar to related results in [19]. And, the boundary condition in system (1.1) is more extensive, which can contain periodic and antiperiodic situations.

Corollaries 2.1 and 2.2 are also two comparative principles for system (1.1), which can be derived from the Lemma 2.1. The proves are similar to relevant conclusions in [19], the details are omitted.

Corollary 2.1. [19] Suppose that $0 \le \zeta_k < 1$ $(k = 1, 2, \cdots, c)$, $\iota_1, \iota_2 \ge 0$, $\xi > 0$, $k < e^{\xi T}$ such that $h \in PC'(I)$ satisfies conditions (2.1) or (2.2), and

$$\frac{(\iota_1\phi_0 + \iota_2\varphi_0)\left(e^{\xi T} - 1\right)e^{\xi T}}{\xi} \le \frac{\prod_{j=1}^{c}\left(1 - \zeta_j\right)^2}{\int_0^T \left(\prod_{s<\tau_k<T} (1 - \zeta_k) \right) ds}, \tag{2.4}$$

where $\phi_0 = \max\{\varpi(\tau, s) : (\tau, s) \in D\}$, $\varphi_0 = \max\{\varsigma(\tau, s) : (\tau, s) \in I \times I\}$. Then, $h \le 0$ for all $\tau \in I$.

Corollary 2.2. [19] Suppose that $0 \le \zeta_k < 1$ $(k = 1, 2, \cdots, c)$, $\iota_1, \iota_2 \ge 0$, $\xi > 0$, $k < e^{\xi T}$ such that $h \in PC'(I)$ satisfies conditions (2.1) or (2.2), and

$$(\xi + \iota_1 T\phi_0 + \iota_2 T\varphi_0)\,\Theta \left(1 + c \prod_{j=1}^{c} \left(1 - \zeta_j\right)^{-1} \right) \le 1, \tag{2.5}$$

where $\Theta = \max\{\tau_k - \tau_{k-1} : k = 1, 2, ..., c + 1\}$, $\phi_0 = \max\{\varpi(\tau, s) : (\tau, s) \in D\}$, $\varphi_0 = \max\{\varsigma(\tau, s) : (\tau, s) \in I \times I\}$. Then, $h \le 0$ for all $\tau \in I$.

3. Existence and uniqueness of solution for a linear system

First, we study the following linear system (LS) [1]:

$$h'(\tau) + \xi h(\tau) = \Upsilon(\tau), \qquad \tau \in I', \tag{3.1}$$

$$h(0) = kh(T), \tag{3.2}$$

$$h\left(\tau_j^+\right) = h\left(\tau_j^-\right) + I_j\left(h\left(\tau_j\right)\right), \qquad j = 1, ..., c, \tag{3.3}$$

where $\Upsilon \in PC(I)$ and $I_j \in C(R, R)$, $j = 1, ..., c$.

Lemma 3.1 gives the expression of solution for system (LS), the prove can refer to [1].

Lemma 3.1. [1] $h \in PC'(I)$ is a solution of (LS) if and only if

$$h(\tau) = \int_0^T \Omega(\tau, s)\Upsilon(s)ds + \sum_{j=1}^{c} \Omega\left(\tau, \tau_j\right) I_j\left(h\left(\tau_j\right)\right), \qquad \tau \in I', \tag{3.4}$$

where

$$\Omega(\tau, s) = \frac{1}{e^{\xi T} - k}\begin{cases} e^{\xi(T-\tau+s)}, & 0 \le s \le \tau \le T, \\ ke^{-\xi(\tau-s)}, & 0 \le \tau < s \le T. \end{cases} \tag{3.5}$$

Now, we define the following operators [22, 23]:

$$M : PC(I) \to PC(I), \qquad [Mu](\tau) = \int_0^T \Omega(\tau, s) h(s)ds, \tau \in I; \tag{3.6}$$

and

$$N : PC(I) \to PC(I), \qquad [Nu](\tau) = \sum_{j=1}^{c} \Omega\left(\tau, \tau_j\right) I_j\left(h\left(\tau_j\right)\right), \tau \in I. \tag{3.7}$$

Then, we can easily obtain the following Lemma 3.2 from Lemma 3.1:

