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Research article

New interpretation of topological degree method of Hilfer fractional neutral functional integro-differential equation with nonlocal condition

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Abstract: This manuscript deals with the concept of Hilfer fractional neutral functional integro-differential equation with a nonlocal condition. The solution representation of a given system is obtained from the strongly continuous operator, linear operator and bounded operator, as well as the Wright type of function. The sufficient and necessary conditions for the existence of a solution are attained using the topological degree method. The uniqueness of the solution is attained by Gronwall's inequality. Finally, we employed some specific numerical computations to examine the effectiveness of the results.

Keywords: Hilfer fractional derivative; nonlocal condition; Kuratowski measure of non-compactness; existence and uniqueness; Gronwall's inequality
Mathematics Subject Classification: 34A08, 34A12, 46B80

1. Introduction

Fractional differential equations are a study of the concept of arbitrary integrals and derivatives of non-integer order. The idea of fractional calculus was first discussed by Gottfried Wilhelm Leibniz in 1695. In real-world applications, fractional calculus excels more than classical calculus. Moreover, it has attracted enormous attention from the scientific communities studying biology, physics, economics, mechanics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data and chemistry. Further, readers can see the following research articles for fractional calculus and its applications: [1–5].

The topological degree method is used to determine whether an equation has a solution or not in a simpler manner than the traditional methods. We need stronger conditions when we deal with problems like differential equations using traditional methods. However, such strong conditions are not required in the topological degree method. This topological degree method is advantageous for finding the existence of solutions using weaker conditions instead of stronger ones. This manuscript examines nonlocal conditions, which have the advantage of producing better results in real-world applications compared to classical local conditions. Nonlocal conditions can be more useful for expressing some physical phenomena than local conditions. The Hilfer fractional derivative was introduced by Hilfer and is a generalisation of the Riemann-Liouville and Caputo fractional derivatives. This generalised fractional derivative of the orders $\vartheta \in [0,1]$ and $\varpi \in (0,1)$ can be reduced to the Riemann-Liouville fractional derivative at $\varpi = 0$ and the Caputo fractional derivative at $\varpi = 1$, respectively. Hilfer proposed the solution of generalised differential equations of fractional order. Compared to the Caputo fractional derivative, the Hilfer fractional derivative parameters allow more degrees of freedom and a variety of stationary states with connections to local as well as nonlocal conditions.

In recent years, many authors have been involved in the field of Hilfer fractional derivatives (see, [6, 7]). In [8, 9], the authors investigated impulsive fractional differential equations and multi-point boundary value problems via the topological degree method. Recently [10], the authors studied the existence of weak solutions for a semilinear fractional elliptic system with Dirichlet boundary conditions by using the Leray-Schauder degree method. In a continuation, the authors introduced the topological degree method for the Caputo-Hadamard type derivatives [11]. The authors extended the topological degree method for existence and uniqueness solutions to ψ-Caputo fractional differential equations in [12]. In the research articles [13, 14], the authors established the topological degree method for the Caputo fractional differential equation, which was also extended to impulsive differential equations and fractional difference equations. The proposed method is to study the existence and uniqueness of the solutions of the Hilfer fractional neutral functional integro-differential equation with nonlocal conditions. Our research is motivated by the fact that, to the best of our knowledge, no previous studies have investigated this topic. As a result, we will demonstrate this concept and consider the following form of Hilfer fractional neutral functional integro-differential equation with a nonlocal condition:

$$\begin{aligned} {}^{H}D^{\vartheta,\varpi}(\Theta(t) - \mathfrak{M}(t,\Theta_t)) &= \mathcal{H}\Theta(t) + \mathcal{P}u(t) + \xi(t,\Theta_t,\int_0^t \chi(t,s,\Theta_s)ds), t \in \mathcal{I} := [0,T], \\ I_{0^+}^{1-\eta}\Theta(t) &= \delta(t) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho),\ t \in (-r,0], \end{aligned} \tag{1.1}$$

where Ξ is a Banach space, ${}^{H}D^{\vartheta,\varpi}$ denotes the Hilfer fractional derivative of order $0 \le \vartheta \le 1$ and $0 < \varpi < 1$, and $I_{0^+}^{1-\eta}$ are generalised fractional derivatives of order $1-\eta = (1-\vartheta)(1-\varpi)$. $\mathcal{H}$ is a closed, linear, and bounded operator in Ξ. The control function $u(t)$ takes values in $L^2(\mathcal{I},\Xi)$. The bounded linear operator $\mathcal{P}$ is a function from Ξ to Ξ. The initial value function $\mathcal{D} := \{\delta : (-r,0] \to \Xi$ is continuous, where $r > 0\}$ and neutral function $\mathfrak{M} : \mathcal{I} \times \mathcal{D} \to \Xi$ are also continuous. The functions $\xi : \mathcal{I} \times \mathcal{D} \times \Xi \to \Xi$ and $\chi : \mathcal{I} \times \Xi \times \mathcal{D} \to \Xi$ are both continuous with respect to t on $\mathcal{I}$. $\sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho)$

is a nonlocal condition, and C_ρ is constant. Let $\Theta_t \in (-r, 0]$ be a continuous function defined by $\Theta_t(s) = \Theta(t+s)$ for $-r \le s < 0$.

The following are the significant features of our suggested work:

- The strongly continuous operator, linear operator, bounded operator and the Wright type function are used to obtain the solution representation of our system.
- Functional differential equations are used to examine the past or the aftereffects of an event.
- The Kuratowski measure of noncompactness is considered in this manuscript.
- The novelty of this manuscript is in finding existence and uniqueness by manipulating the weaker conditions instead of the stronger conditions.
- The originality of this manuscript was that it used the topological degree method for functional fractional differential equation and neutral term in the given system to also discover the existence and uniqueness of a solution by employing specific assumptions. These assumptions have never been manipulated before in any research article. These assumptions will yield a more easy way to get a solution than prior studies [15–19].

The manuscript is organised into five sections. In Section 2, we introduce some preliminary definitions, propositions, and lemmas that can be used to prove the proposed work. Additionally, it provides the solution representation of the Hilfer fractional neutral functional integro-differential equation with a nonlocal condition. In Section 3, we establish the necessary and sufficient conditions for the existence and uniqueness of the Hilfer fractional neutral functional integro-differential equation with a nonlocal condition. In Section 4, we provide two numerical examples. At the end of this manuscript, we discuss the conclusion.

2. Preliminaries and solution representation

In order to attain the objective, we give some propositions and lemmas. Throughout this paper, we use the following notations: $C(\mathcal{I}, \Xi)$ is a continuous function from $\mathcal{I} = [0, T] \to \Xi$. Define the operator $\mathcal{Y} = \{\Theta : t^{(1-\vartheta)(1-\varpi)}\Theta(t) \in C(\mathcal{I}, \Xi)\}$, with the norm $\|.\|_{\mathcal{Y}} = \sup_{t\in\mathcal{I}} |t^{(1-\vartheta)(1-\varpi)}\Theta(t)|$. Obviously, $\mathcal{Y}$ is a Banach space.

Definition 2.1. *[20] The Hilfer fractional derivative is the generalised Riemann-Liouville fractional derivative of order* $0 \le \vartheta \le 1$ *and* $0 < \varpi < 1$*, with lower limit "a" defined as*

$$D_{a+}^{\vartheta,\varpi} f(t) = I_{a+}^{\vartheta(1-\varpi)} \frac{d}{dt} I_{a+}^{(1-\vartheta)(1-\varpi)} f(t).$$

Definition 2.2. *[21] A function* Θ *is said to be a solution of the nonlocal problem if* Θ *satisfies the Eq (1.1) and the condition* $I_{0^+}^{1-\eta}\Theta(t) = \delta(t) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho)$.

