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Research article

Spherical fuzzy rough Hamacher aggregation operators and their application in decision making problem

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Abstract: Aggregation operators are the most effective mathematical tools for aggregating many variables into a single result. The aggregation operators operate to bring together all of the different assessment values offered in a common manner, and they are highly helpful for assessing the options offered in the decision-making process. The spherical fuzzy sets (SFSs) and rough sets are common mathematical tools that are capable of handling incomplete and ambiguous information. We also establish the concepts of spherical fuzzy rough Hamacher averaging and spherical fuzzy rough Hamacher geometric operators. The key characteristics of the suggested operators are thoroughly described. We create an algorithm for a multi-criteria group decision making (MCGDM) problem to cope with the ambiguity and uncertainty. A numerical example of the developed models is shown in the final section. The results show that the specified models are more efficient and advantageous than the other existing approaches when the offered models are contrasted with specific present methods.

Keywords: spherical fuzzy rough set; spherical fuzzy rough Hamacher aggregation operators; decision making
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1. Introduction

The complexity of the social and economic environment makes decision-making (DM) increasingly difficult for people in this continuously changing world. A small decision specialist will have a harder time coming to a clear conclusion in these circumstances. In fact, using group DM models requires merging the viewpoints of a few seasoned authors in order to attain more realistic and desirable goals. In order to produce more rational and sensible DM outcomes, multi-attribute group decision making (MAGDM) considers and assesses a wide range of different attributes across all DM domains. Making decisions is challenging and time-consuming due to the complexity of such a reality, about which DM scenarios have frequently revealed amazing amounts of knowledge.

Zadeh [52] addressed this issue, creating a brand-new kind of fuzzy set that can deal with false data. The fuzzy set information is used to characterize a membership degree (MD), and its value is limited to the adjacent range $[0, 1]$. Atanassov [1] investigated the concept of intuitionistic fuzzy sets (IFS) using the MD and non-membership degree (NMD) functions. The MD and NMD value range for an IFS is $[0, 1]$. Since the inception of IFS, scholars have studied the hybrid effect in great detail, and it is now recognized as a leading area of study. The IFWA (intuitionistic fuzzy weighted average) aggregation operator (AO) was developed by Xu [42]. Intuitionistic fuzzy weighted geometric (IFWG) operators are discussed by Xu and Yager in [41]. Ali et al. [2] presented statistical methods for scoring function and classification accuracy. Averaging aggregation operators based on intuitionistic fuzzy sets were defined by He et al. [19]. He et al. [17] determined by the average factor of the geometric operator and suggested applying it to DM. Zhao et al. [54] established the definitions of the generalized IFWA, IFOWA, and IFHA operators. Wang and Liu describe a variety of averaging and geometric AOs. Seikh and Mandal introduced the idea of intuitionistic fuzzy for the first time in [38]. Dombi enhanced its weighted average and geometric operators and fixed a number of application-specific issues. Huang provided a number of AOs based on the Hamacher t-norm and Hamacher t-conorm in [18]. Xia et al. [43] proposed a variety of AOs based on t-norm and t-conorm.

The Pythagorean fuzzy set (PFS) is another useful technique for illustrating ambiguity in the MADM issues. The PFS model, as opposed to IFS, is better equipped to control the unpredictable parts in the DM problem since it satisfies the conditions $\ddot{a}^2_{\Upsilon}(x) + \ddot{u}^2_{\Upsilon}(x) \leq 1$. The MG and NMG, whose sum of squares are equal to or fewer than 1, are also used for categories PFS. As a result, the PFS is a bigger system than IFS. The PFS is capable of resolving issues that the IFS cannot. As a result, the PFSs is more inclusive and all IFS are seen as being a part of the Pythagorean fuzzy degree. Zhang and Xu [55] invented the Pythagorean fuzzy number (PFN) idea. Additionally, they recommended the PFS detailed mathematical form, the Pythagorean fuzzy TOPSIS methodology, and an order preference strategy that was comparable to the optimal outcome. This strategy was applied within PFNs to address the MCDM issue. Peng and Yang [27] proposed a Pythagorean fuzzy maximum and minimum approach to address the MAGDM problem utilizing PFNs. Additionally, they recommended that PFNs use the division and subtraction procedures. Ren et al. [33] employed the TODIM technique to fix the MCDM problem. Garg [16] were proposed new generalized Pythagorean fuzzy aggregation operators using Einstein's operation.

Ghorabaee et al. [21] were the first to examine the EDAS method for resolving DM issues. This strategy works well in DM situations, especially if there are more conflict criteria present than in MCDM situations. The two most widely used methods for determining separation from PIS and NIS are TOPSIS [47], which are similar to conventional DM procedures. The choice should have been the one that was farthest from the NIS and closest to the PIS. The IFOWA distance and its main characteristics were established and discussed by Zeng and Xiao using the TOPSIS technique in [56]. Wei put out the Grey Relationship Analysis (GRA) approach for MCDM with IFNs data in [36]. Therefore, the ideal applicant ought to have a greater PDAS and a lower NDAS. Ghorabaee et al. [14] used intuitionistic fuzzy data and the EDAS method to choose suppliers.

As $\ddot{a}^2_{\Upsilon}(x) + \ddot{u}^2_{\Upsilon}(x) \leq 1$, the sum of the qth powers of MG and NMG is less than or equal to 1. Yager [48] developed the q-rung orthopair fuzzy sets (q-ROFSs) to represent more decision information in spite of this. PFSs are IFS-specific, while q-ROFS have a wider range of applications. The q-ROF weighted averaging/geometric operators were created by Liu and Wang [24]. Wei et al. [39] defined a number of q-ROF Heronian mean operators. Yang and Pang [50] used q-ROF data. New partitioned Bonferroni mean (BM) operator proposed. Liu and Liu [25] furnished the BM operator with linguistic q-ROF data. Xu et al. [44] created some q-rung dual hesitant orthopair fuzzy Heronian mean operators. An approach for MCDM with q-rung interval-valued orthopair fuzzy information for green supplier selection was presented by Lei and Xu [23] defined a MADM strategy based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers was proposed by Liu and Wang.

Shahzaib et al. [3] first proposed the concept of the spherical fuzzy set (SFS) to address the issue that the picture fuzzy set cannot solve. Then they found a tonne of spherical fuzzy information-based aggregating techniques. Unlike PFSs, where all membership degrees must satisfy the requirement $\ddot{a}_{\Upsilon}(x) + \ddot{u}_{\Upsilon}(x) + \ddot{e}_{\Upsilon}(x) \leq 1$, SFSs require $\ddot{a}^2_{\Upsilon}(x) + \ddot{u}^2_{\Upsilon}(x) + \ddot{e}^2_{\Upsilon}(x) \leq 1$. Ashraf et al. [5], described a few spherical fuzzy aggregation operators and examined their potential use in the Dombi approach to decision-making. Additionally, they looked at how spherical fuzzy t-norm and t-conorm are shown in [4]. Kutlu et al. [22] defined SFSs and the Spherical fuzzy TOPSIS technique were created. Rafiq et al. [34] defined cosine similarity measurements of SFSs and their uses in decision-making. Deli and Agman [11] proposed the concept of spherical fuzzy numbers and an MCDM method using spherical fuzzy set data. Qiyas et al. [29,30] defined spherical fuzzy AOs with sine trigonometry and its use in decision support systems. According to Qiyas et al. [31] Hamacher AOs for Spherical uncertain linguistics were defined, and their use in attaining consistent opinion fusion in group decision-making was examined.

The fuzzy-rough computing model is an important granular computing model that has attracted a lot of attention. Dubois and Prade [10] proposed the fuzzy rough set (FRS) model by combining the advantages of fuzzy sets and rough sets for the first time. Sun et al. [35] defined feature selection with missing labels using multi-label fuzzy neighborhood rough sets and maximum relevance minimum redundancy. Wang et al. [40] developed fuzzy rough set-based attribute reduction using distance measures. Yuan et al. [51] proposed an attribute reduction method in fuzzy rough set theory: an overview, comparative experiments, and new directions. An et al. [7] defined a probability granular distance-based fuzzy rough set model. Hadrani et al. [20] suggested a fuzzy rough set: survey and proposal of an enhanced knowledge representation model based on automatic noisy sample detection. Ahmed and Dai [6] defined the concept of picture fuzzy rough set and rough picture fuzzy set on two different universes and their applications.

The primary objectives of the work are listed as follows:

(1) To construct a new notion of Spherical fuzzy rough sets and analyze their basic operational laws.

(2) The concept of SFRS has been utilized to express the uncertainties in the data.

(3) To develop several weighted aggregation operators to aggregate the collective information.

(4) Some special cases of the proposed operators are deduced under the existing environment.

(5) To establish an MCGDM method based on the proposed operators to solve the problems.

(6) To show the significance and superiority of proposed aggregation operators over existing aggregation operators numerically.

The remainder of the manuscript is organized as follows: Several related definitions are provided in Section 2. By defining a score function and an accuracy function for SFRNs, we have expanded on the idea of SFRS in Section 3. For the suggested technique, some basic SFRN-based operations are offered. Section 4 then examines the idea of average AOs, including SFRHWA, SFRHOWA, and SFRHHA, and their characteristics. In Section 5, the characteristics of geometric AOs including aggregation operators SFRHWG, SFRHOWG, and SFRHHG are comprehensively discussed. Using the provided methodology, a model for MCGDM and the associated step wise algorithm are presented in Section 6. The worth and dependability of the suggested possibilities are demonstrated in Section 7 using the example of selecting the best alternative. The analyzed model is superior to prior possibilities in terms of efficacy and value, as proven by a thorough comparison of the produced models with recent aggregation data in Section 8. The outcomes of our study are then shown.

2. Preliminaries

In this section, which is related to our work, we've provided some fundamental definitions and descriptions of those terms.

Definition 2.1. [52] A fuzzy set $\mathring{U}$ on a fixed set $\mathfrak{R}$ is given as

$$\mathring{U} = \{(r, \ddot{a}_{\mathring{U}}(r))|r \in \mathfrak{R}\}, \tag{2.1}$$

where $\ddot{a}_{\mathring{U}}(r) : \mathfrak{R} \to [0, 1]$ is MD of a fuzzy set $\mathring{U}$.

Definition 2.2. [1] An IFS $\mathring{U}$ on a fixed set $\mathfrak{R}$ is given as

$$\mathring{U} = \{(r, \ddot{a}_{\mathring{U}}(r), \ddot{u}_{\mathring{U}}(r)|r \in \mathfrak{R}\}, \tag{2.2}$$

where $\ddot{a}_{\mathring{U}}(r), \ddot{u}_{\mathring{U}}(r) : \mathfrak{R} \to [0, 1]$ are membership and non-membership functions for every element $r \in \mathfrak{R}$ to the set $\mathfrak{R}$ with the condition $0 \leq \ddot{u}_{\mathring{U}}(r) + \ddot{a}_{\mathring{U}}(r) \leq 1$ for every $r \in \mathfrak{R}$. $\left(\ddot{a}_{\mathring{U}_1}, \ddot{u}_{\mathring{U}_1}\right)$ denotes the intuitionistic fuzzy value (IFV).

Definition 2.3. [24] A q-ROFS $\mathring{U}$ on a fixed set $\mathfrak{R}$ is given as

$$\mathring{U} = \{(r, \ddot{a}_{\mathring{U}}(r), \ddot{u}_{\mathring{U}}(r)|r \in \mathfrak{R}\}, \tag{2.3}$$

where $\ddot{a}_{\mathring{U}}(r), \ddot{u}_{\mathring{U}}(r) : \mathfrak{R} \to [0, 1]$ are the membership degree and non-membership degree functions satisfied the condition $0 \leq \ddot{a}^2_{\mathring{U}}(r) + \ddot{u}^2_{\mathring{U}}(r) \leq 1$ for every $r \in \mathfrak{R}$. $\left(\ddot{a}_{\mathring{U}_1}, \ddot{u}_{\mathring{U}_1}\right)$ denotes the q-ROF number (q-ROFN).

Definition 2.4. [53] Let $\mathfrak{R}$ be a fixed set and $\mathring{U} \in (\mathfrak{R} \times \mathfrak{R})$ represent a crisp relation. Then, the following properties are hold:

(i) $\mathring{U}$ is reflexive if $(\dot{g}, \dot{g}) \in \mathring{U}$, for all $\mathring{U} \in \mathfrak{R}$;

(ii) $\mathring{U}$ is symmetric if $\forall \dot{g}, c \in \mathring{U}, (\dot{g}, c) \in \mathring{U}$. Then, $(c, \dot{g}) \in \mathring{U}$;

(iii) $\mathring{U}$ is transitive if $\forall \dot{g}, c, d \in \mathring{U}$, $(\dot{g}, c) \in \mathring{U}$ and $(c, d) \in \mathring{U}$. Then, $(\dot{g}, d) \in \mathring{U}$.

