Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 17171–17196.
DOI: 10.3934/math.2023877
Received: 03 March 2023
Revised: 27 March 2023
Accepted: 27 March 2023
Published: 17 May 2023

Theory article

Cartesian vector solutions for N-dimensional non-isentropic Euler equations with Coriolis force and linear damping

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Abstract: In this paper, we construct and prove the existence of theoretical solutions to non-isentropic Euler equations with a time-dependent linear damping and Coriolis force in Cartesian form. New exact solutions can be acquired based on this form with examples presented in this paper. By constructing appropriate matrices $A(t)$, and vectors $\mathbf{b}(t)$, special cases of exact solutions, where entropy $s = \ln\rho$, are obtained. This is the first matrix form solution of non-isentropic Euler equations to the best of the authors' knowledge.

Keywords: non-isentropic fluids; Euler equations; Coriolis force; linear-damping symmetric and anti-symmetric matrices; curve integration; Cartesian vector form solutions
Mathematics Subject Classification: 35C05, 35Q31, 76B03, 76M60

1. Introduction

The non-isentropic Euler equations in $\mathbf{R}^N$ in fluid dynamics with a time-dependent linear damping and Coriolis force can be expressed as follows:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \rho J\mathbf{u} + \alpha(t)\rho\mathbf{u} + \nabla p = \mathbf{0}, \tag{1.2}$$

$$S_t + \mathbf{u}\cdot\nabla S = 0, \tag{1.3}$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_N)^T$ is an N-dimensional velocity field, $\rho(\mathbf{x}, t)$ and $p(\mathbf{x}, t) = e^S\rho^\gamma$ represent density and the pressure function respectively, $J^T = -J$ representing Corilis force is an anti-symmetric matrix. The damping term $\alpha(t)\rho\mathbf{u}$ with $\alpha(t) \geq 0$ as a coefficient of friction is proportional to the momentum.

For the special case when $\alpha(t) = 0$, the equations are reduced to Euler equations extended and governed by Coriolis rotational force [1–4]. The theoretical global existence of the Euler equations with rotational forces can be referred to [5–7]. Further studies on stability and tropical cyclones driven by this model can be referred to [8–13].

If $J = 0$, (1.1)–(1.3) are reduced to non-isentropic linear-damped Euler equations, which provide an important model regarding to its physical behaviours. The system can also be used to describe compressible gas dynamics through a porous material driven by a friction force [14–16]. Weak solutions of the damped Euler equations are shown with asymptotic and large-time behavious in [16–19]. Chow, Fan, and Yuen, in 2017, constructed the solutions of Cartesian form with $J = 0$ in [20], which can be regarded as a special case in this article, while taking the parameter γ and 2α in [20] to be $\gamma + 1$ and α respectively. For time-dependent damping, Dong and Li studied a class of analytical solutions with free-boundary [21] in 2022.

For the case with $J = 0$ and $\alpha(t) = 0$, the system (1.1)–(1.3) is reduced to the Euler equations

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.4}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{0}, \tag{1.5}$$

$$S_t + \mathbf{u} \cdot \nabla S = 0. \tag{1.6}$$

There are lots of researches on Euler equations, for example, see [22–26]. Among all the topics, constructing analytical and exact solutions are crucial [27–34] with a common pattern of the velocity function $\mathbf{u}$ in linear form in many previous studies. For non-isentropic Euler equations, Barna and Mátyás presented the analytic solutions for one-dimensional Euler equations and three-dimensional Navier-Stokes equations with polytropic equation of state [34, 35], which can be referred to by taking $n \neq \gamma$ and the viscosities to be zero respectively. Based on the linear form of velocity, An, Fan, and Yuen contributed with Cartesian rotational solutions to the N-dimension isentropic compressible Euler equations (1.4)–(1.6) [36] in 2015:

$$\mathbf{u} = \mathbf{b}(t) + A(t)\mathbf{x}, \tag{1.7}$$

where $\mathbf{b}(t)$ and $A(t)$ are vector and matrix respectively. Further studies have shown the existence of general solutions in Cartesian form to isentropic Euler equations with damping and rotational forces in [20] and [37], respectively.

Referring to the many blowup pheonomena studies [38–40], the global solution is still complicated to look for.

2. Materials and methods

In this article, the existence of a form of Cartesian solutions to non-isentropic Euler equations with rational force and linear damping (1.1)–(1.3) is proven by adopting mainly techniques on matrices, vectors, and curve integration. Enforcing $e^S = \rho$ and regarding velocity field $\mathbf{u}$ as an linear transformation of $\mathbf{x} \in \mathbf{R}^N$, the problem is equivalent to finding the pressure function p, which leads us to a quadratic form and requirments on the matrix A and vector $\mathbf{b}$. With this finding, we can construct some special exact solutions, which could be utilized in benchmarks for testings, simulations of computing flows.

In the following sections, we will prove the existence of the non-isentropic damped Euler equations with Coriolis forces, which admit Cartesian solutions by using appropriate requirements on matrix A and vector $\mathbf{b}$. We will give examples on this first cartesian form solutions to non-isentropic Euler equations based on our finding.

3. Results

In this section, we consider the non-isentropic Euler equations. Suppose that the density ρ and pressure p satisfy the relation

$$p(\rho) = e^S \rho^\gamma, \tag{3.1}$$

where the constant $\gamma = c_p/c_u \geq 1$, and c_p and c_u are the specific heats per unit mass under constant pressure and constant volume, respectively. Then we have the following theorem.

Theorem 3.1. *If matrices A with $tr(A) = 0$ and $B = (A_t + A^2 + JA + \alpha(t)A)/2$ satisfy the matrix differential equations*

$$B^T = B, \tag{3.2}$$

$$B_t + BA + A^T B = 0, \tag{3.3}$$

then the compressible Euler equations with a time-dependent linear damping and Coriolis force (1.1)–(1.3) have explicit solutions in the form

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x}, \tag{3.4}$$

$$\rho = \mu[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]^{\frac{1}{\gamma}}, \tag{3.5}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)], \tag{3.6}$$

where $\mu = (\frac{\gamma}{\gamma+1})^{\frac{1}{\gamma}}$; the vector function $\mathbf{b}(t)$ and scalar function $c(t)$ satisfy the ordinary differential equations:

$$(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + A^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + 2B\mathbf{b} = \mathbf{0}, \tag{3.7}$$

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) = 0. \tag{3.8}$$

Proof. By (3.65), (3.5) and (3.6), $\rho > 0$, $S = \ln\rho$. Let

$$\bar{p} = \frac{\gamma+1}{\gamma}\rho^\gamma, \tag{3.9}$$

$$\frac{\nabla p}{\rho} = \frac{1}{\rho}\nabla(e^s\rho^\gamma) = \frac{1}{\rho}\nabla(\rho^{\gamma+1}) = (\gamma+1)\rho^{\gamma-1}\nabla\rho = \nabla(\frac{\gamma+1}{\gamma}\rho^\gamma) = \nabla\bar{p}. \tag{3.10}$$

With (3.9), the compressible Euler equations (1.2) and (1.3) can then be written as

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{3.11}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + J\mathbf{u} + \alpha(t)\mathbf{u} + \nabla\bar{p} = \mathbf{0}, \tag{3.12}$$

$$S_t + \mathbf{u}\cdot\nabla S = 0. \tag{3.13}$$

Owing to the equivalent relation (3.9) between $\bar{p}$ and ρ, we mainly deal with $\bar{p}$ when solving Eqs (3.11) and (3.12). Substituting Eq (3.4) into Eq (3.12), we have

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + J\mathbf{u} + \alpha(t)\mathbf{u} + \nabla\bar{p} \tag{3.14}$$

$$= \mathbf{b}_t + A_t\mathbf{x} + [(\mathbf{b}+A\mathbf{x})\cdot\nabla](\mathbf{b}+A\mathbf{x}) + JA\mathbf{x} + \alpha(t)A\mathbf{x} + J\mathbf{b} + \alpha(t)\mathbf{b} + \nabla\bar{p} \tag{3.15}$$

$$= \mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A_t\mathbf{x} + (\mathbf{b}\cdot\nabla)A\mathbf{x} + (A\mathbf{x}\cdot\nabla)A\mathbf{x} + JA\mathbf{x} + \alpha(t)A\mathbf{x} + \nabla\bar{p} \tag{3.16}$$

$$= \mathbf{b}_t + (A + J + \alpha(t))\mathbf{b} + (A_t + A^2 + JA + \alpha(t)A)\mathbf{x} + \nabla\bar{p} = \mathbf{0}. \tag{3.17}$$

Let

$$B = (b_{ij})_{N\times N} = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A), \qquad J = (g_{ij})_{N\times N}. \tag{3.18}$$

Then the above equation can be written into a component form

$$Q_i(x_1,\cdots,x_N) \equiv -b_{it} - \alpha(t)b_i - \sum_{k=1}^{N}(a_{ik}b_k + g_{ik}b_k + 2b_{ik}x_k) = \frac{\partial\bar{p}}{\partial x_i}, \quad i = 1,2,\cdots,N. \tag{3.19}$$

