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Injective coloring of planar graphs with girth 5

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Abstract: A k-injective-coloring of a graph G is a mapping $c : V(G) \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ for any two vertices u and v if u and v have a common vertex. The injective chromatic number of G, denoted by $\chi_i(G)$, is the least k such that G has an injective k-coloring. In this paper, we prove that for planar graph G with $g(G) \geq 5$, $\Delta(G) \geq 20$ and without adjacent 5-cycles, $\chi_i(G) \leq \Delta(G) + 2$.

Keywords: injective coloring; planar graph; cycle
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1. Introduction

All graphs considered in this paper are finite simple graphs. For a planar graph G, we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, $\delta(G)$, $g(G)$ and $d_G(u, v)$ to denote its vertex set, edge set, face set, maximum degree, minimum degree, girth and the distance between u and v in graph G, respectively. For a vertex $v \in V(G)$, we use k (k^+ or k^-)-vertex to denote a vertex of degree k (at least k or at most k). A k (k^+ or k^-)-face is defined similarly. A k-neighbor of v is a k-vertex adjacent to v. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [x_1x_2 \cdots x_k]$ if $x_1, x_2, \cdots, x_k$ are the vertices of $b(f)$ in the clockwise order.

An injective k-coloring of a graph G is a vertex coloring $c : V(G) \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if u and v have a common neighbor. The injective chromatic number of G, denoted by $\chi_i(G)$, is the least integer k such that G has an injective k-coloring.

The idea of injective coloring was introduced by Hahn et al. [1]. They proved the inequality $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$ for any planar graph G with maximum degree Δ. For planar graph G with $g(G) \geq 4$, there are fewer results, Bu et al. [2] proved that $\chi_i(G) \leq \Delta + 6$ if $\Delta \geq 20$ and 4-cycle and 4-cycle are disjoint. For planar graph G with $g(G) \geq 5$, Bu and Lu [3] proved that $\chi_i(G) \leq \Delta + 7$; then Dong and

Lin [4] improved this result to $\chi_i(G) \le \Delta + 6$; Lužar et al [5] proved that $\chi_i(G) \le \Delta + 4$ if $\Delta \ge 439$; recently, Bu and Huang [6] showed that $\chi_i(G) \le \Delta + 4$ if $\Delta \ge 11$; Bu and Ye [7] improved this upper bound to $\Delta + 3$ when $\Delta \ge 20$. For planar graph G with $g(G) \ge 6$, Dong and Lin [8] proved that $\chi_i(G) \le \Delta + 2$ if $\Delta \ge 9$ and $\chi_i(G) \le \Delta + 1$ if $\Delta \ge 17$. In [9], Borodin et al proved that for every planar graph G, $\chi_i(G) = \Delta$ in each of the following cases (i-iv): (i) $g(G) = 7$ and $\Delta \ge 16$; (ii) $8 \le g(G) \le 9$ and $\Delta \ge 10$; (iii) $10 \le g(G) \le 11$ and $\Delta \ge 6$; (iv) $g(G) \ge 13$ and $\Delta = 5$.

In this paper, we consider the injective chromatic number of planar graph G with $g(G) \ge 5$ and prove the following theorem.

Theorem 1.1. If G is a planar graph with $g(G) \ge 5$, $\Delta(G) \ge 20$ and without adjacent 5-cycles, then $\chi_i(G) \le \Delta(G) + 2$.

2. Proof of Theorem 1.1

2.1. The properties of minimal counterexample

Let G be a graph. If G can not admit any injective coloring with k colors, but any subgraph of G can, then we call G is injective k-critical. In this section, we assume G is injective k-critical. We give some structural properties of G.

For convenience, we give out some notations. For a k-vertex v, let $N(v) = \{v_1, v_2, \cdots, v_k\}$ with $d(v_1) \le d(v_2) \le \cdots \le d(v_k)$, $D(v) = \sum_{i=1}^{k} d(v_i)$, $N^2(v) = \cup_{1\le i\le k}(N(v_i) \setminus \{v\})$. We use $n_k(v)$ to denote the number k-vertices in $N(v)$. If $d(v_1) = 2$, then let $N(v_1) = \{v, v'_1\}$. A 2-vertex or 4^+-vertex v of G is called a heavy vertex or is heavy if $D(v) \ge \Delta + 2 + d(v)$, otherwise v is called a light vertex or is light. A 3-vertex v of G is called a heavy vertex or is heavy if $D(v) \ge \Delta + 2 + d(v)$ and $n_2(v) = 0$. Conversely, a 3-vertex v is called a light vertex or is light if $D(v) \le \Delta + 1 + d(v)$ and $n_2(v) = 0$. We use $n_k^l(v)$ and $n_k^h(v)$ to denote the number of adjacent light and heavy k-vertices of v, respectively. For a partial vertex coloring c of G, we use $F(v)$ to denote the forbidden colors for v. Let $C = \{1, 2, \cdots, \Delta + 2\}$ be a color set. For integers k and n, a $k(n)$-vertex is a k-vertex adjacent to n 2-vertices. Now, we discuss the structures of G.

