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Research article

Output-based event-triggered control for discrete-time systems with three types of performance analysis

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Abstract: This paper considers an output-based event-triggered control approach for discrete-time systems and proposes three new types of performance measures under unknown disturbances. These measures are motivated by the fact that signals in practical systems are often associated with bounded energy or bounded magnitude, and they should be described in the ℓ_2 and ℓ_∞ spaces, respectively. More precisely, three performance measures from ℓ_q to ℓ_p, denoted by the $\ell_{p/q}$ performances with $(p, q) = (2, 2)$, $(\infty, 2)$ and (∞, ∞), are considered for event-triggered systems (ETSs) in which the corresponding event-trigger mechanism is defined as a function from the measured output of the plant to the input of the dynamic output-feedback controller with the triggering parameter $\sigma(> 0)$. Such a selection of the pair (p, q) represents the $\ell_{p/q}$ performances to be bounded and well-defined, and the three measures are natural extensions of those in the conventional feedback control, such as the H_∞, generalized H_2 and ℓ_1 norms. We first derive the corresponding closed-form representation with respect to the relevant ETSs in terms of a piecewise linear difference equation. The asymptotic stability condition for the ETSs is then derived through the linear matrix inequality approach by developing an adequate piecewise quadratic Lyapunov function. This stability criterion is further extended to compute the $\ell_{p/q}$ performances. Finally, a numerical example is given to verify the effectiveness of the overall arguments in both the theoretical and practical aspects, especially for the trade-off relation between the communication costs and $\ell_{p/q}$ performances.

Keywords: $\ell_{p/q}$ performances; event-triggered systems; linear matrix inequalities; piecewise linear difference equation; output-feedback
Mathematics Subject Classification: 39A06, 93C55, 93D20, 93D25, 93D30

1. Introduction

As the size and complexity of control systems have increased, so have the required communication costs between a plant and a controller in the conventional feedback control approaches. In other words, it is a non-trivial task to maintain consistent control actions such as computing and updating new control inputs for large-scale control systems in real-time since one usually employs limited communication resources, as discussed in [1]. In connection with this, an alternative control scheme, so-called the event-triggered control (ETC), has been introduced [1,2], where reducing the transmission rate based on the information from the plant to the controller is considered. More precisely, the communication between the plant and the controller is activated and the inputs of the controller are updated only when a pre-designed event-trigger mechanism (ETM) shows the corresponding status as 'triggering-on' in the ETC approach. Motivated by the fact that this advantage of ETC can also be effectively used in large-scale systems, its applications to multi-agent systems have been actively considered, as discussed in the recent studies [3, 4].

Regarding discrete-time event-triggered systems (ETSs), the corresponding stability conditions have been discussed in [5–9]. More precisely, the stability arguments in [5–7] are associated with norm-based relative triggering mechanisms. They could also be classified by the asymptotic stability [5], uniformly ultimately bounded stability [6] and Lyapunov function-based input-to-state stability [7]. The asymptotic stability for discrete-time ETSs has been recently discussed in an extended fashion with the switching ETM [8] or the dynamic ETM [9]. However, no arguments on quantitative performance analysis (for example, system gains or induced norms) are discussed in those studies. In connection with this, a linear matrix inequality (LMI)-based approach to the ℓ_2-gain analysis for discrete-time ETSs is proposed [10], in which the corresponding triggering condition is defined as a function of an estimated value from the state observer. However, no discussions on the ℓ_∞-norm analysis for input/output signals is provided in that study; taking the ℓ_∞-norm enables us to take into account some practical applications on reducing peak forces [11] and detecting obstacles [12] since it corresponds to the maximum magnitude of a signal. Furthermore, such an observer-equipped structure for constructing the triggering mechanism as in [10] could reduce the practical applicability because of its relatively high computational cost, even if a performance analysis with respect to the ℓ_∞ norm might be possible.

To resolve these issues, we take an ETM for discrete-time dynamic output-feedback control systems, and develop some relevant analysis methods for both the ℓ_2 and ℓ_∞ norms of input/output signals. To put it another way, the ETM taken in this paper is described by a function from the measured output of the plant to the input of the dynamic output-feedback controller, for which it is not required to consider any additional estimated values from a state observer. For the corresponding ETSs, the three performance measures from ℓ_q to ℓ_p, denoted by the $\ell_{p/q}$ performances, are taken with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞), where such a selection of the pair (p, q) represents the $\ell_{p/q}$ performances to be bounded and well-defined; the $\ell_{p/q}$ performance with $(p, q) = (2, \infty)$ cannot be defined even for the conventional linear time-invariant (LTI) feedback systems.

In connection with this, it should be first required to obtain a tractable form of the ETSs that is tailored to the stability and performance analyses, but this is a quite difficult issue in contrast to the case of conventional feedback control systems. To alleviate this difficulty, we derive a closed-form representation of the corresponding ETSs in terms of piecewise linear difference equations, as in [13, 14], by noting the fact that the overall dynamic behavior of the ETSs can be described by the switching architecture between feedback control and feedforward control. This closed-form representation makes it possible to clarify the stability and performance analyses for the ETSs by developing adequate piecewise quadratic (PWQ) Lyapunov functions.

More precisely, the asymptotic stability of the ETSs is verified by the feasibility of some LMIs, which are derived from the PWQ Lyapunov function associated with the state transition behavior of the ETSs for the case without external disturbances. This LMI-based stability criterion is further extended to the methods for computing the three $\ell_{p/q}$ performances of the ETSs. To put it another way, it is clarified that the $\ell_{p/q}$ performance for fixed (p, q) is less than or equal to a given $\gamma(> 0)$ if some relevant LMI-based conditions are feasible. This is in sharp contrast with the previous studies on discrete-time ETSs [5–9], in which no quantitative performance analysis is discussed. Furthermore, the $\ell_{p/q}$ performance analysis for the ETSs is quite theoretically meaningful because it corresponds to a significant extension of the existing performance measure in the conventional feedback control scheme, such as the H_∞ control [15–17], generalized H_2 control [18–20] and L_1 control schemes [21, 22], for the cases of $(p, q) = (2, 2)$, $(\infty, 2)$ and (∞, ∞), respectively. In this sense, the LMI-based arguments developed in this paper not only lead to computations of the three types of $\ell_{p/q}$ performances of ETSs, but also to the extensive treatment of the H_∞, generalized H_2 and ℓ_1 norms of conventional feedback systems.

Regarding the contributions of this paper, on the other hand, it would be worthwhile to discuss the practical effectiveness of taking the three $\ell_{p/q}$ performances. From the definitions of the ℓ_2 and ℓ_∞ spaces, both the $\ell_{2/2}$ and $\ell_{\infty/2}$ performances assume external disturbances with bounded energies, but they evaluate the energy and the peak magnitude of the output for the worst cases of disturbances, respectively, while the $\ell_{\infty/\infty}$ performance considers the maximum magnitude of the output for the worst bounded disturbance, as summarized in Table 1. If we note that disturbances are generally classified by signals with finite energy and finite magnitude and it is often desired to suppress the energy or peak value of the output in a number of practical systems, then taking the three types of $\ell_{p/q}$ performance could be interpreted as covering all possible cases of quantitative analysis under unknown disturbances occurring from real control systems.

Beyond the aforementioned practical aspect in terms of quantitative performance analyses against unknown exogenous inputs, we are also concerned with the trade-off relation between the communication costs and $\ell_{p/q}$ performances from a numerical standpoint, and we provide another performance measure that is helpful in determining an optimal value of the corresponding triggering parameter $\sigma(> 0)$.

Table 1. The practical characteristics of the $\ell_{p/q}$ performances.

Measure	Disturbance	Output
$\ell_{2/2}$ performance	Finite energy	The energy
$\ell_{\infty/2}$ performance	Finite energy	The peak magnitude
$\ell_{\infty/\infty}$ performance	Finite magnitude	The peak magnitude

To summarize, the contributions of this paper are as follows.

- The closed-form representation of the output-based ETSs is described by a piecewise linear difference equation, and their overall dynamic behavior can be characterized by deriving an adequate PWQ Lyapunov equation.

- The asymptotic stability analysis for the ETSs can be verified by the feasibility of some LMI-based conditions.
- As significant extensions of the H_∞ control, generalized H_2 control and ℓ_1 control for conventional feedback systems, the three types of $\ell_{p/q}$ performances with $(p, q) = (2, 2),\ (\infty, 2)$ and (∞, ∞) are proposed.
- All possible cases of quantitative analyses for unknown disturbances occurring from practical systems can be covered by taking the three $\ell_{p/q}$ performances.
- The trade-off relation between the communication costs and $\ell_{p/q}$ performances is concerned by performing some numerical studies.
- A guideline for determining an adequate value of the triggering parameter $\sigma(> 0)$ in terms of the above trade-off relation is also provided by introducing another performance measure.