Then, the following sufficient and necessary compatible conditions of these N equations,

$$\frac{\partial Q_j(x_1,\cdots,x_N)}{\partial x_i} = \frac{\partial Q_i(x_1,\cdots,x_N)}{\partial x_j}, \quad i,j = 1,2,\cdots,N, \tag{3.20}$$

lead to

$$b_{ji} = b_{ij}, \quad i,j = 1,2,\cdots,N, \tag{3.21}$$

which implies that $B = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A)$ is a symmetric matrix. Under the condition (3.20), $\bar{p}(\mathbf{x})$ is a complete differential function,

$$d\bar{p}(\mathbf{x}) = \sum_{i=1}^{N}\frac{\partial\bar{p}(\mathbf{x})}{\partial x_i}dx_i = \sum_{i=1}^{N}Q_i(x_1,\cdots,x_N)dx_i. \tag{3.22}$$

Therefore we can choose a special integration route to obtain

$$\bar{p}(\mathbf{x},t) = \sum_{i=1}^{N}\int_{(0,0,\cdots,0)}^{(x_1,x_2,\cdots,x_N)} Q_i(x_1,x_2,\cdots,x_N)dx_i \tag{3.23}$$

$$= \int_0^{x_1} Q_1(x_1,0,\cdots,0)dx_1, + \int_0^{x_2} Q_2(x_1,x_2,0,\cdots,0)dx_2$$

$$+\cdots+\int_0^{x_N} Q_N(x_1,x_2,\cdots,x_N)dx_N \tag{3.24}$$

$$= -\sum_{i=1}^{N}\left[b_{i,t} + \sum_{k=1}^{N}(a_{ik}b_k + g_{ik}b_k) + \alpha(t)b_i\right]x_i - \sum_{i=1}^{N}b_{ii}x_i^2 - 2\sum_{i,k=1,\ i<k}^{N}b_{ik}x_ix_k + c(t) \tag{3.25}$$

$$= -\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + A\mathbf{b} + \alpha(t)\mathbf{b}) - \mathbf{x}^TB\mathbf{x} + c(t). \tag{3.26}$$

Next, we show that functions (3.4)–(3.6) satisfy (3.11). By (3.9), we have

$$\rho_t = (\mu\bar{p}^{\frac{1}{\gamma}})_t = \frac{\mu}{\gamma}\bar{p}^{\frac{1}{\gamma}-1}\bar{p}_t, \tag{3.27}$$

$$\rho \text{tr}(A) = \mu \bar{p}^{\frac{1}{\gamma}} \text{tr}(A) = \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \gamma \text{tr}(A) \bar{p}, \tag{3.28}$$

$$\nabla \rho = \nabla(\mu \bar{p}^{\frac{1}{\gamma}}) = \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \nabla \bar{p}, \tag{3.29}$$

$$\mathbf{u} \cdot \nabla \rho = \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \mathbf{u}^T \nabla \bar{p}. \tag{3.30}$$

From Eqs (3.27)–(3.30), we have

$$\begin{aligned}
&\rho_t + \text{div}(\rho \mathbf{u}) = \rho_t + \rho \text{tr}(A) + \mathbf{u} \cdot \nabla \rho \\
= &- \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \{ \mathbf{x}^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + \mathbf{x}^T B_t \mathbf{x} - c_t(t) \\
&+ \gamma \text{tr}(A) [\mathbf{x}^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + \mathbf{x}^T B \mathbf{x} - c(t)] \\
&+ (\mathbf{b} + A\mathbf{x})^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b} + 2B\mathbf{x}) \} \qquad (3.31) \\
= &- \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \{ \mathbf{x}^T (B_t + \gamma \text{tr}(A) B + 2A^T B) \mathbf{x} \\
&+ \mathbf{x}^T [(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + (\gamma \text{tr}(A) I + A^T)(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + 2B\mathbf{b}] \\
&- [c_t + \gamma \text{tr}(A) c - \mathbf{b}^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})] \} \qquad (3.32) \\
= &- \frac{\mu}{\gamma} \bar{p}^{\frac{1}{\gamma}-1} \{ \mathbf{x}^T [B_t + 2A^T B] \mathbf{x} \\
&+ \mathbf{x}^T [(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + A^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + 2B\mathbf{b}] \\
&- [c_t - \mathbf{b}^T (\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})] \} = 0, \qquad (3.33)
\end{aligned}$$

where we use the condition of the first term

$$\mathbf{x}^T (B_t + 2A^T B) \mathbf{x} = 0, \tag{3.34}$$

which is equivalent to

$$(B_t + 2A^T B)^T = -(B_t + 2A^T B), \tag{3.35}$$

that is,

$$B_t + BA + A^T B = 0, \tag{3.36}$$

which is (3.3). The second and third terms are controlled to be 0 with (3.7) and (3.8). By (3.6), we have

$$S = \ln \mu + \frac{1}{\gamma} \ln[-\mathbf{x}^T (\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B \mathbf{x} + c(t)] = \ln \rho. \tag{3.37}$$

From (3.9), (3.37) is equivalent to

$$S = \ln(\mu \bar{p}^{\gamma}) = \ln \mu + \frac{1}{\gamma} \ln \bar{p}, \tag{3.38}$$

$$S_t = \frac{(\ln \bar{p})_t}{\gamma} = \frac{1}{\gamma} \bar{p}^{-1} \bar{p}_t, \tag{3.39}$$

$$\nabla S = \frac{1}{\gamma} \nabla \ln \bar{p} = \frac{1}{\gamma} \bar{p}^{-1} \nabla \bar{p}. \tag{3.40}$$

Substituting (3.4)–(3.6) and (3.38)–(3.40) to (3.13) and using (3.3), (3.7), and (3.8), we obtain by a similar argument used in obtaining Eq (3.33) that

$$S_t + \mathbf{u}\cdot\nabla S = \frac{1}{\gamma}\bar{p}^{-1}(\bar{p}_t + \mathbf{u}^T\nabla\bar{p}) \tag{3.41}$$

$$=\frac{1}{\gamma}\bar{p}^{-1}[-\mathbf{x}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t - \mathbf{x}^T B_t\mathbf{x} + c_t(t)$$
$$-(\mathbf{x}^T A^T + \mathbf{b}^T)(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b} + 2B\mathbf{x})] \tag{3.42}$$

$$=\frac{1}{\gamma\bar{p}}\{-\mathbf{x}^T[(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + A^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + 2B\mathbf{b}]$$
$$-\mathbf{x}^T[B_t + 2A^T B]\mathbf{x} + c_t(t) - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})\} = 0. \tag{3.43}$$

■

We observe that Eq (3.3) is a N^2 matrix differential equation, which demands us to apply special reduction conditions to acquire solutions.

Corollary 3.1. *If A is an anti-symmetric matrix, that is*

$$A^T = -A, \tag{3.44}$$

and the following conditions are satisfied:

$$A_t + \alpha(t)A = 0, \tag{3.45}$$

$$AJ = JA, \tag{3.46}$$

$$B_t = 0, \tag{3.47}$$

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t = \mathbf{0}, \tag{3.48}$$

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) = 0, \tag{3.49}$$

then the compressible Euler equations (3.11)–(3.13) admit a general solution

$$\mathbf{u} = \mathbf{b}(t) + A\mathbf{x}, \tag{3.50}$$

$$\rho = \mu[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]^{\frac{1}{\gamma}}, \tag{3.51}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]. \tag{3.52}$$

Proof. By (3.45) and (3.46),

$$B^T = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A)^T \tag{3.53}$$

$$= \frac{1}{2}[(-A)(-A) + (-A)(-J)] \tag{3.54}$$

$$= \frac{1}{2}(A^2 + JA) = B. \tag{3.55}$$

We can then simplify (3.3), (3.7), and (3.8) into (3.47), (3.48), and (3.49). Since matrix A is anti-symmetric, we have

$$BA + A^T B = 0. \tag{3.56}$$

By (3.47), we have

$$B_t = 0, \tag{3.57}$$

$$B_t + BA + A^T B = 0. \tag{3.58}$$

Thus, Eq (3.3) is ensured.

Since

$$B^T = B, \tag{3.59}$$
$$AJ = JA, \tag{3.60}$$
$$A^T + A = 0, \tag{3.61}$$

we have

$$(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})_t + A^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) + 2B\mathbf{b} \tag{3.62}$$
$$= \mathbf{b}_{tt} + A_t\mathbf{b} + A\mathbf{b}_t - A\mathbf{b}_t - A(A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b})$$
$$+ (A_t + A^2 + JA + \alpha(t)A)\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.63}$$
$$= \mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t = \mathbf{0}. \tag{3.64}$$

Thus, Eq (3.64) is simplified to (3.48). ■

Next, we give the following examples in 2 to N-dimension to demonstrate special cases of this corollary.

Remark 3.1. *As (3.5) and (3.6) demand*

$$-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t) > 0 \tag{3.65}$$

for the positivity of the argument of the logarithm and density, the solutions exist locally.