Lemma 2.1. Let $uv \in E(G)$. If $D(u) \le \Delta + 1 + d(u)$, then $D(v) \ge \Delta + 2 + d(v)$.

Proof. By contradiction, suppose $D(v) \le \Delta + 1 + d(v)$. By the minimality of G, $G - uv$ admits an injective coloring with $\Delta + 2$ colors. Erase the colors on u and v. Since $D(u) \le \Delta + 1 + d(u)$ and $D(v) \le \Delta + 1 + d(v)$, we have $|F(u)| \le \Delta + 1$, $|F(v)| \le \Delta + 1$. Let $c(u) \in C - F(u)$ and $c(v) \in C - F(v)$, then G has an injective $(\Delta + 2)$-coloring, a contradiction.

Based on Lemma 2.1, we can obtain the following two lemmas .

Lemma 2.2. $\delta(G) \ge 2$ and G contains no adjacent 2-vertices.

Lemma 2.3. Let v be a 3(1)-vertex of G with $N(v) = \{v_1, v_2, v_3\}$, where $d(v_1) = 2$, then $d(v_2) + d(v_3) \ge \Delta + 3$.

Lemma 2.4. G contains no adjacent 3(1)-vertices.

Proof. By contradiction, suppose that there exists two adjacent 3(1)-vertices u and v. Let $u_1 \in N(u)$ and $v_1 \in N(v)$ be two 2-vertices. By the minimality of G, $G - uu_1$ admits an injective coloring with $\Delta + 2$ colors. We erase the colors on u, v_1 and u_1. First, we can color u since $|F(u)| \le \Delta + 1$. Then $|F(v_1)| \le \Delta + 1$, $|F(u_1)| \le \Delta + 1$, so we can color v_1 and u_1 in turn to get an injective $(\Delta + 2)$-coloring of G, a contradiction.

Lemma 2.5. If $f = [\cdots uv_1vv_2x\cdots]$, v is a light 3-vertex, v_1 and v_2 are 3(1)-vertices, $d(u) = 2$, then f is a 6^+-face.

Proof. Suppose that the lemma is not true. Assume that $f = [uv_1vv_2x]$ is a 5-face, then $d(x) = \Delta$ by Lemmas 2.1 and 2.2. By the minimality of G, $G - vv_2$ admits an injective coloring with $\Delta + 2$ colors. We erase the colors on v, v_2 and u. Since $|F(v_2)| \le \Delta - 2 + 1 + 2 = \Delta + 1$ and $|F(u)| \le \Delta - 1 + 1 = \Delta$, we can color v_2 and u. Finally, $|F(v)| \le \Delta + 1$ since v is a light 3-vertex, we can color v, a contradiction.

2.2. *Discharging*

In this section, we prove Theorem 1.1 by contradiction. Let G be a minimal counterexample of Theorem 1.1.

Applying Eulers formula $|V| + |F| - |E| = 2$ and the fact $\sum\limits_{v\in V(G)} d(v) = \sum\limits_{f\in F(G)} d(f) = 2|E(G)|$ for a plane graph, we have

$$\sum_{v\in V(G)} (\frac{3}{2}d(v) - 5) + \sum_{f\in F(G)} (d(f) - 5) = -10.$$

Design a weight function $\omega(x)$ such that $\omega(x) = \frac{3}{2}d(x) - 5$ for each $x \in V(G)$, $\omega(x) = d(x) - 5$ for each $x \in F(G)$. Hence, $\sum\limits_{x\in V(G)\bigcup F(G)} \omega(x) = -10$. Next, we shall transfer weight. Our discharging procedure has two steps. Now, let's look at the structural properties of G, while keeping the total weight sum constant, we obtain a new weight $w'(x)$ for all $x \in V \cup F$ by transferring weights after the first step. We shall prove that $w^*(x) \ge 0$ for each $x \in V \cup F$ after the second step and then get the following contradiction:

$$0 \le \sum_{x\in V(G)\bigcup F(G)} \omega^*(x) = \sum_{x\in V(G)\bigcup F(G)} \omega(x) = -10.$$

This contradiction shows that G does not exist and thus Theorem 1.1 is true.

The first step

We define the following discharging rules.

R10. Every vertex v with $d(v) \ge 10$ sends $\frac{3}{2} - \frac{5}{d(v)}$ to each adjacent 9^--vertex.