The organization of this paper is as follows. In Section 2, the considered ETM for dynamic output-feedback control of discrete-time systems and the corresponding closed-form representation are introduced. The asymptotic stability of the associated ETSs is described by the LMI approach in Section 3. For the ETSs, we next introduce three performance measures, i.e., the $\ell_{p/q}$ performances with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞), and develop their LMI-based analysis methods in Section 4. Finally, a numerical example is given in Section 5 to demonstrate the effectiveness of the overall arguments.

The notations used in this paper are as follows. The notations $\mathbb{N}$ and $\mathbb{R}^\nu$ denote the sets of positive integers and ν-dimensional real vectors, respectively, while $\mathbb{N}_0$ and $\mathbb{R}_+$ imply $\mathbb{N} \cup \{0\}$ and the set of non-negative real scalars, respectively. The notation $\mathbb{S}^\nu$ denotes the set of ν-dimensional real symmetric matrices. We further use the symbol $\prec$ $(\preceq)$ to imply the binary relation on $\mathbb{S}^\nu$ such that $A \prec B$ $(A \preceq B)$ for $A, B \in \mathbb{S}^\nu$ means that $B - A$ is positive (semi-)definite. The notations $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a real symmetric matrix $(\cdot)$, respectively. The weighted 2-norm of a finite-dimensional real vector equipped with a positive semidefinite matrix Q is denoted by $|\cdot|_Q$, i.e., $|v|_Q := (v^T Q v)^{1/2}$, while the case of $Q = I$ (i.e., the standard Euclidean norm) is denoted just by $|\cdot|_2$, for a notational simplicity.

Equipped with this symbol, the ℓ_p norms of a real-valued vector sequence are denoted by $\|\cdot\|_p$ $(p = 2, \infty)$, i.e.,

$$\|f(\cdot)\|_2 := \left(\sum_{k=0}^{\infty} |f(k)|_2^2 \right)^{1/2}, \quad \|f(\cdot)\|_\infty := \operatorname*{ess\,sup}_{k \in \mathbb{N}_0} |f(k)|_2,$$

Finally, the normed space of ν-dimensional real-valued vector sequences $f = \{f(k)\}_{k=0}^{\infty}$ such that $\|f\|_p < \infty$ is denoted by $(\ell_p)^\nu$ $(p = 2, \infty)$.

2. Closed-form representation of discrete-time ETSs

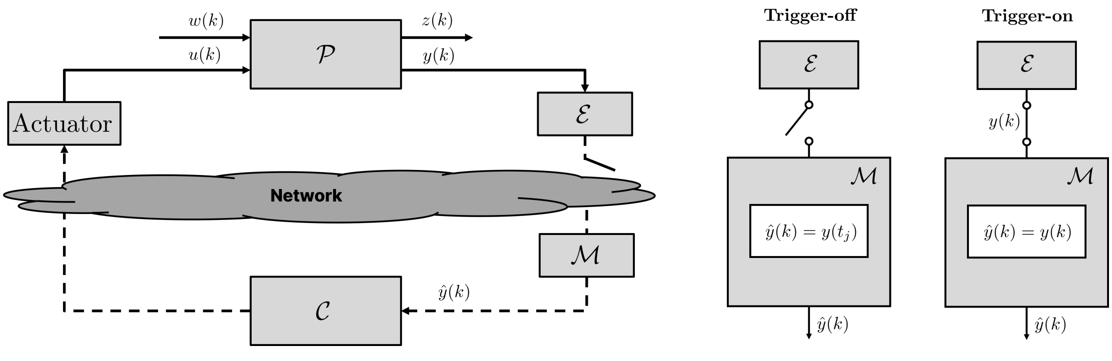
This section is devoted to providing an ETM for dynamic output-feedback control of linear discrete-time systems and deriving the corresponding closed-form representation. More precisely, it is assumed that the state variables of the discrete-time LTI generalized plant cannot be directly measured and the ETM is defined as a function from the measured output of the plant to the input of the dynamic output-feedback controller.

Let us consider the ETS Σ shown in Figure 1, where $\mathcal{P}$ denotes the discrete-time LTI generalized plant, C denotes the discrete-time LTI controller and $\mathcal{E}$ denotes the ETM equipped with the memory

$\mathcal{M}$. The actuator and the ETM $\mathcal{E}$ are connected to the controller C and the memory $\mathcal{M}$, respectively, through the network, in which no packet loss and time delay are assumed. Suppose that $\mathcal{P}$ and C are given respectively by

$$\mathcal{P}: \begin{cases} x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \\ z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k), \\ y(k) = C_2 x(k); \end{cases} \quad C: \begin{cases} x_c(k+1) = A_c x_c(k) + B_c \hat{y}(k), \\ u(k) = C_c x_c(k) + D_c \hat{y}(k), \end{cases} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$, $x_c(k) \in \mathbb{R}^{n_c}$, $u(k) \in \mathbb{R}^{n_u}$, $w(k) \in \mathbb{R}^{n_w}$, $y(k), \hat{y}(k) \in \mathbb{R}^{n_y}$ and $z(k) \in \mathbb{R}^{n_z}$.



(a) Event-triggered system Σ. (b) ETM and the memory.

Figure 1. ETS Σ and the corresponding ETM.

The ETM $\mathcal{E}$ is assumed to be equipped with the triggering instants $\{t_j\}_{j \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ with

$$0 = t_0 < t_1 < t_2 < \cdots < t_j < t_{j+1} < \cdots . \tag{2.2}$$

More precisely, the input signal for the controller C is described by

$$\hat{y}(k) := y(t_j) \quad (t_j \le k < t_{j+1}), \tag{2.3}$$

where the triggering instant $\{t_j\}$ is determined recursively as follows:

$$t_{j+1} = \inf\{k > t_j \mid |y(t_j) - y(k)|_{Q_e} \ge \sigma |y(k)|_{Q_y}\}. \tag{2.4}$$

Here, the positive constant σ is called the triggering parameter for Σ throughout the paper, and the ETS Σ coincides with a conventional dynamic output-feedback control system if $\sigma = 0$ (i.e., $t_j = j$ for all $j \in \mathbb{N}_0$ and thus $\hat{y}(k) = y(k)$ for all $k \in \mathbb{N}_0$). The weighting matrices Q_e and Q_y are associated with determining the triggering conditions for $y(t_j) - y(k)$ and $y(k)$, respectively.

For practical effectiveness of the ETM considered in this paper, on the other hand, taking σ larger naturally leads to a reduction of the communication loads between $\mathcal{P}$ and C since it makes the difference between t_j and t_{j+1} become larger; thus, the transmission of data to C is reduced. Thus, one might argue that the triggering parameter should be taken as large as possible to reduce the corresponding communication cost, but such a selection could result in a performance deterioration of the overall systems. In connection with this, the following is required: the development for developing sophisticated arguments on the stability, as well as a quantitative performance analysis of the overall ETS Σ.

As a preliminary step, we derive a closed-form representation of the ETS Σ as follows. Let us first denote the latest triggered instant less than k by $m(k)$, i.e.,

$$m(k) := \max\{t_j \mid t_j < k\}. \tag{2.5}$$

We next define $x_m(k) := x(m(k))$, with which the ETM as in (2.4) can be equivalently transformed to the following form:

$$\begin{aligned} &\text{ETM is triggered at } k, \text{ i.e., } k = t_j \text{ for some } j \in \mathbb{N}_0 \\ \Leftrightarrow\ & |y(m(k)) - y(k)|_{Q_e} \geq \sigma |y(k)|_{Q_y} \\ \Leftrightarrow\ & \begin{bmatrix} x(k) \\ x_m(k) \end{bmatrix}^T \begin{bmatrix} C_2^T(Q_e - \sigma^2 Q_y)C_2 & -C_2^T Q_e C_2 \\ -C_2^T Q_e C_2 & C_2^T Q_e C_2 \end{bmatrix} \begin{bmatrix} x(k) \\ x_m(k) \end{bmatrix} \geq 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\text{ETM is not triggered at } k, \text{ i.e., } k \neq t_j \text{ for all } j \in \mathbb{N}_0 \\ \Leftrightarrow\ & |y(m(k)) - y(k)|_{Q_e} < \sigma |y(k)|_{Q_y} \\ \Leftrightarrow\ & \begin{bmatrix} x(k) \\ x_m(k) \end{bmatrix}^T \begin{bmatrix} C_2^T(Q_e - \sigma^2 Q_y)C_2 & -C_2^T Q_e C_2 \\ -C_2^T Q_e C_2 & C_2^T Q_e C_2 \end{bmatrix} \begin{bmatrix} x(k) \\ x_m(k) \end{bmatrix} < 0. \end{aligned} \tag{2.7}$$