Example 3.1. *When $\alpha = 0$, we have the following examples:*

2-dimensional Case: We take constant matrix

$$A = J = k_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \mathbf{b} = k_2 \begin{bmatrix} \cos(k_1 t) \\ \sin(k_1 t) \end{bmatrix}, \ c(t) = 0, \tag{3.66}$$

where k_1 and k_2 are arbitrary constants.

By (3.18),

$$B = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A) = \frac{1}{2}(2A^2) = A^2 \tag{3.67}$$
$$= {k_1}^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.68}$$

Since A is a constant matrix, $A_t = 0$, taking $\alpha(t) = 0$,

$$B_t = \frac{d(A_t + A^2 + JA + \alpha(t)A)}{2dt} = 0, \tag{3.69}$$

Eqs (3.45) and (3.47) are satisfied. As J = A, Eq (3.46) is guaranteed. Equation (3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.70}$$

$$= -{k_1}^2\mathbf{b} + \mathbf{0} + J\mathbf{b}_t + \mathbf{0} \tag{3.71}$$

$$= -{k_1}^2 k_2 \begin{bmatrix} \cos(k_1 t) \\ \sin(k_1 t) \end{bmatrix} + {k_1}^2 k_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\sin(k_1 t) \\ \cos(k_1 t) \end{bmatrix} = \mathbf{0}, \tag{3.72}$$

Eq (3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.73}$$

$$= 0 - k_2 \begin{bmatrix} \cos(k_1 t) \\ \sin(k_1 t) \end{bmatrix}^T \left(k_1 k_2 \begin{bmatrix} -\sin(k_1 t) \\ \cos(k_1 t) \end{bmatrix} + 2k_1 k_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(k_1 t) \\ \sin(k_1 t) \end{bmatrix} \right) \tag{3.74}$$

$$= -k_1 {k_2}^2 \begin{bmatrix} \cos(k_1 t) \\ \sin(k_1 t) \end{bmatrix}^T \begin{bmatrix} \sin(k_1 t) \\ -\cos(k_1 t) \end{bmatrix} = 0. \tag{3.75}$$

we obtain the following solution:

$$\mathbf{u}(t) = \begin{bmatrix} k_2 \cos(k_1 t) + k_1 x_2 \\ k_2 \sin(k_1 t) - k_1 x_1 \end{bmatrix}, \tag{3.76}$$

$$\rho = \mu[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]^{\frac{1}{\gamma}} \tag{3.77}$$

$$= \mu[-\mathbf{x}^T(\mathbf{b}_t + 2A\mathbf{b}) - \mathbf{x}^T A^2\mathbf{x}]^{\frac{1}{\gamma}} \tag{3.78}$$

$$= \mu[-\mathbf{x}^T(k_1 k_2 \begin{bmatrix} -\sin(k_1 t) \\ \cos(k_1 t) \end{bmatrix} + 2k_1 k_2 \begin{bmatrix} -\sin(k_1 t) \\ \cos(k_1 t) \end{bmatrix}) - \mathbf{x}^T {k_1}^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}]^{\frac{1}{\gamma}} \tag{3.79}$$

$$= \mu[{k_1}^2({x_1}^2 + {x_2}^2) + k_1 k_2(-\sin(k_1 t)x_1 + \cos(k_1 t)x_2)]^{\frac{1}{\gamma}}, \tag{3.80}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)] \tag{3.81}$$

$$= \ln\mu + \frac{1}{\gamma}\ln[{k_1}^2({x_1}^2 + {x_2}^2) + k_1 k_2(-\sin(k_1 t)x_1 + \cos(k_1 t)x_2)]. \tag{3.82}$$

3-dimensional Case: We take constant matrix

$$A = J = k_1 \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \ \mathbf{b} = k_2 t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ c(t) = \frac{3{k_2}^2}{2}t^2, \tag{3.83}$$

where k_1 and k_2 are arbitrary constants.

Since matrix A is a constant matrix, (3.45)–(3.47) are satisfied. By using of (3.83), (3.48) and (3.49) are ensured. By (3.18),

$$B = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A) = \frac{1}{2}(2A^2) = A^2 \tag{3.84}$$

$$= {k_1}^2 \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{3.85}$$

Since A is a constant matrix, $A_t = 0$, taking $\alpha(t) = 0$,

$$B_t = \frac{d(A_t + A^2 + JA + \alpha(t)A)}{2dt} = 0, \tag{3.86}$$

Eqs (3.45) and (3.47) are satisfied. As $J = A$, Eq (3.46) is guaranteed. Equation (3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.87}$$

$$= \mathbf{0} + \mathbf{0} + J\mathbf{b}_t + \mathbf{0} \tag{3.88}$$

$$= k_1k_2 \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}, \tag{3.89}$$

Eq (3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.90}$$

$$= 3{k_2}^2t - k_2 \begin{bmatrix} t \\ t \\ t \end{bmatrix}^T \left(k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2k_1k_2 \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right) \tag{3.91}$$

$$= 3{k_2}^2t - 3{k_2}^2t + 0 = 0. \tag{3.92}$$

Therefore we obtain the solution:

$$\mathbf{u}(t) = \begin{bmatrix} k_2t + k_1(x_2 - x_3) \\ k_2t + k_1(x_3 - x_1) \\ k_2t + k_1(x_1 - x_2) \end{bmatrix}, \tag{3.93}$$

$$\rho = \mu[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]^{\frac{1}{\gamma}} \tag{3.94}$$

$$= \mu[-\mathbf{x}^T(\mathbf{b}_t + 2A\mathbf{b}) - \mathbf{x}^T A^2\mathbf{x} + \frac{3{k_2}^2}{2}t^2]^{\frac{1}{\gamma}} \tag{3.95}$$

$$= \mu[-\mathbf{x}^T(k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0) - \mathbf{x}^T{k_1}^2 \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\mathbf{x} + \frac{3{k_2}^2}{2}t^2]^{\frac{1}{\gamma}} \tag{3.96}$$

$$= \mu[2{k_1}^2({x_1}^2 + {x_2}^2 + {x_3}^2 - x_1x_2 - x_1x_3 - x_2x_3) - k_2(x_1 + x_2 + x_3) + \frac{3{k_2}^2}{2}t^2]^{\frac{1}{\gamma}}, \tag{3.97}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)] \tag{3.98}$$

$$= \ln\mu + \frac{1}{\gamma}\ln[2{k_1}^2({x_1}^2 + {x_2}^2 + {x_3}^2 - x_1x_2 - x_1x_3 - x_2x_3) - k_2(x_1 + x_2 + x_3) + \frac{3{k_2}^2}{2}t^2]. \tag{3.99}$$

Remark 3.2. *The 3-dimensional example has the same setting with Example 5 in [37], which admits the same* **u** *solution but has different entropy and density.*

4-dimensional Case: We take

$$A = J = k_1 \begin{bmatrix} 0 & -2 & 1 & 1 \\ 2 & 0 & 1 & -3 \\ -1 & -1 & 0 & 2 \\ -1 & 3 & -2 & 0 \end{bmatrix}, \tag{3.100}$$

$$\mathbf{b} = k_2 t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ c(t) = 2{k_2}^2 t^2, \tag{3.101}$$

where k_1 and k_2 are arbitrary constants. By (3.18),

$$B = \frac{1}{2}(A_t + A^2 + JA + \alpha(t)A) = \frac{1}{2}(2A^2) = A^2 \tag{3.102}$$

$$= {k_1}^2 \begin{bmatrix} -6 & 2 & -4 & 8 \\ 2 & -14 & 8 & 4 \\ -4 & 8 & -6 & 2 \\ 8 & 4 & 2 & -14 \end{bmatrix}. \tag{3.103}$$

Since A is a constant matrix, $A_t = 0$, taking $\alpha(t) = 0$,

$$B_t = \frac{d(A_t + A^2 + JA + \alpha(t)A)}{2dt} = 0, \tag{3.104}$$

Eqs (3.45) and (3.47) are satisfied. As $J = A$, Eq (3.46) is guaranteed. Equation (3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.105}$$

$$= \mathbf{0} + \mathbf{0} + J\mathbf{b}_t + \mathbf{0} \tag{3.106}$$

$$= k_1 k_2 \begin{bmatrix} 0 & -2 & 1 & 1 \\ 2 & 0 & 1 & -3 \\ -1 & -1 & 0 & 2 \\ -1 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}, \tag{3.107}$$