Definition 2.5. [53] Let $\mathfrak{R}$ a fixed set and $\mathring{U} \in \mathfrak{R} \times \mathfrak{R}$ be any arbitrary relation on set $\mathfrak{R}$. Then, a mapping $\mathring{U}^* : \mathfrak{R} \to \mathring{U}(\mathfrak{R})$ is defined as

$$\mathring{U}^*(r) = \left\{\hat{c} \in \mathfrak{R} \,\middle|\, (r, \hat{c}) \in \mathring{U}\right\} \text{ for } r \in \mathfrak{R}. \tag{2.4}$$

The crisp approximation space is denoted by the pair $\left(\mathfrak{R}, \mathring{U}\right)$, and the successor neighborhood of the object r under $\mathring{U}$ is denoted by $\mathring{U}^*(r)$. For every $\aleph \subseteq \mathfrak{R}$, the lower and higher approximations of $\aleph$ are defined as follows with regard to the approximation space $\left(\mathfrak{R}, \mathring{U}\right)$:

$$\underline{\mathring{U}}(\aleph) = \left\{r \in \mathfrak{R} \,\middle|\, \mathring{U}^*(r) \subseteq \aleph\right\}, \ \overline{\mathring{U}}(\aleph) = \left\{r \in \mathfrak{R} \,\middle|\, \mathring{U}^*(r) \cap \aleph \neq \phi\right\}. \tag{2.5}$$

And $\left(\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph)\right)$ is called rough set and $\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph) : \mathring{U}(\mathfrak{R}) \to \mathring{U}(\mathfrak{R})$ are lower approximation operator and upper approximation operator.

3. Spherical fuzzy rough set

In order to accomplish the concept of spherical fuzzy rough sets, we provide a hybrid notion of rough sets and spherical fuzzy set (SFRSs).

Definition 3.1. A spherical fuzzy relation $\mathring{U} \in SFS\,(\mathfrak{R} \times \mathfrak{R})$ is stated to exist for any subset of the fixed set $\mathfrak{R}$, and the pair $\left(\mathfrak{R}, \mathring{U}\right)$ is referred to as the spherical fuzzy approximation space. The upper and lower approximations of $\aleph$ in relation $\aleph \subseteq SFS\,(\mathfrak{R})$ to the spherical fuzzy approximation space $\left(\mathfrak{R}, \mathring{U}\right)$ are now two SFSs, denoted by $\overline{\mathring{U}}(\aleph)$ and $\underline{J}(\aleph)$, and are defined as

$$\begin{aligned} \overline{\mathring{U}}(\aleph) &= \{(r, \ddot{a}_{\overline{\mathring{U}}(\aleph)}(r), \ddot{u}_{\overline{\mathring{U}}(\aleph)}(r), \ddot{e}_{\overline{\mathring{U}}(\aleph)}(r) | r \in \mathfrak{R}\}, \\ \underline{\mathring{U}}(\aleph) &= \{(r, \ddot{a}_{\underline{\mathring{U}}(\aleph)}(r), \ddot{u}_{\underline{\mathring{U}}(\aleph)}(r), \ddot{e}_{\underline{\mathring{U}}(\aleph)}(r) | r \in \mathfrak{R}\}. \end{aligned} \tag{3.1}$$

As

$$\begin{aligned} \ddot{a}_{\overline{\mathring{U}}(\aleph)}(r) &= \bigvee_{\hat{c} \in \mathfrak{R}} \{\ddot{a}_{\mathring{U}}(r, \hat{c}) \vee \ddot{a}_{\aleph}(\hat{c})\}, \\ \ddot{u}_{\overline{\mathring{U}}(\aleph)}(r) &= \bigwedge_{\hat{c} \in \mathfrak{R}} \{\ddot{u}_{\mathring{U}}(r, \hat{c}) \wedge \ddot{u}_{\aleph}(\hat{c})\}, \\ \ddot{e}_{\overline{\mathring{U}}(\aleph)}(r) &= \bigwedge_{\hat{c} \in \mathfrak{R}} \{\ddot{e}_{\mathring{U}}(r, \hat{c}) \wedge \ddot{e}_{\aleph}(\hat{c})\}, \\ \ddot{a}_{\underline{\mathring{U}}(\aleph)}(r) &= \bigwedge_{\hat{c} \in \mathfrak{R}} \{\ddot{a}_{\mathring{U}}(r, \hat{c}) \wedge \ddot{a}_{\aleph}(\hat{c})\}, \\ \ddot{u}_{\underline{\mathring{U}}(\aleph)}(r) &= \bigvee_{\hat{c} \in \mathfrak{R}} \{\ddot{u}_{\mathring{U}}(r, \hat{c}) \vee \ddot{u}_{\aleph}(\hat{c})\}, \\ \ddot{e}_{\underline{\mathring{U}}(\aleph)}(r) &= \bigvee_{\hat{c} \in \mathfrak{R}} \{\ddot{e}_{\mathring{U}}(r, \hat{c}) \vee \ddot{e}_{\aleph}(\hat{c})\}. \end{aligned}$$

As $0 \le \ddot{a}^2_{\overline{\mathring{U}}(\aleph)}(r) + \ddot{u}^2_{\overline{\mathring{U}}(\aleph)}(r) + \ddot{e}^2_{\overline{\mathring{U}}(\aleph)}(r) \le 1$ and $0 \le \ddot{a}^2_{\underline{\mathring{U}}(\aleph)}(r) + \ddot{u}^2_{\underline{\mathring{U}}(\aleph)}(r) + \ddot{e}^2_{\underline{\mathring{U}}(\aleph)}(r) \le 1$, $\overline{\mathring{U}}(\aleph)$ and $\underline{\mathring{U}}(\aleph)$ are SFSs and $\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph) : \mathring{U}(\mathfrak{R}) \to \mathring{U}(\mathfrak{R})$ are lower and upper approximation operator. Then,

$$\mathring{U}(\aleph) = \left(\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph)\right) = \left\{\left(\begin{array}{c} r, \left(\ddot{a}^2_{\underline{\mathring{U}}(\aleph)}(r), \ddot{u}^2_{\underline{\mathring{U}}(\aleph)}(r), \ddot{e}^2_{\underline{\mathring{U}}(\aleph)}(r)\right), \\ \left(\ddot{a}^2_{\overline{\mathring{U}}(\aleph)}(r), \overline{\ddot{u}}^2_{\overline{\mathring{U}}(\aleph)}(r), \ddot{e}^2_{\overline{\mathring{U}}(\aleph)}(r)\right) \end{array}\right) \middle| r \in \mathfrak{R}\right\} \tag{3.2}$$

is called SFRS. And $\mathring{U}(\aleph)$ can be written $\left(\left(\underline{\ddot{a}}, \underline{\ddot{u}}, \underline{\ddot{e}}\right), \left(\overline{\ddot{a}}, \overline{\ddot{u}}, \overline{\ddot{e}}\right)\right)$ and called SFRN.

Definition 3.2. Let $\mathring{U}(\aleph_1), \mathring{U}(\aleph_2), \mathring{U}(\aleph_3)$ is three SFRSs and $\gamma, \lambda > 0$. Then, the SFRH operations are given as follows:

$$(1)\ \mathring{U}(\aleph_1) \oplus \mathring{U}(\aleph_2) = \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\underline{\ddot{a}}_1^2+\underline{\ddot{a}}_2^2-\underline{\ddot{a}}_1^2\underline{\ddot{a}}_2^2-(1-\gamma)\underline{\ddot{a}}_1^2\underline{\ddot{a}}_2^2}{1-(1-\gamma)\underline{\ddot{a}}_1^2\underline{\ddot{a}}_2^2}}, \frac{\underline{\ddot{u}}_1^2\underline{\ddot{u}}_2^2}{\sqrt[2]{\gamma+(1-\gamma)\left(\underline{\ddot{u}}_1^2+\underline{\ddot{u}}_2^2-\underline{\ddot{u}}_1^2\underline{\ddot{u}}_2^2\right)}}, \\ \frac{\underline{\ddot{e}}_1^2\underline{\ddot{e}}_2^2}{\sqrt{\gamma+(1-\gamma)\left(\underline{\ddot{e}}_1^2+\underline{\ddot{e}}_2^2-\underline{\ddot{e}}_1^2\underline{\ddot{e}}_2^2\right)}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt{\frac{\overline{\ddot{a}}_1^2+\overline{\ddot{a}}_2^2-\overline{\ddot{a}}_1^2\overline{\ddot{a}}_2^2-(1-\gamma)\overline{\ddot{a}}_1^2\overline{\ddot{a}}_2^2}{1-(1-\gamma)\overline{\ddot{a}}_1^2\overline{\ddot{a}}_2^2}}, \frac{\overline{\ddot{u}}_1^2\overline{\ddot{u}}_2^2}{\sqrt{\gamma+(1-\gamma)\left(\overline{\ddot{u}}_1^2+\overline{\ddot{u}}_2^2-\overline{\ddot{u}}_1^2\overline{\ddot{u}}_2^2\right)}}, \\ \frac{\overline{\ddot{e}}_1^2\overline{\ddot{e}}_2^2}{\sqrt{\gamma+(1-\gamma)\left(\overline{\ddot{e}}_1^2+\overline{\ddot{e}}_2^2-\overline{\ddot{e}}_1^2\overline{\ddot{e}}_2^2\right)}} \end{array} \right) \end{array} \right\};$$

$$(2)\ \mathring{U}(\aleph_1) \otimes \mathring{U}(\aleph_2) = \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{\underline{\ddot{a}}_1^2\underline{\ddot{a}}_2^2}{\sqrt{\gamma+(1-\gamma)\left(\underline{\ddot{a}}_1^2+\underline{\ddot{a}}_2^2-\underline{\ddot{a}}_1^2\underline{\ddot{a}}_2^2\right)}}, \sqrt{\frac{\underline{\ddot{u}}_1^2+\underline{\ddot{u}}_2^2-\underline{\ddot{u}}_1^2\underline{\ddot{u}}_2^2-(1-\gamma)\underline{\ddot{u}}_1^2\underline{\ddot{u}}_2^2}{1-(1-\gamma)\underline{\ddot{u}}_1^2\underline{\ddot{u}}_2^2}}, \\ \sqrt{\frac{\underline{\ddot{e}}_1^2+\underline{\ddot{e}}_2^2-\underline{\ddot{e}}_1^2\underline{\ddot{e}}_2^2-(1-\gamma)\underline{\ddot{e}}_1^2\underline{\ddot{e}}_2^2}{1-(1-\gamma)\underline{\ddot{e}}_1^2\underline{\ddot{e}}_2^2}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\overline{\ddot{a}}_1^2\overline{\ddot{a}}_2^2}{\sqrt{\gamma+(1-\gamma)\left(\overline{\ddot{a}}_1^2+\overline{\ddot{a}}_2^2-\overline{\ddot{a}}_1^2\overline{\ddot{a}}_2^2\right)}}, \sqrt{\frac{\overline{\ddot{u}}_1^2+\overline{\ddot{u}}_2^2-\overline{\ddot{u}}_1^2\overline{\ddot{u}}_2^2-(1-\gamma)\overline{\ddot{u}}_1^2\overline{\ddot{u}}_2^2}{1-(1-\gamma)\overline{\ddot{u}}_1^2\overline{\ddot{u}}_2^2}}, \\ \sqrt{\frac{\overline{\ddot{e}}_1^2+\overline{\ddot{e}}_2^2-\overline{\ddot{e}}_1^2\overline{\ddot{e}}_2^2-(1-\gamma)\overline{\ddot{e}}_1^2\overline{\ddot{e}}_2^2}{1-(1-\gamma)\overline{\ddot{e}}_1^2\overline{\ddot{e}}_2^2}} \end{array} \right) \end{array} \right\};$$

$$(3)\ \lambda\mathring{U}(\aleph_1) = \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{a}}_1^2\right)^\lambda-\left(1-\underline{\ddot{a}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\underline{\ddot{a}}_1^2\right)^\lambda+(\gamma-1)\left(1-\underline{\ddot{a}}_1^2\right)^\lambda}}, \frac{\sqrt{\gamma}\left(\underline{\ddot{u}}_1\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{u}}_1^2\right)^\lambda+(\gamma-1)\left(\underline{\ddot{u}}_1^2\right)^\lambda}}, \\ \frac{\sqrt{\gamma}\left(\underline{\ddot{e}}_1\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{e}}_1^2\right)^\lambda+(\gamma-1)\left(\underline{\ddot{e}}_1^2\right)^\lambda}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt{\frac{\left(1+(\gamma-1)\overline{\ddot{a}}_1^2\right)^\lambda-\left(1-\overline{\ddot{a}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\overline{\ddot{a}}_1^2\right)^\lambda+(\gamma-1)\left(1-\overline{\ddot{a}}_1^2\right)^\lambda}}, \frac{\sqrt{\gamma}\left(\overline{\ddot{u}}_1^2\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{u}}_1^2\right)^\lambda+(\gamma-1)\left(\overline{\ddot{u}}_1^2\right)^\lambda}}, \\ \frac{\sqrt{\gamma}\left(\overline{\ddot{e}}_1^2\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{e}}_1^2\right)^\lambda+(\gamma-1)\left(\overline{\ddot{e}}_1^2\right)^\lambda}} \end{array} \right) \end{array} \right\};$$

