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Research article

On a class of three coupled fractional Schrödinger systems with general nonlinearities

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Abstract: In this paper, a class of systems of three-component coupled nonlinear fractional Schrödinger equations with general nonlinearities is investigated. Without any monotonicity condition and the Ambrosetti-Rabinowitz growth condition, we obtain some novel existence results of least energy solutions by using variational arguments and a Pohozaev manifold method.

Keywords: three coupled fractional Schrödinger system; variational methods; general nonlinearities; least energy solution; fully nontrivial solution
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1. Introduction

In this paper, we study the following three-component coupled fractional Schrödinger system:

$$\begin{cases} (-\Delta)^\alpha u_1 + \omega_1 u_1 = f_1(u_1) + \lambda u_2 u_3 & \text{in } \mathbb{R}^d, \\ (-\Delta)^\alpha u_2 + \omega_2 u_2 = f_2(u_2) + \lambda u_1 u_3 & \text{in } \mathbb{R}^d, \\ (-\Delta)^\alpha u_3 + \omega_3 u_3 = f_3(u_3) + \lambda u_1 u_2 & \text{in } \mathbb{R}^d, \\ u_j \in H^\alpha(\mathbb{R}^d), \ j = 1, 2, 3, \end{cases} \tag{1.1}$$

where $\alpha \in (0, 1)$, $d > 2\alpha$, $\omega_j > 0$, $j = 1, 2, 3$, $\lambda > 0$ is a coupling parameter and the fractional Laplacian $(-\Delta)^\alpha$ is given by

$$(-\Delta)^\alpha w(x) = C_{d,\alpha}\text{P.V.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha}} \mathrm{dy} = C_{d,\alpha} \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha}} dy,$$

where

$$C_{d,\alpha} = \alpha(1-\alpha)4^{\alpha}\frac{\Gamma(\frac{d}{2}+\alpha)}{\pi^{\frac{d}{2}}\Gamma(2-\alpha)}$$

is a normalization constant, P.V. is the Cauchy principal value. We are dedicated to the existence of least energy solutions for system (1.1).

The fractional Laplacian operator $(-\Delta)^{\alpha}$ arises in several physical phenomena like fractional quantum mechanics and flames propagation, in population dynamics and geophysical fluid dynamics. In addition, $(-\Delta)^{\alpha}$ also arises in modeling diffusion and transport in a highly heterogeneous medium, or is used as an effective diffusion in a limiting advection-diffusion equation with a random velocity field. In [9], Laskin introduced the fractional Laplacian equation by expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. For more motivations and backgrounds, we refer the interested readers to [9, 14] and references therein. From the mathematicians point of view, one of the main difficulties lies in that the fractional Laplacian $(-\Delta)^{\alpha}$ is a nonlocal operator.

Over the past few years, the following fractional Schrödinger equation has drawn many researchers' a great deal of attention

$$(-\Delta)^{\alpha}u + u = f(x,u), \tag{1.2}$$

where $\alpha \in (0,1)$. Equation (1.2) arises in looking for standing wave solutions for the fractional Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = (-\Delta)^{\alpha}\Psi + \Psi - f(x,\Psi), \tag{1.3}$$

where i and Ψ denote the imaginary unit and the wave function, respectively. Different approaches have been applied to deal with problem (1.2) under various hypotheses on the nonlinearity f, and several existence and nonexistence results via variational methods are obtained, see for example, [1–3] and the references therein. Recently, Guo and He [6] considered the system

$$\begin{cases} (-\Delta)^{\alpha}u + u = (|u|^{2q} + b|u|^{q-1}|v|^{q+1})u & \text{in } \mathbb{R}^d, \\ (-\Delta)^{\alpha}v + \omega^{2\alpha}v = (|v|^{2q} + b|v|^{q-1}|u|^{q+1})v & \text{in } \mathbb{R}^d, \\ u, v \in H^{\alpha}(\mathbb{R}^d), \end{cases} \tag{1.4}$$

where $\alpha \in (0,1)$, $\omega > 0$, $b > 0$, $2q+2 \in (2, 2^*_{\alpha})$. They, with the application of Nehari manifold method, proved (1.4) has a least energy solution. They also proved that if b is large enough, system (1.4) has a positive least energy solution with both nontrivial components by using the similar arguments as in [13]. In [12], Lü and Peng investigated the following two-component coupled fractional Schrödinger system

$$\begin{cases} (-\Delta)^{\alpha}u + u = f(u) + \beta v & \text{in } \mathbb{R}^d, \\ (-\Delta)^{\alpha}v + v = g(v) + \beta u & \text{in } \mathbb{R}^d, \\ u, v \in H^{\alpha}(\mathbb{R}^d). \end{cases} \tag{1.5}$$

Under some suitable assumptions on the nonlinear terms f and g, they obtained the existence of positive solutions with both nontrivial components and least energy solutions with both nontrivial components for (1.5) by using variational methods. They also proved the asymptotic behavior of these solutions as the coupling parameter $\beta \to 0$. More results concerning the fractional Schrödinger systems (1.4) and (1.5), can be seen in [5, 8, 17–19].