Lemma 3.2. [22, 23] The solution of system (LS) is the fixed point of the operator

$$X : PC(I) \to PC(I), Xu = M\Upsilon + Nu.$$

Similarly, the following Lemma 3.3 can be obtained from Lemma 3.1:

Lemma 3.3. Let $\zeta_k < 1$ $(k = 1, 2, \cdots, c)$, $\iota_1, \iota_2 \ge 0$, $k < e^{\xi T}$, $\xi > 0$, $\Upsilon \in PC(I)$ and $\Xi \in PC'(I)$. Then, a function $h(\tau) \in PC'(I)$ is a solution of the boundary value system

$$\begin{cases} h'(\tau) + \xi h(\tau) + \iota_1[\Gamma h](\tau) + \iota_2[\delta h](\tau) = \Upsilon(\tau), & \tau \in I', \\ \triangle h(\tau_k) = -\zeta_k h(\tau_k) + I_k(\Xi(\tau_k)) + \zeta_k(\Xi(\tau_k)), & k = 1, 2, \cdots, c, \\ h(0) = kh(T), \end{cases} \tag{3.8}$$

when $h(\tau) \in PC(I)$ satisfies

$$h(\tau) = \int_0^T \Omega(\tau, s)\{\iota_1 [Tu](s) + \iota_2 [\delta h](s) + \Upsilon(s)\} ds$$
$$+ \sum_{0<\tau_k<T} \Omega(\tau, \tau_k)(-\zeta_k h(\tau_k) + I_k(\Xi(\tau_k)) + \zeta_k(\Xi(\tau_k))), \quad (3.9)$$

where

$$\Omega(\tau, s) = \frac{1}{e^{\xi T} - k} \begin{cases} e^{\xi(T-\tau+s)}, & 0 \le s \le \tau \le T, \\ e^{-\xi(\tau-s)}, & 0 \le \tau < s \le T. \end{cases}$$

Lemma 3.4. Let $\zeta_k < 1$ $(k = 1, 2, \cdots, c)$, $\iota_1, \iota_2 \ge 0$, $k < e^{\xi T}$, $\xi > 0$, $\Upsilon \in PC(I)$ and $\Xi \in PC'(I)$, $I_k \in C(I)$ and

$$\sup_{\tau \in I} \int_0^T \Omega(\tau, s) \left\{ \iota_1 \int_0^s \varpi(s, r)\, dr + \iota_2 \int_0^T \varsigma(s, r) dr \right\} ds + \frac{1}{1 - ke^{-\xi T}} \sum_{j=1}^{c} |\zeta_k| < 1. \quad (3.10)$$

Then, the solution of system (3.8) is unique in space $PC'(I)$.

Proof. Define the operator $Q : PC(I) \to PC(I)$, where Qh is given by the right-hand term in (3.9). Then,

$$\begin{aligned}
& \|Qh - Q\beta\| \\
= & \sup_{\tau \in I} \left| \int_0^T \Omega(\tau, s)\{\iota_1 \{[\Gamma h](s) - [\Gamma \beta](s)\} + \iota_2 \{[\delta h](s) - [\delta \beta](s)\} ds\} \right. \\
& \left. - \sum_{0<\tau_k<T} \Omega(\tau, \tau_k)\zeta_k (h(\tau_k) - \beta(\tau_k)) \right| \\
\le & \sup_{\tau \in I} \left\{ \int_0^T \Omega(\tau, s) \left[\iota_1 \int_0^s \varpi(s, r) |h(r) - \beta(r)| dr \right. \right. \\
& \left. + \iota_2 \int_0^T \varsigma(s, r) |h(r) - \beta(r)| dr \right] ds + \sum_{0<\tau_k<T} \Omega(\tau, \tau_k) |\zeta_k| |(h(\tau_k) - \beta(\tau_k))| \} \\
\le & \|h - \beta\| \left(\sup_{\tau \in I} \int_0^T \Omega(\tau, s) \left\{ \iota_1 \int_0^s \varpi(s, r)\, dr + \iota_2 \int_0^T \varsigma(s, r) dr \right\} ds + \frac{1}{1 - ke^{-\xi T}} \sum_{j=1}^{c} |\zeta_k| \right).
\end{aligned}$$

We can see that Q is a contractive mapping from condition (3.10), so the proof is complete.