$$(4)\ \mathring{U}(\aleph_1)^\lambda = \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{\sqrt{\gamma}\left(\underline{\ddot{a}}_1\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{a}}_1^2\right)^\lambda+(\gamma-1)\left(\underline{\ddot{a}}_1^2\right)^\lambda}}, \sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{u}}_1^2\right)^\lambda-\left(1-\underline{\ddot{u}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\underline{\ddot{u}}_1^2\right)^\lambda+(\gamma-1)\left(1-\underline{\ddot{u}}_1^2\right)^\lambda}}, \\ \sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{e}}_1^2\right)^\lambda-\left(1-\underline{\ddot{e}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\underline{\ddot{e}}_1^2\right)^\lambda+(\gamma-1)\left(1-\underline{\ddot{e}}_1^2\right)^\lambda}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{\gamma}\left(\overline{\ddot{a}}_1\right)^\lambda}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{a}}_1^2\right)^\lambda+(\gamma-1)\left(\overline{\ddot{a}}_1^2\right)^\lambda}}, \sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{u}}_1^2\right)^\lambda-\left(1-\underline{\ddot{u}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\underline{\ddot{u}}_1^2\right)^\lambda+(\gamma-1)\left(1-\underline{\ddot{u}}_1^2\right)^\lambda}}, \\ \sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{e}}_1^2\right)^\lambda-\left(1-\underline{\ddot{e}}_1^2\right)^\lambda}{\left(1+(\gamma-1)\underline{\ddot{e}}_1^2\right)^\lambda+(\gamma-1)\left(1-\underline{\ddot{e}}_1^2\right)^\lambda}} \end{array} \right) \end{array} \right\}.$$

Definition 3.3. Let $\mathring{U}(\aleph) = \left(\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph)\right) = \left(\left(\underline{\ddot{a}}, \underline{\ddot{u}}, \underline{\ddot{e}}\right), \left(\overline{\ddot{a}}, \overline{\ddot{u}}, \overline{\ddot{e}}\right)\right)$ be a SFRN. Then, the score function

$S(\mathring{U}(\aleph))$ is given as

$$S(\mathring{U}(\aleph)) = \left[\left(\underline{\ddot{a}}^2 + \overline{\ddot{a}}^2 - \underline{\ddot{u}}^2 - \overline{\ddot{u}}^2 - \underline{\ddot{e}}^2 - \overline{\ddot{e}}^2\right)/3\right], \tag{3.3}$$

where $S(\mathring{U}(\aleph)) \in [-1, 1]$.

Definition 3.4. Let $\mathring{U}(\aleph) = \left(\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph)\right) = \left(\left(\underline{\ddot{a}}, \underline{\ddot{u}}, \underline{\ddot{e}}\right), \left(\overline{\ddot{a}}, \overline{\ddot{u}}, \overline{\ddot{e}}\right)\right)$ be an SFRN. The accuracy function $H(\mathring{U}(\aleph))$ is given as

$$H(\mathring{U}(\aleph)) = [\left(\underline{\ddot{a}}^2 + \overline{\ddot{a}}^2 + \underline{\ddot{u}}^2 + \overline{\ddot{u}}^2 + \underline{\ddot{e}}^2 + \overline{\ddot{e}}^2\right)/3],$$

where $H(\mathring{U}(\aleph)) \in [0, 1]$.

Definition 3.5. Let $\mathring{U}(\aleph_1) = \left(\underline{\mathring{U}}(\aleph_1), \overline{\mathring{U}}(\aleph_1)\right)$, and $\mathring{U}(\aleph_2) = \left(\underline{\mathring{U}}(\aleph_2), \overline{\mathring{U}}(\aleph_2)\right)$ are two SFRNs. Then, the comparison rules are defined as

i) $S(\mathring{U}(\aleph_2)) < S(\mathring{U}(\aleph_1)) \Longrightarrow \mathring{U}(\aleph_2) < \mathring{U}(\aleph_1)$.

ii) $S(\mathring{U}(\aleph_2)) = S(\mathring{U}(\aleph_1))$, and

a) $H(\mathring{U}(\aleph_2)) < H(\mathring{U}(\aleph_1)) \Longrightarrow \mathring{U}(\aleph_2) < \mathring{U}(\aleph_1)$,

b) $H(\mathring{U}(\aleph_2)) = H(\mathring{U}(\aleph_1)) \Longrightarrow \mathring{U}(\aleph_2) = \mathring{U}(\aleph_1)$.

4. Spherical fuzzy rough Hamacher averaging aggregation operators

Here, we define the spherical fuzzy rough Hamacher aggregation operators and go through some of its fundamental features.

4.1. Spherical fuzzy rough Hamacher weighted averaging operator

Definition 4.1. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a collection of SFRNs with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, and $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, the SFRHWA operator is described as

$$SFRHWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \bigoplus_{i=1}^{n} \varpi_i \mathring{U}(\aleph_i), \tag{4.1}$$

where the weight vector of $\mathring{U}(\aleph_i)$ are $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\varpi_i \in [0, 1]$ and $\sum_{i=1}^n \varpi_i = 1$.

Theorem 4.1. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a collection of SFRNs. Then, aggregated value utilizing SFRHWA operator is again a SFRN, and

$$\begin{aligned} & SFRHWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \\ = & \left(\bigoplus_{i=1}^{n} \varpi_i \underline{\mathring{U}}(\aleph_i), \bigoplus_{i=1}^{n} \varpi_i \overline{\mathring{U}}(\aleph_i)\right) \\ = & \left\{ \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \right\}. \end{aligned} \tag{4.2}$$

Proof. We'll use mathematical induction to prove this theorem.

(i) When $n = 1$, we have

$$
\begin{aligned}
& SFRHWA\left(\mathring{U}(\aleph)\right) \\
= & \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{1+(\gamma-1)\underline{\ddot{a}}^2-\left(1-\underline{\ddot{a}}^2\right)}{1+(\gamma-1)\underline{\ddot{a}}^2+(\gamma-1)\left(1-\underline{\ddot{a}}^2\right)}}, \frac{\sqrt{\gamma}\left(\underline{\ddot{u}}\right)}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{u}}^2\right)+(\gamma-1)\underline{\ddot{u}}^2}}, \\ \frac{\sqrt{\gamma}\left(\underline{\ddot{e}}\right)}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{e}}^2\right)+(\gamma-1)\underline{\ddot{e}}^2}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt{\frac{1+(\gamma-1)\overline{\ddot{a}}^2-\left(1-\overline{\ddot{a}}^2\right)}{1+(\gamma-1)\overline{\ddot{a}}^2+(\gamma-1)(1-\overline{\ddot{a}}^2)}}, \frac{\sqrt{\gamma}\left(\overline{\ddot{u}}\right)}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{u}}^2\right)+(\gamma-1)\overline{\ddot{u}}^2}}, \\ \frac{\sqrt{\gamma}\left(\overline{\ddot{e}}\right)}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{e}}^2\right)+(\gamma-1)\overline{\ddot{e}}^2}} \end{array} \right) \end{array} \right\}.
\end{aligned}
$$

Thus, for $n = 1$, Eq (4.2) is hold.

(ii) Let $n = k$, Eq (4.2) be hold, then from Eq (4.2),

$$
\begin{aligned}
& SFRHWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_k)\right) \\
= & \bigoplus_{i=1}^{k} \varpi_i \mathring{U}(\aleph_i) \\
= & \left\{ \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^k\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^k\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^k\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^2\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^2\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^k\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^k\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^k\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^2\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^2\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \right\}.
\end{aligned}
$$

Now, for $n = k + 1$, we have

$$
\begin{aligned}
& SFRHWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_k), \mathring{U}(\aleph_{k+1})\right) \\
= & \bigoplus_{i=1}^{k} \varpi_i \mathring{U}(\aleph_i) \oplus \varpi_{k+1} \mathring{U}(\aleph_{k+1}) \\
= & \left\{ \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^k\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^k\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^k\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^k\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^k\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^k\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^k\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^k(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^k\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \right\}
\end{aligned}
$$

$$\oplus\left\{\begin{array}{l}\left(\begin{array}{c}\sqrt{\frac{\left(1+(\gamma-1)\underline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}-\left(1-\underline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}}{\left(1+(\gamma-1)\underline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)\left(1-\underline{\ddot{a}}_{\mathring{U}_{k+1}}^{2}\right)^{\varpi_{k+1}}}},\\ \frac{\sqrt{\gamma}\left(\underline{\ddot{u}}_{k+1}\right)^{\varpi_{k+1}}}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{u}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)(\underline{\ddot{u}}_{k+1}^{2})^{\varpi_{k+1}}}},\\ \frac{\sqrt{\gamma}\left(\underline{\ddot{e}}_{k+1}\right)^{\varpi_{k+1}}}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{e}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)(\underline{\ddot{e}}_{k+1}^{2})^{\varpi_{k+1}}}}\end{array}\right),\left(\begin{array}{c}\sqrt{\frac{\left(1+(\gamma-1)\overline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}-\left(1-\overline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}}{\left(1+(\gamma-1)\overline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)\left(1-\overline{\ddot{a}}_{k+1}^{2}\right)^{\varpi_{k+1}}}},\\ \frac{\sqrt{\gamma}\left(\overline{\ddot{u}}_{k+1}\right)^{\varpi_{k+1}}}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{u}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)(\overline{\ddot{u}}_{k+1}^{2})^{\varpi_{k+1}}}},\\ \frac{\sqrt{\gamma}\left(\overline{\ddot{e}}_{k+1}\right)^{\varpi_{k+1}}}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{e}}_{k+1}^{2}\right)^{\varpi_{k+1}}+(\gamma-1)(\overline{\ddot{e}}_{k+1}^{2})^{\varpi_{k+1}}}}\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left(\begin{array}{c}\sqrt{\frac{\Pi_{i=1}^{k+1}\left(1+(\gamma-1)\underline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{k+1}\left(1-\underline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{k+1}\left(1+(\gamma-1)\underline{\ddot{a}}_{\mathring{U}_i}^{2}\right)^{\varpi\mathring{U}}+(\gamma-1)\Pi_{i=1}^{k+1}\left(1-\underline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{k+1}\left(\underline{\ddot{u}}_{i}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{k+1}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_{i}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{k+1}\left(\underline{\ddot{u}}_{i}^{2}\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{k+1}\left(\underline{\ddot{e}}_{i}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{k+1}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_{i}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{k+1}\left(\underline{\ddot{e}}_{i}^{2}\right)^{\varpi_i}}}\end{array}\right),\left(\begin{array}{c}\sqrt{\frac{\Pi_{i=1}^{k+1}\left(1+(\gamma-1)\overline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{k+1}\left(1-\overline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{k+1}\left(1+(\gamma-1)\overline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{k+1}\left(1-\overline{\ddot{a}}_{i}^{2}\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{k+1}\left(\overline{\ddot{u}}_{i}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{k+1}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_{i}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{k+1}\left(\overline{\ddot{u}}_{i}^{2}\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{k+1}\left(\overline{\ddot{u}}_{i}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{k+1}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_{i}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{k+1}\left(\overline{\ddot{e}}_{i}^{2}\right)^{\varpi_i}}}\end{array}\right)\end{array}\right\}.$$

Thus, step (i) and (ii) shows that (4.2) is holds for all values of n.

Effect of parameter γ on SFRHWA operator

When the parameter γ has the values 1 or 2, there are two special cases in which the SFRHWA operator is implemented.

Case 1. The SFRHWA operator's structure is reduced to the SFRWA operator if $\gamma = 1$.

$$\begin{aligned}&SFRWA\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right)\\ =&\left\{\begin{array}{l}\left(\sqrt{1-\Pi_{i=1}^{n}(1-\underline{\ddot{a}}_i^2)^{\varpi_i}},\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i},\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i\right)^{\varpi_i}\right),\\ \left(\sqrt{1-\Pi_{i=1}^{n}(1-\overline{\ddot{a}}_i^2)^{\varpi_i}},\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i},\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i\right)^{\varpi_i}\right)\end{array}\right\}.\end{aligned} \tag{4.3}$$

Case 2. The SFRHWA operator's structure is reduced to the spherical fuzzy rough Einstein weighted averaging (SFREWA) operator if $\gamma = 2$.

$$SFREWA_{\varpi}\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right)=\left\{\begin{array}{l}\left(\begin{array}{c}\sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{a}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}},\frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\underline{\ddot{u}}_i^2)^{\varpi_i}+\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\underline{\ddot{e}}_i^2)^{\varpi_i}+\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}}\end{array}\right),\\ \left(\begin{array}{c}\sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{a}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}},\frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}(\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}(\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}}\end{array}\right)\end{array}\right\}. \tag{4.4}$$

Based on the Theorem 4.1, the SFRHWA operator has the following characteristics.