Eq (3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.108}$$

$$= 4{k_2}^2 t - k_2 \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}^T \left(k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2k_1k_2 \begin{bmatrix} 0 & -2 & 1 & 1 \\ 2 & 0 & 1 & -3 \\ -1 & -1 & 0 & 2 \\ -1 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} \right) \tag{3.109}$$

$$= 4{k_2}^2 t - 4{k_2}^2 t + 0 = 0. \tag{3.110}$$

We have the following solutions:

$$\mathbf{u} = k_2 t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + k_1\begin{bmatrix} -2x_2 + x_3 + x_4 \\ 2x_1 + x_3 - 3x_4 \\ -x_1 - x_2 + 2x_4 \\ -x_1 + 3x_2 - 2x_3 \end{bmatrix}, \tag{3.111}$$

$$\rho = \mu[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)]^{\frac{1}{\gamma}} \tag{3.112}$$

$$= \mu[-\mathbf{x}^T(\mathbf{b}_t + 2A\mathbf{b}) - \mathbf{x}^T A^2\mathbf{x} + 2{k_2}^2t^2]^{\frac{1}{\gamma}} \tag{3.113}$$

$$= \mu[-\mathbf{x}^T(k_2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0) - \mathbf{x}^T {k_1}^2\begin{bmatrix} -6 & 2 & -4 & 8 \\ 2 & -14 & 8 & 4 \\ -4 & 8 & -6 & 2 \\ 8 & 4 & 2 & -14 \end{bmatrix}\mathbf{x} + 2{k_2}^2t^2]^{\frac{1}{\gamma}} \tag{3.114}$$

$$= \mu[-k_2(x_1 + x_2 + x_3 + x_4) + k_1^2(6{x_1}^2 + 14{x_2}^2 + 6{x_3}^2 + 14{x_4}^2$$

$$- 4x_1x_2 + 8x_1x_3 - 16x_1x_4 - 16x_2x_3 - 8x_2x_4 - 4x_3x_4) + 2{k_2}^2t^2]^{\frac{1}{\gamma}}, \tag{3.115}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[-\mathbf{x}^T(\mathbf{b}_t + J\mathbf{b} + \alpha(t)\mathbf{b} + A\mathbf{b}) - \mathbf{x}^T B\mathbf{x} + c(t)] \tag{3.116}$$

$$= \ln\mu + \frac{1}{\gamma}\ln[-k_2(x_1 + x_2 + x_3 + x_4) + k_1^2(6{x_1}^2 + 10{x_2}^2 + 6{x_3}^2 + 14{x_4}^2$$

$$- 4x_1x_2 + 8x_1x_3 - 16x_1x_4 - 16x_2x_3 - 8x_2x_4 - 4x_3x_4) + 2{k_2}^2t^2]. \tag{3.117}$$

Example 3.2. *When α is a constant, we have the following examples.*

2-dimensional Case: We take

$$A = -J = k_1e^{-\alpha t}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \mathbf{b} = k_2e^{-\alpha t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{3.118}$$

$c(t) = m > 0$*, where* k_1*,* k_2*, and* m *are arbitrary constants. Then we get a solution*

$$\mathbf{u}(t) = e^{-\alpha t}\begin{bmatrix} k_1x_2 + k_2 \\ -k_1x_1 + k_2 \end{bmatrix}, \tag{3.119}$$

$$\rho = \mu m^{\frac{1}{\gamma}}, \tag{3.120}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln m. \tag{3.121}$$

3-dimensional Case: We take

$$A = -J = k_1e^{-\alpha t}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \ \mathbf{b} = k_2e^{-\alpha t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.122}$$

$c(t) = m > 0$*, where* k_1*,* k_2*, and* m *are arbitrary constants. Then we get a solution*

$$\mathbf{u}(t) = e^{-\alpha t}\begin{bmatrix} k_1(x_2 + x_3) + k_2 \\ k_1(x_3 - x_1) + k_2 \\ -k_1(x_1 + x_2) + k_2 \end{bmatrix}, \tag{3.123}$$

$$\rho = \mu m^{\frac{1}{\gamma}}, \tag{3.124}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln m. \tag{3.125}$$

4-dimensional Case: We take

$$A = -J = k_1 e^{-\alpha t}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix},\ \mathbf{b} = k_2 e^{-\alpha t}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.126}$$

$c(t) = m > 0$, where k_1, k_2, and m are arbitrary constants. Then we get a solution

$$\mathbf{u}(t) = e^{-\alpha t}\begin{bmatrix} k_1(x_2 + x_3 + x_4) + k_2 \\ k_1(x_3 + x_4 - x_1) + k_2 \\ k_1(x_4 - x_1 - x_2) + k_2 \\ -k_1(x_1 + x_2 + x_3) + k_2 \end{bmatrix}, \tag{3.127}$$

$$\rho = \mu m^{\frac{1}{\gamma}}, \tag{3.128}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln m. \tag{3.129}$$

N-dimensional Case: We take

$$A = k_1 e^{-\alpha t}\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & \ddots & & 1 \\ \vdots & \vdots & & \ddots & 1 \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix},\ \mathbf{b} = k_2 e^{-\alpha t}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.130}$$

$$J = -A, \quad c(t) = m > 0, \tag{3.131}$$

where k_1, k_2, and m are arbitrary constants. Then we get a solution

$$u_i = e^{-\alpha t}[k_1(\sum_{k=i+1}^{N} x_k - \sum_{k=1}^{i-1} x_k) + k_2], \tag{3.132}$$

$$\rho = \mu m^{\frac{1}{\gamma}}, \tag{3.133}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln m. \tag{3.134}$$

Proof. Since N-dimensional case covers 2 to 4-dimensional cases, here gives the verification of N-dimensional case. (3.46) is guaranteed by $J = -A$, with

$$A_t = -\alpha k_1 e^{-\alpha t} = -\alpha A, \tag{3.135}$$

(3.45) is satisfied. Therefore,

$$B = \frac{1}{2}(A_t + A^2 + JA + \alpha A) = 0, \ \ B_t = 0, \tag{3.136}$$

(3.47) is ensured. Substituting (3.130) and (3.131) into (3.48) and (3.49) produces

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.137}$$
$$= \alpha^2\mathbf{b} + 2\alpha A\mathbf{b} + (\alpha\mathbf{b} - A\mathbf{b})_t \tag{3.138}$$
$$= \alpha^2\mathbf{b} + 2\alpha A\mathbf{b} - \alpha^2\mathbf{b} - 2\alpha A\mathbf{b} = \mathbf{0}, \tag{3.139}$$

and

$$c_t - \mathbf{b}^T(\mathbf{b}_t + \alpha\mathbf{b} + A\mathbf{b} + J\mathbf{b}) \tag{3.140}$$
$$=0 - \mathbf{b}^T(-\alpha\mathbf{b} + \alpha\mathbf{b} - J\mathbf{b} + J\mathbf{b}) = 0. \tag{3.141}$$

■ When $\alpha(t)$ is not a constant, we have the following examples.

Example 3.3. *(2-dimensional case) We take*

$$A = t^{k_1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ J = (t^{k_1} - \frac{k_2}{t^{k_1}})\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \mathbf{b} = t^{k_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ c(t) = \beta,\ \alpha(t) = -\frac{k_1}{t}, \tag{3.142}$$

where $k_1 < 0$, k_2, and β are arbitrary constants. As

$$AJ = JA = (t^{2k_1} - k_2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.143}$$

(3.46) is satisfied.

Denoting

$$Q = (q_{ij})_{N\times N} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{3.144}$$

it is easy to see

$$A_t = k_1 t^{k_1-1}Q = -\alpha(t)A, \tag{3.145}$$

and,

$$B = \frac{A^2 + JA}{2} = \frac{(A+J)A}{2} = \frac{k_2}{2t^{k_1}}QA = -\frac{k_2}{2}I = B^T, \tag{3.146}$$
$$B_t = (-\frac{k_2}{2}I)_t = 0, \tag{3.147}$$

therefore, (3.47) is satisfied. Since

$$\mathbf{b}_t = -\alpha(t)\mathbf{b}, J = \frac{k_2}{t^{k_1}}Q - A, \tag{3.148}$$

(3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.149}$$
$$= -(\alpha(t)\mathbf{b})_t - 2\alpha(t)A\mathbf{b} + [(\frac{k_2}{t^{k_1}}Q - A)t^{k_1}\mathbf{w}]_t + (\alpha(t)\mathbf{b})_t \tag{3.150}$$
$$= -2\alpha(t)A\mathbf{b} - (A\mathbf{b})_t \tag{3.151}$$

$$= -2\alpha(t)A\mathbf{b} - A\mathbf{b}_t - A_t\mathbf{b} \tag{3.152}$$

$$= -2\alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} = \mathbf{0}, \tag{3.153}$$

(3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.154}$$

$$=0 - t^{k_1}\mathbf{w}^T[k_1 t^{k1-1}\mathbf{w} + t^{k_1}Qt^{k_1}\mathbf{w} + (\frac{k_2}{t^{k_1}} - t^{k_1})Qt^{k_1}\mathbf{w} - k_1 t^{k1-1}\mathbf{w}] \tag{3.155}$$

$$=0 - t^{k_1}\mathbf{w}^T(\frac{k_2}{t^{k_1}}Qt^{k_1}\mathbf{w}) \tag{3.156}$$

$$= -k_2 t^{k_1}\mathbf{w}^T Q\mathbf{w} \tag{3.157}$$

$$= -k_2 t^{k_1}\sum_{i=1,j=1}^{N} q_{ij} = 0. \tag{3.158}$$

Then we get a solution

$$\mathbf{u}(t) = t^{k_1}\begin{bmatrix} 1 + x_2 \\ 1 - x_1 \end{bmatrix}, \tag{3.159}$$

$$\rho = \mu[k_2(\frac{{x_1}^2 + {x_2}^2}{2} - x_1 - x_2) + \beta]^{\frac{1}{\gamma}}, \tag{3.160}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[k_2(\frac{{x_1}^2 + {x_2}^2}{2} - x_1 - x_2) + \beta]. \tag{3.161}$$

Example 3.4 (3-dimensional case)**.** *We take*

$$A = t^{k_1}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \ J = (t^{k_1} - \frac{k_2}{t^{k_1}})\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \ \mathbf{b} = t^{k_1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, c(t) = \beta, \alpha(t) = -\frac{k_1}{t}, \tag{3.162}$$

where $k_1 < 0$, k_2, and β are arbitrary constants. As

$$AJ = JA = (t^{2k_1} - k_2)\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \tag{3.163}$$

(3.46) is satisfied.