To our best knowledge, there is no result in the literature on the existence result for three coupled fractional Schrödinger systems with general nonlinearities. We will prove some existence results for system (1.1). On a broader scale, this paper presumes that $f_j (j = 1, 2, 3)$ meet the conditions below:
(A_1) $f_j \in C^1(\mathbb{R}, \mathbb{R})$ and $f_j(t) = o(t)(t \to 0^+)$.
(A_2) There exist $q_j \in (2, 2^*_\alpha)$ such that $\lim_{|t|\to+\infty} \frac{f_j(t)}{|t|^{q_j-1}} = 0$, where $2^*_\alpha = \frac{2d}{d-2\alpha}$ is the fractional critical exponent.
(A_3) There exist $T_j > 0$ such that $F_j(T_j) > \frac{\omega_j}{2} T_j^2$, where $F_j(t) := \int_0^t f_j(s)\mathrm{d}s$.

When $\lambda = 0$, system (1.1) is converted to three uncoupled equations

$$(-\Delta)^\alpha u_j + \omega_j u_j = f_j(u_j), \ \ u_j \in H^\alpha(\mathbb{R}^d), \ j = 1, 2, 3. \tag{1.6}$$

In [2, 3], it is proved that if f_j satisfy (A_1)–(A_3), (1.6) possesses a positive least energy solution $u_j^\star$ for $j = 1, 2, 3$. Hence, for all $\lambda \in \mathbb{R}$, the pairs $(u_1^\star, 0, 0)$, $(0, u_2^\star, 0)$ and $(0, 0, u_3^\star)$ solve system (1.1). In this paper this sort of solutions (i.e., solutions with at least one trivial component) is called as semi-trivial solutions. An appealing question is whether the system (1.1) contains solutions (u_1, u_2, u_3) such that $u_1, u_2, u_3 \not\equiv 0$ under the conditions (A_1)–(A_3), such kind of solutions will be called fully nontrivial solutions. The major results are the following:

Theorem 1.1. *Suppose that f_j satisfy $(A_1) - (A_3)$ for $j = 1, 2, 3$ and $2\alpha < d < 6\alpha$. Then*

(i) *for any $\lambda > 0$, the system (1.1) has a least energy solution,*
(ii) *there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, the system (1.1) has a fully nontrivial least energy solution.*

Remark 1.1. *Theorem 1.1 can be thought of as an extension of the results in [3, 6, 16]. We note that in our assumptions (A_1)–(A_3) neither any monotonicity condition nor any Ambrosetti-Rabinowitz growth condition is required, and we need a new method different from those used in [3, 6, 16].*

The structure of the other parts of the paper is as follows. In Section 2, some notations and preliminary results are proposed. In Section 3, we conclude the proof of Theorem 1.1.

2. Preliminaries

Throughout this paper, C, C_i will signify different kinds of positive constants; the strong convergence is denoted by $\to$, and the weak convergence denoted by $\rightharpoonup$; $B_\rho(y)$ denotes a ball centered at y with radius $\rho > 0$; $\|u\|_{L^q(\mathbb{R}^d)} = (\int_{\mathbb{R}^d} |u|^q \mathrm{d}x)^{\frac{1}{q}}$ denote the norm of $L^q(\mathbb{R}^d)$. The fractional Sobolev space $H^\alpha(\mathbb{R}^d)$ is marked as

$$H^\alpha(\mathbb{R}^d) = \left\{ w \in L^2(\mathbb{R}^d) : \frac{|w(x) - w(y)|}{|x - y|^{\frac{d}{2}+\alpha}} \in L^2(\mathbb{R}^{2d}) \right\},$$

endowed with the norm

$$\|w\|_{H^\alpha(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|w(x) - w(y)|^2}{|x - y|^{d+2\alpha}} \mathrm{d}x\mathrm{d}y + \|w\|^2_{L^2(\mathbb{R}^d)} \right)^{\frac{1}{2}},$$

where

$$[w]_{H^\alpha(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|w(x)-w(y)|^2}{|x-y|^{d+2\alpha}}\mathrm{d}xdy\right)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of w. Via Fourier transform, we have

$$\widehat{(-\Delta)^\alpha u}(\xi) = |\xi|^{2\alpha}\widehat{u}(\xi) \ \text{ for } \xi \in \mathbb{R}^d,$$

where the symbol $\widehat{}$ stands for Fourier transform. Therefore, by the Fourier transform, $H^\alpha(\mathbb{R}^d)$ can be equivalently defined as follows

$$H^\alpha(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d}|\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi < \infty\right\},$$

and the norm can be equivalently written

$$\|u\|_{H^\alpha(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}|\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi + \|u\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}.$$

From Propositions 3.4 and 3.6 in [14], for any $w \in H^\alpha(\mathbb{R}^d)$, we have

$$\|(-\Delta)^{\frac{\alpha}{2}}w\|^2_{L^2(\mathbb{R}^d)} = \frac{C_{d,\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|w(x)-w(y)|^2}{|x-y|^{d+2\alpha}}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^d}|\xi|^{2\alpha}|\widehat{w}(\xi)|^2\mathrm{d}\xi.$$