Note 2.3. *[21] The function* $\omega : \mathcal{Q} \to \mathbb{R}_+$ *(where* $\mathbb{R}_+$ *is the set of positive real numbers) is defined by*

$$\omega(\mathfrak{B}) = inf\{\epsilon > 0 : \mathfrak{B} \subset \bigcup_{j=1}^{m} M_j \text{ and } diam(M_j) \le \epsilon\},$$

where $diam(M_j)$ *is defined by* $diam(M_j) = sup\{|x_1 - x_2| : x_1, x_2 \in M_j,\ j = 1, 2, 3, .., m\}$, *and* $\mathfrak{B} \in \mathcal{Q}$ *(*$\mathcal{Q}$ *is a family of all bounded sets) is called the Kuratowski measure of non compactness. The notation of* ω *in the above definition will be considered in the following theorems, lemmas and propositions of this manuscript.*

Definition 2.4. *[21] A map* $\mathfrak{F}^* : \psi \subset \Xi \to \Xi$ *is said to be* ω*-Lipschitz if there exists* $\kappa \geq 0$ *such that* $\omega(\mathfrak{F}^*(\mathfrak{B})) \leq \kappa\omega(\mathfrak{B})$, $\forall \mathfrak{B} \subset \psi$.

Definition 2.5. *[21] A map* $\mathfrak{F}^* : \psi \subset \Xi \to \Xi$ *is said to be* ω*- condensing if* $\omega(\mathfrak{F}^*(\mathfrak{B})) < \omega(\mathfrak{B})$, $\forall\ \mathfrak{B} \subset \psi$ *with* $\omega(\mathfrak{B}) > 0$.

Proposition 2.1. *[21] If* $\mathfrak{F}^*, \mathfrak{Y}^* : \psi \to \Xi$ *are* ω*-Lipschitz mappings with constants* κ, κ'*, respectively, then* $\mathfrak{F}^* + \mathfrak{Y}^* : \psi \to \Xi$ *are* ω*-Lipschitz with constants* $\kappa + \kappa'$.

Proposition 2.2. *[21] If* $\mathfrak{F}^* : \psi \to \Xi$ *is compact, then* $\mathfrak{F}^*$ *is* ω*-Lipschitz with constant value equal to zero.*

Proposition 2.3. *[21] If* $\mathfrak{F}^* : \psi \to \Xi$ *is Lipschitz with constant* κ*, then* $\mathfrak{F}^*$ *is* ω*-Lipschitz with the same constant* κ.

Proposition 2.4. *[21] Let* $\tau = \{(I - \mathfrak{F}^*, \psi, \Theta) : \psi \subset \Xi$ *open and bounded,* $\mathfrak{F}^* \in C_\omega(\psi)$, $\Theta \in \Xi \backslash (I - \mathfrak{F}^*)(\partial\psi)\}$*, and then there exists one degree function* $\mathfrak{D} : \omega \to \mathbb{N}_0$ *which satisfies the following properties:*

- $\mathfrak{D}(I, \psi, \Theta) = 1\ \forall\ \Theta \in \psi$ *(Normalization).*
- *For every disjoint, open sets* $\psi_1, \psi_2 \subset \psi$ *and every* Θ *does not belong to* $(I - \mathfrak{F}^*)(\overline{\psi}\backslash(\psi_1 \cup \psi_2))$ *we have* $\mathfrak{D}(I, \psi, \Theta) = \mathfrak{D}(I, \psi_1, \Theta) + \mathfrak{D}(I, \psi_2, \Theta)$ *(Additive on domain).*
- $(I - \mathcal{H}(t, .), \psi, \Theta(t))$ *is independent of* $t \in [0, 1]$ *for every continuous, bounded mapping* $\mathcal{H} : [0, 1] \times \overline{\psi} \to \Xi$ *which satisfies* $\mathfrak{F}^*(\mathcal{H}([0, 1] \times \mathfrak{B})) < \mathfrak{F}^*(\mathfrak{B})$, $\forall \mathfrak{B} \subset \overline{\psi}$ *with* $\mathfrak{F}^*(\mathfrak{B}) > 0$*, and every continuous function* $\Theta : [0, 1] \to \Lambda$ *which satisfies* $\Theta(t) \neq \Lambda - \mathcal{H}(t, \Lambda), \forall t \in [0, 1], \forall \Lambda \in \partial\psi$. *(Invariance under homotopy).*
- $\mathfrak{D}(I - \mathfrak{F}^*, \psi, \Theta) \neq 0 \Rightarrow \Theta \in (I - \mathfrak{F}^*)(\psi)$ *(Existence).*
- $\mathfrak{D}(I - \mathfrak{F}^*, \psi, \Theta) = \mathfrak{D}(I - \mathfrak{F}^*, \psi_1, \Theta)$ *for every open set* $\psi_1 \subset \psi$*, and every* Θ *does not belong to* $(I - \mathfrak{F}^*)(\overline{\psi}\backslash\psi_1)$ *(Excision).*

Lemma 2.6. *[21] Let* $\mathfrak{F}^* : \Xi \to \Xi$ *be* ω*-condensing, and*

$$\mathcal{W} = \{\Theta \in \Xi : \exists\Omega \in [0, 1] \text{ such that } \Theta = \Omega\mathfrak{F}^*\Theta\}.$$

If $\mathcal{W}$ *is a bounded set in* Ξ*, so there exists* $\mathfrak{E} > 0$ *such that* $\mathcal{W} \subset \mathfrak{B}_{\mathfrak{E}}(0)$*, and then*

$$deg(I - \Omega\mathfrak{F}^*, \mathfrak{B}_{\mathfrak{E}}(0), 0) = 1,\ \forall\ \Omega \in [0, 1].$$

Then, $\mathfrak{F}^*$ *has at least one fixed point.*

Lemma 2.7. (Gronwall's inequality) *[22] Assume that* $\ell(t)$ *and* $\wp(t)$ *are non negative continuous functions for* $t \geq t_0$*. Let* $\mathcal{K} > 0$ *be a constant. Then, the inequality*

$$\ell(t) \leq \mathcal{K} + \int_{t_0}^{t} \wp(s)\ell(s)ds,$$

implies the inequality

$$\ell(t) \leq \mathcal{K}\, exp\left(\int_{t_0}^{t} \wp(s)ds\right).$$

Note 2.8. *[22] If $\ell(t) \leq \mathcal{K} \int_{t_0}^{t} \ell(s)ds$ where ℓ and $\mathcal{K}$ are as above in Lemma 2.7, then $\ell(t) = 0$ for $t \geq t_0$.*

Lemma 2.9. *[23] The operators $S_{\vartheta,\varpi}(t)$ and $\Psi_{\varpi}(t)$ have the following properties:*

- *$\Psi_{\varpi}(t)$ is continuous in the uniform operator topology.*
- *For any fixed $t > 0$, $S_{\vartheta,\varpi}(t)$ and $\Psi_{\varpi}(t)$ are linear, strongly continuous, and bounded operators,*

$$\|\Psi_{\varpi}(t)\| \leq \frac{\mathcal{V}t^{\varpi-1}}{\Gamma(\varpi)} \text{ and } \|S_{\vartheta,\varpi}(t)\| \leq \frac{\mathcal{V}t^{(\vartheta-1)(\varpi-1)}}{\Gamma(\vartheta(1-\varpi)+\varpi)}. \tag{2.1}$$