With this in mind, we can derive the closed-form representation of Σ by the piecewise linear difference equation

$$\begin{cases} \xi(k+1) = F_i \xi(k) + G w(k) \\ z(k) = H_i \xi(k) + D_{11} w(k) \end{cases}, \quad \text{if } \xi(k) \in \mathcal{N}_i, \ \ (i = 0, 1) \tag{2.8}$$

with $\xi(k) := [x^T(k)\ \ x_m^T(k)\ \ x_c^T(k)]^T \in \mathbb{R}^{2n+n_c}$, the matrices

$$\begin{aligned} F_0 &= \begin{bmatrix} A & B_2 D_c C_2 & B_2 C_c \\ 0 & I & 0 \\ 0 & B_c C_2 & A_c \end{bmatrix}, \ F_1 = \begin{bmatrix} A + B_2 D_c C_2 & 0 & B_2 C_c \\ I & 0 & 0 \\ B_c C_2 & 0 & A_c \end{bmatrix}, \\ Q &= \begin{bmatrix} C_2^T(Q_e - \sigma^2 Q_y)C_2 & -C_2^T Q_e C_2 & 0 \\ -C_2^T Q_e C_2 & C_2^T Q_e C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ G = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, \\ H_0 &= \begin{bmatrix} C_1 & D_{12} D_c C_2 & D_{12} C_c \end{bmatrix}, \ H_1 = \begin{bmatrix} C_1 + D_{12} D_c C_2 & 0 & D_{12} C_c \end{bmatrix} \end{aligned} \tag{2.9}$$

and the sets $\mathcal{N}_0$ and $\mathcal{N}_1$ described respectively by

$$\begin{aligned} \mathcal{N}_0 &= \{\xi(k) \mid \xi(k) \in \mathbb{R}^{2n+n_c}, \ \xi^T(k) Q \xi(k) < 0\}, \\ \mathcal{N}_1 &= \{\xi(k) \mid \xi(k) \in \mathbb{R}^{2n+n_c}, \ \xi^T(k) Q \xi(k) \geq 0\}. \end{aligned}$$

Note that the subscripts '0' and '1' stand for the conditions for the triggering-off and triggering-on, respectively, and the matrix Q in (2.9) is essentially equivalent to the symmetric matrix in the triggering condition described by (2.6) and (2.7).

With the piecewise linear difference equation mentioned above, we can construct a PWQ Lyapunov function for the ETS Σ, by which its asymptotic stability, as well as three types of the performances from ℓ_2 to ℓ_2, from ℓ_2 to ℓ_∞ and from ℓ_∞ to ℓ_∞, can be described in terms of LMIs, and the details will be given in the following sections.

3. Stability analysis for Σ through its LMI-based representation

This section is devoted to establishing a stability condition for the ETS Σ with the general case of $\sigma > 0$ in terms of LMI-based representations. The stability argument is essential for defining the three types of performance measures in the following section, and the relevant arguments will become key in deriving parallel LMI-based conditions for the associated performance analysis. In connection with this, we first introduce the definition of the asymptotic stability of Σ for a given $\sigma(> 0)$ as follows.

Definition 1. *Assume that $w = 0$. Then, the ETS Σ for a given $\sigma(> 0)$ is said to be asymptotically stable if $\xi(k) \to 0$ $(k \to \infty)$ for an arbitrary $\xi(0) \in \mathbb{R}^{2n+n_c}$.*

This asymptotic stability could be interpreted as a weaker version of the global exponential stability discussed in [23], for which a PWQ function is introduced. Hence, it is quite important to develop a parallel approach to the asymptotic stability of Σ with respect to deriving feasible conditions, and let us consider the PWQ function described by

$$V(\xi(k)) = \begin{cases} \xi^T(k)P_0\xi(k), & \text{if } \xi^T(k)Q\xi(k) < 0, \\ \xi^T(k)P_1\xi(k), & \text{if } \xi^T(k)Q\xi(k) \geq 0, \end{cases} \tag{3.1}$$

with $P_0,\ P_1 \in \mathbb{S}^{2n+n_c}$. Note that it is not required for P_0 and P_1 to be positive (semi-)definite, but $V(\xi(k))$ should be positive for all $\xi(k) \neq 0$ in terms of the quadratic constraints in (3.1) to establish the asymptotic stability. Considering this, the case when P_0 and P_1 are negative (semi-)definite is out of our consideration. Based on this PWQ function, we are led to the following result.

Theorem 1. *The ETS Σ for a given $\sigma(> 0)$ is asymptotically stable if there exist matrices $P_0,\ P_1 \in \mathbb{S}^{2n+n_c}$ and scalars $\kappa_i,\ \mu,\ \alpha_{ij},\ \beta_{ij} \in \mathbb{R}_+$ with $i,\ j \in \{0, 1\}$ such that*

$$Z_i \succ 0,\ X_{ij}^{[\xi\xi]} \succeq 0,\ \forall i, j \in \{0, 1\}, \tag{3.2}$$

where

$$Z_i := P_i + (-1)^i\kappa_i Q, \tag{3.3}$$
$$X_{ij}^{[\xi\xi]} := P_i - F_i^T P_j F_i - \mu I + (-1)^i\alpha_{ij}Q + (-1)^j\beta_{ij}F_i^T QF_i. \tag{3.4}$$

Remark 1. *The first and second LMI-based conditions in (3.2) correspond to the positiveness of $V(\xi(k))$ for any $\xi(k) \neq 0$ and the property that $V(\xi(k)) \to 0$ $(k \to \infty)$, respectively. In a comparison to the parallel results in [23], the second LMI might be interpreted as a weaker version of the relevant results in that study since the considered stability definitions are different from each other, although the first LMI is exactly provided in that study.*

Remark 2. *P_i and P_j in (3.4) are used to describe the status of $\mathcal{E}$ between the triggering-off and triggering-on at the current time-index k and the next time index $k + 1$, respectively. In accordance with (3.1), taking i or j by 0 refers to the triggering-off, while that by 1 implies the triggering-on. For example, if we take $i = 0$ and $j = 1$, this means that the status of $\mathcal{E}$ is changed from the triggering-off at k to the triggering-on at $k + 1$.*

Remark 3. *These implications in Remarks 1 and 2 for the notations are equivalently applied to the LMI-based representations corresponding to the performance analysis in Section* 4 *(i.e., Theorems 5–7).*

Proof. We first show that $V(\xi(k)) > 0$ $(\forall \xi(k) \neq 0)$. Substituting $i = 0$ into the first inequality in (3.2) leads to

$$\begin{aligned} V(\xi(k)) &= \xi^T(k)P_0\xi(k) = \xi^T(k)(P_0 + \kappa_0 Q)\xi(k) - \kappa_0\xi^T(k)Q\xi(k) \\ &\geq \lambda_{\min}(P_0 + \kappa_0 Q)|\xi(k)|_2^2 - \kappa_0\xi^T(k)Q\xi(k) \\ &\geq \lambda_{\min}(P_0 + \kappa_0 Q)|\xi(k)|_2^2 \end{aligned}$$

for all $\xi(k)$ such that $\xi^T(k)Q\xi(k) < 0$. On the other hand, substituting $i = 1$ into the first inequality in (3.2) also implies that

$$\begin{aligned} V(\xi(k)) &= \xi^T(k)P_1\xi(k) = \xi^T(k)(P_1 - \kappa_1 Q)\xi(k) + \kappa_1\xi^T(k)Q\xi(k) \\ &\geq \lambda_{\min}(P_1 - \kappa_1 Q)|\xi(k)|_2^2 + \kappa_1\xi^T(k)Q\xi(k) \\ &\geq \lambda_{\min}(P_1 - \kappa_1 Q)|\xi(k)|_2^2 \end{aligned}$$

for all $\xi(k)$ such that $\xi^T(k)Q\xi(k) \geq 0$. In this sense, if we define

$$c := \min\{\lambda_{\min}(P_0 + \kappa_0 Q), \lambda_{\min}(P_1 - \kappa_1 Q)\} > 0,$$

then we obtain

$$0 < c|\xi(k)|_2^2 \leq V(\xi(k)), \quad \forall \xi(k) \neq 0,$$

and this clearly implies that $V(\xi(k)) > 0$ $(\forall \xi(k) \neq 0)$.