We also use the following notations:
(1) $\mathcal{D}^{\alpha,2}(\mathbb{R}^d)$ is completion of $C_0^\infty(\mathbb{R}^d)$ concerning the norm

$$\|w\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}|(-\Delta)^{\frac{\alpha}{2}}w|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

(2) For $\omega_j > 0$, $j = 1, 2, 3$, we use the notation

$$\|w\|_{\omega_i} = \left(\int_{\mathbb{R}^d}(|(-\Delta)^{\frac{\alpha}{2}}w|^2 + \omega_i w^2)\mathrm{d}x\right)^{\frac{1}{2}},$$

which is an equivalent norm to $\|w\|_{H^\alpha(\mathbb{R}^d)}$.
(3)

$$\mathscr{H} = H^\alpha(\mathbb{R}^d)\times H^\alpha(\mathbb{R}^d)\times H^\alpha(\mathbb{R}^d)$$

with the norm

$$\|(u_1,u_2,u_3)\|^2 = \|u_1\|^2_{\omega_1} + \|u_2\|^2_{\omega_2} + \|u_3\|^2_{\omega_3}$$

and

$$\mathscr{H}_r = H^\alpha_r(\mathbb{R}^d)\times H^\alpha_r(\mathbb{R}^d)\times H^\alpha_r(\mathbb{R}^d),$$

where

$$H^\alpha_r(\mathbb{R}^d) = \left\{w \in H^\alpha(\mathbb{R}^d) : w(x) = w(|x|)\right\}.$$

For the fractional Sobolev spaces, the embedding results below can be got in [14].

Lemma 2.1. *If $\alpha \in (0,1)$ and $d > 2\alpha$, then*

(i) $\mathcal{D}^{\alpha,2}(\mathbb{R}^d)$ *is continuously embedded into* $L^{2^*_\alpha}(\mathbb{R}^d)$, *i.e.,*

$$\|w\|^2_{L^{2^*_\alpha}(\mathbb{R}^d)} \le C\|(-\Delta)^{\frac{\alpha}{2}} w\|^2_{L^2(\mathbb{R}^d)}$$

for any $w \in \mathcal{D}^{\alpha,2}(\mathbb{R}^d)$, *where constant* C *depending only on* d, α.

(ii) $H^\alpha(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *is continuous for any* $q \in [2, 2^*_\alpha]$.

(iii) $H^\alpha(\mathbb{R}^d) \hookrightarrow L^q_{loc}(\mathbb{R}^d)$ *is compact for any* $q \in [1, 2^*_\alpha)$; $H^\alpha_r(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *is compact for any* $q \in (2, 2^*_\alpha)$.

3. Proof of Theorem 1.1

This section is devoted to proving the Theorem 1.1. Define a functional related to system (1.1) by

$$\Phi_\lambda(u_1, u_2, u_3) = \sum_{j=1}^{3} I_j(u_j) - \lambda \int_{\mathbb{R}^d} u_1 u_2 u_3 \mathrm{d}x, \tag{3.1}$$

where for $j = 1, 2, 3$,

$$I_j(u_j) = \frac{1}{2}\int_{\mathbb{R}^d} \left(|(-\Delta)^{\frac{\alpha}{2}} u_j|^2 + \omega_j |u_j|^2\right)\mathrm{d}x - \int_{\mathbb{R}^d} F_j(u_j)\mathrm{d}x.$$

On the basis of the conditions (A_1)–(A_3), one can easily verify that Φ_λ is well defined and C^1. Now, we define the Pohozaev set by

$$\mathcal{N}_\lambda = \{(u_1, u_2, u_3) \in \mathcal{H} \setminus \{(0,0,0)\} : \mathcal{N}(u_1, u_2, u_3) = 0\},$$

where

$$\mathcal{N}(u_1, u_2, u_3) = \sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} u_j\|^2_{L^2(\mathbb{R}^d)} - 2^*_\alpha \int_{\mathbb{R}^d} \left(\lambda u_1 u_2 u_3 - \frac{1}{2}\sum_{j=1}^{3} \omega_j |u_j|^2 + \sum_{j=1}^{3} F_j(u_j)\right)\mathrm{d}x.$$

From (A_1)–(A_3), if $(u_1, u_2, u_3) \in \mathcal{H}$ is a weak solution to system (1.1), using the similar regularity arguments as in [2], we can get $u_j \in C^1(\mathbb{R}^d)$ for $j = 1, 2, 3$. Then it is classical to confirm that each nontrivial solution of (1.1) belongs to $\mathcal{N}_\lambda$. Moreover, we have

Lemma 3.1. *Let the conditions* (A_1)–(A_3) *hold, then*

(i) $\mathcal{N}_\lambda$ *is a* C^1 *manifold,*

(ii) *for any* $(u_1, u_2, u_3) \in \mathcal{N}_\lambda$, *there exists constant* $\varrho_0 > 0$ *such that* $\|(u_1, u_2, u_3)\| \ge \varrho_0$,

(iii) *if* $u_i \in H^\alpha(\mathbb{R}^d) \setminus \{0\}$ *and* $\mathcal{N}(u_1, u_2, u_3) \le 0$, *then exists a unique* $\bar{t} \in (0, 1]$ *such that*

$$(u_1^{\bar{t}}, u_2^{\bar{t}}, u_3^{\bar{t}}) \in \mathcal{N}_\lambda,$$

where $u_i^t(x) = u_i(t^{-1}x)$.