Denoting

$$Q = (q_{ij})_{N\times N} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{3.164}$$

it is easy to see

$$A_t = -\alpha(t)A, \tag{3.165}$$

$$B = \frac{A^2 + JA}{2} = \frac{(A + J)A}{2} = \frac{k_2}{2t^{k_1}}QA = \frac{k_2}{2}Q^2 = B^T, \tag{3.166}$$

$$B = \frac{A^2 + JA}{2} = \frac{k_2}{2}\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}, \tag{3.167}$$

therefore

$$B^T = B,\ B_t = 0, \tag{3.168}$$

(3.47) is satisfied.

Since

$$\mathbf{b}_t = -\alpha(t)\mathbf{b},\ J = \frac{k_2}{t^{k_1}}Q - A, \tag{3.169}$$

(3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.170}$$

$$= -(\alpha(t)\mathbf{b})_t - 2\alpha(t)A\mathbf{b} + [(\frac{k_2}{t^{k_1}}Q - A)t^{k_1}\mathbf{w}]_t + (\alpha(t)\mathbf{b})_t \tag{3.171}$$

$$= -2\alpha(t)A\mathbf{b} - (A\mathbf{b})_t \tag{3.172}$$

$$= -2\alpha(t)A\mathbf{b} - A\mathbf{b}_t - A_t\mathbf{b} \tag{3.173}$$

$$= -2\alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} = \mathbf{0}, \tag{3.174}$$

(3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.175}$$

$$=0 - t^{k_1}\mathbf{w}^T[k_1t^{k1-1}\mathbf{w} + t^{k_1}Qt^{k_1}\mathbf{w} + (\frac{k_2}{t^{k_1}} - t^{k_1})Qt^{k_1}\mathbf{w} - k_1t^{k1-1}\mathbf{w}] \tag{3.176}$$

$$=0 - t^{k_1}\mathbf{w}^T(\frac{k_2}{t^{k_1}}Qt^{k_1}\mathbf{w}) \tag{3.177}$$

$$= -k_2t^{k_1}\mathbf{w}^TQ\mathbf{w} \tag{3.178}$$

$$= -k_2t^{k_1}\sum_{i=1,j=1}^{N} q_{ij} = 0. \tag{3.179}$$

We then get a solution

$$\mathbf{u}(t) = t^{k_1}\begin{bmatrix} x_2 + x_3 + 1 \\ x_3 - x_1 + 1 \\ -x_1 - x_2 + 1 \end{bmatrix}, \tag{3.180}$$

$$\rho = \mu[k_2({x_1}^2 + {x_2}^2 + {x_3}^2 + x_1x_2 + x_2x_3 - x_1x_3 - 2x_1 + 2x_3) + \beta]^{\frac{1}{\gamma}}, \tag{3.181}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln[k_2({x_1}^2 + {x_2}^2 + {x_3}^2 + x_1x_2 + x_2x_3 - x_1x_3 - 2x_1 + 2x_3) + \beta]. \tag{3.182}$$

Remark 3.3 (N-dimensional case)**.** *We can abtain N-dimensional solutions denoting*

$$Q = (q_{ij})_{N\times N} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & \ddots & & 1 \\ \vdots & \vdots & & \ddots & 1 \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.183}$$

and taking

$$A = f(t)Q,\ \ J = (\frac{k_1}{f(t)} - f(t))Q,\ \ \mathbf{b} = f(t)\mathbf{w},\ \ \alpha(t) = -\frac{\dot{f}(t)}{f(t)},\ c(t) = \beta, \tag{3.184}$$

where $\dot{f}(t)f(t) \leq 0$, k_1 *and* β *are arbitrary constants. As*

$$AJ = JA = (k_1 - f(t)^2)Q^2, \tag{3.185}$$

(3.46) is satisfied. It is easy to see

$$A_t = -\alpha(t)A, \tag{3.186}$$

$$B = \frac{A^2 + JA}{2} = \frac{(A + J)A}{2} = \frac{k_1}{2f(t)}Qf(t)Q = \frac{k_1}{2}Q^2, \tag{3.187}$$

therefore

$$B^T = B,\ B_t = 0, \tag{3.188}$$

(3.47) are satisfied. Since

$$\mathbf{b}_t = -\alpha(t)\mathbf{b},\ J = \frac{k_1}{f(t)}Q - A, \tag{3.189}$$

(3.48) is satisfied by

$$\mathbf{b}_{tt} + 2A_t\mathbf{b} + (J\mathbf{b} + \alpha(t)\mathbf{b})_t \tag{3.190}$$

$$= -\ (\alpha(t)\mathbf{b})_t - 2\alpha(t)A\mathbf{b} + [(\frac{k_1}{f(t)} - f(t))Qf(t)\mathbf{w}]_t + (\alpha(t)\mathbf{b})_t \tag{3.191}$$

$$= -\ 2\alpha(t)A\mathbf{b} + (k_1Q\mathbf{w} - A\mathbf{b})_t \tag{3.192}$$

$$= -\ 2\alpha(t)A\mathbf{b} - (A\mathbf{b})_t \tag{3.193}$$

$$= -\ 2\alpha(t)A\mathbf{b} - A\mathbf{b}_t - A_t\mathbf{b} \tag{3.194}$$

$$= -\ 2\alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} + \alpha(t)A\mathbf{b} = \mathbf{0}, \tag{3.195}$$

(3.49) is satisfied by

$$c_t - \mathbf{b}^T(\mathbf{b}_t + A\mathbf{b} + J\mathbf{b} + \alpha(t)\mathbf{b}) \tag{3.196}$$

$$=0 - f(t)\mathbf{w}^T[\dot{f}(t)\mathbf{w} + f(t)Qf(t)\mathbf{w} + (\frac{k_1}{f(t)} - f(t))Qf(t)\mathbf{w} - \dot{f}(t)\mathbf{w}] \tag{3.197}$$

$$=0 - f(t)\mathbf{w}^T(\frac{k_1}{f(t)}Qf(t)\mathbf{w}) \tag{3.198}$$

$$= -\ k_1f(t)\mathbf{w}^TQ\mathbf{w} \tag{3.199}$$

$$= -\ k_1f(t)\sum_{i=1,j=1}^{N} q_{ij} = 0. \tag{3.200}$$

4. Conclusions and discussion

In this paper, we construct the Cartesian solutions

$$\mathbf{u} = \mathbf{b}(t) + A(t)\mathbf{x}$$

for the non-isentropic Euler equations with a time-dependent linear damping and a rotational force. By constructing appropriate matices $A(t)$ and vectors $\mathbf{b}(t)$, we obtain new theoretical new exact solutions, which are obtained under the requirement of entropy $S = \ln \rho$. We then invite the scientific community to provide solutions with other forms of or more general form of entropy. The global existence of the solutions remains open, while the blowup phenomena are complicated to higher dimensional cases due to the existence of many temporal variables and the multiple requirements imposed on them.

Conflict of interest

The author declares there is no interest in relation to this article.