We next show that $V(\xi(k)) \to 0$ $(k \to \infty)$ for any initial condition $\xi(0)$. Note from (2.8), together with the arguments in Remark 2, that

$$V(\xi(k+1)) = (F_i\xi(k))^T P_j(F_i\xi(k)), \tag{3.5}$$

$$V(\xi(k)) = \xi^T(k)P_i\xi(k). \tag{3.6}$$

Combining (3.2)–(3.6) leads to

$$\begin{aligned} V(\xi(k+1)) - V(\xi(k)) &= \xi^T(k)(-P_i + F_i^T P_j F_i)\xi(k) \\ \leq &\xi^T(k)(-\mu I + (-1)^i\alpha_{ij}Q + (-1)^j\beta_{ij}F_i^T Q F_i)\xi(k) \\ = &-\mu\xi^T(k)\xi(k) + (-1)^i\alpha_{ij}\xi^T(k)Q\xi(k) + (-1)^j\beta_{ij}\xi^T(k+1)Q\xi(k+1). \end{aligned} \tag{3.7}$$

Because α_{ij} and β_{ij} are positive for all $i,\ j \in \{0, 1\}$, the second and third terms in (3.7) cannot be positive for all $i,\ j \in \{0, 1\}$ as mentioned in Remark 2. Hence, (3.7) admits the representation

$$V(\xi(k+1)) - V(\xi(k)) \leq -\mu\xi^T(k)\xi(k). \tag{3.8}$$

This, together with the fact that $\mu > 0$, obviously implies that $V(\xi(k)) \to 0$ $(k \to \infty)$. To summarize, $V(\xi(k))$ tends to 0 as k becomes larger for an arbitrary $\xi(0)$, and this means that $\xi(k) \to 0$ $(k \to \infty)$. This completes the proof. □

Here, it would be worthwhile to note that the asymptotic stability of Σ should be concerned with after the controller synthesis for the standard feedback systems (i.e., the case with $\sigma = 0$). Thus, it is quite important to characterize the relation between the cases of $\sigma = 0$ and $\sigma > 0$ under the feasibility conditions of the LMIs in (3.2), and we provide the following results.

Theorem 2. *Both the LMIs in (3.2) for* $\sigma = 0$ *and* $(i, j) = (1, 1)$ *are feasible if and only if the matrix* F_1 *given in (2.9) is Schur-stable. Furthermore,* F_1 *is Schur-stable only if the pairs* (A, B_2) *and* (C_2, A) *are stabilizable and detectable, respectively.*

Proof. Let us first denote Q by

$$Q = \begin{bmatrix} C_2^T Q_e C_2 & -C_2^T Q_e C_2 & 0 \\ -C_2^T Q_e C_2 & C_2^T Q_e C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \sigma^2 \begin{bmatrix} C_2^T Q_y C_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: Q_0 - \sigma^2 L. \tag{3.9}$$

If the LMIs in (3.2) for $\sigma = 0$ and $(i, j) = (1, 1)$ are feasible, then it immediately follows from $Q_0 \geq 0$ that

$$P_1 > \kappa_1 Q_0 \geq 0, \quad P_1 - F_1^T P_1 F_1 \geq \mu I + \alpha_{11} Q_0 + \beta_{11} F_1^T Q_0 F_1 > 0. \tag{3.10}$$

These inequalities undoubtedly imply the Lyapunov inequalities $P_1 > 0$ and $P_1 - F_1^T P_1 F_1 > 0$; thus, F_1 is Schur-stable. We next assume that F_1 is Schur-stable. Because there exists a matrix P_1 such that the Lyapunov inequalities hold, there also exist sufficiently small $\kappa_1, \mu, \alpha_{11}, \beta_{11} > 0$ such that (3.10) is satisfied. This is equivalent to the LMIs in (3.2) for $\sigma = 0$ and $(i, j) = (1, 1)$ being feasible. Finally, we note that F_1 is Schur-stable if and only if the matrix $\tilde{F}_1$ defined as

$$\tilde{F}_1 := \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \tag{3.11}$$

is Schur-stable since F_1 can be described by

$$F_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}^T \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c & 0 \\ B_c C_2 & A_c & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}. \tag{3.12}$$

This, together with the definitions of stabilizability and detectability, completes the proof of the last assertion. □

Theorem 3. *If both the LMIs in (3.2) for* $\sigma = 0$ *and* $(i, j) = (1, 1)$ *are feasible, then these LMIs for* $\sigma > 0$ *and* $(i, j) = (1, 1)$ *are also feasible.*

Proof. We first note from (3.9) that

$$P_1 - \kappa_1 Q = (P_1 - \kappa_1 Q_0) + \kappa_1 \sigma^2 L, \tag{3.13}$$

$$\begin{aligned} &P_1 - F_1^T P_1 F_1 - \mu I - \alpha_{11} Q - \beta_{11} F_1^T Q F_1 \\ &= (P_1 - F_1^T P_1 F_1 - \mu I - \alpha_{11} Q_0 - \beta_{11} F_1^T Q_0 F_1) + \alpha_{11} \sigma^2 L + \beta_{11} \sigma^2 F_1^T L F_1. \end{aligned} \tag{3.14}$$

If the LMIs in (3.2) for $\sigma = 0$ and $(i, j) = (1, 1)$ are feasible, then both (3.13) and (3.14) are positive semi-definite since $L \geq 0$. This completes the proof. □

Theorem 4. *Assume that $P_0 = P_1$ and $\kappa_i = \alpha_{ij} = \beta_{ij}$. Then, the LMIs in (3.2) for $(i, j) = (1, 0)$ are feasible if and only if F_1 is Schur-stable.*

Proof. Let us first define $\tilde{P} := P_1 - \kappa_1 Q$. If the LMIs in (3.2) for $(i, j) = (1, 0)$ are feasible with the assumptions, then we readily see that

$$\tilde{P} > 0, \quad \tilde{P} - F_1^T \tilde{P} F_1 \geq \mu I > 0. \tag{3.15}$$

This clearly implies that F_1 is Schur-stable. Next, if F_1 is Schur-stable, then there exist $\tilde{P} > 0$ and a sufficiently small $\mu > 0$ such that the inequalities in (3.15) hold. Then, $P_1 := \tilde{P} + \kappa_1 Q$ for some $\kappa_1 > 0$ becomes a solution of the LMIs in (3.2) with the assumptions. This completes the proof. □

Theorem 2 establishes the necessary and sufficient condition for the feasibility of both LMIs in (3.2) for the specific case of $\sigma = 0$ and $(i, j) = (1, 1)$, and it clarifies that the relevant stabilizability and detectability are required for the necessary and sufficient condition. It is shown in Theorem 3 that these two properties (i.e., stabilizability and detectability) corresponding to the conventional feedback systems (i.e., $\sigma = 0$) could lead to the feasibility of the LMIs in (3.2) with $(i, j) = (1, 1)$ for the general case of $\sigma > 0$. Furthermore, the stabilizability and detectability of P are closely related to the availability of the LMIs in (3.2) based on the arguments in Theorem 4, since the LMIs are taken by reducing the corresponding variables in this theorem.

4. Three performance measures for Σ with their LMI-based analyses

This section is concerned with quantitative performance analysis for the ETS Σ. To deal with the effect of the unknown disturbance w on the regulated output z, more precisely, we first introduce the definition of the $\ell_{p/q}$ performance of Σ as follows.

Definition 2. *Assume that Σ is asymptotically stable. If there exists a class $\mathcal{K}$ function $\beta(\cdot)$ and a scalar $\gamma_{p/q}(\geq 0)$ such that*

$$||z||_p \leq \beta(|\xi(0)|_2) + \gamma_{p/q} \cdot ||w||_q, \tag{4.1}$$

then the $\ell_{p/q}$ performance of Σ is said to be less than or equal to $\gamma_{p/q}$.

Regarding the quantitative performance measure for Σ introduced in Definition 2, we consider the three types of the $\ell_{p/q}$ performance with $(p, q) = (2, 2)$, $(\infty, 2)$ and (∞, ∞) in this section. Here, it would be worthwhile to note that these three $\ell_{p/q}$ performances coincide with the induced norms from $w \in \ell_q$ to $z \in \ell_p$, respectively, when we confine ourselves to the standard discrete-time LTI systems (i.e., $\sigma = 0$).