Proof. (i) From (A_1)–(A_3), we know that $\mathcal{N}(u_1,u_2,u_3)$ is a C^1 functional, in order to prove $\mathscr{N}_\lambda$ is a C^1 manifold, it suffices to prove that $\mathcal{N}'(u_1,u_2,u_3)\neq 0$ for all $(u_1,u_2,u_3)\in\mathscr{N}_\lambda$. Indeed, assume by contradiction that $\mathcal{N}'(u_1,u_2,u_3)=0$ for some $(u_1,u_2,u_3)\in\mathscr{N}_\lambda$. Then in a weak sense, (u_1,u_2,u_3) can be seen as a solution of the system

$$\begin{cases} (-\Delta)^\alpha u_1+\frac{2_\alpha^*}{2}\omega_1u_1=\frac{2_\alpha^*}{2}f_1(u_1)+\frac{2_\alpha^*}{2}\lambda u_2u_3 & \text{in } \mathbb{R}^d,\\ (-\Delta)^\alpha u_2+\frac{2_\alpha^*}{2}\omega_2u_2=\frac{2_\alpha^*}{2}f_2(u_2)+\frac{2_\alpha^*}{2}\lambda u_1u_3 & \text{in } \mathbb{R}^d,\\ (-\Delta)^\alpha u_3+\frac{2_\alpha^*}{2}\omega_3u_3=\frac{2_\alpha^*}{2}f_3(u_3)+\frac{2_\alpha^*}{2}\lambda u_1u_2 & \text{in } \mathbb{R}^d. \end{cases} \tag{3.2}$$

As a consequence, we see that (u_1,u_2,u_3) satisfies the Pohozaev type identity referred to (3.2), that is

$$\begin{aligned} &\int_{\mathbb{R}^d}|(-\Delta)^{\frac{\alpha}{2}}u_1|^2\mathrm{d}x+\int_{\mathbb{R}^d}|(-\Delta)^{\frac{\alpha}{2}}u_2|^2\mathrm{d}x+\int_{\mathbb{R}^d}|(-\Delta)^{\frac{\alpha}{2}}u_3|^2\mathrm{d}x\\ &\qquad=\frac{(2_\alpha^*)^2}{2}\int_{\mathbb{R}^d}\Big(\lambda u_1u_2u_3-\frac{1}{2}\sum_{j=1}^3\omega_j|u_j|^2+\sum_{j=1}^3F_j(u_j)\Big)\mathrm{d}x. \end{aligned} \tag{3.3}$$

Since $\mathcal{N}(u_1,u_2,u_3)=0$, by (3.3) we deduce that

$$\left(1-\frac{2}{2_\alpha^*}\right)\sum_{j=1}^3\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2=0,$$

which implies that $u_j=0$ for all $j=1,2,3$, which is a contradiction since $(u_1,u_2,u_3)\in\mathscr{N}_\lambda$. Thus $\mathscr{N}_\lambda$ is a C^1 manifold.

(ii) Let $(u_1,u_2,u_3)\in\mathscr{N}_\lambda$, then we have

$$\sum_{j=1}^3\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2+\frac{2_\alpha^*}{2}\sum_{j=1}^3\int_{\mathbb{R}^d}\omega_j|u_j|^2\mathrm{d}x=2_\alpha^*\int_{\mathbb{R}^d}\Big(\lambda u_1u_2u_3+\sum_{j=1}^3F_j(u_j)\Big)\mathrm{d}x. \tag{3.4}$$

From the conditions (A_1) and (A_2), we know that, for any $\epsilon>0$, there is $C_\epsilon>0$ such that

$$|f_j(t)|\le\epsilon|t|+C_\epsilon|t|^{q_j-1},\quad |F_j(t)|\le\epsilon|t|^2+C_\epsilon|t|^{q_j},\ j=1,2,3. \tag{3.5}$$

Then by (3.4), (3.5) and the Sobolev embedding inequality, one has

$$\begin{aligned} \|(u_1,u_2,u_3)\|^2&=\|u_1\|_{\omega_1}^2+\|u_2\|_{\omega_2}^2+\|u_3\|_{\omega_3}^2\\ &\le C_1\int_{\mathbb{R}^d}(|u_1|^{q_1}+|u_2|^{q_2}+|u_3|^{q_3}+u_1u_2u_3)\mathrm{d}x\\ &\le C_2(\|u_1\|_{\omega_1}^{q_1}+\|u_2\|_{\omega_2}^{q_2}+\|u_3\|_{\omega_3}^{q_3}+\|u_1\|_{\omega_1}^3+\|u_2\|_{\omega_2}^3+\|u_3\|_{\omega_3}^3), \end{aligned}$$

which implies that $\|(u_1,u_2,u_3)\|\ge\varrho_0$ for some positive constant $\varrho_0>0$ since $q_1,q_2,q_3>2$.