Lemma 2.10. *Let $0 \leq \vartheta \leq 1$, $0 < \varpi < 1$ and $\rho = 1, 2, .., \mathcal{M}$. Then, the Eq (1.1) can be equivalent in the form of*

$$\Theta(t) = \begin{cases} \delta(t) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho) & t \in (-r, 0]; \\ S_{\vartheta,\varpi}(t)(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho) - \mathfrak{M}(0, \Theta_0) + \mathfrak{M}(t, \Theta_t)) & \\ \quad + \int_0^t \mathcal{H}\Psi_{\varpi}(t-s)\mathfrak{M}(s, \Theta_s)ds & \\ \quad + \int_0^t \Psi_{\varpi}(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s, \int_0^s \kappa(s, m, \Theta_m)dm))ds & t \in [0, T]; \end{cases}$$

where $S_{\vartheta,\varpi} = I_{0^+}^{\vartheta(1-\varpi)}\Psi_{\varpi}(t)$, $\Psi_{\varpi}(t) = t^{\varpi-1}\mathcal{T}_{\varpi}(t)$, and $\mathcal{T}_{\varpi}(t) = \int_0^\infty \varpi\sigma\wp_{\varpi}(\sigma)S(t^{\varpi}\sigma)d\sigma$.

$$\wp_{\varpi}(\sigma) = \sum_{n=1}^{\infty} \frac{(-\sigma)^{n-1}}{(n-1)!\Gamma(1-n\varpi)}, \sigma \in (0, \infty),$$

where $\wp_{\varpi}(\sigma)$ is a function of Wright type which satisfies $\int_0^\infty \sigma^{\delta}\wp_{\varpi}(\sigma)d\sigma = \frac{\Gamma(1+\delta)}{\Gamma(1+\varpi\delta)}, \sigma \geq 0$.

3. Main results

This section's vital focus is to explore the existence and uniqueness of solutions of Eq (1.1). Our approach is primarily based on the following assumptions:

$\mathcal{R}(1)$. For arbitrary $\Theta^*, \Theta^{**} \in C(\mathcal{I}, \Xi)$, where $\mathcal{I} =: [0, T]$, there exist constants $C_\xi, \mathcal{B}_\xi^\rho \in (0, 1)$ such that

(i) $|\mathfrak{M}(t, \Theta_t^*) - \mathfrak{M}(t, \Theta_t^{**})| \leq C_\xi |\Theta_t^* - \Theta_t^{**}|$.

(ii) $\sum_{\rho=1}^{\mathcal{M}} |C_\rho \Theta^*(t_\rho) - C_\rho \Theta^{**}(t_\rho)| \leq \mathcal{B}_\xi^\rho |\Theta^*(t_\rho) - \Theta^{**}(t_\rho)|$, where, $\sum_{\rho=1}^{\mathcal{M}} |C_\rho| \leq \mathcal{B}_\xi^\rho$.

$\mathcal{R}(2)$. The functions $\xi : \mathcal{I} \times \mathcal{D} \times \Xi \to \Xi$ and $\kappa : \mathcal{I} \times \Xi \times \mathcal{D} \to \Xi$ are both continuous with respect to t on $\mathcal{I}$ and there exists a constants $\mathcal{B}_1 > 0, \mathcal{B}_2 > 0, \mathcal{L}_\xi > 0$ such that

(i) $|\xi(t, \Theta_t^*, \mathcal{Z}_t^*) - \xi(t, \Theta_t^{**}, \mathcal{Z}_t^{**})| \leq \mathcal{B}_1 |\Theta_t^* - \Theta_t^{**}| + \mathcal{B}_2 |\mathcal{Z}_t^* - \mathcal{Z}_t^{**}|$,

(ii) $|\mathcal{Z}_t^* - \mathcal{Z}_t^{**}| \leq \mathcal{L}_\xi |\Theta_t^* - \Theta_t^{**}|$,

where $\mathcal{Z}_t^* = \int_0^t \chi(t, s, \Theta_s^*)ds$, and $\mathcal{Z}_t^{**} = \int_0^t \chi(t, s, \Theta_s^{**})ds$.

Theorem 3.1. *The existence of a solution for the Hilfer fractional neutral functional integro-differential equation with nonlocal condition (1.1) is equivalent to the existence of a fixed point operator $\mathfrak{R}$.*

Proof. First, we have to define the operator $\mathfrak{U} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ in the following form

$$\mathfrak{U}(\Theta(t)) = S_{\vartheta,\varpi}(t)(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho) - \mathfrak{M}(0, \Theta_0) + \mathfrak{M}(t, \Theta_t)). \tag{3.1}$$

Next, we define the operator $\mathfrak{P} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ by

$$\begin{aligned} \mathfrak{P}(\Theta(t)) &= \int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s)ds \\ &\quad + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s, \int_0^s \chi(s, m, \Theta_m)dm))ds, \ \forall\, t \in [0, T]. \end{aligned}$$

Finally, the operator $\mathfrak{R} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ is defined for every $t \in [0, T]$ and is given by

$$\begin{aligned} \mathfrak{R}(\Theta(t)) &= \mathfrak{U}(\Theta(t)) + \mathfrak{P}(\Theta(t)) \\ &= S_{\vartheta,\varpi}(t)(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta(t_\rho) - \mathfrak{M}(0, \Theta_0) + \mathfrak{M}(t, \Theta_t)) \\ &\quad + \int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s)ds \\ &\quad + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s, \int_0^s \chi(s, m, \Theta_m)dm))ds \\ &= \Theta(t). \end{aligned}$$

Thus, the existence of a solution to the Hilfer fractional neutral functional integro-differential equation with nonlocal condition (1.1) is equivalent to the existence of a fixed point operator $\mathfrak{R}$. □

Theorem 3.2. *The operator $\mathfrak{U} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ is Lipschitz with constant $\mathcal{G}_\xi^\rho$, where $\mathcal{G}_\xi^\rho = \frac{\mathcal{V}(C_\xi + \mathcal{B}_\xi^\rho)}{\Gamma(\vartheta(1-\varpi)+\varpi)}$. Consequently, $\mathfrak{U}$ is ω-Lipschitz with same constant $\mathcal{G}_\xi^\rho$.*