For a given $\gamma_{p/q}(\geq 0)$, the following arguments construct LMI-based conditions for the $\ell_{p/q}$ performance of Σ that is less than or equal to $\gamma_{p/q}$, where the asymptotic stability of Σ is implicitly assumed. First of all, we are led to the following result, in which an LMI condition for the $\ell_{2/2}$ performance of Σ is established based on the PWQ function given by (3.1).

Theorem 5. *The $\ell_{2/2}$ performance of Σ is less than or equal to $\gamma_{2/2}$ if there exist matrices $P_0,\ P_1 \in \mathbb{S}^{2n+n_c}$ and scalars $\kappa_i,\ \mu,\ \alpha_{ij},\ \beta_{ij} \in \mathbb{R}_+$ with $i,\ j \in \{0, 1\}$ such that*

$$Z_i > 0, \quad \begin{bmatrix} X_{ij}^{[\xi\xi]} & X_{ij}^{[\xi w]} & H_i^T \\ (*) & X_{ij}^{[ww]} + \gamma_{2/2}^2 \cdot I & D_{11}^T \\ (*) & (*) & I \end{bmatrix} \geq 0, \quad \forall i, j \in \{0, 1\}, \tag{4.2}$$

where $X_{ij}^{[\xi w]}$ and $X_{ij}^{[ww]}$ are given respectively by

$$X_{ij}^{[\xi w]} = -F_i^T P_j G + (-1)^j \beta_{ij} F_i^T QG, \tag{4.3}$$

$$X_{ij}^{[ww]} = -G^T P_j G + (-1)^j \beta_{ij} G^T QG. \tag{4.4}$$

Remark 4. *Because the matrices defined as (3.3), (3.4), (4.3) and (4.4) can be regarded as affine functions of the variables P_0, P_1, κ_i, μ, α_{ij} and β_{ij}, the inequalities (4.2) obviously construct LMI conditions. This interpretation can be equivalently applied to the inequalities in Theorems 6 and 7.*

Proof. As with the proof of Theorem 1, $V(\xi)$ becomes a positive definite function of ξ from (4.2), and we show that the $\ell_{2/2}$ performance of Σ is less than or equal to $\gamma_{2/2}$ based on this positive definite function. To this end, let us note that (3.5) is converted into

$$V(\xi(k+1)) = (F_i\xi(k) + Gw(k))^T P_j (F_i\xi(k) + Gw(k)) \tag{4.5}$$

in accordance with the effect of w. Then, we obtain from (3.6) and (4.5) that

$$V(\xi(k+1)) - V(\xi(k)) = \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -P_i + F_i^T P_j F_i & F_i^T P_j G \\ G^T P_j F_i & G^T P_j G \end{bmatrix} \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix}. \tag{4.6}$$

Regarding an upper bound on (4.6), taking the Schur complement of the second LMI in (4.2) leads to

$$\begin{bmatrix} X_{ij}^{[\xi\xi]} & X_{ij}^{[\xi w]} \\ (*) & X_{ij}^{[ww]} + \gamma_{2/2}^2 \cdot I \end{bmatrix} - \begin{bmatrix} H_i^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} H_i & D_{11} \end{bmatrix} \geq 0, \tag{4.7}$$

and this admits from (3.3), (3.4), (4.3) and (4.4) the representation

$$\begin{bmatrix} -P_i + F_i^T P_j F_i & F_i^T P_j G \\ G^T P_j F_i & G^T P_j G \end{bmatrix} \leq \begin{bmatrix} (-1)^i \alpha_{ij} Q + (-1)^j \beta_{ij} F_i^T Q F_i - \mu I - H_i^T H_i & (-1)^j \beta_{ij} F_i^T QG - H_i^T D_{11} \\ (-1)^j \beta_{ij} G^T Q F_i - D_{11}^T H_i & (-1)^j \beta_{ij} G^T QG + \gamma_{2/2}^2 I - D_{11}^T D_{11} \end{bmatrix}. \tag{4.8}$$

By combining (4.6) and (4.8) together with (2.8), we obtain that

$$\begin{aligned} V(\xi(k+1)) - V(\xi(k)) &\leq (-1)^i \alpha_{ij} \xi^T(k) Q \xi(k) - \mu \xi^T(k) \xi(k) + \gamma_{2/2}^2 w^T(k) w(k) \\ &\quad + (-1)^j \beta_{ij} (F_i \xi(k) + Gw(k))^T Q (F_i \xi(k) + Gw(k)) \\ &\quad - (H_i \xi(k) + D_{11} w(k))^T (H_i \xi(k) + D_{11} w(k)) \\ &= (-1)^i \alpha_{ij} \xi^T(k) Q \xi(k) + (-1)^j \beta_{ij} \xi^T(k+1) Q \xi(k+1) - \mu \xi^T(k) \xi(k) \\ &\quad + \gamma_{2/2}^2 w^T(k) w(k) - z^T(k) z(k) \leq \gamma_{2/2}^2 w^T(k) w(k) - z^T(k) z(k), \end{aligned} \tag{4.9}$$

where the last term follows from the arguments relevant to (3.7) and (3.8), i.e.,

$$(-1)^i \alpha_{ij} \xi^T(k) Q \xi(k) \leq 0, (-1)^j \beta_{ij} \xi^T(k+1) Q \xi(k+1) \leq 0, \ -\mu \xi^T(k) \xi(k) \leq 0. \tag{4.10}$$

On the other hand, if we define

$$J(k) := \begin{cases} -V(\xi(0)), & \text{if } k = 0, \\ -|\xi(0)|_2^2 + \displaystyle\sum_{n=0}^{k-1} \left(|z(n)|_2^2 - \gamma_{2/2}^2 |w(n)|_2^2 \right), & \text{if } k \geq 1, \end{cases} \tag{4.11}$$

then it readily follows from (4.9) that

$$V(\xi(k+1)) - V(\xi(k)) \le \gamma_{2/2}^2 |w(k)|_2^2 - |z(k)|_2^2 = -(J(k+1) - J(k)). \tag{4.12}$$

Because $J(0) = -V(\xi(0))$, (4.12) clearly implies that $0 \le V(\xi(k)) \le -J(k)$ for all $k \in \mathbb{N}_0$. To put it another way, we obtain

$$\begin{aligned}\sum_{n=0}^{k-1} |z(n)|_2^2 \le |\xi(0)|_2^2 + \gamma_{2/2}^2 \sum_{n=0}^{k-1} |w(n)|_2^2 \\ \le |\xi(0)|_2^2 + \gamma_{2/2}^2 \sum_{n=0}^{\infty} |w(n)|_2^2 = |\xi(0)|_2^2 + \gamma_{2/2}^2 \|w\|_2^2 \\ \le (|\xi(0)|_2 + \gamma_{2/2}\|w\|_2)^2, \quad \forall k \in \mathbb{N}_0. \end{aligned} \tag{4.13}$$

This, together with defining $\beta(|\xi(0)|_2) := |\xi(0)|_2$, leads to

$$\|z\|_2 \le \beta(|\xi(0)|_2) + \gamma_{2/2}\|w\|_2. \tag{4.14}$$

This completes the proof. □

Furthermore, the LMI conditions in Theorem 5 in fact establish the necessary and sufficient condition for the $\ell_{2/2}$ performance of Σ to be less than or equal to $\gamma_{2/2}$, especially for the specific case with $\sigma = 0$ (i.e., the H_∞ norm of discrete-time LTI systems [16]), although we omit the details.

Next, we derive an LMI condition for the $\ell_{\infty/2}$ performance of theETS Σ to be less than or equal to $\gamma_{\infty/2}(\ge 0)$. In other words, we consider the disturbance $w \in (\ell_2)^{n_w}$ and the regulated output $z \in (\ell_\infty)^{n_z}$ and establish the following result, parallel to Theorem 5.