(iii) Let $u_1,u_2,u_3\in H^\alpha(\mathbb{R}^d)\setminus\{0\}$ satisfy $\mathcal{N}(u_1,u_2,u_3)\le 0$, then

$$0<\sum_{j=1}^3\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2\le 2_\alpha^*\int_{\mathbb{R}^d}\Big(\lambda u_1u_2u_3-\frac{1}{2}\sum_{j=1}^3\omega_j|u_j|^2+\sum_{j=1}^3F_j(u_j)\Big)\mathrm{d}x. \tag{3.6}$$

For $t>0$, let $u_j^t=u_j(t^{-1}x)$, $j=1,2,3$, we define

$$\begin{aligned}h(t):=&\Phi_\lambda(u_1^t,u_2^t,u_3^t)\\=&\frac{t^{d-2\alpha}}{2}\sum_{j=1}^{3}\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2-t^d\int_{\mathbb{R}^d}\Big(\lambda u_1u_2u_3-\frac{1}{2}\sum_{j=1}^{3}\omega_j|u_j|^2+\sum_{j=1}^{3}F_j(u_j)\Big)\mathrm{d}x.\end{aligned}$$

Obviously, $h(t)\to-\infty$ as $t\to+\infty$. Moreover, $h(t)>0$ for $t>0$ small enough. In fact, from (A_1) and (A_2), for any $\epsilon>0$, there is $C_\epsilon>0$ such that for all $\tau\in\mathbb{R}$,

$$|F_j(\tau)|\le\epsilon|\tau|^2+C_\epsilon|\tau|^{2_\alpha^*},\ \ j=1,2,3. \tag{3.7}$$

Then by (3.7), Hölder inequality and Lemma 2.1(i), we get

$$\begin{aligned}h(t)\ge&\frac{t^{d-2\alpha}}{2}\sum_{j=1}^{3}\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2-Ct^d\sum_{j=1}^{3}\int_{\mathbb{R}^d}|u_j|^{2_\alpha^*}\mathrm{d}x\\\ge&\frac{t^{d-2\alpha}}{2}\sum_{j=1}^{3}\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2-C_0t^d\sum_{j=1}^{3}\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^{2_\alpha^*},\end{aligned}$$

which yields that $h(t)>0$ when $t>0$ sufficiently small. Hence, we can find $\bar{t}>0$ such that $h(t)$ has a positive maximum and $h'(\bar{t})=0$. Notice that

$$\mathcal{N}(u_1^t,u_2^t,u_3^t)=th'(t)=t\frac{\mathrm{d}\Phi_\lambda(u_1^t,u_2^t,u_3^t)}{\mathrm{d}t},$$

so we get $\mathcal{N}(u_1^{\bar{t}},u_2^{\bar{t}},u_3^{\bar{t}})=0$. Furthermore, by $\mathcal{N}(u_1^{\bar{t}},u_2^{\bar{t}},u_3^{\bar{t}})=0$, we can deduce that

$$\bar{t}=\left(\frac{\sum\limits_{j=1}^{3}\|(-\Delta)^{\frac{\alpha}{2}}u_j\|_{L^2(\mathbb{R}^d)}^2}{2_\alpha^*\int_{\mathbb{R}^d}(\lambda u_1u_2u_3-\frac{1}{2}\sum\limits_{j=1}^{3}\omega_j|u_j|^2+\sum\limits_{j=1}^{3}F_j(u_j))\mathrm{d}x}\right)^{\frac{1}{2\alpha}}.$$

By (3.6), we know that $\bar{t}\in(0,1]$. Hence, $\bar{t}\in(0,1]$ is the unique critical point of $h(t)$ corresponding to its maximum. □

Let us define the least energy

$$c_\lambda=\inf_{(u_1,u_2,u_3)\in\mathcal{N}_\lambda}\Phi_\lambda(u_1,u_2,u_3). \tag{3.8}$$

We call that a minimizer on $\mathcal{N}_\lambda$ is a least energy solution for system (1.1). By the proof of Lemma 3.1(ii) and (iii), it is clear that the functional Φ_λ satisfies the mountain-pass geometry. Let c^* be the minmax mountain-pass level for the functional Φ_λ given by

$$c^*=\inf_{\gamma\in\Lambda}\sup_{0\le t\le1}\Phi_\lambda(\gamma(t)),$$

where

$$\Lambda=\{\gamma\in C([0,1],\mathscr{H}):\gamma(0)=0,\Phi_\lambda(\gamma(1))<0\}.$$

Arguing as in the proof of Lemma 4.2 in [10], we can get that $c_\lambda=c^*>0$.