Proposition 4.1. Let $\mathring{U}(\aleph_i)=\left(\underline{\mathring{U}}(\aleph_i),\overline{\mathring{U}}(\aleph_i)\right)(i=1,...,n)$ be a set of SFRNs with the weights are $\varpi=(\varpi_1,...,\varpi_n)^T$, with $\sum_{i=1}^{n}\varpi_i=1$ and $\varpi_i>0$. Here are some examples of properties.

Idempotency: For all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are equal, i.e., $\mathring{U}(\aleph) = \mathring{U}(\aleph_i)$, then,

$$SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{4.5}$$

Proof. Since $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, i.e., $\underline{\ddot{a}}_i = \underline{\ddot{a}}$, $\overline{\ddot{a}}_i = \overline{\ddot{a}}$, $\underline{\ddot{u}}_i = \underline{\ddot{u}}$, $\overline{\ddot{e}}_i = \overline{\ddot{e}}$,

$$\begin{aligned}
& SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \\
= & \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\} \\
= & \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{1+(\gamma-1)\underline{\ddot{a}}_i^2\left(1-\underline{\ddot{a}}\right)}{1+(\gamma-1)\underline{\ddot{a}}^2+(\gamma-1)\left(1-\underline{\ddot{a}}^2\right)}}, \frac{\sqrt{\gamma}\left(\underline{\ddot{u}}\right)}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{u}}^2\right)+(\gamma-1)\underline{\ddot{u}}^2}}, \\ \frac{\sqrt{\gamma}\left(\underline{\ddot{e}}\right)}{\sqrt{1+(\gamma-1)\left(1-\underline{\ddot{e}}^2\right)+(\gamma-1)\underline{\ddot{e}}^2}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt{\frac{1+(\gamma-1)\overline{\ddot{a}}^2-(1-\overline{\ddot{a}}^2)}{1+(\gamma-1)\overline{\ddot{a}}^2+(\gamma-1)\left(1-\overline{\ddot{a}}^2\right)}}, \frac{\sqrt{\gamma}\left(\overline{\ddot{u}}\right)}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{u}}^2\right)+(\gamma-1)\overline{\ddot{u}}^2}}, \\ \frac{\sqrt{\gamma}\left(\overline{\ddot{u}}\right)}{\sqrt{1+(\gamma-1)\left(1-\overline{\ddot{e}}^2\right)+(\gamma-1)\overline{\ddot{e}}^2}} \end{array} \right) \end{array} \right\} \\
= & \left(\left(\underline{\ddot{a}}, \underline{\ddot{u}}, \underline{\ddot{e}}\right), \left(\overline{\ddot{a}}, \overline{\ddot{u}}, \overline{\ddot{e}}\right)\right) = \mathring{U}(\aleph),
\end{aligned}$$

proved.

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are the set of SFRNs, and $\mathring{U}(\aleph)^{+} = \left(\left(\max \underline{\ddot{a}}_i, \max \overline{\ddot{a}}_i\right), \left(\min \underline{\ddot{u}}_i, \min \overline{\ddot{u}}_i\right), \left(\min \underline{\ddot{e}}_i, \min \overline{\ddot{e}}_i\right)\right)$, $\mathring{U}(\aleph)^{-} = \left(\left(\min \underline{\ddot{a}}_i, \min \overline{\ddot{a}}_i\right), \left(\max \underline{\ddot{u}}_i, \max \overline{\ddot{u}}_i\right), \left(\max \underline{\ddot{e}}_i, \max \overline{\ddot{e}}_i\right)\right)$. Then,

$$\mathring{U}(\aleph)^{-} \leq SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^{+}. \tag{4.6}$$

Monotonicity: Let $\mathring{U}(\aleph^*) = \left(\left(\underline{\ddot{a}}_i^*, \underline{\ddot{u}}_i^*, \underline{\ddot{e}}_i^*\right), \left(\overline{\ddot{a}}_i^*, \overline{\ddot{u}}_i^*, \overline{\ddot{e}}_i^*\right)\right)(i = 1, ..., n)$, be a set of SFRNs, if $\underline{\ddot{a}}_i \leq \underline{\ddot{a}}_i^*$, $\underline{\ddot{u}}_i^* \geq \underline{\ddot{u}}_i$, $\underline{\ddot{e}}_i^* \geq \underline{\ddot{e}}_i$, $\overline{\ddot{a}}_i \leq \overline{\ddot{a}}_i^*$, $\overline{\ddot{u}}_i \geq \overline{\ddot{u}}_i^*$, $\overline{\ddot{e}}_i \geq \overline{\ddot{e}}_i^*$ for all i. Then,

$$SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1^*), ..., \mathring{U}(\aleph_n^*)\right). \tag{4.7}$$

Proof.

$$SFRHWA_{\varpi}\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right)=\left\{\begin{array}{c}\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^2\right)^{\varpi_i}}}\\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^2\right)^{\varpi_i}}},\\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^2\right)^{\varpi_i}}}\end{array}\right\},$$

as

$$\begin{aligned}&\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_{\mathring{U}}}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_{\mathring{U}}}}}\\ \leq\;&\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}},\\ &\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\,1-\underline{\ddot{u}}_i^{2*})^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\underline{\ddot{u}}_i^{2*})^{\varpi_i}}}\\ \geq\;&\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\underline{\ddot{u}}_i^{2})^{\varpi_i}}},\\ &\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\,1-\underline{\ddot{e}}_i^{2*})^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\underline{\ddot{e}}_i^{2*})^{\varpi_i}}}\\ \geq\;&\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\underline{\ddot{e}}_i^{2})^{\varpi_i}}},\end{aligned}\tag{4.8}$$

$$\begin{aligned}&\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}}\\ \leq\;&\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}}},\end{aligned}\tag{4.9}$$

$$
\begin{aligned}
& \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^{2}\right)^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\overline{\ddot{u}}_i^{2})^{\varpi_i}}} \\
\geq\ & \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\overline{\ddot{u}}_i^{2*})^{\varpi_i}}}, \\
& \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\overline{\ddot{e}}_i^{2})^{\varpi_i}}} \\
\geq\ & \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\,\Pi_{i=1}^{n}(\overline{\ddot{e}}_i^{2*})^{\varpi_i}}}.
\end{aligned}
$$

Now, from Eqs (4.8) and (4.9), we get

$$
\left\{
\begin{array}{c}
\left(
\begin{array}{c}
\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}(\underline{\ddot{u}}_i)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}(\underline{\ddot{e}}_i)^{\varpi_i}}}
\end{array}
\right), \\
\left(
\begin{array}{c}
\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^{2}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^{2}\right)^{\varpi_i}}}
\end{array}
\right)
\end{array}
\right\}
\leq
\left\{
\begin{array}{c}
\left(
\begin{array}{c}
\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_i^{2*}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^{*}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_i^{2}\right))^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_i^{2*}\right)^{\varpi_i}}}
\end{array}
\right), \\
\left(
\begin{array}{c}
\sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_i^{2*}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^{*}\right)^{\varpi i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_i^{2*}\right)^{\varpi_i}}}, \\
\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^{*}\right)^{\varpi i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_i^{2*}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_i^{2*}\right)^{\varpi_i}}}
\end{array}
\right)
\end{array}
\right\},
$$

i.e.,

$$
SFRHWA_{\varpi}\left(\mathring{U}\left(\aleph_1\right),...,\mathring{U}\left(\aleph_n\right)\right) \leq SFRHWA_{\varpi}\left(\mathring{U}\left(\aleph_1^{*}\right),...,\mathring{U}\left(\aleph_n^{*}\right)\right).
$$

Proved.

4.2. Spherical fuzzy rough Hamacher ordered weighted averaging operator

Here, we've discuss the SFRHOWA operator and its key characteristics, including idempotency, boundedness and monotonicity.

Definition 4.2. Let $\mathring{U}\left(\aleph_i\right) = \left(\underline{\mathring{U}}\left(\aleph_i\right), \overline{\mathring{U}}\left(\aleph_i\right)\right) (i = 1, ..., n)$ be a set of SFRNs with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, such as $\sum_{i=1}^{n}\varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, SFRHOWA operator is described as

$$
SFRHOWA\left(\mathring{U}\left(\aleph_1\right),...,\mathring{U}\left(\aleph_n\right)\right) = \bigoplus_{i=1}^{n}\varpi_i\mathring{U}\left(\aleph_{\sigma(i)}\right). \tag{4.10}
$$

Theorem 4.2. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a collection of SFRNs. Then, aggregated value utilizing SFRHOWA operator is again a SFRN, and

$$
\begin{aligned}
& SFRHOWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \qquad (4.11)\\
= & \left(\bigoplus_{i=1}^{n} \varpi_i \underline{\mathring{U}}(\aleph_{\sigma(i)}), \bigoplus_{i=1}^{n} \varpi_i \overline{\mathring{U}}(\aleph_{\sigma(i)})\right)\\
= & \left\{ \begin{pmatrix} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} - \Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} - \Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{pmatrix} \right\}.
\end{aligned}
$$

Effect of parametern γ on SFRHOWA operator

Here, we looked at two specific cases of the SFRHOWA operator using various values of the parameter γ.

Case 1. SFRHOWA operator reduced to the SFROWA operator if $\gamma = 1$.

$$
\begin{aligned}
& SFROWA_{\varpi}(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) \qquad (4.12)\\
= & \left\{ \begin{array}{c} \left(\sqrt{1-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}, \Pi_{i=1}^{n}\underline{\ddot{u}}_{\sigma(i)}^{\varpi_i}, \Pi_{i=1}^{n}\underline{\ddot{e}}_{\sigma(i)}^{\varpi_i}\right), \\ \left(\sqrt{1-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}, \Pi_{i=1}^{n}\overline{\ddot{u}}_{\sigma(i)}^{\varpi_i}, \Pi_{i=1}^{n}\overline{\ddot{e}}_{\sigma(i)}^{\varpi_i}\right) \end{array} \right\}.
\end{aligned}
$$

Case 2. The SFRHOWA operator's structure is reduced to the spherical fuzzy rough Einstein ordered weighted averaging (SFREOWA) operator if $\gamma = 2$.

$$
\begin{aligned}
& SFREOWA_{\varpi}(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) \qquad (4.13)\\
= & \left\{ \begin{array}{c} \begin{pmatrix} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} - \Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} + \Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\underline{\ddot{u}}_{\sigma(i)}^{2})^{\varpi_i} + \Pi_{i=1}^{n}\left(\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\underline{\ddot{e}}_{\sigma(i)}^{2})^{\varpi_i} + \Pi_{i=1}^{n}\left(\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{pmatrix} \\ \begin{pmatrix} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} - \Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i} + (\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(\left(2-\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i} + \Pi_{i=1}^{n}\left(\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(\left(2-\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i} + \Pi_{i=1}^{n}\left(\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{pmatrix} \end{array} \right\}.
\end{aligned}
$$

Proposition 4.2. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs in $\mathfrak{R}$ and the weight vector of $\mathring{U}(\aleph_i)$ be $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, the properties listed below are established.

Idempotency: If all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are equal, i.e., $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, then

$$SFRHOWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{4.14}$$

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs, and $\mathring{U}(\aleph)^+ = \left(\max \underline{\ddot{a}}_{\sigma(i)}, \max \overline{\ddot{a}}_{\sigma(i)}\right), \left(\min \underline{\ddot{u}}_{\sigma(i)}, \min \overline{\ddot{u}}_{\sigma(i)}\right), \left(\min \underline{\ddot{e}}_{\sigma(i)}, \min \overline{\ddot{e}}_{\sigma(i)}\right)$, $\mathring{U}(\aleph)^- = \left(\max \underline{\ddot{a}}_{\sigma(i)}, \max \overline{\ddot{a}}_{\sigma(i)}\right), \left(\min \underline{\ddot{u}}_{\sigma(i)}, \min \overline{\ddot{u}}_{\sigma(i)}\right), \left(\min \underline{\ddot{e}}_{\sigma(i)}, \min \overline{\ddot{e}}_{\sigma(i)}\right)$. Then,

$$\mathring{U}(\aleph)^- \leq SFRHOWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^+. \tag{4.15}$$

Monotonicity: Let $\mathring{U}(\aleph^*) = \left(\underline{\ddot{a}}^*_{\sigma(i)}, \underline{\ddot{u}}^*_{\sigma(i)}, \underline{\ddot{e}}^*_{\sigma(i)}\right), \left(\overline{\ddot{a}}^*_{\sigma(i)}, \overline{\ddot{u}}^*_{\sigma(i)}, \overline{\ddot{e}}^*_{\sigma(i)}\right)(i = 1, ..., n)$, be a group of SFRNs, if $\underline{\ddot{a}}_{\sigma(i)} \leq \underline{\ddot{a}}^*_{\sigma(i)}$, $\underline{\ddot{u}}^*_{\sigma(i)} \geq \underline{\ddot{u}}_{\sigma(i)}$, $\underline{\ddot{e}}^*_{\sigma(i)} \geq \underline{\ddot{e}}_{\sigma(i)}$, $\overline{\ddot{a}}_{\sigma(i)} \leq \overline{\ddot{a}}^*_{\sigma(i)}$, $\overline{\ddot{u}}_{\sigma(i)} \geq \overline{\ddot{u}}^*_{\sigma(i)}$, $\overline{\ddot{e}}_{\sigma(i)} \geq \overline{\ddot{e}}^*_{\sigma(i)}$ for all i. Then,

$$SFRHOWA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq SFRHOWA\left(\mathring{U}(\aleph^*_1), ..., \mathring{U}(\aleph^*_n)\right). \tag{4.16}$$

4.3. Spherical fuzzy rough Hamacher hybrid averaging operator

In this section, we define the SFRHHA operator and look at its characteristics.