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Supplementary

Verification of examples on Euler equations

For simplicity, we use the same $\bar{p}$ defined in (3.9), solutions of ρ and S in all dimensions are equivalent to

$$\rho = \mu\bar{p}^{\frac{1}{\gamma}}, \tag{4.1}$$

$$S = \ln\mu + \frac{1}{\gamma}\ln\bar{p}. \tag{4.2}$$

It is clear that from the theorem (3.5) and (3.6) and can be easily verified from substitution that all solutions satisfy $S = \ln\rho$. Dividing ρ from both sides of (1.2), we rewrite the Euler equations (1.1)–(1.3) as

$$\rho_t + \sum_{k=1}^{N}\frac{\partial}{\partial x_k}\rho u_k = 0, \tag{4.3}$$

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{N}u_k(\frac{\partial u_i}{\partial x_k} + j_{ik}) + \alpha u_i + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_i}\rho^{\gamma} = 0, \tag{4.4}$$

$$S_t + \sum_{k=1}^{N}u_k\frac{\partial}{\partial x_k}S = 0. \tag{4.5}$$

Example 1

For 2-dimension case: Substituting (3.76)–(3.82) and (4.1) into (4.3) produces

$$\rho_t + \frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) \tag{4.6}$$

$$= \mu(\bar{p}^{\frac{1}{\gamma}})_t + \frac{\partial}{\partial x_1}(\mu\bar{p}^{\frac{1}{\gamma}}u_1) + \frac{\partial}{\partial x_2}(\mu\bar{p}^{\frac{1}{\gamma}}u_2) \tag{4.7}$$

$$= \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\bar{p}_t + (k_2\cos(k_1t) + k_1x_2)\frac{\partial}{\partial x_1}\mu\bar{p}^{\frac{1}{\gamma}} + (k_2\cos(k_1t) - k_1x_1)\frac{\partial}{\partial x_2}\mu\bar{p}^{\frac{1}{\gamma}} \tag{4.8}$$

$$\begin{aligned} &= \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[-k_1^2k_2\cos(k_1t)x_1 - k_1^2k_2\sin(k_1t)x_2] \\ &\quad + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\frac{\partial}{\partial x_1}[\bar{p}(k_2\cos(k_1t) + k_1x_2)] + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\frac{\partial}{\partial x_2}[\bar{p}(k_2\cos(k_1t) - k_1x_1)] \end{aligned} \tag{4.9}$$

$$\begin{aligned} &= \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\{-k_1^2k_2\cos(k_1t)x_1 - k_1^2k_2\sin(k_1t)x_2 \\ &\quad + \frac{\partial}{\partial x_1}[{k_1}^2({x_1}^2 + {x_2}^2) + k_1k_2(-\sin(k_1t)x_1 + \cos(k_1t)x_2)](k_2\cos(k_1t) + k_1x_2) \\ &\quad + \frac{\partial}{\partial x_2}[{k_1}^2({x_1}^2 + {x_2}^2) + k_1k_2(-\sin(k_1t)x_1 + \cos(k_1t)x_2)](k_2\sin(k_1t) - k_1x_1)\} \end{aligned} \tag{4.10}$$

$$= \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[-k_1^2k_2\cos(k_1t)x_1 - k_1^2k_2\sin(k_1t)x_2$$

$$+ (2k_1^2 x_1 - k_1 k_2 \sin(k_1 t))(k_2 \cos(k_1 t) + k_1 x_2)$$
$$+ (2k_1^2 x_2 + k_1 k_2 \cos(k_1 t))(k_2 \sin(k_1 t) - k_1 x_1)] = 0. \tag{4.11}$$

Substituting (3.76)–(3.82) into (4.4), the first momentum gives

$$\frac{\partial u_1}{\partial t} + u_1(\frac{\partial u_1}{\partial x_1} + j_{11}) + u_2(\frac{\partial u_1}{\partial x_2} + j_{12}) + \alpha u_1 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_1}\rho^\gamma \tag{4.12}$$
$$=- k_1 k_2 \sin(k_1 t) + u_1(0+0) + (k_2 \sin(k_1 t) - k_1 x_1)(k_1 + k_1) + 2k_1^2 x_1 - k_1 k_2 \sin(k_1 t) \tag{4.13}$$
$$=0, \tag{4.14}$$

the second momentum gives

$$\frac{\partial u_2}{\partial t} + u_1(\frac{\partial u_2}{\partial x_1} + j_{11}) + u_2(\frac{\partial u_2}{\partial x_2} + j_{12}) + \alpha u_2 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_2}\rho^\gamma \tag{4.15}$$
$$=k_1 k_2 \sin(k_1 t) + (k_2 \cos(k_1 t) - k_1 x_2)(-k_1 - k_1) + u_2(0+0) + 2k_1^2 x_2 - k_1 k_2 \cos(k_1 t) \tag{4.16}$$
$$=0. \tag{4.17}$$

Substituting (3.76)–(3.82) into (4.5) gives

$$S_t + u_1 \frac{\partial}{\partial x_1} S + u_2 \frac{\partial}{\partial x_2} S \tag{4.18}$$
$$=\frac{1}{\gamma \bar{p}}(-k_1^2 k_2 \cos(k_1 t) x_1 - k_1^2 k_2 \sin(k_1 t) x_2)$$
$$+ \frac{1}{\gamma \bar{p}}(2k_1^2 x_1 - k_1 k_2 \sin(k_1 t))(k_2 \cos(k_1 t) + k_1 x_2)$$
$$+ \frac{1}{\gamma \bar{p}}(2k_1^2 x_2 + k_1 k_2 \cos(k_1 t))(k_2 \sin(k_1 t) - k_1 x_1) \tag{4.19}$$
$$=\frac{1}{\gamma \bar{p}}(-k_1^2 k_2 \cos(k_1 t) x_1 - k_1^2 k_2 \sin(k_1 t) x_2)$$
$$+ \frac{1}{\gamma \bar{p}}(k_1^2 k_2 \sin(k_1 t) x_2 + k_1^2 k_2 \cos(k_1 t) x_1) = 0. \tag{4.20}$$

For 3-dimensional case: Substituting (3.93)–(3.99) and (4.1) into (4.3) produces

$$\rho_t + \frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3) \tag{4.21}$$
$$=\mu(\bar{p}^{\frac{1}{\gamma}})_t + \frac{\partial}{\partial x_1}(\mu \bar{p}^{\frac{1}{\gamma}} u_1) + \frac{\partial}{\partial x_2}(\mu \bar{p}^{\frac{1}{\gamma}} u_2) + \frac{\partial}{\partial x_3}(\mu \bar{p}^{\frac{1}{\gamma}} u_3) \tag{4.22}$$
$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\bar{p}_t + [k_2 t + k_1(x_2 - x_3)]\frac{\partial}{\partial x_1}\mu \bar{p}^{\frac{1}{\gamma}}$$
$$+ [k_2 t + k_1(x_3 - x_1)]\frac{\partial}{\partial x_2}\mu \bar{p}^{\frac{1}{\gamma}} + [k_2 t + k_1(x_1 - x_2)]\frac{\partial}{\partial x_3}\mu \bar{p}^{\frac{1}{\gamma}} \tag{4.23}$$
$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}} 3k_2^2 t + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2 t + k_1(x_2 - x_3)]\frac{\partial \bar{p}}{\partial x_1}$$
$$+ \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2 t + k_1(x_3 - x_1)]\frac{\partial \bar{p}}{\partial x_2} + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2 t + k_1(x_1 - x_2)]\frac{\partial \bar{p}}{\partial x_3} \tag{4.24}$$

$$
\begin{aligned}
&=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\{3k_2^2t+[k_2t+k_1(x_2-x_3)]\frac{\partial\bar{p}}{\partial x_1}\\
&\quad+[k_2t+k_1(x_3-x_1)]\frac{\partial\bar{p}}{\partial x_2}+[k_2t+k_1(x_1-x_2)]\frac{\partial\bar{p}}{\partial x_3}\} \qquad (4.25)\\
&=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\{3k_2^2t+[k_2t+k_1(x_2-x_3)][2k_2^2(2x_1-x_2-x_3)-k_2]\\
&\quad+[k_2t+k_1(x_3-x_1)][2k_2^2(2x_2-x_1-x_3)-k_2]\\
&\quad+[k_2t+k_1(x_1-x_2)][2k_2^2(2x_3-x_1-x_2)-k_2]\}=0. \qquad (4.26)
\end{aligned}
$$

Substituting (3.93)–(3.99) into (4.4), the first momentum gives

$$
\frac{\partial u_1}{\partial t}+u_1(\frac{\partial u_1}{\partial x_1}+j_{11})+u_2(\frac{\partial u_1}{\partial x_2}+j_{12})+u_3(\frac{\partial u_1}{\partial x_3}+j_{13})+\alpha u_1+\frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_1}\rho^{\gamma} \qquad (4.27)
$$

$$
\begin{aligned}
&=k_2+u_1(0+0)+[k_2t+k_1(x_3-x_1)](k_1+k_1)+[k_2t+k_1(x_1-x_2)](-k_1-k_1)\\
&\quad+2k_1^2(2x_1-x_2-x_3)-k_2=0, \qquad (4.28)
\end{aligned}
$$

the second momentum gives

$$
\frac{\partial u_2}{\partial t}+u_1(\frac{\partial u_2}{\partial x_1}+j_{21})+u_2(\frac{\partial u_2}{\partial x_2}+j_{22})+u_3(\frac{\partial u_2}{\partial x_3}+j_{23})+\alpha u_2+\frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_2}\rho^{\gamma} \qquad (4.29)
$$

$$
\begin{aligned}
&=k_2+[k_2t+k_1(x_2-x_3)](-k_1-k_1)+u_2(0+0)+[k_2t+k_1(x_1-x_2)](k_1+k_1)\\
&\quad+2k_1^2(2x_2-x_1-x_3)-k_2=0, \qquad (4.30)
\end{aligned}
$$

the third momentum gives

$$
\frac{\partial u_3}{\partial t}+u_1(\frac{\partial u_3}{\partial x_1}+j_{31})+u_2(\frac{\partial u_3}{\partial x_2}+j_{32})+u_3(\frac{\partial u_3}{\partial x_3}+j_{33})+\alpha u_3+\frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_3}\rho^{\gamma} \qquad (4.31)
$$

$$
\begin{aligned}
&=k_2+[k_2t+k_1(x_2-x_3)](k_1+k_1)+[k_2t+k_1(x_3-x_1)](-k_1-k_1)+u_3(0+0)\\
&\quad+2k_1^2(2x_3-x_1-x_2)-k_2=0. \qquad (4.32)
\end{aligned}
$$