Proof. We have to show that the operator $\mathfrak{U}$ is ω-Lipschitz with the same constant $\mathcal{G}_\xi^\rho$ for every $t \in [0, T]$ by using the assumption $\mathcal{R}(1)$,

$$\begin{aligned} \mathfrak{U}(\Theta^*(t)) &= S_{\vartheta,\varpi}(t)(\delta(0) + \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^*(t_\rho) - \mathfrak{M}(0, \Theta_0) + \mathfrak{M}(t, \Theta_t^*)). \\ \mathfrak{U}(\Theta^{**}(t)) &= S_{\vartheta,\varpi}(t)(\delta(0) + \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^{**}(t_\rho) - \mathfrak{M}(0, \Theta_0) + \mathfrak{M}(t, \Theta_t^{**})). \\ \|\mathfrak{U}(\Theta^*(t)) - \mathfrak{U}(\Theta^{**}(t))\|_{\mathcal{Y}} &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)} \Bigg\{ \Bigg| S_{\vartheta,\varpi}(t)(\delta(0) + \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^*(t_\rho) - \mathfrak{M}(0, \Theta_0) \\ &\quad + \mathfrak{M}(t, \Theta_t^*)) - S_{\vartheta,\varpi}(t)(\delta(0) + \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^{**}(t_\rho) - \mathfrak{M}(0, \Theta_0) \end{aligned}$$

$$
\begin{aligned}
& +\mathfrak{M}(t, \Theta_t^{**}))\Big|\Big\} \\
\leq\ & \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{|S_{\vartheta,\varpi}(t)| \times \left|\sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^*(t_\rho) - \sum_{\rho=1}^{\mathcal{M}} C_\rho \Theta^{**}(t_\rho)\right| \\
& +|S_{\vartheta,\varpi}(t)| \times |\mathfrak{M}(t, \Theta_t^*) + \mathfrak{M}(t, \Theta_t^{**})|\Big\} \\
\leq\ & \frac{\mathcal{V}(C_\xi + \mathcal{B}_\xi^\rho)}{\Gamma(\vartheta(1-\varpi)+\varpi)}|\Theta_t^* - \Theta_t^{**}| \\
\leq\ & \mathcal{G}_\xi^\rho|\Theta_t^* - \Theta_t^{**}|,\ \rho = 1, 2, ..., \mathcal{M}.
\end{aligned}
$$

Therefore, $\mathfrak{U}$ is Lipschitz with constant $\mathcal{G}_\xi^\rho$ and by using proposition (2.3), we obtained $\mathfrak{U}$ is ω-Lipschitz with the same constant $\mathcal{G}_\xi^\rho$. □

Theorem 3.3. *The operator $\mathfrak{P} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ is continuous. Moreover, $\mathfrak{P}$ satisfies the following condition,*

$$
\|\mathfrak{P}(\Theta(t))\|_{\mathcal{Y}} \leq \mu_\xi,\ \forall\ t \in \mathcal{I} \text{ and } \Theta \in C(\mathcal{I}, \Xi). \tag{3.2}
$$

For brevity let us take

$$
\begin{aligned}
\mathcal{Z}_t &= \int_0^t \chi(t, s, \Theta_s)ds, \\
\mathcal{Z}_t^\nu &= \int_0^t \chi(t, s, \Theta_s^\nu)ds, \\
\mu_\xi &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)} \left\{\left|\int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s)ds \right.\right. \\
&\quad \left.\left. + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s, \int_0^s \chi(s, m, \Theta_m)dm))ds\right|\right\}.
\end{aligned}
$$

Proof. From Eq (3.2) the operator $\mathfrak{P}$ is uniformly bounded. Next, we have to show that, the operator $\mathfrak{P}$ is continuous for every $t \in [0, T]$

$$
\begin{aligned}
\mathfrak{P}(\Theta^\nu)(t) &= \int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s^\nu)ds \\
&\quad + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s^\nu, \mathcal{Z}_s^\nu)ds. \\
\mathfrak{P}(\Theta)(t) &= \int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s)ds \\
&\quad + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s, \mathcal{Z}_s))ds.
\end{aligned}
$$

$$
\begin{aligned}
\text{Consider, } \|\mathfrak{P}(\Theta^\nu)(t) - \mathfrak{P}(\Theta)(t)\|_{\mathcal{Y}} &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)} \left\{\left|\int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s, \Theta_s^\nu)ds \right.\right. \\
&\quad + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s, \Theta_s^\nu, \mathcal{Z}_s^\nu))ds
\end{aligned}
$$

$$
\begin{aligned}
& -\int_0^t \mathcal{H}\Psi_{\varpi}(t-s)\mathfrak{M}(s,\Theta_s)ds \\
& -\int_0^t \Psi_{\varpi}(t-s)(\mathcal{P}u(s)+\xi(s,\Theta_s,\mathcal{Z}_s))ds\Bigg|\Bigg\} \\
\leq\ & \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{(\mathcal{B}_1+\mathcal{B}_2\mathcal{L}_{\xi})\int_0^t |\Psi_{\varpi}(t-s)ds| \\
& \times|\Theta^{\nu}(s)-\Theta(s)| + C_{\xi}\int_0^t |\Psi_{\varpi}(t-s)ds||\mathcal{H}| \\
& \times|\Theta^{\nu}(s)-\Theta(s)|\Big\}.
\end{aligned}
$$

Since as $\nu \to \infty$, $\Theta^{\nu}(s) \to \Theta(s)$, we obtained the operator $\|\mathfrak{P}(\Theta^{\nu})(t) - \mathfrak{P}(\Theta(t))\|_{\mathcal{Y}} \to 0$ as $\nu \to \infty$ for every $t \in [0, T]$. From this we can say $\mathfrak{P}(\Theta(t))$ is continuous. On the other hand, it is easy to see that $\xi(s, \Theta^{\nu}_{s}, \mathcal{Z}^{\nu}_{s}) \to \xi(s, \Theta_s, \mathcal{Z}_s)$ as $\nu \to \infty$ due to the continuity of ξ. □

Theorem 3.4. *The operator $\mathfrak{P} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ is compact. Consequently, $\mathfrak{P}$ is ω–Lipschitz with zero constant.*

Proof. Let us consider the two arbitrary elements $\tau_1, \tau_2 \in [0, T]$, and the relation between τ_1, τ_2 is $\tau_1 < \tau_2$. Then, we have to show that, the $\mathfrak{P} : C(\mathcal{I}, \Xi) \to C(\mathcal{I}, \Xi)$ is equicontinuous on every $t \in \mathcal{I}$,

$$
\begin{aligned}
\|\mathfrak{P}(\Theta(\tau_2)) - \mathfrak{P}(\Theta(\tau_1))\|_{\mathcal{Y}} =\ & \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Bigg\{\Bigg|\int_0^{\tau_2} \mathcal{H}\Psi_{\varpi}(\tau_2-s)\mathfrak{M}(s,\Theta_s)ds \\
& + \int_0^{\tau_2} \Psi_{\varpi}(\tau_2-s)(\mathcal{P}u(s)+\xi(s,\Theta_s,\mathcal{Z}_s))ds \\
& - \int_0^{\tau_1} \mathcal{H}\Psi_{\varpi}(\tau_1-s)\mathfrak{M}(s,\Theta_s)ds \\
& - \int_0^{\tau_1} \Psi_{\varpi}(\tau_1-s)(\mathcal{P}u(s)+\xi(s,\Theta_s,\mathcal{Z}_s))ds\Bigg|\Bigg\} \\
\leq\ & \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{\Big(\int_0^{\tau_2} |\mathcal{H}\Psi_{\varpi}(\tau_2-s)|ds - \int_0^{\tau_1} |\mathcal{H}\Psi_{\varpi}(\tau_1-s)|ds\Big) \\
& \times|\mathfrak{M}(s,\Theta_s)| + \Big(\int_0^{\tau_2} |\Psi_{\varpi}(\tau_2-s)|ds - \int_0^{\tau_1} |\Psi_{\varpi}(\tau_1-s)|ds\Big) \\
& \times|(\mathcal{P}u(s)+\xi(s,\Theta_s,\mathcal{Z}_s)|ds\Big\}. \qquad (3.3)
\end{aligned}
$$