Definition 4.3. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, such as $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, SFRHHA is operator defined as

$$SFRHHA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \bigoplus_{n=1}^{n} \varpi_i \mathring{U}\left(\aleph'_{\sigma(i)}\right), \tag{4.17}$$

where $\mathring{U}\left(\aleph'_{\sigma(i-1)}\right) \geq \mathring{U}\left(\aleph'_{\sigma(i)}\right)$. The SFRHHA operator change into the following form by induction on the fundamental SFRH operational laws of SFRNs. Here, $\mathring{U}\left(\aleph'_{\sigma(i)}\right)$ is the i^{th} biggest value of the weighted SFRNs. Also, $w = (w_1, ..., w_n)^T$ is the weight of $\mathring{U}(\aleph_i)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, i.e.,

$$\mathring{U}\left(\aleph'_{\sigma(i)}\right) = nw_i \mathring{U}(\aleph_{\sigma(i)}) = \left(\left(\underline{\ddot{a}}'_{\sigma(i)}, \underline{\ddot{u}}'_{\sigma(i)}, \underline{\ddot{e}}'_{\sigma(i)}\right), \left(\overline{\ddot{a}}'_{\sigma(i)}, \overline{\ddot{u}}'_{\sigma(i)}, \overline{\ddot{e}}'_{\sigma(i)}\right)\right)(i = 1, ..., n),$$

$$\begin{aligned}
\underline{\ddot{a}}'_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\underline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \underline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\underline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \underline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i}}}, \\
\overline{\ddot{a}}'_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\overline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \overline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\overline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \overline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i}}}, \\
\underline{\ddot{u}}'_{\sigma(i)} &= \frac{\left(1 + (\gamma - 1)\underline{\ddot{u}}_{\sigma(i)}\right)^{nw_i} - \left(1 - \underline{\ddot{u}}_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\underline{\ddot{u}}_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \underline{\ddot{u}}_{\sigma(i)}\right)^{nw_i}},
\end{aligned}$$

$$\overline{\ddot{u}}^{/}_{\sigma(i)} = \frac{\left(1+(\gamma-1)\,\overline{\ddot{u}}_{\mathring{U}_{\sigma(i)}}\right)^{nw_i}-\left(1-\overline{\ddot{u}}_{\sigma(i)}\right)^{nw_i}}{\left(1+(\gamma-1)\,\overline{\ddot{u}}_{\sigma(i)}\right)^{nw_i}+(\gamma-1)\left(1-\overline{\ddot{u}}_{\sigma(i)}\right)^{nw_i}},$$

$$\underline{b}^{/}_{\sigma(i)} = \frac{\left(1+(\gamma-1)\,\underline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}-\left(1-\underline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}}{\left(1+(\gamma-1)\,\underline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}+(\gamma-1)\left(1-\underline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}},$$

$$\overline{\ddot{e}}^{/}_{\sigma(i)} = \frac{\left(1+(\gamma-1)\,\overline{\ddot{e}}_{\mathring{U}_{\sigma(i)}}\right)^{nw_i}-\left(1-\overline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}}{\left(1+(\gamma-1)\,\overline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}+(\gamma-1)\left(1-\overline{\ddot{e}}_{\sigma(i)}\right)^{nw_i}}.$$

The SFRHWA and SFRHOWA operators are referred to as a particular case of the SFRHH averaging operator, when $\varpi = (1/n, ..., 1/n)^T$.

Theorem 4.3. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs. Then, aggregated value utilizing SFRHHWA operator is again a SFRN , and

$$SFRHHA\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \tag{4.18}$$
$$= \left(\bigoplus_{i=1}^{n} \varpi_i \underline{\mathring{U}}\left(\aleph^{/}_{\sigma(i)}\right), \bigoplus_{i=1}^{n} \varpi_i \overline{\mathring{U}}\left(\aleph^{/}_{\sigma(i)}\right)\right)$$
$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{u}}^{/}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\underline{\ddot{u}}^{/2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{u}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{e}}^{/}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\underline{\ddot{e}}^{/2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{e}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{u}}^{2}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\overline{\ddot{u}}^{/2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{u}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{e}}^{2}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\overline{\ddot{e}}^{/2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{e}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\}.$$

Effect of parameter γ on SFRHHA operator

Here, we looked at two specific cases of the SFRHHA operator using various values of the parameter γ.

Case 1. The SFRHHA operator reduced to the SFRHHA operator if $\gamma = 1$.

$$SFRHA(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) \tag{4.19}$$
$$= \left\{ \begin{array}{c} \left(\sqrt{1-\Pi_{i=1}^{n}\left(1-\underline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}, \Pi_{i=1}^{n}\left(\underline{\ddot{u}}^{/}_{\sigma(i)}\right)^{\varpi_i}, \Pi_{i=1}^{n}\left(\underline{\ddot{e}}^{/}_{\sigma(i)}\right)^{\varpi_i}\right) \\ \left(\sqrt{1-\Pi_{i=1}^{n}\left(1-\overline{\ddot{a}}^{/2}_{\sigma(i)}\right)^{\varpi_i}}, \Pi_{i=1}^{n}\left(\overline{\ddot{u}}^{/}_{\sigma(i)}\right)^{\varpi_i}, \Pi_{i=1}^{n}\left(\overline{\ddot{e}}^{/}_{\sigma(i)}\right)^{\varpi_i}\right) \end{array} \right\}.$$

Case 2. The SFRHHA operator's structure is reduced to the spherical fuzzy rough Einstein hybrid averaging (SFREHA) operator if $\gamma = 2$.

$$SFREHA(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)) \tag{4.20}$$
$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^n\left(1+\underline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+\underline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}+\Pi_{i=1}^n\left(1-\underline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^n\left(\underline{\ddot{u}}_{\sigma(i)}^{/}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n\left(2-\underline{\ddot{u}}_{\sigma(i)}^{/2}\right)^{\varpi_i}+\Pi_{i=1}^n\left(\underline{\ddot{u}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}}, \frac{\sqrt{2}\Pi_{i=1}^n\left(\underline{\ddot{e}}_{\sigma(i)}^{/}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n\left(2-\underline{\ddot{e}}_{\sigma(i)}^{/2}\right)^{\varpi_i}+\Pi_{i=1}^n\left(\underline{\ddot{e}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}} \end{array} \right), \\ \left(\begin{array}{c} \sqrt{\frac{\Pi_{i=1}^n\left(1+\overline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\overline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+\overline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\overline{\ddot{a}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}}, \\ \frac{\sqrt{2}\Pi_{i=1}^n\left(\overline{\ddot{u}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(2-\overline{\ddot{u}}_{\sigma(i)}^{/2})^{\varpi_i}+\Pi_{i=1}^n\left(\overline{\ddot{u}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}}, \frac{\sqrt{2}\Pi_{i=1}^n\left(\overline{\ddot{e}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(2-\overline{\ddot{e}}_{\sigma(i)}^{/2})^{\varpi_i}+\Pi_{i=1}^n\left(\overline{\ddot{e}}_{\sigma(i)}^{/2}\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\}.$$

Proposition 4.3. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1,...,n)$ be a set of SFRNs and $\varpi = (\varpi_1,...,\varpi_n)^T$ be the weight vector of $\aleph_i$, with $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then, the below characteristics are defined.

Idempotency: If all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1,...,n)$ are equal, i.e., $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, then,

$$SFRHHA\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{4.21}$$

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1,...,n)$ be a set of SFRNs, and $\mathring{U}(\aleph)^+ = \left(\left(\max \underline{\ddot{a}}_{\sigma(i)}^{/}, \min \underline{\ddot{u}}_{\sigma(i)}^{/}, \min \underline{\ddot{e}}_{\sigma(i)}^{/}\right), \left(\max \overline{\ddot{a}}_{\sigma(i)}^{/}, \min \overline{\ddot{u}}_{\sigma(i)}^{/}, \min \overline{\ddot{e}}_{\sigma(i)}^{/}\right)\right)$, $\mathring{U}(\aleph)^- = \left(\left(\min \overline{\ddot{a}}_{\sigma(i)}^{/}, \max \underline{\ddot{u}}_{\sigma(i)}^{/}, \max \underline{\ddot{e}}_{\sigma(i)}^{/}\right), \left(\min \overline{\ddot{a}}_{\mathring{U}_{\sigma(i)}^{\prime}}, \max \overline{\ddot{u}}_{\sigma(i)}^{/}, \max \overline{\ddot{e}}_{\sigma(i)}^{/}\right)\right)$. Then,

$$\mathring{U}(\aleph)^- \leq SFRHHA\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^+. \tag{4.22}$$

Monotonicity: Suppose $\mathring{U}(\aleph^*) = \left(\left(\underline{\ddot{a}}_{\sigma(i)}^{/*}, \underline{\ddot{u}}_{\sigma(i)}^{/*}, \underline{\ddot{e}}_{\sigma(i)}^{/*}\right), \left(\overline{\ddot{a}}_{\sigma(i)}^{/*}, \overline{\ddot{u}}_{\sigma(i)}^{/*}, \overline{\ddot{e}}_{\sigma(i)}^{/*}\right)\right)(i = 1,...,n)$ be a set of SFRNs, if $\underline{\ddot{a}}_{\sigma(i)}^{/} \leq \underline{\ddot{a}}_{\sigma(i)}^{/*}$, $\underline{\ddot{u}}_{\sigma(i)}^{/*} \leq \underline{\ddot{u}}_{\sigma(i)}^{/}$, $\underline{\ddot{e}}_{\sigma(i)}^{/*} \leq \underline{\ddot{e}}_{\sigma(i)}^{/}$, $\overline{\ddot{a}}_{\sigma(i)}^{/} \leq \overline{\ddot{a}}_{\sigma(i)}^{/*}$, $\overline{\ddot{u}}_{\sigma(i)}^{/} \geq \overline{\ddot{u}}_{\sigma(i)}^{/*}$, $\overline{\ddot{e}}_{\sigma(i)}^{/} \geq \overline{\ddot{e}}_{\sigma(i)}^{/*}$ for all i. Then,

$$SFRHHA\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right) \leq SFRHHA\left(\mathring{U}(\aleph_1^*),...,\mathring{U}(\aleph_n^*)\right). \tag{4.23}$$

5. Spherical fuzzy rough Hamacher geometric aggregation operators

This section, introduces several spherical fuzzy rough Hamacher geometric aggregation operators and describes some of their characteristics.

5.1. Spherical fuzzy rough Hamacher weighted geometric operator

Definition 5.1. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1,...,n)$ be a set of SFRNs with weight vector $\varpi = (\varpi_1,...,\varpi_n)^T$, such as $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0,1]$. Then, SFRHWG operator is described as

$$SFRHWG\left(\mathring{U}(\aleph_1),...,\mathring{U}(\aleph_n)\right) = \bigotimes_{i=1}^n \varpi_i \mathring{U}(\aleph_i), \tag{5.1}$$

where the weights of $\mathring{U}(\aleph_i)\,(i = 1, ...n)$ be $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0, 1]$.

Theorem 5.1. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs. Then, aggregated value utilizing SFRHWG operator is again a SFRN, and

$$SFRHWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \tag{5.2}$$
$$= \left(\bigotimes_{i=1}^{n}\left(\underline{\mathring{U}}(\aleph_i)\right)^{\varpi_i}, \bigotimes_{i=1}^{n}\left(\overline{\mathring{U}}(\aleph_i)\right)^{\varpi_i}\right)$$
$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\underline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{u}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{e}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\underline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(1+(\gamma-1)\left(1-\overline{\ddot{a}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{u}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{u}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{e}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\overline{\ddot{e}}_i^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\}.$$

Effect of parameter γ on SFRHWG operator

Here, we looked at two specific cases of the SFRHWG operator using various values of the parameter γ.