Lemma 3.2. *(u_1, u_2, u_3) is a solution of system (1.1) provided c_λ is attained at $(u_1, u_2, u_3) \in \mathscr{N}_\lambda$, where c_λ is defined in (3.8).*

Proof. Suppose that $(u_1, u_2, u_3) \in \mathscr{N}_\lambda$ such that $\Phi_\lambda(u_1, u_2, u_3) = c_\lambda$. Then by the theory of Lagrange multipliers, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\Phi'_\lambda(u_1, u_2, u_3) - \mu \mathcal{N}'(u_1, u_2, u_3) = 0.$$

As a consequence, (u_1, u_2, u_3) satisfies the following Pohozaev type identity

$$\mathcal{N}(u_1, u_2, u_3) = \mu \left[\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} u_j\|^2_{L^2(\mathbb{R}^d)} - \frac{(2^*_\alpha)^2}{2} \int_{\mathbb{R}^d} \Big(\lambda u_1 u_2 u_3 - \frac{1}{2} \sum_{j=1}^{3} \omega_j |u_j|^2 + \sum_{j=1}^{3} F_j(u_j) \Big) \mathrm{d}x \right]. \quad (3.9)$$

Notice that $\mathcal{N}(u_1, u_2, u_3) = 0$, from (3.4) and (3.9) we obtain

$$\mu \Big(1 - \frac{2^*_\alpha}{2} \Big) \sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} u_j\|^2_{L^2(\mathbb{R}^d)} \equiv 0,$$

which implies that $\mu = 0$. Thus $\Phi'_\lambda(u_1, u_2, u_3) = 0$, and so (u_1, u_2, u_3) is a solution of system (1.1). □

Lemma 3.3. *Let $\alpha \in (0, 1)$ and $d > 2\alpha$. Suppose that $\{w_n\}$ is a bounded sequence in $H^\alpha(\mathbb{R}^d)$ and*

$$\lim_{n\to\infty} \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} |w_n|^2 \mathrm{d}x = 0 \text{ for some } R > 0.$$

Then $w_n \to 0$ in $L^q(\mathbb{R}^d)$ for every $q \in (2, 2^*_\alpha)$.

The above vanishing lemma has been proved in [4] (see Lemma 2.2). Then we have:

Lemma 3.4. *If $\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathscr{N}_\lambda$ is a bounded sequence, there exists a sequence $\{z_n\} \subset \mathbb{R}^d$ and constants $R, \eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(z_n)} (|u_{1,n}|^2 + |u_{2,n}|^2 + |u_{3,n}|^2) \mathrm{d}x \geq \eta > 0.$$

Proof. Arguing from the reversed point, suppose that the conclusion does not hold, then for every $R > 0$, one has

$$\sup_{z \in \mathbb{R}^d} \int_{B_R(z)} |u_{j,n}|^2 \mathrm{d}x \to 0 (n \to \infty), \ j = 1, 2, 3. \quad (3.10)$$

By (3.10) and Lemma 3.3, we obtain that for all $r \in (2, 2^*_\alpha)$, $u_{j,n} \to 0$ in $L^r(\mathbb{R}^d)$ for $j = 1, 2, 3$. Furthermore, notice that $\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathscr{N}_\lambda$, then we can deduce that $(u_{1,n}, u_{2,n}, u_{3,n}) \to (0, 0, 0)$ in $\mathscr{H}$. On the other hand, from Lemma 3.1(ii), we have $\|(u_{1,n}, u_{2,n}, u_{3,n})\| \geq \varrho_0$ for some $\varrho_0 > 0$, so we get a contradiction, the proof is finished. □

Lemma 3.5. *Suppose that f_j satisfy (A_1)–(A_3) for $j = 1, 2, 3$ and $2\alpha < d < 6\alpha$. Then for each $\lambda > 0$, there exists*

$$(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) \in \mathscr{N}_\lambda,$$

such that

$$\Phi_\lambda(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) = c_\lambda.$$

Proof. Suppose $\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathcal{N}_\lambda$ such that

$$\Phi_\lambda(u_{1,n}, u_{2,n}, u_{3,n}) \to c_\lambda (n \to \infty).$$

Let $(u^*_{1,n}, u^*_{2,n}, u^*_{3,n})$ be the Schwarz symmetrization of $(u_{1,n}, u_{2,n}, u_{3,n})$, by the fractional Polya-Szegö inequality (see Theorem 2.1 in [7] or Theorem 1.1 in [15]) and the properties of the Schwarz symmetrization (see Lieb-Loss [11]) and Lemma 3.1(iii), there exists $\tilde{t}_n \in (0, 1]$ such that

$$(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}) := \left(u^*_{1,n}\left(\frac{x}{\tilde{t}_n}\right), u^*_{2,n}\left(\frac{x}{\tilde{t}_n}\right), u^*_{3,n}\left(\frac{x}{\tilde{t}_n}\right)\right) \in \mathcal{N}_\lambda$$

and

$$\Phi_\lambda(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}) \le \Phi_\lambda(u_{1,n}, u_{2,n}, u_{3,n}).$$