As $\tau_2 \to \tau_1$, we have $\|\mathfrak{P}(\Theta(\tau_2)) - \mathfrak{P}(\Theta(\tau_1))\|_{\mathcal{Y}} \to 0$, and from this we can say the operator $\mathfrak{P}(\Theta(t))$ is equicontinuous on $[0, T]$. Hence, $\mathfrak{P}(\Theta(t))$ satisfies the hypothesis of the Arzela-Ascoli theorem and then by using proposition (2.2), $\mathfrak{P}$ is ω-Lipschitz with zero constant. □

Theorem 3.5. *The set of solutions to the Hilfer fractional neutral functional integro-differential equation with nonlocal condition (1.1) is bounded on $C(\mathcal{I}, \Xi)$, and there is at least one solution on $\Theta \in C(\mathcal{I}, \Xi)$.*

Proof. From preceding Theorems 3.2–3.4, the operator $\mathfrak{U}, \mathfrak{P}$ is continuous, equicontinuous, uniformly bounded, and compact. We must exhibit that the set of solution of Eq (1.1) is bounded and has a solution on $C(\mathcal{I},\Xi)$. Let $\Re : C(\mathcal{I},\Xi) \to C(\mathcal{I},\Xi)$ be an ω-condensing mapping and assume that the specified set is defined as

$$\mathcal{W} = \{\Theta \in C(\mathcal{I},\Xi) : \exists\ \Omega \in [0,1] \text{ such that } \Theta(t) = \Omega\Re\Theta\}. \tag{3.4}$$

Initially, we want to demonstrate that $\mathcal{W}$ is bounded for every $t \in [0,T]$,

$$\begin{aligned} \|\Theta(t)\|_{\mathcal{Y}} &= \|\Omega\Re\Theta(t)\|_{\mathcal{Y}} \\ &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}|\Omega(\mathfrak{U}(\Theta(t)) + \mathfrak{P}(\Theta(t)))| \\ &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{\Omega\Big|S_{\vartheta,\varpi}(t)(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta(t_\rho) - \mathfrak{M}(0,\Theta_0) + \mathfrak{M}(t,\Theta_t)) \\ &\quad + \int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s,\Theta_s)ds + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) \\ &\quad + \xi(s,\Theta_s,\int_0^s \chi(s,m,\Theta_m)dm))ds\Big|\Big\}. \end{aligned} \tag{3.5}$$

By using Eq (2.1), we have obtained the following inequality:

$$\begin{aligned} \|\Theta(t)\|_{\mathcal{Y}} \leq\ & \frac{\Omega\mathcal{V}}{\Gamma(\vartheta(1-\varpi)+\varpi)}|(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta(t_\rho) - \mathfrak{M}(0,\Theta_0) + \mathfrak{M}(t,\Theta_t)| \\ & + \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{\int_0^t |\mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s,\Theta_s)ds \\ & + \int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) + \xi(s,\Theta_s,\int_0^s \chi(s,m,\Theta_m)dm))ds|\Big\}. \end{aligned} \tag{3.6}$$

By using Eqs (2.1) and (3.2), we have obtained the following:

$$\|\Theta(t)\|_{\mathcal{Y}} \leq \frac{\Omega\mathcal{V}}{\Gamma(\vartheta(1-\varpi)+\varpi)}|(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta(t_\rho) - \mathfrak{M}(0,\Theta_0) + \mathfrak{M}(t,\Theta_t)| + \mu_\xi \tag{3.7}$$

$$\leq \Big\{\frac{\Omega\mathcal{V}}{\Gamma(\vartheta(1-\varpi)+\varpi)}\sigma_\rho\Big\} + \mu_\xi, \tag{3.8}$$

where $\sigma_\rho = |(\delta(0) - \sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta(t_\rho) - \mathfrak{M}(0,\Theta_0) + \mathfrak{M}(t,\Theta_t)|$. We can deduce from inequality (3.8) that the solution $\Theta(t)$ is bounded for every t on $C(\mathcal{I},\Xi)$. If this is not the case, we assume by contradiction, $\mathcal{N} := \|\Theta\| \to \infty$, dividing both sides of (3.8) by $\mathcal{N}$, and we have

$$1 \leq \lim_{\mathcal{N}\to\infty} \frac{\Big\{\frac{\Omega\mathcal{V}}{\Gamma(\vartheta(1-\varpi)+\varpi)} \times \sigma_\rho\Big\} + \mu_\xi}{\mathcal{N}} = 0. \tag{3.9}$$

This is a contradiction. In this regard, we can conclude that $\Re$ has at least one solution by using Lemma 2.6. □

Theorem 3.6. *Assume that $\mathcal{R}(1)(i)$, $\mathcal{R}(1)(ii)$, $\mathcal{R}(2)(i)$ and $\mathcal{R}(2)(ii)$ hold. Then, the Hilfer fractional neutral functional integro-differential equation with nonlocal condition (1.1) have a unique solution on $C(\mathcal{I},\Xi)$.*

Proof. Let us take $\Theta^*(t)$ and $\Theta^{**}(t)$ as the two solutions of the Hilfer fractional neutral functional integro-differential equation with nonlocal condition (1.1). Then, we have to examine the unique solution for every $t \in [0,T]$ on $C(\mathcal{I},\Xi)$.

$$\begin{aligned}
\|\Theta^*(t)-\Theta^{**}(t)\|_{\mathcal{Y}} &= \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Bigg\{\Bigg|S_{\vartheta,\varpi}(t)(\delta(0)+\sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta^*(t_\rho)-\mathfrak{M}(0,\Theta_0)+\mathfrak{M}(t,\Theta_t^*)) \\
&\quad +\int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s,\Theta_s^*)ds+\int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) \\
&\quad +\mathfrak{M}(t,\Theta_t^{**}))+\int_0^t \mathcal{H}\Psi_\varpi(t-s)\mathfrak{M}(s,\Theta_s^{**})ds+\int_0^t \Psi_\varpi(t-s)(\mathcal{P}u(s) \\
&\quad +\xi(s,\Theta_s^{**},\int_0^s \chi(s,m,\Theta_m^{**})dm))ds\Bigg|\Bigg\}\Bigg\} \\
&\le \sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Bigg\{\Bigg|S_{\vartheta,\varpi}(t)\Bigg(\sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta^*(t_\rho)-\sum_{\rho=1}^{\mathcal{M}} C_\rho\Theta^{**}(t_\rho)\Bigg)\Bigg| \\
&\quad +|S_{\vartheta,\varpi}(t)(\mathfrak{M}(t,\Theta_t^*)-\mathfrak{M}(t,\Theta_t^{**}))| \\
&\quad +\int_0^t |\mathcal{H}||\Psi_\varpi(t-s)|ds\times|\mathfrak{M}(s,\Theta_s^*)-\mathfrak{M}(s,\Theta_s^{**})| \\
&\quad +\int_0^t |\Psi_\varpi(t-s)|ds\times|\xi(s,\Theta_s^*,\mathcal{Z}_s^*)-\xi(s,\Theta_s^{**},\mathcal{Z}_s^{**})|\Bigg\} \\
&\le \frac{\mathcal{V}}{\Gamma(\vartheta(1-\varpi)+\varpi)}\Big\{(C_\xi+\mathcal{B}_\xi^\rho)|\Theta^*(t)-\Theta^{**}(t)|\Big\} \\
&\quad +\sup_{t\in\mathcal{I}} t^{(1-\vartheta)(1-\varpi)}\Big\{\int_0^t |\mathcal{H}||\Psi_\varpi(t-s)|\times C_\xi|\Theta^*(s)-\Theta^{**}(s)|ds \\
&\quad +\int_0^t |\Psi_\varpi(t-s)|\times(\mathcal{B}_1+\mathcal{B}_2\mathcal{L}_\xi)|\Theta^*(s)-\Theta^{**}(s)|ds\Big\}.
\end{aligned}$$