Case 1. The SFRHWG operator reduced to the SFRWG operator if $\gamma = 1$.

$$SFRWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \tag{5.3}$$
$$= \left\{ \begin{array}{c} \left(\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_i\right)^{\varpi_i}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}}\right), \\ \left(\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_i\right)^{\varpi_i}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}}\right) \end{array} \right\}.$$

Case 2. The SFRHWG operator's structure is reduced to the spherical fuzzy rough Einstein weighted geometric (SFREWG) operator if $\gamma = 2$.

$$SFRWG_{\varpi}\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \tag{5.4}$$
$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(2-\underline{\ddot{a}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{u}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{u}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{e}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{e}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_i\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\left(\overline{\ddot{a}}_i^2\right))^{\varpi_i}+\Pi_{i=1}^{n}(\left(\overline{\ddot{a}}_i^2\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{u}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{u}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_i^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{e}}_i^2\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{e}}_i^2\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_i^2\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\}.$$

Based on the Theorem 5.1 we have discuses basic properties of SFRHWG operator.

Proposition 5.1. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs and weight vector of $\mathring{U}(\aleph_i)$ be $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i > 0$. Then, the subsequent characteristics hold.

Idempotency: For all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are equal, i.e., $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, then,

$$SFRHWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{5.5}$$

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs, and $\mathring{U}(\aleph)^+ = \left(\max \underline{\ddot{a}}_i, \min \underline{\ddot{u}}_i, \min \underline{\ddot{e}}_i\right), \left(\max \overline{\ddot{a}}_i, \min \overline{\ddot{u}}_i, \min \overline{\ddot{e}}_i\right)$, $\mathring{U}(\aleph)^- = \left(\min \underline{\ddot{a}}_i, \max \underline{\ddot{u}}_i, \max \underline{\ddot{e}}_i\right), \left(\min \overline{\ddot{a}}_i, \max \overline{\ddot{u}}_i, \max \overline{\ddot{e}}_i\right)$. Then,

$$\mathring{U}(\aleph)^- \leq SFRHWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^+. \tag{5.6}$$

Monotonicity: Let $\mathring{U}(\aleph^*) = \left(\left(\underline{\ddot{a}}_i^*, \underline{\ddot{u}}_i^*, \underline{\ddot{e}}_i^*\right), \left(\overline{\ddot{a}}_i^*, \overline{\ddot{u}}_i^*, \overline{\ddot{e}}_i^*\right)\right)(i = 1, ..., n)$ be a set of SFRNs, if $\underline{\ddot{a}}_i^* \leq \underline{\ddot{a}}_i$, $\underline{\ddot{u}}_i \geq \underline{\ddot{u}}_i^*$, $\underline{\ddot{e}}_i \geq \underline{\ddot{e}}_i^*$, $\overline{\ddot{a}}_i \leq \overline{\ddot{a}}_i^*$, $\overline{\ddot{u}}_i \geq \overline{\ddot{u}}_i^*$, $\overline{\ddot{e}}_i \geq \overline{\ddot{e}}_i^*$ for all i. Then,

$$SFRHWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq SFRHWG\left(\mathring{U}(\aleph_1^*), ..., \mathring{U}(\aleph_n^*)\right). \tag{5.7}$$

5.2. Spherical fuzzy rough Hamacher ordered weighted geometric operator

In this section, we examined two specific cases of the SFRHOWG operator with regard to various parameter γ values.

Definition 5.2. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, with the $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, SFRHOWG operator is defined as

$$SFRHOWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \bigotimes_{i=1}^{n} \varpi_i \mathring{U}(\aleph_{\sigma(i)}), \tag{5.8}$$

where the weight vector of $\mathring{U}(\aleph_i)(i = 1, ...n)$ be $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\sum_{i=1}^n \varpi_i = 1$ and $\varpi_i \in [0, 1]$, $\mathring{U}(\aleph_{\sigma(i-1)}) \geq \mathring{U}(\aleph_{\sigma(i)})$, for all i.

Theorem 5.2. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs. Then, aggregated value utilizing SFRHOWG operator is again a SFRN, and

$$\begin{aligned} & SFRHOWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \qquad (5.9) \\ = & \left(\bigotimes_{i=1}^{n}\left(\underline{\mathring{U}}(\aleph_{\sigma(i)})\right)^{\varpi_i}, \bigotimes_{i=1}^{n}\left(\overline{\mathring{U}}(\aleph_{\sigma(i)})\right)^{\varpi_i}\right) \\ = & \left\{ \left(\begin{array}{c} \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\underline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\underline{\ddot{a}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\underline{\ddot{a}}_{\sigma(i)}^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\underline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\underline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\underline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\underline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\underline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}}} \end{array} \right), \left(\begin{array}{c} \frac{\sqrt{\gamma}\Pi_{i=1}^n\left(\overline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^n(1+(\gamma-1)\left(1-\overline{\ddot{a}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(\overline{\ddot{a}}_{\sigma(i)}^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\overline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\overline{\ddot{u}}_{\sigma(i)}^2\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}-\Pi_{i=1}^n\left(1-\overline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}}{\Pi_{i=1}^n\left(1+(\gamma-1)\overline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^n\left(1-\overline{\ddot{e}}_{\sigma(i)}^2\right)^{\varpi_i}}} \end{array} \right) \right\}. \end{aligned}$$

Effect of parameter γ on SFRHOWG operator

Here, we looked at two specific cases of the SFRHOWG operator using various values of the parameter γ.

Case 1. The SFRHOWG operator reduced to the SFROWG operator if $\gamma = 1$.

$$SFROWG(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) = \left\{ \begin{array}{c} \left(\begin{array}{c} \Pi_{i=1}^{n} \left(\underline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}, \sqrt{1 - \Pi_{i=1}^{n} \left(1 - \underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}, \\ \sqrt{1 - \Pi_{i=1}^{n} \left(1 - \underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}} \end{array} \right), \\ \left(\begin{array}{c} \Pi_{i=1}^{n} \left(\overline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}, \sqrt{1 - \Pi_{i=1}^{n} \left(1 - \overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}, \\ \sqrt{1 - \Pi_{i=1}^{n} \left(1 - \overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}} \end{array} \right) \end{array} \right\}. \tag{5.10}$$

Case 2. The SFRHOWG operator's structure is reduced to the spherical fuzzy rough Einstein ordered weighted geometric (SFREOWG) operator if $\gamma = 2$.

$$\begin{aligned} & SFREOWG(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) \\ = & \left\{ \begin{array}{c} \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(2-\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(\underline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\underline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\underline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(2-\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(\overline{\ddot{a}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{u}}_{\sigma(i)}^{2}\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\overline{\ddot{e}}_{\sigma(i)}^{2}\right)^{\varpi_i}}} \end{array} \right) \end{array} \right\}. \end{aligned} \tag{5.11}$$

Proposition 5.3. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs and the weight vector of $\mathring{U}(\aleph_i)$ be $\varpi = (\varpi_1, ..., \varpi_n)^T$, with $\varpi_i \in [0, 1]$ and $\sum_{i=1}^{n} \varpi_i = 1$. Then, the below characteristics are established.

Idempotency: If all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are equal, i.e., $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, then,

$$SFRHOWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{5.12}$$

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs, and $\mathring{U}(\aleph)^- = \left(\min \underline{\ddot{a}}_{\sigma(i)}, \max \underline{\ddot{u}}_{\sigma(i)} \max \underline{\ddot{e}}_{\sigma(i)}\right), \left(\min \overline{\ddot{a}}_{\sigma(i)}, \max \overline{\ddot{u}}_{\sigma(i)}, \max \overline{\ddot{e}}_{\sigma(i)}\right)$, $\mathring{U}(\aleph)^+ = \left(\max \underline{\ddot{a}}_{\sigma(i)}, \min \underline{\ddot{u}}_{\sigma(i)}, \min \underline{\ddot{e}}_{\sigma(i)}\right), \left(\max \overline{\ddot{a}}_{\sigma(i)}, \min \overline{\ddot{u}}_{\sigma(i)}, \min \overline{\ddot{e}}_{\sigma(i)}\right)$. Then,

$$\mathring{U}(\aleph)^- \leq SFRHOWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^+. \tag{5.13}$$

Monotonicity: Let $\mathring{U}(\aleph^*) = \left(\left(\underline{\ddot{a}}_{\sigma(i)}^*, \underline{\ddot{u}}_{\sigma(i)}^*, \underline{\ddot{e}}_{\sigma(i)}^*\right), \left(\overline{\ddot{a}}_{\sigma(i)}^*, \overline{\ddot{u}}_{\sigma(i)}^*, \overline{\ddot{e}}_{\sigma(i)}^*\right)\right)(i = 1, ..., n)$, be a set of SFRNs, if $\underline{\ddot{a}}_{\sigma(i)}^* \geq \underline{\ddot{a}}_{\sigma(i)}$, $\underline{\ddot{u}}_{\sigma(i)} \geq \underline{\ddot{u}}_{\sigma(i)}^*$, $\underline{\ddot{e}}_{\sigma(i)} \geq \underline{\ddot{e}}_{\sigma(i)}^*$, $\overline{\ddot{a}}_{\sigma(i)}^* \geq \overline{\ddot{a}}_{\sigma(i)}$, $\overline{\ddot{u}}_{\sigma(i)} \geq \overline{\ddot{u}}_{\sigma(i)}^*$, $\overline{\ddot{e}}_{\sigma(i)} \geq \overline{\ddot{e}}_{\sigma(i)}^*$ for all i. Then,

$$SFRHOWG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq SFRHOWG\left(\mathring{U}(\aleph_1^*), ..., \mathring{U}(\aleph_n^*)\right). \tag{5.14}$$

5.3. Spherical fuzzy rough Hamacher hybrid geometric operator

In this section, we describe the SFRHHG operator and examined at its fundamental characteristics, i.e., the boundedness, monotonicity, and idempotency properties.

Definition 5.3. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, such that $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, SFRHHG operator is described as

$$SFRHHG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \bigotimes_{i=1}^{n} \varpi_i \mathring{U}\left(\aleph^{/}_{\sigma(i)}\right), \tag{5.15}$$

where $\varpi = (\varpi_1, ..., \varpi_n)^T$ be the weight vector of $\mathring{U}(\aleph_i)(i = 1, ...n)$, with $\sum_{i=1}^{n} \varpi_i = 1$ and $\varpi_i \in [0, 1]$. And $\mathring{U}\left(\aleph^{/}_{\sigma(i-1)}\right) \geq \mathring{U}\left(\aleph^{/}_{\sigma(i)}\right)$.

Using the q-ROFRHHG operator and induction on n based on the fundamental spherical fuzzy laws, the following structure can be produced. Here, $\mathring{U}\left(\aleph^{/}_{\sigma(i)}\right)$ be the i^{th} largest value of the weighted SFRNs. Also, $w = (w_1, ..., w_n)^T$ is the weight vector of $\mathring{U}(\aleph_i)$, with $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$, i.e.,

$$\mathring{U}\left(\aleph^{/}_{\sigma(i)}\right) = \left(\mathring{U}(\aleph_{\sigma(i)})\right)^{nw_i} = \left((\underline{\ddot{a}}^{/}_{\sigma(i)}, \underline{\ddot{u}}^{/}_{\sigma(i)}), (\overline{\ddot{a}}^{/}_{\sigma(i)}, \overline{\ddot{u}}^{/}_{\sigma(i)})\right)(i = 1, ..., n),$$

$$\begin{aligned}
\underline{\ddot{a}}^{/}_{\sigma(i)} &= \frac{\left(1 + (\gamma - 1)\underline{\ddot{a}}_{\sigma(i)}\right)^{nw_i} - \left(1 - \underline{\ddot{a}}_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\underline{\ddot{a}}_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \underline{\ddot{a}}_{\sigma(i)}\right)^{nw_i}}, \\
\overline{\ddot{a}}^{/}_{\sigma(i)} &= \frac{\left(1 + (\gamma - 1)\overline{\ddot{a}}_{\sigma(i)}\right)^{nw_i} - \left(1 - \overline{\ddot{a}}_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\overline{\ddot{a}}_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \overline{\ddot{a}}_{\sigma(i)}\right)^{nw_i}}, \\
\underline{\ddot{u}}^{/}_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\underline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \underline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\underline{\ddot{a}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \underline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i}}}, \\
\overline{\ddot{u}}^{/}_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\overline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \overline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\overline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \overline{\ddot{u}}^2_{\sigma(i)}\right)^{nw_i}}}, \\
\underline{\ddot{e}}^{/}_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\underline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \underline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\underline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \underline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i}}}, \\
\overline{\ddot{e}}^{/}_{\sigma(i)} &= \sqrt{\frac{\left(1 + (\gamma - 1)\overline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i} - \left(1 - \overline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i}}{\left(1 + (\gamma - 1)\overline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i} + (\gamma - 1)\left(1 - \overline{\ddot{e}}^2_{\sigma(i)}\right)^{nw_i}}}.
\end{aligned}$$

When $\varpi = (1/n, ..., 1/n)^T$, the balancing coefficient n keeps the balance in the position. The SFRHWG and SFRHOWG operators are therefore thought of as a specific case of the SFRHHG operators.