4. Existence result of solution for system (1.1)

In this section, we prove the existence results for system (1.1) by using Lemmas 2.1, 3.3 and 3.4.

Theorem 4.1. Set parameters $\zeta_k < 1$ $(k = 1, 2, \cdots, c)$, $\iota_1, \iota_2 \ge 0$, $k < e^{\xi T}$ and $\xi > 0$ such that

(1) Expressions (2.3) and (3.10) hold.

(2) Suppose that functions $\Psi, \Phi \in PC'(I)$ satisfy $\Psi \ge \Phi$ on I and

$$\begin{cases} \Psi'(\tau) \ge f(\tau, \Psi(\tau), [\Gamma\Psi](\tau), [\delta\Psi](\tau)), & \tau \in I' = I - \{\tau_1, \tau_2, \ldots, \tau_c\}, \\ \triangle\Psi(\tau_k) \ge I_k(\Psi(\tau_k)), & k = 1, 2, \cdots, c, \\ \Psi(0) \ge k\Psi(T), \end{cases}$$

or

$$\begin{cases} \Psi'(\tau) \geq f(\tau, \Psi(\tau), [\Gamma\Psi](\tau), [\delta\Psi](\tau)) + a_{\Psi}(\tau), \quad \tau \in I' = I - \{\tau_1, \tau_2, \ldots, \tau_c\}, \\ \triangle\Psi(\tau_k) \geq I_k(\Psi(\tau_k)) + \varrho_{\Psi k}, \quad k = 1, 2, \cdots, c, \\ \Psi(0) < k\Psi(T), \end{cases}$$

where $a_{\Psi}(\tau) = \kappa_2'(\tau) + \xi\kappa_2(\tau) + \iota_1[\Gamma\kappa_2](\tau) + \iota_2[\delta\kappa_2](\tau)$, $\varrho_{\Psi k} = \Delta\kappa_2(\tau_k) + \zeta_k\kappa_2(\tau_k)$ for some $\kappa_2 \in PC'(I)$ with $\kappa_2 \geq 0,\ k\kappa_2(T) - \kappa_2(0) \geq k\Psi(T) - \Psi(0) > 0$. And,

$$\begin{cases} \Phi'(\tau) \leq f(\tau, \Phi(\tau), [\Gamma\Phi](\tau), [\delta\Phi](\tau)), \quad \tau \in I' = I - \{\tau_1, \tau_2, \ldots, \tau_c\}, \\ \triangle\Phi(\tau_k) \leq I_k(\Phi(\tau_k)), \quad k = 1, 2, \cdots, c, \\ \Phi(0) \leq k\Phi(T), \end{cases}$$

or

$$\begin{cases} \Phi'(\tau) \leq f(\tau, \Phi(\tau), [\Gamma\Phi](\tau), [\delta\Phi](\tau)) - a_{\Phi}(\tau), \quad \tau \in I' = I - \{\tau_1, \tau_2, \ldots, \tau_c\}, \\ \triangle\Phi(\tau_k) \leq I_k(\Phi(\tau_k)) - \varrho_{\Phi k}, \quad k = 1, 2, \cdots, c, \\ \Phi(0) > k\Phi(T), \end{cases}$$

where $a_{\Phi}(\tau) = \kappa_1'(\tau) + \xi\kappa_1(\tau) + \iota_1[\Gamma\kappa_1](\tau) + \iota_2[\delta\kappa_1](\tau)$, $\varrho_{\Phi k} = \Delta\kappa_1(\tau_k) + \zeta_k\kappa_1(\tau_k)$ for some $\kappa_1 \in PC'(I)$ with $\kappa_1 \geq 0,\ k\kappa_1(T) - \kappa_1(0) \geq \Phi(0) - k\Phi(T) > 0$.