R11. Let v be a 2-vertex with $N(v) = \{v_1, v_2\}$. If $d(v_1) \le d(v_2) \le 9$, then $v_i\,(i = 1, 2)$ sends $\frac{11}{12}$ to v. If $d(v_1) \le 9 < d(v_2)$, then v_1 sends $\frac{1}{3} + \frac{5}{d(v_2)}$ to v.

R12. Let v be a 3(1)-vertex with $N(v) = \{v_1, v_2, v_3\}$. If $d(v_1) = 2$, $3 \le d(v_2) \le 9$, and $d(v_3) \le 20$, then v_2 sends $\frac{5}{d(v_3)} - \frac{1}{4}$ to v.

R13. Every heavy vertex v with $3 \le d(v) \le 9$ sends $\frac{1}{3}$ to each adjacent light 3-vertex.

R14. Each of 8-vertex and 9-vertex sends $\frac{1}{6}$ to every adjacent heavy 3-vertex.

R15. Every 6^+-face sends $\frac{d(f)-5}{d(f)}$ to each incident 9^--vertex.

Let $\omega'(x)$ be the new weight of each $x \in V \cup F$ by applying the above rules. Let v be a k-vertex. Note that $k \ge 2$ by Lemma 2.2. By $R10$, every 10^+-vertex sends at least 1 to each adjacent 9^--vertex.

(1) $k = 2$, $w(v) = -2$.

If $d(v_1) \le d(v_2) \le 9$, then $v_i (i = 1, 2)$ sends $\frac{11}{12}$ to v. If $d(v_1) \le 9 < d(v_2)$, then v_1 sends $\frac{1}{3} + \frac{5}{d(v_2)}$ to v, v_2 sends $\frac{3}{2} - \frac{5}{d(v_2)}$ to v by R10 and R11. If $10 \le d(v_1) \le d(v_2)$, then v_i sends $\frac{3}{2} - \frac{5}{d(v_i)}(i = 1, 2)$ to v by

R10. By R15, 6^+-face sends at least $\frac{1}{6}$ to v. Hence,

$$\omega'(v) \geq -2 + \min\left\{\frac{11}{12} + \frac{11}{12}, \frac{1}{3} + \frac{5}{d(v_2)} + \frac{3}{2} - \frac{5}{d(v_2)}, \frac{3}{2} - \frac{5}{d(v_1)} + \frac{3}{2} - \frac{5}{d(v_2)}\right\} + \frac{1}{6} = 0.$$

(2) $k = 3$, $\omega(v) = -\frac{1}{2}$.

Case 1. Suppose that v is a 3(1)-vertex. By Lemma 2.3, $d(v_2) + d(v_3) \geq \Delta + 3 \geq 23$. Note that v is adjacent to at least a 12^+-vertex and $d(v_2) \geq 3$. If $3 \leq d(v_2) \leq 9$, then $d(v_3) \geq \Delta - 6$. By R10, R11, R12 and R15, $\omega'(v) \geq -\frac{1}{2} - \frac{11}{12} + \frac{1}{6} \times 2 + \frac{5}{d(v_3)} - \frac{1}{4} + \frac{3}{2} - \frac{5}{d(v_3)} = \frac{1}{6}$ when $d(v_3) \leq 20$; $\omega'(v) \geq -\frac{1}{2} - \frac{11}{12} + \frac{1}{6} \times 2 + \frac{3}{2} - \frac{5}{d(v_3)} \geq \frac{5}{28}$ when $d(v_3) \geq 21$. If $d(v_3) \geq d(v_2) \geq 10$, then $\omega'(v) \geq -\frac{1}{2} - \frac{11}{12} + \frac{3}{2} - \frac{5}{d(v_2)} + \frac{3}{2} - \frac{5}{d(v_3)} + \frac{1}{6} \times 2 > 0$ by R10, R11 and R15.

Case 2. Supose that v is a heavy 3-vertex. Note that $D(v) \geq \Delta + 2 + 3 \geq 25$, v is a (8,8,9)-vertex or (d_1,9,9)-vertex($7 \leq d_1 \leq 9$) or (d_1,d_2,10^+)-vertex($d_1 \leq d_2 \leq 10$). By R13, a heavy 3-vertex sends at most $\frac{1}{3}$ to a light 3-vertex. Hence, $\omega'(v) \geq -\frac{1}{2} + \min\left\{\frac{1}{6} \times 3, \frac{1}{6} \times 2, 1 - \frac{1}{3} \times 2\right\} + \frac{1}{6} \times 2 > 0$ by R10, R14 and R15.