Theorem 6. *The $\ell_{\infty/2}$ performance of Σ is less than or equal to $\gamma_{\infty/2}$ if there exist matrices $P_0,\ P_1 \in \mathbb{S}^{2n+n_c}$ and scalars $\kappa_i,\ \mu,\ \alpha_{ij},\ \beta_{ij} \in \mathbb{R}_+$ with $i,\ j \in \{0,1\}$ such that*

$$Z_i \succ 0, \begin{bmatrix} X_{ij}^{[\xi\xi]} & X_{ij}^{[\xi w]} \\ (*) & X_{ij}^{[ww]} + I \end{bmatrix} \ge 0, \begin{bmatrix} -P_i & 0 & H_i^T \\ (*) & 0 & D_{11}^T \\ (*) & (*) & -\gamma_{\infty/2}^2 \cdot I \end{bmatrix} \le 0,\ \forall i, j \in \{0,1\}. \tag{4.15}$$

Proof. Because the proof is essentially equivalent to that of Theorem 5, we briefly outline the process. To derive an upper bound on (4.6), we note from the second LMI in (4.15) and (4.10) (as well as (3.4), (4.3) and (4.4)) that

$$\begin{aligned} V(\xi(k+1)) - V(\xi(k)) &= \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -P_i + F_i^T P_j F_i & F_i^T P_j G \\ G^T P_j F_i & G^T P_j G \end{bmatrix} \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix} \\ &\le \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -\mu I + (-1)^i \alpha_{ij} Q + (-1)^j \beta_{ij} F_i^T Q F_i & (-1)^j \beta_{ij} F_i^T Q G \\ (-1)^j \beta_{ij} G^T Q F_i & I + (-1)^j \beta_{ij} G^T Q G \end{bmatrix} \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix} \\ &= -\mu \xi^T(k)\xi(k) + (-1)^i \alpha_{ij} \xi^T(k) Q \xi(k) + w^T(k) w(k) \\ &\quad + (-1)^j \beta_{ij} \xi^T(k+1) Q \xi(k+1) \le w^T(k) w(k). \end{aligned} \tag{4.16}$$

Taking summation with respect to k from 0 to $k-1$ in (4.16) leads to

$$V(\xi(k)) \le V(\xi(0)) + \sum_{n=0}^{k-1} |w(n)|_2^2. \tag{4.17}$$

On the other hand, we obtain from the Schur complement of the third LMI in (4.15) that

$$\begin{bmatrix} H_i^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} H_i & D_{11} \end{bmatrix} \le \gamma_{\infty/2}^2 \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.18}$$

This, together with the fact that $z(k) = H_i\xi(k) + D_{11}w(k)$, allows us to arrive at

$$|z(k)|_2^2 = \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} H_i^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} H_i & D_{11} \end{bmatrix} \begin{bmatrix} \xi(k) \\ w(k) \end{bmatrix} \le \gamma_{\infty/2}^2 \cdot V(\xi(k)). \tag{4.19}$$

Then, combining (4.17) with (4.19) leads to the following for all $k \in \mathbb{N}_0$:

$$\begin{aligned} |z(k)|_2^2 &\le \gamma_{\infty/2}^2 \cdot \left(V(\xi(0)) + \sum_{n=0}^{k-1} |w(n)|_2^2 \right) \le \gamma_{\infty/2}^2 \cdot (V(\xi(0)) + \|w\|_2^2) \\ &\le \gamma_{\infty/2}^2 \cdot (\max\{\lambda_{\max}(P_0), \lambda_{\max}(P_1)\}) \, |\xi(0)|_2^2 + \gamma_{\infty/2}^2 \|w\|_2^2, \end{aligned} \tag{4.20}$$

because $\lambda_{\max}(A+B) \le \lambda_{\max}(A) + \lambda_{\max}(B)$ for arbitrary $A,\ B \in \mathbb{S}^{\nu}$. This, together with the fact that $\max\{\lambda_{\max}(P_0), \lambda_{\max}(P_1)\} > 0$, allows us to define

$$\beta(|\xi(0)|_2) := \gamma_{\infty/2} \left(\max\{\lambda_{\max}(P_0), \lambda_{\max}(P_1)\}\right)^{1/2} |\xi(0)|_2. \tag{4.21}$$

This completes the proof. □

Even though we omit the details for a limited space, the LMI conditions in this theorem are also equivalent to the necessary and sufficient condition for the induced norm from ℓ_2 to ℓ_∞ to be less than or equal to $\gamma_{\infty/2}$, especially for the specific case with $\sigma = 0$ (i.e., the generalized H_2 norm of discrete-time LTI systems [18]).

On the other hand, let us consider the disturbance $w \in (\ell_\infty)^{n_w}$ with the regulated output $z \in (\ell_\infty)^{n_z}$ and provide the following LMI condition for the $\ell_{\infty/\infty}$ performance of Σ to be less than or equal to $\gamma_{\infty/\infty} (\ge 0)$.

Theorem 7. *If there exist matrices $P_0,\ P_1 \in \mathbb{S}^{2n+n_c}$ and scalars $\kappa_i,\ \mu,\ \alpha_{ij},\ \beta_{ij} \in \mathbb{R}_+$ with $i,\ j \in \{0,\ 1\}$ such that*

$$Z_i \succ 0, \begin{bmatrix} \tilde{X}_{ij}^{[\xi\xi]} & X_{ij}^{[\xi w]} & H_i^T \\ (*) & X_{ij}^{[ww]} + I & D_{11}^T \\ (*) & (*) & \tilde{\gamma}^2 \cdot I \end{bmatrix} \ge 0,\ \forall i, j \in \{0, 1\}, \tag{4.22}$$

where

$$\tilde{X}_{ij}^{[\xi\xi]} := cP_i - F_i^T P_j F_i + (-1)^i \alpha_{ij} Q + (-1)^j \beta_{ij} F_i^T Q F_i - \mu I, \tag{4.23}$$

$$\tilde{\gamma} := \gamma_{\infty/\infty} \sqrt{1-c} \tag{4.24}$$

for $c \in (0, 1)$; then, the $\ell_{\infty/\infty}$ performance of Σ is less than or equal to $\gamma_{\infty/\infty}$.

Remark 5. *The statement with respect to c is only for deriving a class $\mathcal{K}$ function $\beta(\cdot)$, and it is not required to modify the overall LMI conditions in Theorem 7, except for setting $c = 0$ when the arguments are confined to $\xi(0) = 0$.*

Proof. As with the proofs of Theorems 5 and 6, we obtain from (3.6) and (4.5) that

$$V(\xi(k+1))-cV(\xi(k))=\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}^T\begin{bmatrix}-cP_i+F_i^TP_jF_i & F_i^TP_jG\\ G^TP_jF_i & G^TP_jG\end{bmatrix}\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}. \tag{4.25}$$

Regarding an upper bound of (4.25), it immediately follows from taking the Schur complement of the second LMI in (4.22) that

$$\begin{bmatrix}\tilde{X}_{ij}^{[\xi\xi]} & X_{ij}^{[\xi w]}\\ (*) & X_{ij}^{[ww]}+I\end{bmatrix}-\begin{bmatrix}H_i^T\\ D_{11}^T\end{bmatrix}(\tilde{\gamma}^2I)^{-1}\begin{bmatrix}H_i & D_{11}\end{bmatrix}\geq 0. \tag{4.26}$$

Then, combining (4.26) with (4.10) (as well as (4.3), (4.4) and (4.23)) leads to

$$\begin{aligned}
V(\xi(k+1))-cV(\xi(k))&=\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}^T\begin{bmatrix}-cP_i+F_i^TQF_i & F_i^TP_jG\\ G^TP_jF_i & G^TP_jG\end{bmatrix}\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}\\
\leq\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}^T&\begin{bmatrix}(-1)^i\alpha_{ij}Q+(-1)^j\beta_{ij}F_i^TQF_i-\mu I-\tilde{\gamma}^{-2}H_i^TH_i & (-1)^j\beta_{ij}F_i^TQG-\tilde{\gamma}^{-2}H_i^TD_{11}\\ (-1)^j\beta_{ij}G^TQF_i-\tilde{\gamma}^{-2}D_{11}^TH_i & (-1)^j\beta_{ij}G^TQG+I-\tilde{\gamma}^{-2}D_{11}^TD_{11}\end{bmatrix}\begin{bmatrix}\xi(k)\\ w(k)\end{bmatrix}\\
=(-1)^i\alpha_{ij}\xi^T(k)Q\xi(k)&+(-1)^j\beta_{ij}\xi(k+1)^TQ\xi(k+1)-\tilde{\gamma}^{-2}z^T(k)z(k)\\
&-\mu\xi^T(k)\xi(k)+w^T(k)w(k)\leq\tilde{\gamma}^{-2}z^T(k)z(k)+w^T(k)w(k).
\end{aligned} \tag{4.27}$$

Here, we note from (4.27) that

$$V(\xi(k+1))-cV(\xi(k))\leq|w(k)|_2^2 \tag{4.28}$$

and multiply $c^{-k}(>0)$ on both sides of (4.28), i.e.,

$$c^{-k}V(\xi(k+1))-c^{-(k-1)}V(\xi(k))\leq c^{-k}|w(k)|_2^2. \tag{4.29}$$

Taking summation with respect to k from 0 to $k-1$ in (4.29) leads to

$$c^{-(k-1)}V(\xi(k))-cV(\xi(0))\leq\sum_{n=0}^{k-1}c^{-n}|w(n)|_2^2. \tag{4.30}$$