From above inequality, for each $t > 0$ and let us consider an arbitrary element $\zeta > 0$ such that

$$\begin{aligned}
\|\Theta^*(t)-\Theta^{**}(t)\|_{\mathcal{Y}} &\le \zeta+\frac{\mathcal{V}(C_\xi+\mathcal{B}_\xi^\rho)}{\Gamma(\vartheta(1-\varpi)+\varpi)}|\Theta^*(s)-\Theta^{**}(s)| \\
&\quad +\Big\{C_\xi\int_0^t |\mathcal{H}||\Psi_\varpi(t-s)|ds+(\mathcal{B}_1+\mathcal{B}_2\mathcal{L}_\xi)\int_0^t |\Psi_\varpi(t-s)|ds\Big\} \\
&\quad \times|\Theta^*(s)-\Theta^{**}(s)|.
\end{aligned}$$

Applying Gronwall's inequality yields,

$$\begin{aligned}
\|\Theta^*(t)-\Theta^{**}(t)\|_{\mathcal{Y}} &\le \zeta\times exp\Big\{\frac{\mathcal{V}(C_\xi+\mathcal{B}_\xi^\rho)}{\Gamma(\vartheta(1-\varpi)+\varpi)}|\Theta^*(t)-\Theta^{**}(t)| \\
&\quad +\Big\{C_\xi\int_0^t |\mathcal{H}||\Psi_\varpi(t-s)|ds+(\mathcal{B}_1+\mathcal{B}_2\mathcal{L}_\xi)\int_0^t |\Psi_\varpi(t-s)|ds\Big\} \\
&\quad \times|\Theta^*(s)-\Theta^{**}(s)|\Big\}.
\end{aligned}$$

Let $\zeta \to 0$, and then $\|\Theta^*(t)-\Theta^{**}(t)\|_{\mathcal{Y}} = 0 \implies \Theta^*(t) = \Theta^{**}(t)$ for every $t \in [0,T]$. □

Remark 3.7. *From the above theorems, one can observe that existence and uniqueness results are attained without using strong conditions compared to other traditional methods. The results presented in this paper give a simple explanation of the existence of a solution using our new assumptions $\mathcal{R}(1)$ and $\mathcal{R}(2)$. The monographs [24, 25] demonstrate the no-flux boundary condition and the multi-boundary value problem with the topological degree method. In [26] Luo et al. discussed a novel result on the averaging principle of stochastic Hilfer type with a non-Lipschitz condition. In [27], Ma et al. discussed the Hilfer fractional neutral differential systems in Hilbert spaces with strong conditions. In [28], Ouaarabi et al. proved the existence of a weak solution to a class of nonlinear parabolic problems using the topological degree method. Most recently, Shah et al. [29] extended the degree theory for non-monotone-type fractional order delay differential equations. With the motivation from the above works, we suggest the present new interpretation of the topological degree method for the Hilfer fractional neutral functional integro-differential equation with the nonlocal condition.*

4. Numerical examples

Example 4.1. *Consider the existence of a solution for the given Hilfer fractional neutral functional integro-differential equation with a nonlocal condition defined in the form of*

$$
\begin{aligned}
{}^{H}D^{\frac{1}{4},\frac{1}{7}}(\Theta(t)-\frac{8}{7}cos(\frac{\pi}{2}+t)) &= \mathcal{H}\Theta(t)+\mathcal{P}u(t)+\frac{1}{9}\int_0^5 (sin(\frac{\pi}{2}+s))ds, t\in[0,5],\\
I_{0^+}^{0.6428}\Theta(t) &= \delta(t)-\sum_{\rho=1}^{4} C_\rho\Theta(t_\rho),\ \rho=1,2,3,4,\ t\le 0.
\end{aligned}
\tag{4.1}
$$

Let us consider the matrix of $\mathcal{H}=\begin{bmatrix}0.874 & 1.093\\ 0.572 & 2.344\end{bmatrix}$, $\mathcal{P}=\begin{bmatrix}0.916 & 0.200\\ 1.414 & 0.110\end{bmatrix}$ and $u(t)=\begin{bmatrix}1\\1\end{bmatrix}$. Let us take $\delta(0)=0$, $m(0,\Theta_0)=0$, $m(t,\Theta_t))=\frac{8}{7}cos(\frac{\pi}{2}+t)$, $\int_0^5 \psi_{\frac{1}{7}}(5-s)ds\le 1$, $\mathcal{V}=0.7$ and $\sum_{\rho=1}^{4} C_\rho\Theta(t_\rho)=9$. Since $\sum_{\rho=1}^{4} C_\rho=0.9$ and $\sum_{\rho=1}^{4} t_\rho=10$, the solution representation for Eq (4.1) for every $t\in(0,5]$ is defined in the form of

$$
\begin{aligned}
\|\Theta(t)\|_{\mathcal{Y}} &= \sup_{t\in[0,5]} t^{(0.6428)}\Big\{\Big|S_{\frac{1}{4},\frac{1}{7}}(t)(\frac{8}{7}cos(\frac{\pi}{2}+t)+9)\\
&\quad+\begin{bmatrix}0.874 & 1.093\\ 0.572 & 2.344\end{bmatrix}\frac{8}{7}\int_0^5 \Psi_{\frac{1}{7}}(5-s)cos(\frac{\pi}{2}+s)ds\\
&\quad+\int_0^5 \Psi_{\frac{1}{7}}(5-s)\begin{bmatrix}0.916 & 0.200\\ 1.414 & 0.110\end{bmatrix}\times\begin{bmatrix}1\\1\end{bmatrix}ds\\
&\quad+\int_0^5 \Psi_{\frac{1}{7}}(5-s)\times\frac{1}{9}\int_0^5\{sin(\frac{\pi}{2}+m)dm\}ds\Big|\Big\}\\
&\le 0.2808\times 10.0959-5^{(1-1/4)(1-1/7)}\times\{0.8187-1.8889+0.1065\}\\
&= 5.5969,\\
\|\Theta(t)\|_{\mathcal{Y}} &\le 5.5969.
\end{aligned}
$$

By using the Theorems (3.3) and (3.5) from this, we can say Eq (4.1) has a solution on $C(\mathcal{I}, \Xi)$. Figures 1–4 represent the existence solutions for different parameters with a finite time interval for Eq (4.1).

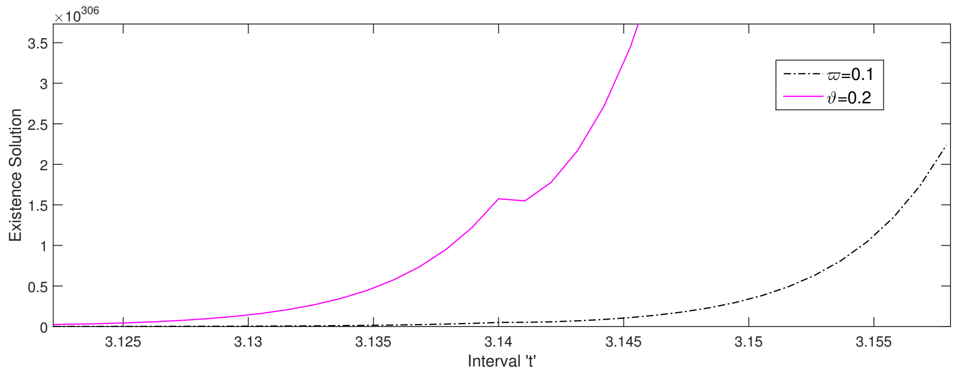

Figure 1. Graphical representation of existence solutions of the Hilfer ($\vartheta = 0.2, \varpi = 0.1$) form.