Substituting (3.93)–(3.99) into (4.5) gives

$$
S_t+u_1\frac{\partial}{\partial x_1}S+u_2\frac{\partial}{\partial x_2}S+u_3\frac{\partial}{\partial x_3}S \qquad (4.33)
$$

$$
\begin{aligned}
&=\frac{1}{\gamma\bar{p}}3k_2^2t+[k_2t+k_1(x_2-x_3)]\frac{\partial}{\partial x_1}\ln\bar{p}\\
&\quad+[k_2t+k_1(x_3-x_1)]\frac{\partial}{\partial x_2}\ln\bar{p}+[k_2t+k_1(x_1-x_2)]\frac{\partial}{\partial x_3}\ln\bar{p} \qquad (4.34)\\
&=\frac{1}{\gamma\bar{p}}\{3k_2^2t+[k_2t+k_1(x_2-x_3)][2k_1^2(2x_1-x_2-x_3)-k_2]\\
&\quad+[k_2t+k_1(x_3-x_1)][2k_1^2(2x_2-x_1-x_3)-k_2]\\
&\quad+[k_2t+k_1(x_1-x_2)][2k_1^2(2x_3-x_1-x_2)-k_2]\}=0. \qquad (4.35)
\end{aligned}
$$

For 4-dimensional case: Substituting (3.111)–(3.117) and (4.1) into (4.3) produces

$$
\rho_t+\frac{\partial}{\partial x_1}(\rho u_1)+\frac{\partial}{\partial x_2}(\rho u_2)+\frac{\partial}{\partial x_3}(\rho u_3)+\frac{\partial}{\partial x_4}(\rho u_4) \qquad (4.36)
$$

$$=\mu(\bar{p}^{\frac{1}{\gamma}})_t + \frac{\partial}{\partial x_1}(\mu\bar{p}^{\frac{1}{\gamma}}u_1) + \frac{\partial}{\partial x_2}(\mu\bar{p}^{\frac{1}{\gamma}}u_2) + \frac{\partial}{\partial x_3}(\mu\bar{p}^{\frac{1}{\gamma}}u_3) + \frac{\partial}{\partial x_4}(\mu\bar{p}^{\frac{1}{\gamma}}u_4) \tag{4.37}$$

$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\bar{p}_t + [k_2t + k_1(-2x_2 + x_3 + x_4)]\frac{\partial}{\partial x_1}\mu\bar{p}^{\frac{1}{\gamma}} + [k_2t + k_1(2x_1 + x_3 - 3x_4)]\frac{\partial}{\partial x_2}\mu\bar{p}^{\frac{1}{\gamma}}$$
$$+ [k_2t + k_1(-x_1 - x_2 + 2x_4)]\frac{\partial}{\partial x_3}\mu\bar{p}^{\frac{1}{\gamma}} + [k_2t + k_1(-x_1 + 3x_2 - 2x_3)]\frac{\partial}{\partial x_4}\mu\bar{p}^{\frac{1}{\gamma}} \tag{4.38}$$

$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}4k_2^2t + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2t + k_1(-2x_2 + x_3 + x_4)]\frac{\partial\bar{p}}{\partial x_1} + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2t + k_1(2x_1 + x_3 - 3x_4)]\frac{\partial\bar{p}}{\partial x_2}$$
$$+ \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2t + k_1(-x_1 - x_2 + 2x_4)]\frac{\partial\bar{p}}{\partial x_3} + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[k_2t + k_1(-x_1 + 3x_2 - 2x_3)]\frac{\partial\bar{p}}{\partial x_4} \tag{4.39}$$

$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\{4k_2^2t + [k_2t + k_1(-2x_2 + x_3 + x_4)]\frac{\partial\bar{p}}{\partial x_1} + [k_2t + k_1(2x_1 + x_3 - 3x_4)]\frac{\partial\bar{p}}{\partial x_2}$$
$$+ [k_2t + k_1(-x_1 - x_2 + 2x_4)]\frac{\partial\bar{p}}{\partial x_3} + [k_2t + k_1(-x_1 + 3x_2 - 2x_3)]\frac{\partial\bar{p}}{\partial x_4}\} \tag{4.40}$$

$$=\frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}\{4k_2^2t + [k_2t + k_1(-2x_2 + x_3 + x_4)][-k_2 + k_1^2(12x_1 - 4x_2 + 8x_3 - 16x_4)]$$
$$+ [k_2t + k_1(2x_1 + x_3 - 3x_4)][-k_2 + k_1^2(28x_2 - 4x_1 - 16x_3 - 8x_4)]$$
$$+ [k_2t + k_1(-x_1 - x_2 + 2x_4)][-k_2 + k_1^2(12x_3 + 8x_1 - 16x_2 - 4x_4)]$$
$$+ [k_2t + k_1(-x_1 + 3x_2 - 2x_3)][-k_2 + k_1^2(28x_4 - 16x_1 - 8x_2 - 4x_3)]\} = 0. \tag{4.41}$$

Substituting (3.111)–(3.117) into (4.4), the first momentum gives

$$\frac{\partial u_1}{\partial t} + u_1(\frac{\partial u_1}{\partial x_1} + j_{11}) + u_2(\frac{\partial u_1}{\partial x_2} + j_{12}) + u_3(\frac{\partial u_1}{\partial x_3} + j_{13}) + u_4(\frac{\partial u_1}{\partial x_4} + j_{14}) + \alpha u_1 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_1}\rho^\gamma \tag{4.42}$$
$$=k_2 + u_1(0 + 0) + [k_2t + k_1(2x_1 + x_3 - 3x_4)](-2k_1 - 2k_1)$$
$$+ [k_2t + k_1(-x_1 - x_2 + 2x_4)](k_1 + k_1) + [k_2t + k_1(-x_1 + 3x_2 - 2x_3)](k_1 + k_1)$$
$$- k_2 + k_1^2(12x_1 - 4x_2 + 8x_3 - 16x_4) = 0, \tag{4.43}$$

the second momentum gives

$$\frac{\partial u_2}{\partial t} + u_1(\frac{\partial u_2}{\partial x_1} + j_{21}) + u_2(\frac{\partial u_2}{\partial x_2} + j_{22}) + u_3(\frac{\partial u_2}{\partial x_3} + j_{23}) + u_4(\frac{\partial u_2}{\partial x_4} + j_{24}) + \alpha u_2 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_2}\rho^\gamma \tag{4.44}$$
$$=k_2 + [k_2t + k_1(-2x_2 + x_3 + x_4)](2k_1 + 2k_1) + u_2(0 + 0)$$
$$+ [k_2t + k_1(-x_1 - x_2 + 2x_4)](k_1 + k_1) + [k_2t + k_1(-x_1 + 3x_2 - 2x_3)](-3k_1 - 3k_1)$$
$$- k_2 + k_1^2(-4x_1 + 28x_2 - 16x_3 - 8x_4) = 0, \tag{4.45}$$

the third momentum gives

$$\frac{\partial u_3}{\partial t} + u_1(\frac{\partial u_3}{\partial x_1} + j_{31}) + u_2(\frac{\partial u_3}{\partial x_2} + j_{32}) + u_3(\frac{\partial u_3}{\partial x_3} + j_{33}) + u_4(\frac{\partial u_3}{\partial x_4} + j_{34}) + \alpha u_3 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_3}\rho^\gamma \tag{4.46}$$
$$=k_2 + [k_2t + k_1(-2x_2 + x_3 + x_4)](-k_1 - k_1) + [k_2t + k_1(2x_1 + x_3 - 3x_4)](-k_1 - k_1)$$
$$+ u_3(0 + 0) + [k_2t + k_1(-x_1 + 3x_2 - 2x_3)](2k_1 + 2k_1)$$
$$- k_2 + k_1^2(8x_1 - 16x_2 + 12x_3 - 4x_4) = 0, \tag{4.47}$$

the fourth momentum gives

$$\frac{\partial u_4}{\partial t} + u_1(\frac{\partial u_4}{\partial x_1} + j_{41}) + u_2(\frac{\partial u_4}{\partial x_2} + j_{42}) + u_3(\frac{\partial u_4}{\partial x_3} + j_{43}) + u_4(\frac{\partial u_4}{\partial x_4} + j_{44}) + \alpha u_4 + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_4}\rho^\gamma \tag{4.48}$$

$$\begin{aligned}=&k_2 + [k_2 t + k_1(-2x_2 + x_3 + x_4)](-k_1 - k_1)\\ &+ [k_2 t + k_1(2x_1 + x_3 - 3x_4)](3k_1 + 3k_1) + [k_2 t + k_1(-x_1 - x_2 + 2x_4)](-2k_1 - 2k_1)\\ &+ u_4(0+0) - k_2 + k_1^2(-16x_1 - 8x_2 - 4x_3 + 28x_4) = 0.\end{aligned} \tag{4.49}$$