Theorem 5.4. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be a set of SFRNs. Then, aggregated value utilizing SFRHHWG operator is again a SFRN , and

$$SFRHHG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \tag{5.16}$$

$$= \left(\bigotimes_{i=1}^{n}\left(\underline{\mathring{U}}\left(\aleph'_{\sigma(i)}\right)\right)^{\varpi_i}, \bigotimes_{i=1}^{n}\left(\overline{\mathring{U}}\left(\aleph'_{\sigma(i)}\right)\right)^{\varpi_i}\right)$$

$$= \left\{\begin{array}{c}\left(\begin{array}{c}\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\ddot{\underline{a}}'_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\ddot{\underline{a}}'^{2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\ddot{\underline{a}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}\end{array}\right), \left(\begin{array}{c}\frac{\sqrt{\gamma}\Pi_{i=1}^{n}\left(\ddot{\overline{a}}'_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(1+(\gamma-1)\left(1-\ddot{\overline{a}}'^{2}_{\sigma(i)}\right)\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(\ddot{\overline{a}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+(\gamma-1)\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}\end{array}\right)\end{array}\right\}.$$

Effect of parameter γ on SFRHHG operator

Here, we looked at two specific cases of the SFRHHG operator using various values of the parameter γ.

Case 1. The SFRHHG operator reduced to the SFRHG operator if $\gamma = 1$.

$$SFRHG(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) = \left\{\begin{array}{c}\left(\begin{array}{c}\Pi_{i=1}^{n}\left(\ddot{\underline{a}}'_{\sigma(i)}\right)^{\varpi_i}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}, \\ \sqrt{1-\Pi_{i=1}^{n}\left(1-\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}\end{array}\right) \\ \left(\begin{array}{c}\Pi_{i=1}^{n}\left(\ddot{\overline{a}}'_{\sigma(i)}\right)^{\varpi_i}, \sqrt{1-\Pi_{i=1}^{n}\left(1-\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}, \\ \sqrt{1-\Pi_{i=1}^{n}\left(1-\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}\end{array}\right)\end{array}\right\}. \tag{5.17}$$

Case 2. The SFRHHG operator's structure is reduced to the spherical fuzzy rough Einstein hybrid geometric (SFREHG) operator if $\gamma = 2$.

$$SFREHG(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)) \tag{5.18}$$

$$= \left\{\begin{array}{c}\frac{\sqrt{2}\Pi_{i=1}^{n}\left(\ddot{\underline{a}}'_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}(2-\ddot{\underline{a}}'^{2}_{\sigma(i)})^{\varpi_i}+\Pi_{i=1}^{n}\left(\ddot{\underline{a}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\ddot{\underline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(1-\ddot{\underline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}} \\ \frac{\sqrt{2}\Pi_{i=1}^{n}\left(\ddot{\overline{a}}'_{\sigma(i)}\right)^{\varpi_i}}{\sqrt{\Pi_{i=1}^{n}\left(2-\ddot{\overline{a}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+\Pi_{i=1}^{n}\left(\ddot{\overline{a}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\overline{u}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}, \\ \sqrt{\frac{\Pi_{i=1}^{n}\left(1+\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}-\Pi_{i=1}^{n}\left(1-\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}{\Pi_{i=1}^{n}\left(1+\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}+(\gamma-1)\Pi_{i=1}^{n}\left(1-\ddot{\overline{e}}'^{2}_{\sigma(i)}\right)^{\varpi_i}}}\end{array}\right\}.$$

Proposition 5.4. Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be SFRNs and $\varpi = (\varpi_1, ..., \varpi_n)^T$ be the weight vector of $\mathring{U}(\aleph_i)$ with $\sum_{i=1}^{n}\varpi_i = 1$ and $\varpi_i \in [0, 1]$. Then, the below characteristics are defined.

Idempotency: If all $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ are equal, i.e., $\mathring{U}(\aleph_i) = \mathring{U}(\aleph)$, then

$$SFRHHG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) = \mathring{U}(\aleph). \tag{5.19}$$

Boundedness: Let $\mathring{U}(\aleph_i) = \left(\underline{\mathring{U}}(\aleph_i), \overline{\mathring{U}}(\aleph_i)\right)(i = 1, ..., n)$ be SFRNs, and $\mathring{U}(\aleph)^- = \left(\left(\min \underline{\ddot{a}}^{/}_{\sigma(i)}, \max \underline{\ddot{u}}^{/}_{\sigma(i)}, \max \underline{\ddot{e}}^{/}_{\sigma(i)}\right), \left(\min \overline{\ddot{a}}^{/}_{\sigma(i)}, \max \overline{\ddot{u}}^{/}_{\sigma(i)}, \max \overline{\ddot{e}}^{/}_{\sigma(i)}\right)\right)$, $\mathring{U}(\aleph)^+ = \left(\left(\max \underline{\ddot{a}}^{/}_{\sigma(i)}, \min \underline{\ddot{u}}^{/}_{\sigma(i)}, \min \underline{\ddot{e}}^{/}_{\sigma(i)}\right), \left(\max \overline{\ddot{a}}^{/}_{\sigma(i)}, \min \overline{\ddot{u}}^{/}_{\sigma(i)}, \min \overline{\ddot{e}}^{/}_{\sigma(i)}\right)\right)$. Then,

$$\mathring{U}(\aleph)^- \leq SFRHHG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq \mathring{U}(\aleph)^+. \tag{5.20}$$

Monotonicity: Let $\mathring{U}(\aleph^*) = \left(\left(\underline{\ddot{a}}^{/*}_{\sigma(i)}, \underline{\ddot{u}}^{/*}_{\sigma(i)}, \underline{\ddot{e}}^{/*}_{\sigma(i)}\right), \left(\overline{\ddot{a}}^{/*}_{\sigma(i)}, \overline{\ddot{u}}^{/*}_{\sigma(i)}, \overline{\ddot{e}}^{/*}_{\sigma(i)}\right)\right)(i = 1, ..., n)$ be a set of SFRNs, if $\underline{\ddot{a}}^{/*}_{\sigma(i)} \geq \underline{\ddot{a}}^{/}_{\sigma(i)}$, $\underline{\ddot{u}}^{/}_{\sigma(i)} \geq \underline{\ddot{u}}^{/*}_{\sigma(i)}$, $\underline{\ddot{e}}^{/}_{\sigma(i)} \geq \underline{\ddot{e}}^{/*}_{\sigma(i)}$, $\overline{\ddot{a}}^{/}_{\sigma(i)} \leq \overline{\ddot{a}}^{/*}_{\sigma(i)}$, $\overline{\ddot{u}}^{/}_{\sigma(i)} \geq \overline{\ddot{u}}^{/*}_{\sigma(i)}$, $\overline{\ddot{e}}^{/}_{\sigma(i)} \geq \overline{\ddot{e}}^{/*}_{\sigma(i)}$ for all i. Then,

$$SFRHHG\left(\mathring{U}(\aleph_1), ..., \mathring{U}(\aleph_n)\right) \leq SFRHHG\left(\mathring{U}(\aleph_1^*), ..., \mathring{U}(\aleph_n^*)\right). \tag{5.21}$$

6. An approach for MCGDM problem using SFRH aggregation operators

In this section, a MCGDM problem is solved using proposed operators. Suppose n alternatives $\mathbb{R} = \{\mathbb{R}_1, ..., \mathbb{R}_n\}$ and m criteria $\mathbb{Z} = \{\mathbb{Z}_1, ..., \mathbb{Z}_m\}$ are evaluated with weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$, such as $\varpi_j \in [0, 1]$ and $\sum_{j=1}^{n} \varpi_j = 1$. To assess the accomplishment on the basis of criteria $\mathbb{Z}_j$ of the alternatives $\mathbb{R}_i$, the experts give the data about the alternative $\mathbb{R}_i$, satisfy the criteria and also about the alternative $\mathbb{R}_i$, not satisfying the criteria $\mathbb{Z}_j$. Let that rating of alternatives $\mathbb{R}_i$ on attribute $\mathbb{Z}_j$ is given by experts in the form of SFRNs as; $\mathfrak{R} : \mathring{U}\left(\aleph_j\right) = \left(\underline{\mathring{U}}\left(\aleph_j\right), \overline{\mathring{U}}\left(\aleph_j\right)\right)(j = 1, ..., m)$. Where $\overline{\mathring{U}}(\aleph) = \{(r, \overline{\ddot{a}}(r) + \overline{\ddot{u}}(r) + \overline{\ddot{e}}(r) | r \in \mathfrak{R}\}$ and $\underline{\mathring{U}}(\aleph) = \{(r, \underline{\ddot{a}}(r) + \underline{\ddot{u}}(r) + \underline{\ddot{e}}(r) | r \in \mathfrak{R}\}$. Let C_{ij} shows the grade of alternative $\mathbb{R}_i$ satisfying the criteria $\mathbb{Z}_{\mathring{U}}$ and c'_{ij} show the grade of alternative $\mathbb{R}_i$ not satisfying the criteria $\mathbb{Z}_j$, such as $C_{ij} = \overline{\mathring{U}}\left(\aleph_j\right)$, and $C'_{ij} = \underline{\mathring{U}}\left(\aleph_j\right)$ have the condition $0 \leq \underline{\ddot{a}}(r) + \underline{\ddot{u}}(r) + \underline{\ddot{e}}(r) \leq 1$ and $0 \leq \overline{\ddot{a}}(r) + \overline{\ddot{u}}(r) + \overline{\ddot{e}}(r) \leq 1$, $\overline{\mathring{U}}(\aleph)$ and $\underline{\mathring{U}}(\aleph)$ are SFRNs and $\underline{\mathring{U}}(\aleph), \overline{\mathring{U}}(\aleph) : \mathring{U}(\mathfrak{R}) \to \mathring{U}(\mathfrak{R})$ are lower approximation operator and upper approximation operator. This method involved the following steps.

Step 1. Construct decision matrix based on spherical fuzzy rough information.

$$D = (\mathring{U}\left(\aleph_{ij}\right))_{n\times m} = \left(\mathring{U}\left(\aleph_{ij}\right), \mathring{U}\left(\aleph_{ij}^n\right)\right)_{n\times m} \ (j = 1, ..., m; i = 1, ..., n). \tag{6.1}$$

Step 2. Normalize the given decision matrix using the following normalization formula.

$$D^n = (\mathring{U}\left(\aleph_{ij}^n\right))_{n\times m} = \begin{cases} \mathring{U}\left(\aleph_{ij}\right) = \left(\left(\underline{\ddot{a}}_{ij}, \underline{\ddot{u}}_{ij}, \underline{\ddot{e}}_{ij}\right), \left(\overline{\ddot{a}}_{ij}, \overline{\ddot{u}}_{ij}, \overline{\ddot{e}}_{ij}\right)\right) \text{ for benefit type,} \\ \mathring{U}\left(\aleph_{ij}^n\right) = \left(\left(\underline{\ddot{e}}_{ij}, \underline{\ddot{u}}_{ij}, \underline{\ddot{a}}_{ij}\right), \left(\overline{\ddot{e}}_{ij}, \overline{\ddot{u}}_{ij}, \overline{\ddot{a}}_{ij}\right)\right) \text{ for cost type.} \end{cases} \tag{6.2}$$

As $\mathring{U}\left(\aleph_{ij}^n\right)$ is the complement of $\mathring{U}\left(\aleph_{ij}\right)$. As a result, we were able to obtain a normalized decision matrix.

$$D^n = \left(\mathring{U}\left(\aleph_{ij}\right), \mathring{U}\left(\aleph_{ij}^n\right)\right)_{n\times m} \ (j = 1, ..., m;\ i = 1, ..., n).$$

Step 3. Using the defined aggregation operator to find the SFRNs $\mathring{U}(\aleph_i)$ $(i = 1, ..., n)$, for alternatives $\mathbb{R}_i$ $(i = 1, ..., n)$, using expert weight vector $\omega = (\omega_1, ..., \omega_n)^T$.

Step 4. Using the defined aggregation operator to find the SFRNs $\mathring{U}(\aleph_i)$ $(i = 1, ..., n)$, for alternatives $\mathbb{R}_i$ $(i = 1, ..., n)$, using criteria weight vector $\varpi = (\varpi_1, ..., \varpi_n)^T$.

Step 5. Using the score equation, find $S(\mathring{U}(\aleph_i))$ $(i = 1, ..., n)$, for ranking the alternatives $\mathbb{R}_i$ $(i = 1, ..., n)$.

Step 6. Select the best option after ranking the alternatives $\mathbb{R}_i$ $(i = 1, ..., n)$.

7. Illustrative example

Using a numerical example to choose the best railway train for service out of four options, the proposed MCGDM method is illustrated (adapted from [32]).