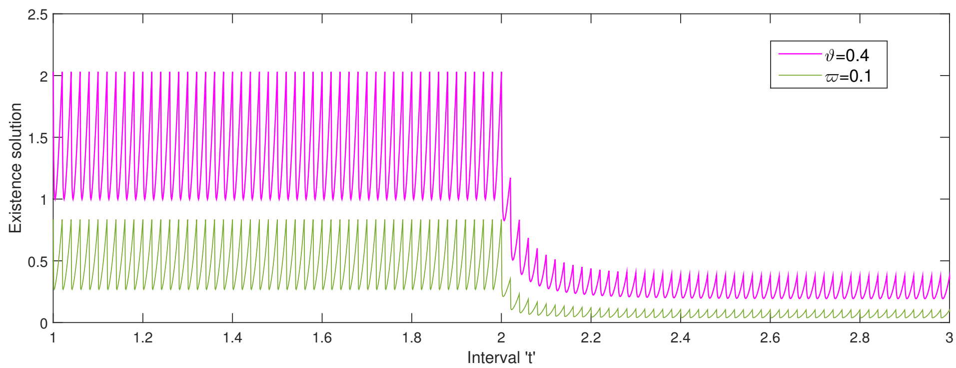

Figure 2. Graphical representation of existence solutions of the Hilfer ($\vartheta = 0.4, \varpi = 0.1$) form.

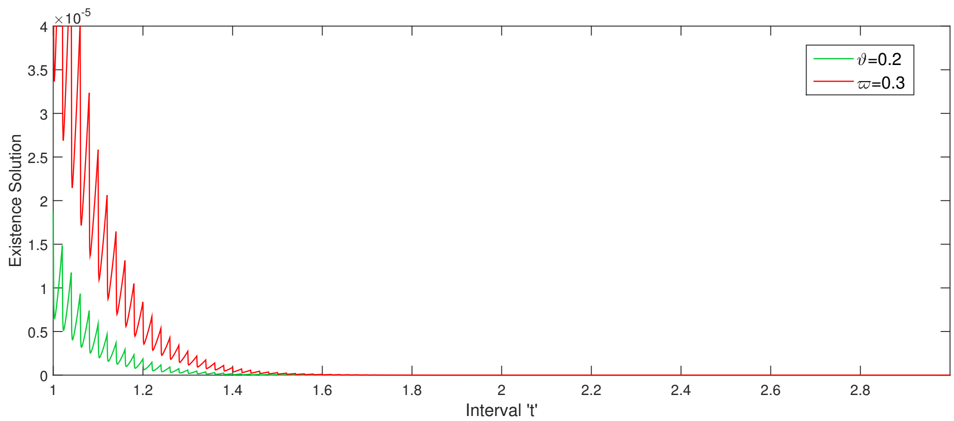

Figure 3. Graphical representation of existence solutions of the Hilfer ($\vartheta = 0.2, \varpi = 0.3$) form.

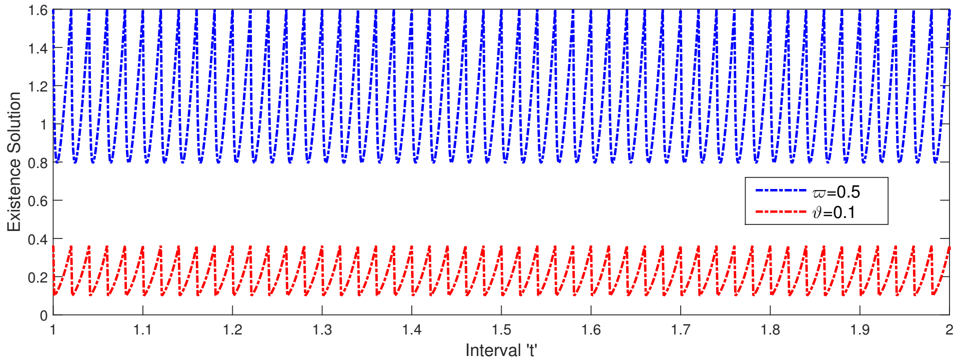

Figure 4. Graphical representation of existence solutions of the Hilfer ($\vartheta = 0.1, \varpi = 0.5$) form.

Example 4.2. *Consider the unique solution for the given Hilfer fractional neutral functional integro-differential equation with a nonlocal condition defined as the following form:*

$$\begin{aligned} {}^{H}D^{\frac{1}{2},\frac{1}{3}}(\Theta(t) - \frac{4}{7}cos\Theta(\pi+t)) &= \mathcal{H}\Theta(t) + \mathcal{P}u(t) + \frac{2}{7}cosec(\pi+t),\ t \in [0,5], \\ I^{0.3333}_{0^+}\Theta(t) &= \delta(t) - \sum_{\rho=1}^{4} C_\rho \Theta(t_\rho),\ \rho = 1,2,3,4 \ and\ t \le 0. \end{aligned} \tag{4.2}$$

Let us consider the matrix $\mathcal{H}=\begin{bmatrix}0.589 & 0.276\\ 0.729 & 0.913\end{bmatrix}$, $\mathcal{P}=\begin{bmatrix}3.21 & 2.17\\ 6.11 & 8.72\end{bmatrix}$ and $u(t)=\begin{bmatrix}1\\1\end{bmatrix}$. Let us take $\delta(t) = 2$, $\mathfrak{M}(0,\Theta_0) = 0$, $\mathfrak{M}(t,\Theta_t) = \frac{4}{7}cos\Theta(\pi+t)$, $\mathcal{V} = 0.7$, and $\sum_{\rho=1}^{4} C_\rho\Theta(t_\rho) = 9$, where $\sum_{\rho=1}^{4} C_\rho = 0.9$ and $\sum_{\rho=1}^{4} t_\rho = 10$.

$$\begin{aligned} \|\Theta^*(t) - \Theta^{**}(t)\|_{\mathcal{Y}} &= \sup_{t\in[0,5]} t^{0.3333} \Bigg\{ \Bigg| S_{\frac{1}{2},\frac{1}{3}}(\frac{4}{7}(t)cos\,\rho^*(\pi+t)) + \frac{4}{7}\int_0^5 \begin{bmatrix}0.589 & 0.276\\ 0.729 & 0.913\end{bmatrix} \\ &\quad \times \Psi_{\frac{1}{3}}(5-s)cos\Theta^*(\pi+s)ds + \int_0^5 \Psi_{\frac{1}{3}}(5-s)\begin{bmatrix}3.21 & 2.17\\ 6.11 & 8.72\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}ds \\ &\quad + \frac{2}{7}\int_0^5 \Psi_{\frac{1}{7}}(5-s)cosec(\rho^*(\pi+s))ds - \Bigg\{ S_{\frac{1}{2},\frac{1}{3}}(\frac{4}{7}cos\,\rho^{**}(\pi+t)) \\ &\quad + \frac{4}{7}\int_0^5 \begin{bmatrix}0.589 & 0.276\\ 0.729 & 0.913\end{bmatrix}\Psi_{\frac{1}{3}}(5-s)cos\,\Theta^{**}(\pi+s)ds \\ &\quad + \int_0^5 \Psi_{\frac{1}{3}}(5-s)\begin{bmatrix}3.21 & 2.17\\ 6.11 & 8.72\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}ds \\ &\quad + \frac{2}{7}\int_0^5 \Psi_{\frac{1}{3}}(5-s)cosec(\rho^{**}(\pi+s))ds \Bigg| \Bigg\} \Bigg\} \\ &\le 0. \end{aligned}$$

By using Theorem 3.6, Lemma 2.7 and Note 2.8, we obtained $\|\Theta^*(t) - \Theta^{**}(t)\|_{\mathcal{Y}} = 0 \implies \Theta^*(t) = \Theta^{**}(t), \forall t \in [0,5]$. Eventually we acquired the uniqueness solution for the given problem (4.2). Figures 5–8 reflect the unique solutions of different parameters with finite time interval for Eq (4.2).