Multiplying c^{k-1} again on both sides of (4.30) yields

$$\begin{aligned}
V(\xi(k))&\leq c^kV(\xi(0))+c^{k-1}\sum_{n=0}^{k-1}c^{-n}|w(n)|_2^2=c^kV(\xi(0))+\sum_{n=0}^{k-1}c^n|w(k-1-n)|_2^2\\
&\leq V(\xi(0))+\left(\sum_{n=0}^{k-1}c^n\right)\cdot\|w\|_\infty^2\leq V(\xi(0))+\left(\sum_{n=0}^{\infty}c^n\right)\cdot\|w\|_\infty^2.
\end{aligned} \tag{4.31}$$

On the other hand, we obtain from (4.27) that

$$\tilde{\gamma}^{-2}|z(k)|_2^2 \leq -V(\xi(k+1)) + cV(\xi(k)) + |w(k)|_2^2 \leq cV(\xi(k)) + |w(k)|_2^2, \tag{4.32}$$

where the last assertion readily follows from $V(\xi) \geq 0,\ \forall \xi \in \mathbb{R}^{2n+n_c}$, and this further leads to

$$|z(k)|_2^2 \leq c\tilde{\gamma}^2 V(\xi(k)) + \tilde{\gamma}^2 |w(k)|_2^2. \tag{4.33}$$

This, together with (4.31), allows us, for all $k \in \mathbb{N}_0$, to arrive at

$$\begin{aligned}
|z(k)|_2^2 &\leq c\tilde{\gamma}^2 \cdot \left(V(\xi(0)) + \left(\sum_{n=0}^{\infty} c^n \right) \|w\|_\infty^2 \right) + \tilde{\gamma}^2 |w(k)|_2^2 \\
&\leq c\tilde{\gamma}^2 V(\xi(0)) + \frac{c\tilde{\gamma}^2}{1-c}\|w\|_\infty^2 + \tilde{\gamma}^2\|w\|_\infty^2 = c\tilde{\gamma}^2 V(\xi(0)) + \gamma_{\infty/\infty}^2 \|w\|_\infty^2 \\
&\leq c\tilde{\gamma}^2 \cdot (\max\{\lambda_{\max}(P_0), \lambda_{\max}(P_1)\})\, |\xi(0)|_2^2 + \gamma_{\infty/\infty}^2 \|w\|_\infty^2.
\end{aligned} \tag{4.34}$$

This, together with the definition

$$\beta(|\xi(0)|_2) := \sqrt{c}\tilde{\gamma}\, (\max\{\lambda_{\max}(P_0), \lambda_{\max}(P_1)\})^{1/2}\, |\xi(0)|_2, \tag{4.35}$$

completes the proof. □

In contrast to the cases of the $\ell_{2/2}$ and $\ell_{\infty/2}$ performances, no explicit closed-form representation of the $\ell_{\infty/\infty}$ performance is studied, even for the specific case with $\sigma = 0$ (i.e., the ℓ_∞-induced norm of discrete-time LTI systems equipped with the spatial 2-norm). Thus, the assertions in Theorem 7 are quite meaningful given that the $\ell_{\infty/\infty}$ performance for discrete-time LTI systems, as well as ETSs is described through the LMI-based representations for the first time.

Before ending this section, it should be remarked that it is not clarified in these theorems whether or not a relation between the $\ell_{p/q}$ performances and the triggering parameter σ could be derived. Thus, let us provide somewhat intuitive interpretations of the LMI conditions in Theorems 5–7, by which a tendency of the $\ell_{p/q}$ performances with respect to σ could be observed. More precisely, it could be expected that the $\ell_{p/q}$ performances would be increased by taking σ larger, because such a selection procedure of σ obviously causes the transmission rate between the plant and the controller to be small.

For the $\ell_{2/2}$ performance, it immediately follows from the $(2,2)$th block of the second LMI in (4.2) with $j = 0$, (4.4) and (3.9) that

$$-G^T P_0 G + \beta_{i0} G^T Q_0 G - \sigma^2 \beta_{i0} G^T L G + \gamma_{2/2}^2 \cdot I \geq 0. \tag{4.36}$$

It could be naturally expected that $\gamma_{2/2}$ would be increased as σ becomes larger to ensure that (4.36) is feasible. Regarding the $\ell_{\infty/\infty}$ performance, applying essentially the same arguments to the $(2,2)$th block of (4.26) leads to

$$-G^T P_0 G + \beta_{i0} G^T Q_0 G - \sigma^2 \beta_{i0} G^T L G + I - \frac{1}{\tilde{\gamma}^2} D_{11}^T D_{11} \geq 0. \tag{4.37}$$

Thus, taking $\tilde{\gamma}$ larger leads to making (4.37) feasible when σ is increasing. In connection with the case of the $\ell_{\infty/2}$ performance, we can obtain by applying the same arguments to the $(2,2)$th block in the second LMI in (4.15) with $j = 0$, that

$$-G^T P_0 G + \beta_{i0} G^T Q_0 G - \sigma^2 \beta_{i0} G^T L G + I \geq 0. \tag{4.38}$$

For the feasibility of (4.38), it would be expected that P_0 becomes smaller in terms of positive definiteness as σ becomes larger. Here, if we note from the (1, 1)th block of (4.18) for $i = 0$ that

$$H_0^T H_0 \leq \gamma_{\infty/2}^2 \cdot P_0, \tag{4.39}$$

$\gamma_{\infty/2}$ also becomes larger by taking σ larger.

These expectations relevant to the $\ell_{p/q}$ performances with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞) will also be confirmed from the numerical results in the following section.

5. Numerical example

This section presents a numerical example to verify the effectiveness of the overall arguments. To this end, let us consider the linear inverted pendulum model (LIPM) of biped walking systems described by

$$\frac{d}{dt}\begin{bmatrix} c \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix}\begin{bmatrix} c \\ \dot{c} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ -\omega^2 \end{bmatrix} u, \tag{5.1}$$

where ω, $c_{\mathbf{x}}$, $\dot{c}_{\mathbf{x}}$, w and u denote the natural frequency, the position of the center of mass (CoM), the velocity of the CoM, the unknown disturbance force and the control input, respectively.

For $\omega^2 = 14$ [1/s^2], we also take both the measurement and regulated outputs by $y = z = c$. Then, applying the zero-order hold discretization [24] with the sampling time $h = 0.1$ [s] leads to the discrete-time plant given by the following matrices.

$$\begin{aligned} &A = \begin{bmatrix} 1.0708 & 0.1023 \\ 1.4329 & 1.0708 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.0051 \\ 0.1023 \end{bmatrix}, \ B_2 = \begin{bmatrix} -0.0708 \\ -1.4329 \end{bmatrix}, \\ &C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D_{11} = D_{12} = 0. \end{aligned} \tag{5.2}$$

With this discrete-time plant, we consider the standard (i.e., the case with $\sigma = 0$) stabilizing controller described by the following matrices.

$$A_c = \begin{bmatrix} -2.6258 & 0.0753 \\ -33.7707 & 0.5242 \end{bmatrix}, \ B_c = \begin{bmatrix} 3.5416 \\ 32.0671 \end{bmatrix}, C_c = \begin{bmatrix} 2.1889 & 0.3815 \end{bmatrix}, \ D_c = 0. \tag{5.3}$$

Remark 6. *We omit the details for the synthesis procedure of (5.3) since these parameters are readily obtained by following the arguments in Theorem 11.1.1 in [24], in which the necessary and sufficient condition for the asymptotic stability of the resulting discrete-time systems is discussed.*

For the discrete-time LIPM for (5.2) and (5.3), we take the triggering parameter σ ranging from 0 to 0.35 and the weighting matrices $Q_e = Q_y = I$. Based on the arguments in Theorems 5–7, we compute the minimum values of $\gamma_{p/q}$, for which the LMI conditions relevant to each theorem are feasible, and the corresponding results are shown in Figure 2. Because the results in Figure 2 show that all $\ell_{p/q}$ performances of Σ with $(p, q) = (2, 2)$, $(\infty, 2)$ and (∞, ∞) have finite values for all $\sigma \in [0, 0.35]$, it is obvious that Σ is asymptotically stable for all $\sigma \in [0, 0.35]$. It is also observed from Figure 2 that all three $\ell_{p/q}$ performances of Σ are increased by taking a larger σ, and this tendency is discussed in the previous section. In this sense, it would be worthwhile to provide a guideline to take the triggering parameter σ.