Hence we can assume that $u_{1,n}, u_{2,n}$ and $u_{3,n}$ are radial, i.e.,

$$\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathcal{N}_\lambda \cap \mathscr{H}_r.$$

First, we note that $\{(u_{1,n}, u_{2,n}, u_{3,n})\}$ is bounded in $\mathscr{H}_r$. Indeed, since $\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathcal{N}_\lambda$, we have $\mathcal{N}(u_{1,n}, u_{2,n}, u_{3,n}) = 0$, then we infer that

$$\Phi_\lambda(u_{1,n}, u_{2,n}, u_{3,n}) = \frac{\alpha}{d}\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} u_{j,n}\|^2_{L^2(\mathbb{R}^d)} = c_\lambda + o_n(1). \tag{3.11}$$

By (3.11), we get that $\{u_{j,n}\}$ are bounded in $\mathcal{D}^{\alpha,2}(\mathbb{R}^d)$ for all $j = 1, 2, 3$. On the other hand, since $\{(u_{1,n}, u_{2,n}, u_{3,n})\} \subset \mathcal{N}_\lambda$ and note that $2^*_\alpha > 3$, then by (3.4), (3.7), Young inequality and Lemma 2.1, we can deduce that $\|u_{j,n}\|_{L^2(\mathbb{R}^d)}$ are bounded for $j = 1, 2, 3$. Therefore, $\{(u_{1,n}, u_{2,n}, u_{3,n})\}$ is bounded in $\mathscr{H}_r$, and then there exist $u_1, u_2, u_3 \in H^\alpha_r(\mathbb{R}^d)$ such that for $j = 1, 2, 3$,

$$\begin{cases} u_{j,n} \rightharpoonup u_j, & \text{in } H^\alpha_r(\mathbb{R}^d), \\ u_{j,n} \to u_j, & \text{a.e. in } \mathbb{R}^d, \\ u_{j,n} \to u_j, & \text{in } L^r(\mathbb{R}^d), 2 < r < 2^*_\alpha. \end{cases}$$

From Lemma 3.4, we know that there exists $\{z_n\} \subset \mathbb{R}^d$ and constants $R, \eta > 0$ satisfying

$$\int_{B_R(z_n)} (|u_{1,n}|^2 + |u_{2,n}|^2 + |u_{3,n}|^2)\mathrm{d}x \ge \eta > 0. \tag{3.12}$$

Now we define

$$(\widetilde{u}_{1,n}(x), \widetilde{u}_{2,n}(x), \widetilde{u}_{3,n}(x)) = (u_{1,n}(x + z_n), u_{2,n}(x + z_n), u_{3,n}(x + z_n)),$$

from the invariance of $\mathbb{R}^d$ by translations, then $\{(\widetilde{u}_{1,n}(x), \widetilde{u}_{2,n}(x), \widetilde{u}_{3,n}(x))\}$ is also a minimizing sequence for c_λ. Hence, by arguing as we did for $\{(u_{1,n}, u_{2,n}, u_{3,n})\}$, passing to a subsequence, we can assume that

$$(\widetilde{u}_{1,n}, \widetilde{u}_{2,n}, \widetilde{u}_{3,n}) \rightharpoonup (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$$

in $\mathscr{H}_r$, $(\widetilde{u}_{1,n}, \widetilde{u}_{2,n}, \widetilde{u}_{3,n}) \to (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ in $L^2_{loc}(\mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d)$. Additionally, by (3.12),

$$\liminf_{n\to\infty} \int_{B_R(0)} (|\widetilde{u}_{1,n}|^2 + |\widetilde{u}_{2,n}|^2 + |\widetilde{u}_{3,n}|^2)\mathrm{d}x \ge \eta > 0. \tag{3.13}$$

From (3.13), one has

$$\int_{B_R(0)} (|\tilde{u_1}|^2 + |\tilde{u_2}|^2 + |\tilde{u_3}|^2)\mathrm{d}x \geq \eta > 0,$$

and so $(\tilde{u_1}, \tilde{u_2}, \tilde{u_3}) \neq (0, 0, 0)$. On the other hand, since

$$\{(\widetilde{u}_{1,n}(x), \widetilde{u}_{2,n}(x), \widetilde{u}_{3,n}(x))\} \subset \mathscr{N}_\lambda$$

passing to the limit, we get

$$\mathcal{N}(\tilde{u_1}, \tilde{u_2}, \tilde{u_3}) \leq \liminf_{n\to\infty} \mathcal{N}(\widetilde{u}_{1,n}, \widetilde{u}_{2,n}, \widetilde{u}_{3,n}) = 0,$$

then by Lemma 3.1(iii) there is $\tilde{t} \in (0, 1]$ such that

$$(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) := \left(\tilde{u_1}\Big(\frac{x}{\tilde{t}}\Big), \tilde{u_2}\Big(\frac{x}{\tilde{t}}\Big), \tilde{u_3}\Big(\frac{x}{\tilde{t}}\Big)\right) \in \mathscr{N}_\lambda \cap \mathscr{H}_r.$$