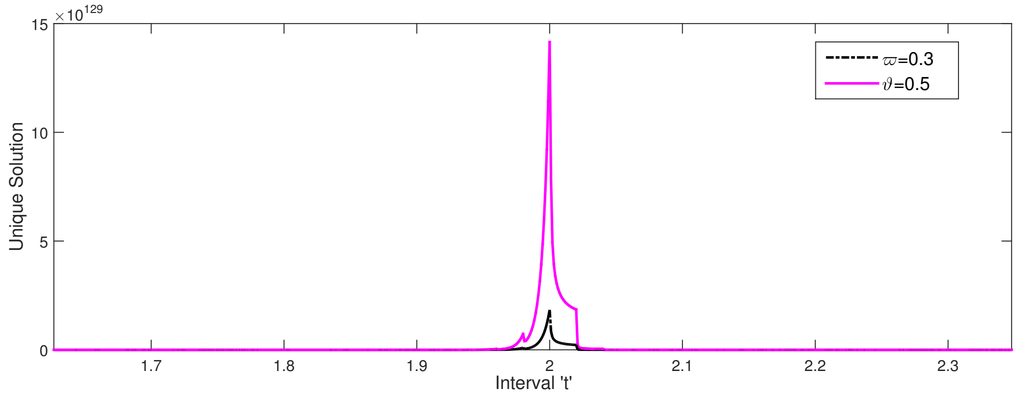

Figure 5. Graphical representation of unique solutions of the Hilfer ($\vartheta = 0.5, \varpi = 0.3$) form.

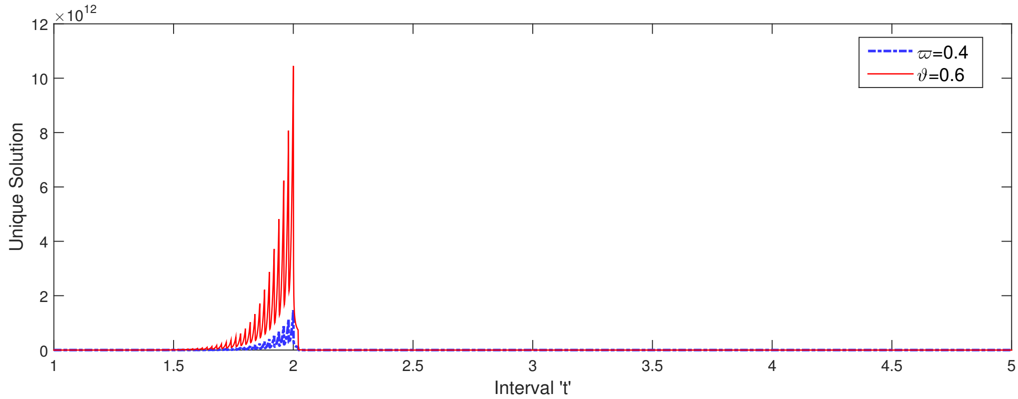

Figure 6. Graphical representation of unique solutions of the Hilfer ($\vartheta = 0.6, \varpi = 0.4$) form.

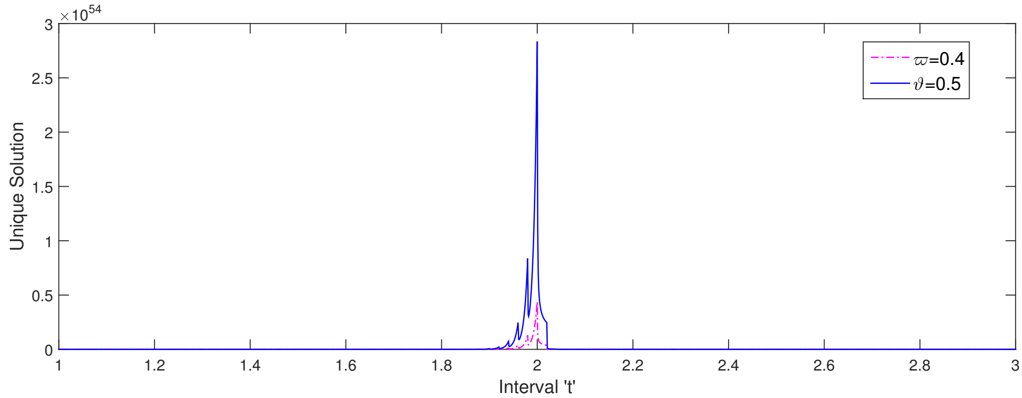

Figure 7. Graphical representation of unique solutions of the Hilfer ($\vartheta = 0.5, \varpi = 0.4$) form.

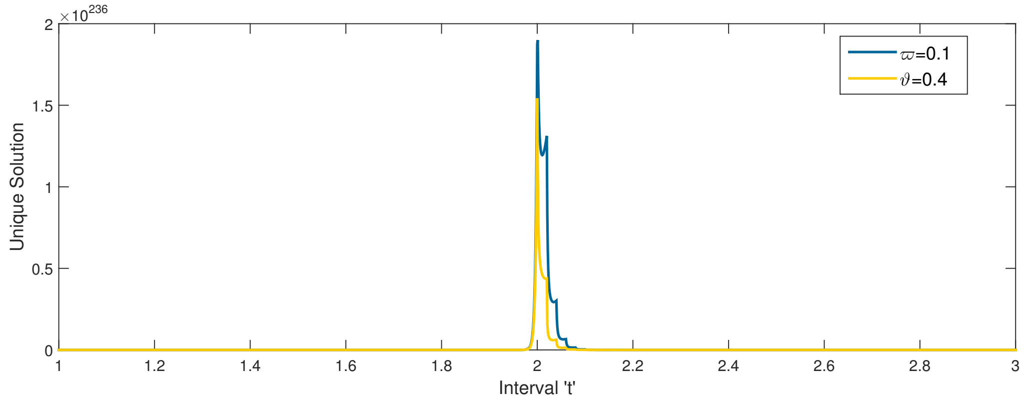

Figure 8. Graphical representation of unique solutions of the Hilfer ($\vartheta = 0.4$, $\varpi = 0.1$) form.

5. Conclusions

In this study, we explored the existence and uniqueness of the Hilfer fractional neutral functional integro-differential equation with a nonlocal condition. We obtained these results using the topological degree method and Gronwall's inequality. To illustrate and demonstrate the applicability of the obtained results, two numerical computations with several graphical representations of different parameters are provided.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

Data availability statement

Not applicable.

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