To develop the service quality of domestic railway trains, the Ministry of Railways (MOR) of the government of Pakistan needs to know which railways train is the most excellent in Pakistan. After initial information, four main domestic railway trains which are represented by $\mathbb{R}_i$ $(i = 1, ..., 4)$ are reminded on the applicant record. They are: $\mathbb{R}_1$: Allama Iqbal Express, $\mathbb{R}_2$: Badar Express, $\mathbb{R}_3$: Hazara Express, $\mathbb{R}_4$: Jinnah Express. To select the most excellent alternatives,four main domestic railway trains are evaluated from four major characteristics (attributes): $\mathbb{Z}_1$: ticketing and booking service; $\mathbb{Z}_2$: better and poor condition; $\mathbb{Z}_3$: cabin service; $\mathbb{Z}_4$: responsiveness; assume that decision-makers provide the rating values by utilizing SFRNs, and the SFR decision matrix is presented in Table 1–3.

We used a group of three experts with weight vectors of $\omega = (0.2, 0.3, 0.5)^T$ to apply and validate our own proposed methods. Using the suggested aggregation operators, we also have a criteria weight vector with the notation $\varpi = (0.26, 0.24, 0.28, 0.22)^T$. Additionally, in order to select the best option from all of the available options, we will use the score and accuracy functions, respectively. Additionally, the specifics of the criteria and alternatives were discussed above, and in this case, the only aggregation operators that can produce the best results are those that have been suggested.

Table 1. Spherical fuzzy information given by expert one.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$
$\mathbb{R}_1$	$\begin{pmatrix} \langle 0.8, 0.4, 0.2 \rangle, \\ \langle 0.3, 0.9, 0.1 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.6, 0.4 \rangle, \\ \langle 0.6, 0.3, 0.7 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.8, 0.4, 0.4 \rangle, \\ \langle 0.5, 0.7, 0.3 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.7, 0.5 \rangle, \\ \langle 0.8, 0.4, 0.3 \rangle \end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix} \langle 0.7, 0.6, 0.3 \rangle, \\ \langle 0.5, 0.8, 0.2 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.3, 0.5 \rangle, \\ \langle 0.6, 0.6, 0.2 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.6, 0.4 \rangle, \\ \langle 0.8, 0.3, 0.3 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.5, 0.4 \rangle, \\ \langle 0.5, 0.8, 0.1 \rangle \end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix} \langle 0.7, 0.5, 0.5 \rangle, \\ \langle 0.5, 0.6, 0.4 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.4, 0.5 \rangle, \\ \langle 0.8, 0.3, 0.4 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.4, 0.6 \rangle, \\ \langle 0.6, 0.2, 0.7 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.7, 0.3 \rangle, \\ \langle 0.5, 0.8, 0.2 \rangle \end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix} \langle 0.7, 0.3, 0.4 \rangle, \\ \langle 0.6, 0.5, 0.5 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.3, 0.5 \rangle, \\ \langle 0.5, 0.8, 0.2 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.8, 0.3, 0.2 \rangle, \\ \langle 0.9, 0.2, 0.3 \rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.4, 0.6 \rangle, \\ \langle 0.7, 0.5, 0.4 \rangle \end{pmatrix}$

Table 2. Spherical fuzzy information given by expert two.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$
$\mathbb{R}_1$	$\begin{pmatrix} \langle 0.5, 0.3, 0.4\rangle, \\ \langle 0.6, 0.4, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.5, 0.3\rangle, \\ \langle 0.5, 0.7, 0.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.7, 0.3\rangle, \\ \langle 0.6, 0.4, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, .6, 0.6\rangle, \\ \langle 0.4, .8, 0.3\rangle \end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix} \langle 0.6, 0.7, 0.2\rangle, \\ \langle 0.5, 0.2, 0.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.7, 0.1\rangle, \\ \langle 0.7, 0.4, 0.2\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.7, 0.4\rangle, \\ \langle 0.8, 0.3, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.5, 0.4\rangle, \\ \langle 0.5, 0.7, 0.3\rangle \end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix} \langle 0.8, 0.4, 0.3\rangle, \\ \langle 0.7, 0.6, 0.2\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.5, 0.5\rangle, \\ \langle 0.7, 0.4, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.3, 0.6\rangle, \\ \langle 0.4, 0.9, 0.1\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.4, 0.6\rangle, \\ \langle 0.6, 0.5, 0.6\rangle \end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix} \langle 0.6, 0.5, 0.1\rangle, \\ \langle 0.7, 0.6, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.4, 0.5\rangle, \\ \langle 0.6, 0.2, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.8, 0.5, 0.2\rangle, \\ \langle 0.3, 0.9, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.5, 0.5\rangle, \\ \langle 0.7, 0.4, 0.2\rangle \end{pmatrix}$

Table 3. Spherical fuzzy information given by expert three.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$
$\mathbb{R}_1$	$\begin{pmatrix} \langle 0.6, 0.5, 0.4\rangle, \\ \langle 0.7, 0.6, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.4, 0.5\rangle, \\ \langle 0.6, 0.5, 0.6\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.8, 0.2\rangle, \\ \langle 0.6, 0.4, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.6, 0.5\rangle, \\ \langle 0.4, 0.9, 0.2\rangle \end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix} \langle 0.5, 0.5, 0.6\rangle, \\ \langle 0.7, 0.4, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.4, 0.7\rangle, \\ \langle 0.6, 0.5, 0.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.5, 0.3, 0.7\rangle, \\ \langle 0.8, 0.4, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.5, 0.3\rangle, \\ \langle 0.5, 0.7, 0.4\rangle \end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix} \langle 0.6, 0.7, 0.2\rangle, \\ \langle 0.6, 0.4, 0.4\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.8, 0.4, 0.3\rangle, \\ \langle 0.7, 0.6, 0.2\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.7, 0.1, 0.6\rangle, \\ \langle 0.4, 0.6, 0.3\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.5, 0.3\rangle, \\ \langle 0.7, 0.4, 0.5\rangle \end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix} \langle 0.7, 0.5, 0.3\rangle, \\ \langle 0.5, 0.7, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.3, 0.7\rangle, \\ \langle 0.7, 0.4, 0.5\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.6, 0.3, 0.5\rangle, \\ \langle 0.4, 0.9, 0.2\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.8, 0.5, 0.3\rangle, \\ \langle 0.3, 0.2, 0.9\rangle \end{pmatrix}$

Step 1. Construct decision matrix based on spherical fuzzy rough information in Table 1–3.
Step 2. There is no need to normalize the data in Table 1–3, because every criteria are a similar benefit criteria.
Step 3. Let $\gamma = 2$ and $\omega = (0.2, 0.3, 0.5)^T$ as weights of experts. Then, using SFRHWA operator, and data from Tables 1–3. The aggregated data given in Table 4.

Table 4. Aggregated values using SFRHWA operator.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$
$\mathbb{R}_1$	$\begin{pmatrix} \langle 0.62, 0.27, 0.25\rangle, \\ \langle 0.61, 0.35, 0.22\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.64, 0.30, 0.28\rangle, \\ \langle 0.57, 0.32, 0.33\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.61, 0.39, 0.22\rangle, \\ \langle 0.58, 0.29, 0.30\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.50, 0.36, 0.32\rangle, \\ \langle 0.51, 0.44, 0.21\rangle \end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix} \langle 0.58, 0.34, 0.27\rangle, \\ \langle 0.61, 0.27, 0.29\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.58, 0.30, 0.28\rangle, \\ \langle 0.62, 0.31, 0.23\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.51, 0.30, 0.33\rangle, \\ \langle 0.80, 0.25, 0.26\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.65, 0.31, 0.25\rangle, \\ \langle 0.50, 0.41, 0.23\rangle \end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix} \langle 0.69, 0.34, 0.22\rangle, \\ \langle 0.61, 0.31, 0.24\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.71, 0.29, 0.27\rangle, \\ \langle 0.72, 0.30, 0.22\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.64, 0.18, 0.35\rangle, \\ \langle 0.44, 0.36, 0.22\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.57, 0.32, 0.26\rangle, \\ \langle 0.64, 0.31, 0.29\rangle \end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix} \langle 0.67, 0.30, 0.20\rangle, \\ \langle 0.58, 0.36, 0.28\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.63, 0.27, 0.31\rangle, \\ \langle 0.64, 0.39, 0.28\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.71, 0.33, 0.24\rangle, \\ \langle 0.53, 0.28, 0.29\rangle \end{pmatrix}$	$\begin{pmatrix} \langle 0.67, 0.34, 0.28\rangle, \\ \langle 0.53, 0.25, 0.21\rangle \end{pmatrix}$

Step 4. Using SFRHOWA operator, $\gamma = 2$ and the aggregated value of Table 4, with the weight vector $\varpi = (0.26, 0.24, 0.28, 0.22)^T$. The total values of the alternatives $\mathbb{R}_i$ $(i = 1, ..., 4)$ is given in Table 5.

Table 5. Total values of the alternatives using SFRHOWA operator.

Alternatives	Aggregated values
$\mathbb{R}_1$	$(\langle 0.5978, 0.1744, 0.1545\rangle, \langle 0.5710, 0.1788, 0.1547\rangle)$
$\mathbb{R}_2$	$(\langle 0.5779, 0.1637, 0.1449\rangle, \langle 0.6565, 0.1794, 0.1621\rangle)$
$\mathbb{R}_3$	$(\langle 0.5673, 0.1726, 0.1509\rangle, \langle 0.6051, 0.1654, 0.1871\rangle)$
$\mathbb{R}_4$	$(\langle 0.6726, 0.1943, 0.1389\rangle, \langle 0.5711, 0.1842, 0.2201\rangle)$

Step 5. Determine the score $S(\mathbb{R}_i)$ of alternatives $\mathbb{R}_i$ $(i = 1, ..., 4)$ as

$$S(\mathbb{R}_1) = 0.1691,\ S(\mathbb{R}_2) = 0.1947,\ S(\mathbb{R}_3) = 0.1655,\ S(\mathbb{R}_4) = 0.1687.$$

Step 6. Ranking the alternatives based on their score values as follows:

$$\mathbb{R}_2 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_3.$$

Comparative analysis

In order to demonstrate the superiority of our investigated approach, a comparison study was undertaken in the context of a few current methods [3, 22, 29, 34]. The precision of operators is very great when it comes to actual manipulation. We contrasted our plan with those that are already in use. Table 6 presents the results of the comparison. The best alternative generated by the suggested strategy is compatible with the existing methods, according to the assessment of Table 6. As a result, while resolving problems with spherical fuzzy information, our suggested emergency, alternative selection methodology based on the suggested operators is more adaptable and successful than other existing techniques.

Table 6. Ranking of the alternatives using different methods.

Methods	Ranking
Ashraf et al. [3]	$\mathbb{R}_2 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_3$
Qiyas et al. [29]	$\mathbb{R}_2 > \mathbb{R}_1 > \mathbb{R}_3 > \mathbb{R}_4$
Rafiq et al. [34]	$\mathbb{R}_2 > \mathbb{R}_4 > \mathbb{R}_3 > \mathbb{R}_1$
Kutlu et al. [22]	$\mathbb{R}_2 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_3$

It is evident from the study above that the researched operator and the current operators provide the same ranking values, with $\mathbb{R}_2$ being the best. Additionally, it is evident that the approaches suggested in this paper are more versatile than SFSs. As a result, the strategies suggested in this paper are better suited to solving MCGDM problems. Therefore, the multiple attribute group decision-making models developed in this study are more generalization and adaptable than the existing multiple attribute group decision-making models under a spherical fuzzy rough environment, making them applicable in a wider range of settings where the multi-criteria group decision-making (MCDM) procedure is used.

8. Conclusions

According to the concept of the spherical fuzzy rough set (SFRS) and the Hamacher aggregation operator, we extended the Hamacher operations to the spherical fuzzy rough set information in this study. In order to completely analyse the applications of smart grading systems and handle the uncertainty problem, a novel MCGDM approach based on SFRS was developed in this study, which is both innovative and significant. A helpful tool for handling ambiguous data is the SFS. Since, SFSs can express information more freely and represent a larger span of space. We present an innovative method for average and geometric aggregation operators based on SFS data in this research. We started by creating the SFRHS operational laws. Moreover, the notions of SFRHWA, SFRHOWA, SFRHHA and SFRHWG, SFRHOWG and SFRHHG aggregation operators are proposed. The main essential attributes of the evolved operator are covered in detail. For the suggested operators, new score and accuracy functions have been defined. Therefore, a unique approach for MCGDM in the Spherical fuzzy rough environment was proposed based on the described operators. For the purpose of demonstrating the efficacy and use of the specified method, a real-world example is given. The final step is to perform a comparison analysis of the suggested method and the existing approaches. The method described in this study may contribute a new approach to resolving MCGDM issues. The Hamacher aggregation operators based on the spherical fuzzy rough set are a significant addition to related research.

In the future, we will use this method could involve fractional orthotriple fuzzy operators, Maclaurin's symmetric mean operators, Dombi operations, and power aggregation operators. The application of the proposed methods in different fields such as data mining, decision making, construction of anomaly detection models and pattern recognition is also a potential research topic.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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