(3) For $\Phi(\tau_k) \leq y \leq x \leq \Psi(\tau_k)$, $I_k\,(k = 1, 2, ..., c)$ satisfy

$$I_k(x) - I_k(y) \geq -\zeta_k(x - y).$$

(4) For $\tau \in I,\ \Phi \leq \overline{h} \leq h \leq \Psi,\ [\Gamma\Phi] \leq \overline{\beta} \leq \beta \leq [\Gamma\Psi],\ [\delta\Phi] \leq \overline{w} \leq w \leq [\delta\Psi]$, suppose that f satisfies

$$f(\tau, h, \beta, w) - f\left(\tau, \overline{h}, \overline{\beta}, \overline{w}\right) \geq -\xi\left(h - \overline{h}\right) - \iota_1\left(\beta - \overline{\beta}\right) - \iota_2(w - \overline{w}).$$

Then, we can find two monotone sequences Φ_n and Ψ_n satisfy $\Phi = \Phi_0 \leq \Phi_n \leq ... \leq \Psi_n \leq \Psi_0 = \Psi$, and they can uniformly converge to the maximum and minimum solutions of system (1.1) in interval

$$[\Phi, \Psi] = \{h \in PC(I) : \Phi(\tau) \leq h(\tau) \leq \Psi(\tau), \tau \in I\}.$$

Proof. Let $[\Phi, \Psi] = \{h \in PC(I) : \Phi(\tau) \leq h(\tau) \leq \Psi(\tau), \tau \in I\}$. For any $\Xi \in [\Phi, \Psi]$, we consider system (3.8) with

$$\Upsilon(\tau) = f(\tau, \Xi(\tau), [\Gamma\Xi](\tau), [\delta\Xi](\tau)) + \xi\Xi(\tau) + \iota_1[\Gamma\Xi](\tau) + \iota_2[\delta\Xi](\tau).$$

By Lemma 3.4, we know that system (3.8) has an unique solution $h \in PC'(I)$. We define an operator A by $h = A\Xi$, then the operator A satisfies: (i) $\Phi \leq A\Phi, A\Psi \leq \Psi$; (ii) A is monotone nondecressing in $[\Phi, \Psi]$, i.e., for any $\Xi_1, \Xi_2 \in [\Phi, \Psi]$, $\Xi_1 \leq \Xi_2$ implies $A\Xi_1 \leq A\Xi_2$.

To prove (i), let $c = \Phi_0 - \Phi_1 = \Phi - A\Phi_0$, then $c(0) - kc(T) = \Phi_0(0) - k\Phi_0(T)$. Since $\Phi_1(0) = k\Phi_1(T)$, then,

$$\begin{aligned} c'(\tau) &= \Phi_0'(\tau) - \Phi_1'(\tau) = \Phi'(\tau) - \Phi_1'(\tau) \\ &\leq f(\tau, \Phi(\tau), [\Gamma\Phi](\tau), [\delta\Phi](\tau)) + \xi\Phi_1(\tau) + \iota_1[\Gamma\Phi_1](\tau) + \iota_2[\delta\Phi_1](\tau) \\ &\quad - f(\tau, \Phi(\tau), [\Gamma\Phi](\tau), [\delta\Phi](\tau)) - \xi\Phi(\tau) - \iota_1[\Gamma\Phi](\tau) - \iota_2[\delta\Phi](\tau) \\ &= -\xi c(\tau) - \iota_1[\Gamma c](\tau) - \iota_2[\delta c](\tau), \qquad (\tau \neq \tau_k) \quad \tau \in I, \end{aligned}$$

and

$$\begin{aligned}\Delta c(\tau_k) &= \Delta\Phi\,(\tau_k) - \Delta\Phi_1\,(\tau_k) \le I_k(\Phi(\tau_k)) + \zeta_k(\Phi_1(\tau_k)) - I_k(\Phi(\tau_k)) - \zeta_k(\Phi(\tau_k))\\ &= -\zeta_k\,(c\,(\tau_k))\,, \qquad k = 1,2,...,c.\end{aligned}$$

It is clear that

$$c\,(0) - kc\,(T) = \Phi\,(0) - k\Phi\,(T) \le 0,$$

so $c(0) \le kc(T)$. By Lemma 2.1, we have $c(\tau) \le 0$ on I, i.e., $\Phi \le A\Phi$. Similarly, we can prove $A\Psi \le \Psi$.