Thus we have

$$\begin{aligned} c_\lambda \leq \Phi_\lambda(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) =& \frac{\alpha}{d}\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} \widetilde{u}_{j\lambda}\|^2_{L^2(\mathbb{R}^d)} \\ =& \frac{\alpha \tilde{t}^{d-2\alpha}}{d}\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}_j\|^2_{L^2(\mathbb{R}^d)} \\ \leq& \frac{\alpha}{d}\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}_j\|^2_{L^2(\mathbb{R}^d)} \\ \leq& \liminf_{n\to\infty} \frac{\alpha}{d}\sum_{j=1}^{3} \|(-\Delta)^{\frac{\alpha}{2}} \widetilde{u}_{j,n}\|^2_{L^2(\mathbb{R}^d)} \\ =& \lim_{n\to\infty} \Phi_\lambda(\widetilde{u}_{1,n}, \widetilde{u}_{2,n}, \widetilde{u}_{3,n}) = c_\lambda, \end{aligned}$$

hence $\Phi_\lambda(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) = c_\lambda$ and $(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda})$ is a minimizer of Φ_λ restricted to $\mathscr{N}_\lambda$. □

Now the proof of Theorem 1.1 will be presented.

Proof of Theorem 1.1. (i) From Lemma 3.5, we know that there exists

$$(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) \in \mathscr{N}_\lambda \cap \mathscr{H}_r,$$

such that $\Phi_\lambda(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) = c_\lambda$. Then by Lemma 3.2, we have that $\Phi'_\lambda(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda}) = 0$, that is, $(\widetilde{u}_{1\lambda}, \widetilde{u}_{2\lambda}, \widetilde{u}_{3\lambda})$ is a least energy solution for the system (1.1). The Theorem 1.1(i) is proved.
(ii) For $j = 1, 2, 3$, let $u_j^\star \in H_r^\alpha(\mathbb{R}^d)$ be the positive least energy solutions respectively for Eq (1.6). Then it is easy to see that $\mathcal{N}(u_1^\star, u_2^\star, u_3^\star) \leq 0$, from Lemma 3.1(iii), there is $t^* > 0$ such that

$$(u_1^\star(\frac{x}{t^*}), u_2^\star(\frac{x}{t^*}), u_3^\star(\frac{x}{t^*})) \in \mathscr{N}_\lambda.$$

Notice that the pairs $(u_1^\star, 0, 0)$, $(0, u_2^\star, 0)$ and $(0, 0, u_3^\star)$ solve system (1.1), and system (1.1) has no solutions with exactly one trivial component. Now, on the basis of idea from [13] (or see [6]), to indicate the radial least energy solution of system (1.1) is a fully nontrivial least energy solution, we just need to prove that, for $\lambda > 0$ sufficiently large,

$$\Phi_\lambda\Big(u_1^\star(\frac{x}{t^*}), u_2^\star(\frac{x}{t^*}), u_3^\star(\frac{x}{t^*})\Big) < \min\{\Phi_\lambda(u_1^\star,0,0), \Phi_\lambda(0,u_2^\star,0), \Phi_\lambda(0,0,u_3^\star)\}. \tag{3.14}$$

Indeed, by some calculations, we can infer that

$$\Phi_\lambda\Big(u_1^\star(\frac{x}{t^*}), u_2^\star(\frac{x}{t^*}), u_3^\star(\frac{x}{t^*})\Big) = \frac{\frac{\alpha}{d}\Big(\sum\limits_{j=1}^{3}\int_{\mathbb{R}^d}|(-\Delta)^{\frac{\alpha}{2}}u_j^\star|^2\mathrm{d}x\Big)^{\frac{d}{2\alpha}}}{\Big(2_\alpha^*\int_{\mathbb{R}^d}(\sum\limits_{j=1}^{3}F_j(u_j^\star) + \lambda|u_1^\star u_2^\star u_3^\star| - \frac{1}{2}\sum\limits_{j=1}^{3}\omega_j|u_j^\star|^2)\mathrm{d}x\Big)^{\frac{d-2\alpha}{2\alpha}}}.$$

Therefore, when $\lambda > 0$ large enough, (3.14) holds. Thus the Theorem 1.1(ii) follows. □

4. Conclusions

In this paper, we are interested in studying a class of systems of three-component coupled nonlinear fractional Schrödinger equations with general nonlinearities. In our assumptions (A_1)–(A_3) neither any monotonicity condition nor any Ambrosetti-Rabinowitz growth condition is required, so we need to overcome several difficulties when using variational methods. By using a Pohozaev manifold method and variational arguments, we establish some novel existence results of least energy solutions for the three-component coupled fractional Schrödinger system (1.1). We believe that the proposed approach in this paper can also be applied to study other related equations and systems.

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Conflict of interest

The authors declare that they have no competing interests.

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