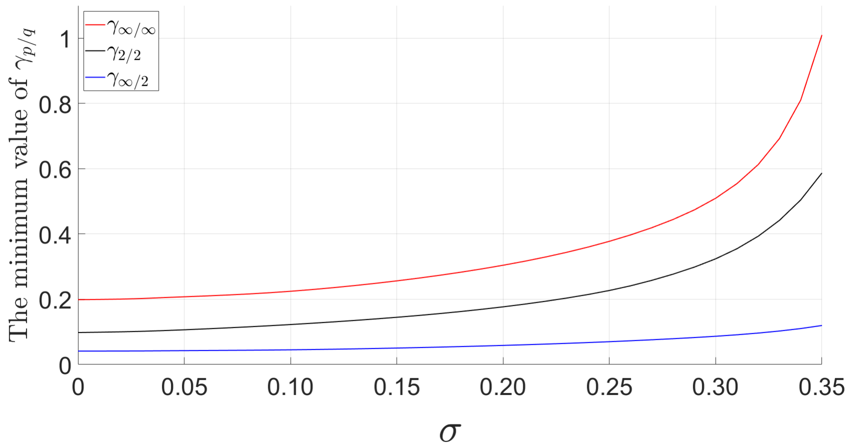
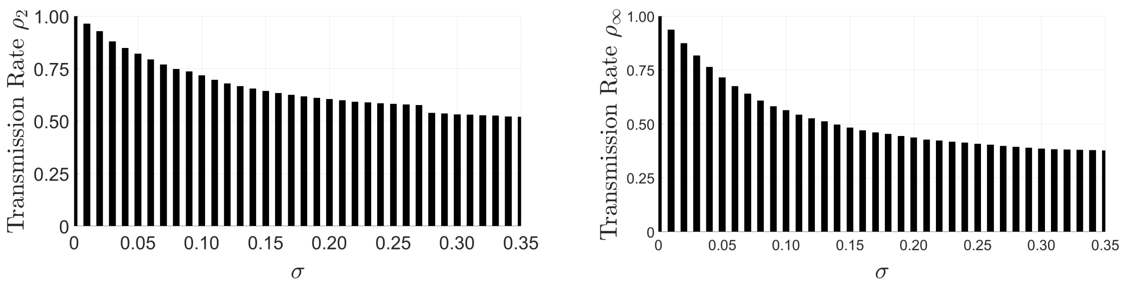


Figure 2. The results for the $\ell_{p/q}$ performances.

In connection with this, we first introduce some parameters to describe the practical effectiveness of the ETS Σ. We take ρ_q ($q = 2, \infty$) to denote the transmission rates between the plant and the memory, which are obtained by computing the mean values of 1000 numbers of random sequences $\{w(k)\}_{k=0}^{\infty}$ for $w \in \ell_2$ and $w \in \ell_\infty$. The results for ρ_2 and ρ_∞ are shown in Figure 3(a) and Figure 3(b), respectively.



(a) The results for ρ_2.

(b) The results for ρ_∞.

Figure 3. The results for $\rho_q(q = 2, \infty)$.

We can observe in Figure 3 that the transmission rates for both the disturbances such that $w \in \ell_2$ and $w \in \ell_\infty$ decrease by taking the triggering parameter σ larger. To put it another way, we can ascertain from Figures 2 and 3 that there exists a trade-off between $\gamma_{p/q}$ and ρ_q.

Remark 7. *Even though the transmission rates ρ_2 and ρ_∞ are not always monotonously decreasing by taking the triggering parameter σ larger, such a decreasing tendency can be observed in Figure 3 by considering the average. This is in contrast to the results observed in Figure 2; all of the $\ell_{p/q}$ performances obtained from the LMI-based conditions in Theorems 5–7 are monotonously increasing as σ becomes larger.*

Regarding this, we call $\gamma_{p/q} \times \rho_q$ given the same triggering parameter σ with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞), the penalized performances, which could be used as effective criteria for selecting the triggering parameter σ. More precisely, taking σ such that the penalized performance for fixed (p, q) is minimized could be regarded as the most effective decision with respect to the trade-off between $\gamma_{p/q}$ and ρ_q. For ease of understanding, the meanings of σ, $\gamma_{p/q}$, ρ_q and $\gamma_{p/q} \times \rho_q$ are summarized as in Table 2.

Table 2. The meanings of some notations.

σ	Triggering parameter associated with (2.4)
$\gamma_{p/q}$	The $\ell_{p/q}$ performance obtained through Theorems 5–7
ρ_q	The transmission rate between the plant and the memory
$\gamma_{p/q} \times \rho_q$	The penalized performance for the trade-off between $\gamma_{p/q}$ and ρ_q

The results for the penalized performances with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞) are shown in Figure 4, and we can observe in this figure that $\gamma_{p/q} \times \rho_q$ achieves its minimum values at $\sigma = 0.08, 0.12$ and 0.13 for $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞), respectively.

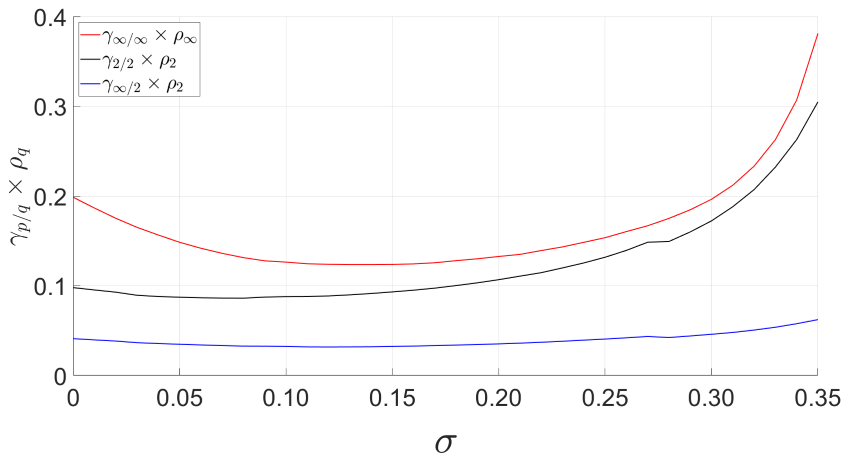


Figure 4. The results for penalized performances (i.e., $\gamma_{p/q} \times \rho_q$).

For a comparison between the ETC with these values of σ and the conventional output-feedback control (i.e., $\sigma = 0$), the results for σ, $\gamma_{p/q}$, ρ_q and $\gamma_{p/q} \cdot \rho_q$ are shown in Table 3. It can be ascertained from this table that the rate of increase from the conventional output-feedback control to the developed ETC in the $\ell_{p/q}$ performance is quite smaller than that of decrease from the conventional output-feedback control to the developed ETC in terms of the transmission rate ρ_q for all cases with $(p, q) = (2, 2), (\infty, 2)$ and (∞, ∞). This clearly implies that the developed ETC with the aforementioned values of the triggering parameter σ is practically superior to the conventional output-feedback control.

Table 3. The $\ell_{p/q}$ performances and transmission rates.

	Output-feedback control	Event-triggered control
σ	0.00	0.08
$\gamma_{2/2}$	0.0979	0.1154
ρ_2	1.00	0.7701
$\gamma_{2/2} \times \rho_2$	0.0979	0.0863
σ	0.00	0.12
$\gamma_{\infty/2}$	0.0411	0.0468
ρ_2	1.00	0.6789
$\gamma_{\infty/2} \times \rho_2$	0.0411	0.0318
σ	0.00	0.13
$\gamma_{\infty/\infty}$	0.1986	0.2418
ρ_∞	1.00	0.507
$\gamma_{\infty/\infty} \times \rho_\infty$	0.1986	0.1228

6. Conclusions

This paper presents new performance measures for discrete-time ETSs, and their computational methods in terms of LMIs. Regarding the objective of reducing the communication loads between a plant and a feedback controller, the ETM taken in this paper determines whether or not the current measurement output from the plant is transmitted to the controller. Regarding the input/output behavior of the ETSs obtained by connecting the plant, ETM and feedback controller, we derived their closed-form representation via a piecewise linear difference equation. Based on this expression, the asymptotic stability condition for the ETSs was established through the LMI approach. Towards a further sophisticated argument on the considered ETM, the $\ell_{2/2}$, $\ell_{\infty/2}$ and $\ell_{\infty/\infty}$ performances of the ETSs were also described through the LMI approach. The theoretical validity and the practical effectiveness of the developed methods on performance analysis were verified through a numerical example. Extending the $l_{p/q}$ performance analysis to recently developed ETC schemes [8, 9] is left as interesting future work.

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Conflict of interest

All authors declare no conflicts of interest regarding the publication of this paper.

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