To prove (ii), let $\Xi_1, \Xi_2 \in [\Phi, \Psi]$ satisfies $\Xi_1 \le \Xi_2$ on I and set $c = h_1 - h_2$, where $h_1 = A\Xi_1,\ h_2 = A\Xi_2$. Using conditions (2), (3) and (3.8), we have

$$\begin{aligned}c'(\tau) &= h_1'\,(\tau) - h_2'\,(\tau)\\ &= -\xi h_1(\tau) - \iota_1[\Gamma h_1](\tau) - \iota_2[\delta h_1](\tau) + f(\tau, \Xi_1(\tau), [\Gamma\Xi_1](\tau), [\delta\Xi_1](\tau)) + \xi\Xi_1(\tau)\\ &\quad +\iota_1[\Gamma\Xi_1](\tau) + \iota_2[\delta\Xi_1](\tau) + \xi h_2(\tau) + \iota_1[\Gamma h_2](\tau) + \iota_2[\delta h_2](\tau)\\ &\quad -f(\tau, \Xi_2(\tau), [\Gamma\Xi_2](\tau), [\delta\Xi_2](\tau)) - \xi\Xi_2(\tau) - \iota_1[\Gamma\Xi_2](\tau) - \iota_2[\delta\Xi_2](\tau)\\ &\le -\xi c(\tau) - \iota_1[\Gamma c](\tau) - \iota_2[\delta c](\tau), \qquad (\tau \ne \tau_k) \qquad \tau \in I,\end{aligned}$$

and

$$\begin{aligned}\Delta c(\tau_k) &= \Delta h_1(\tau_k) - \Delta h_2(\tau_k) = -\zeta_k(h_1(\tau_k) + I_k(\Xi_1(\tau_k)) + \zeta_k(\Xi_1(\tau_k))\\ &+ \zeta_k(h_2(\tau_k)) - I_k(\Xi_2(\tau_k)) - \zeta_k(\Xi_2(\tau_k))\\ &\le -\zeta_k(c(\tau_k)), \qquad k = 1,2,...,c.\end{aligned}$$

It is clear that $c(0) \le kc(T)$. Again by Lemma 2.1, we have $c(\tau) \le 0$ on I, that is, $h_1 \le h_2$ on I.

Now, we define sequences Φ_n, Ψ_n with $\Phi_0 = \Phi, \Psi_0 = \Psi$ such that $\Phi_{n+1} = A\Phi_n, \Psi_{n+1} = A\Psi_n (n = 0,1,2,\cdots)$. Clearly, from (i) and (ii), we can get

$$\Phi = \Phi_0 \le \Phi_1 \le \cdots \Phi_n \le \cdots \le \Psi_n \le \cdots \le \Psi_1 \le \Psi_0 = \Psi,$$

and each $\Phi_n, \Psi_n \in PC'(I)$ $(n = 0,1,2,\cdots)$ satisfies

$$\begin{aligned}\Phi_n(\tau) = &\int_0^T G(\tau,s)\,\{\iota_1[\Gamma\Phi_n](s) + \iota_2[\delta\Phi_n](s) + \Upsilon_{n-1}(s)\}\,ds\\ &+ \sum_{0<\tau_k<T} \Omega(\tau,\tau_k)\,(-\zeta_k\Phi_n(\tau_k) + I_k(\Phi_{n-1}(\tau_k)) + \zeta_k(\Phi_{n-1}(\tau_k)))\,, \qquad \tau \in I,\\ \Psi_n(\tau) = &\int_0^T G(\tau,s)\left\{\iota_1[\Gamma\Psi_n](s) + \iota_2[\delta\Psi_n](s) + \overline{\Upsilon_{n-1}}(s)\right\}ds\\ &+ \sum_{0<\tau_k<T} \Omega(\tau,\tau_k)\,(-\zeta_k\Psi_n(\tau_k) + I_k(\Psi_{n-1}(\tau_k)) + \zeta_k(\Psi_{n-1}(\tau_k)))\,, \qquad \tau \in I,\end{aligned}$$

where

$$\begin{aligned}\Upsilon_{n-1}(\tau) &= f(\tau, \Phi_{n-1}(\tau), [\Gamma\Phi_{n-1}](\tau), [\delta\Phi_{n-1}](\tau)) + \xi\Phi_{n-1}(\tau)\\ &\quad +\iota_1[\Gamma\Phi_{n-1}](\tau) + \iota_2[\delta\Phi_{n-1}](\tau),\end{aligned}$$

$$\overline{\Upsilon_{n-1}}(\tau) = f(\tau, \Psi_{n-1}(\tau), [\Gamma\Psi_{n-1}](\tau), [\delta\Psi_{n-1}](\tau)) + \xi\Psi_{n-1}(\tau) + \iota_1[\Gamma\Psi_{n-1}](\tau) + \iota_2[\delta\Psi_{n-1}](\tau).$$