Substituting (3.111)–(3.117) into (4.5) gives

$$S_t + u_1\frac{\partial}{\partial x_1}S + u_2\frac{\partial}{\partial x_2}S + u_3\frac{\partial}{\partial x_3}S + u_4\frac{\partial}{\partial x_4}S \tag{4.50}$$

$$=\frac{\bar{p}_t}{\gamma\bar{p}} + u_1\frac{\partial}{\partial x_1}\frac{\ln\bar{p}}{\gamma} + u_2\frac{\partial}{\partial x_2}\frac{\ln\bar{p}}{\gamma} + u_3\frac{\partial}{\partial x_3}\frac{\ln\bar{p}}{\gamma} + u_4\frac{\partial}{\partial x_4}\frac{\ln\bar{p}}{\gamma} \tag{4.51}$$

$$\begin{aligned}=&\frac{1}{\gamma\bar{p}}\{4k_2^2 t + [k_2 t + k_1(-2x_2 + x_3 + x_4)][-k_2 + k_1^2(12x_1 - 4x_2 + 8x_3 - 16x_4)]\\ &+ [k_2 t + k_1(2x_1 + x_3 - 3x_4)][-k_2 + k_1^2(28x_2 - 4x_1 - 16x_3 - 8x_4)]\\ &+ [k_2 t + k_1(-x_1 - x_2 + 2x_4)][-k_2 + k_1^2(12x_3 + 8x_1 - 16x_2 - 4x_4)]\\ &+ [k_2 t + k_1(-x_1 + 3x_2 - 2x_3)][-k_2 + k_1^2(28x_4 - 16x_1 - 8x_2 - 4x_3)]\} = 0.\end{aligned} \tag{4.52}$$

Example 2

Since N-dimensional case covers 2 to 4-dimensional cases, here gives the verification of N-dimensional case. Substituting solutions into Euler equations, as S is a constant, (4.5) is guaranteed. Since ρ is also a constant, by

$$\rho_t + \sum_{k=1}^{N}\frac{\partial}{\partial x_k}\rho u_k \tag{4.53}$$

$$=0 + \rho\sum_{k=1}^{N}\frac{\partial}{\partial x_k}u_k \tag{4.54}$$

$$=\rho e^{-\alpha t}\sum_{k=1}^{N}\frac{\partial}{\partial x_k}[k_1(\sum_{g=k+1}^{N}x_g - \sum_{g=1}^{k-1}x_g) + k_2] = 0, \tag{4.55}$$

Eq (4.3) is verified.

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial x_k}e^{-\alpha t}[k_1(\sum_{k=i+1}^{N}x_k - \sum_{k=1}^{i-1}x_k) + k_2] \tag{4.56}$$

$$= \begin{Bmatrix} -k_1 e^{-\alpha t}, & \text{for } k < i \\ 0, & \text{for } k = i \\ k_1 e^{-\alpha t}, & \text{for } k > i \end{Bmatrix} = -j_{ik}, \tag{4.57}$$

therefore,

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{N}u_k(\frac{\partial u_i}{\partial x_k} + j_{ik}) + \alpha u_i + \frac{\gamma+1}{\gamma}\frac{\partial}{\partial x_i}\rho^\gamma \tag{4.58}$$

$$= -\alpha u_i + 0 + \alpha u_i + 0 = 0, \tag{4.59}$$

the n-th momentum Eq (4.4) is satisfied.

Example 3

Substituting (3.159)–(3.161) into (4.3) produces

$$\rho_t + \frac{\partial}{\partial x_1}\rho u_1 + \frac{\partial}{\partial x_2}\rho u_2 \tag{4.60}$$

$$=0 + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}} t^{k_1}(1 + x_2)k_2(x_1 - 1) + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}} t^{k_1}(1 - x_1)k_2(x_2 + 1) = 0. \tag{4.61}$$

Substituting (3.159)–(3.161) into (4.4) gives

$$\frac{\partial u_i}{\partial t} + u_1(\frac{\partial u_i}{\partial x_k} + j_{i1}) + u_2(\frac{\partial u_i}{\partial x_k} + j_{i2}) + \alpha u_i + \frac{\gamma + 1}{\gamma}\frac{\partial}{\partial x_i}\rho^{\gamma} \tag{4.62}$$

$$=u_1(\frac{\partial u_i}{\partial x_k} + j_{i1}) + u_2(\frac{\partial u_i}{\partial x_k} + j_{i2}) + 0 \tag{4.63}$$

$$=k_1 t^{k_1-1}\begin{bmatrix} 1 + x_1 \\ 1 - x_2 \end{bmatrix} + t^{k_1}(1 + x_2)(t^{k_1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ t^{k_1} - \frac{k_2}{t^{k_1}} \end{bmatrix}) +$$

$$t^{k_1}(1 - x_1)(t^{k_1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -t^{k_1} + \frac{k_2}{t^{k_1}} \\ 0 \end{bmatrix}) - \frac{k_1}{t}t^{k_1}\begin{bmatrix} 1 + x_1 \\ 1 - x_2 \end{bmatrix} + \begin{bmatrix} k_2(x_1 - 1) \\ k_2(x_2 + 1) \end{bmatrix} = 0. \tag{4.64}$$

Substituting (3.159)–(3.161) into (4.5) gives

$$S_t + u_1\frac{\partial}{\partial x_1}S + u_2\frac{\partial}{\partial x_2}S \tag{4.65}$$

$$=0 + t_{k_1}(1 + x_2)\frac{k_2(2x_1 - 1)}{\gamma\bar{p}} + t_{k_1}(1 - x_1)\frac{k_2(2x_2 + 1)}{\gamma\bar{p}} = 0. \tag{4.66}$$

Example 4

Substituting (3.180)–(3.182) into (4.3) produces

$$\rho_t + \frac{\partial}{\partial x_1}\rho u_1 + \frac{\partial}{\partial x_2}\rho u_2 + \frac{\partial}{\partial x_3}\rho u_3 \tag{4.67}$$

$$=0 + \frac{\mu}{\gamma}\bar{p}^{\frac{1-\gamma}{\gamma}}[t^{k_1}(x_2 + x_3 + 1)k_2(2x_1 + x_2 - x_3 - 2) + t^{k_1}(x_3 - x_1 + 1)k_2(2x_2 + x_1 + x_3)$$

$$+ t^{k_1}(-x_1 - x_2 + 1)k_2(2x_3 - x_1 + x_2 + 2)] = 0. \tag{4.68}$$

Substituting (3.180)–(3.182) into (4.4), since

$$u_t = k_1 t^{k_1-1} = \frac{k_1}{t}t^{k_1} = -\alpha(t)u, \tag{4.69}$$

we have

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k(\frac{\partial u_i}{\partial x_k} + j_{ik}) + \alpha u_i + \frac{\gamma + 1}{\gamma}\frac{\partial}{\partial x_i}\rho^{\gamma} \tag{4.70}$$

$$=u_1(\frac{\partial u_i}{\partial x_k} + j_{i1}) + u_2(\frac{\partial u_i}{\partial x_k} + j_{i2}) + u_3(\frac{\partial u_i}{\partial x_k} + j_{i3}) + 0 \tag{4.71}$$

$$=t^{k_1}(x_2 + x_3 + 1)(t^{k_1}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ t^{k_1} - \frac{k_2}{t^{k_1}} \\ t^{k_1} - \frac{k_2}{t^{k_1}} \end{bmatrix}) + t^{k_1}(x_3 - x_1 + 1)(t^{k_1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -t^{k_1} + \frac{k_2}{t^{k_1}} \\ 0 \\ t^{k_1} - \frac{k_2}{t^{k_1}} \end{bmatrix})+$$
$$t^{k_1}(-x_1 - x_2 + 1)(t^{k_1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -t^{k_1} + \frac{k_2}{t^{k_1}} \\ -t^{k_1} + \frac{k_2}{t^{k_1}} \\ 0 \end{bmatrix} + \begin{bmatrix} k_2(2x_1 + x_2 - x_3 - 2) \\ k_2(2x_2 + x_1 + x_3) \\ k_2(2x_3 - x_1 + x_2 - 2) \end{bmatrix}) = 0. \tag{4.72}$$

Substituting (3.180)–(3.182) into (4.5) gives

$$S_t + u_1\frac{\partial}{\partial x_1}S + u_2\frac{\partial}{\partial x_2}S + u_3\frac{\partial}{\partial x_3}S \tag{4.73}$$

$$=0 + \frac{1}{\gamma\bar{p}}[t^{k_1}(x_2 + x_3 + 1)k_2(2x_1 + x_2 - x_3 - 2) + t^{k_1}(x_3 - x_1 + 1)k_2(2x_2 + x_1 + x_3)$$
$$+ t^{k_1}(-x_1 - x_2 + 1)k_2(2x_3 - x_1 + x_2 + 2)] = 0. \tag{4.74}$$