So, there exist functions b and ρ such that $\lim_{n\to\infty}\Phi_n(\tau) = b(\tau), \lim_{n\to\infty}\Psi_n(\tau) = \rho(\tau)$ uniformly on I. From the expressions of $\Phi_n(\tau)$ and $\Psi_n(\tau)$, it is clear that b and ρ satisfy system (1.1).

Now, we prove b and ρ are extreme solutions of system (1.1).

Suppose that $E(\tau)$ is an any solution of system (1.1) and $E \in [\Phi, \Psi]$. Set $\Phi_n(\tau) \le E(\tau) \le \Psi_n(\tau)$ for a positive integer n. Then, let $c = \Phi_{n+1} - E$, we can obtain

$$\begin{aligned} c'(\tau) &= \Phi'_{n+1}(\tau) - E'(\tau) \\ &= -\xi\Phi_{n+1}(\tau) - \iota_1[\Gamma\Phi_{n+1}](\tau) - \iota_2[\delta\Phi_{n+1}](\tau) \\ &\quad + f(\tau, \Phi_n(\tau), [\Gamma\Phi_n](\tau), [\delta\Phi_n](\tau)) + \xi\Phi_n(\tau) + \iota_1[\Gamma\Phi_n](\tau) + \iota_2[\delta\Phi_n](\tau) \\ &\quad - f(\tau, E(\tau), [\Gamma E](\tau), [\delta E](\tau)) \\ &\le -\xi c(\tau) - \iota_1[\Gamma c](\tau) - \iota_2[\delta c](\tau), \qquad (\tau \neq \tau_k) \qquad \tau \in I, \\ \Delta c(\tau_k) &= \Delta\Phi_{n+1}(\tau_k) - \Delta E(\tau_k) - \zeta_k(\Phi_{n+1}(\tau_k)) + I_k(\Phi_n(\tau_k)) + \zeta_k(\Phi_n(\tau_k)) - I_k(E(\tau_k)) \\ &\le -\zeta_k(c(\tau_k)), \qquad k = 1, 2, ..., c, \end{aligned}$$

and $c(0) \le kc(T)$.

By Lemma 2.1, we have $c(\tau) \le 0$ on I, that is, $\Phi_{n+1}(\tau) \le E(\tau)$ on I. Similarly, we can prove that $E(\tau) \le \Psi_{n+1}(\tau)$ on I. Therefore, through induction, we can get that $\Phi_n(\tau) \le E(\tau) \le \Psi_n$ on I. Finally, by taking $n \to \infty$, we get $b(\tau) \le E(\tau) \le \rho(\tau)$ on I.

Obviously, when $\Phi(0) > k\Phi(T)$ and $\Psi(0) > k\Psi(T)$, the proof is similar and processes are omitted. Then end the proof.

5. Conclusions

In this paper, we build a class of impulsive integro-differential systems with improved boundary conditions, and the existence of solution are discussed. First, some new comparison lemma and corollaries are obtained. Then, we discuss the existence and uniqueness of solutions for a class of linearized equations. Finally, the existence of maximum and minimum solutions for the new boundary value system are obtained by using the monotone iterative technique together with upper and lower solutions. The boundary condition "$h(0) = kh(T),\ k < e^{\xi T}$" is more general than traditional periodic and anti-periodic boundary value conditions. It contains cases k=1 (periodic) and k=-1 (anti-periodic) and can find a wider application in this field.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work was supported by the National Science Foundation of China (Grant No. 11602092), the Natural Science Foundation of Hubei Province, China (Grant No. 2018CFB628) and the China Postdoctoral Science Foundation (No. 2018M632184).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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