Case 3. Suppose that v is a light 3-vertex. If v is adjacent to a 3(1)-vertex u, then u is adjacent to a Δ-vertex by Lemma 2.3. If $\Delta = 20$, then v sends $\frac{5}{\Delta} - \frac{1}{4} = 0$ to u. Otherwise, v sends nothing to u. If v is adjacent to at least one 10^+-vertex, then $\omega'(v) \geq -\frac{1}{2} - \left(\frac{5}{\Delta} - \frac{1}{4}\right) + 1 + \frac{1}{6} \times 2 > 0$ by R10, R12 and R15. If v is adjacent to at most two 3(1)-vertices and not adjacent to 10^+-vertex, then $\omega'(v) \geq -\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \times 2 > 0$ by R12, R13 and R15. Finally, v is adjacent to three 3(1)-vertices. If v is incident with three 6^+-faces or a 5-face, two 7^+-faces or a 5-face, at least a 8^+-face, then $\omega'(v) \geq -\frac{1}{2} + \min\left\{\frac{1}{6} \times 3, \frac{2}{7} \times 2, \frac{1}{6} + \frac{3}{8}\right\} > 0$ by R15. If v is incident with a 5-face, two 6-faces or a 5-face, a 6-face and a 7-face, then $\omega'(v) = -\frac{1}{6}$ or $\omega'(v) = -\frac{1}{12}$ by R15.

(3) $k = 4$, $\omega(v) = 1$.

Case 1. $n_2(v) = 0$. By Lemma 2.3, the 3(1)-neighbor u of v is adjacent to a $(\Delta - 1)^+$-vertex. If u is adjacent to a 21^+-vertex, then v sends 0 to u. Suppose v is adjacent to a l-vertex with $19 \leq l \leq 20$. By R12, v sends at most $\frac{5}{19} - \frac{1}{4}$ to 3(1)-vertex. If $D(v) \leq \Delta + 1 + 4$, then v is light. Therefore, $\omega'(v) \geq 1 - 4 \times \left(\frac{5}{19} - \frac{1}{4}\right) + \frac{1}{6} \times 2 > 0$ by R15. If $D(v) \geq \Delta + 6$, then v is heavy. If $n_3^l(v) = 1$, then $\omega'(v) \geq 1 - 2 \times \left(\frac{5}{19} - \frac{1}{4}\right) - \frac{1}{3} + \frac{1}{6} \times 2 > 0$ by R12, R13 and R15. If $n_3^l(v) \geq 2$, then $d(v_3) + d(v_4) \geq \Delta$. Hence, v_4 is a 10^+-vertex. This implies that $\omega'(v) \geq 1 - \frac{1}{3} \times 3 + \left(\frac{3}{2} - \frac{5}{d(v_4)}\right) + \frac{1}{6} \times 2 > 0$ by R10, R13 and R15.

Case 2. $n_2(v) = 1$. If $D(v) \geq \Delta + 6$, then v is heavy. If $n_3(v) = 0$, then v only sends weight to 2-vertex. By R10, R11 and R15, $\omega'(v) \geq 1 - \frac{11}{12} + \frac{1}{6} \times 2 > 0$. If $n_3(v) = 1$, then $d(v_3) + d(v_4) \geq \Delta + 1$. Therefore, v_4 is a 11^+-vertex. It follows from R10, R11, R13 and R15 that $\omega'(v) \geq 1 - \frac{11}{12} - \frac{1}{3} + \left(\frac{3}{2} - \frac{5}{d(v_4)}\right) + \frac{1}{6} \times 2 > 0$. If $n_3(v) = 2$, then $d(v_4) \geq \Delta - 2$. This together with R10, R11, R13 and R15 implies that $\omega'(v) \geq 1 - \frac{11}{12} - \frac{1}{3} \times 2 + (\frac{3}{2} - \frac{5}{\Delta - 2}) + \frac{1}{6} \times 2 > 0$. Otherwise, $D(v) \leq \Delta + 5$, then v is light and the vertices adjacent to v are all heavy. So the 2-neighbor u of v is adjacent to a Δ-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. This implies that $\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) - 3 \times \left(\frac{5}{\Delta - 1} - \frac{1}{4}\right) + \frac{1}{6} \times 2 > 0$ by R11, R12 and R15.

Case 3. $n_2(v) = 2$. If $D(v) \geq \Delta + 6$, then v is heavy. It follows from Lemma 2.1 that $d(v_3) + d(v_4) \geq \Delta + 2$, which means that v_4 is a 11^+-vertex. Therefore, $\omega'(v) \geq 1 - \frac{11}{12} \times 2 + \min\left\{-\frac{1}{3} + \frac{3}{2} - \frac{5}{\Delta - 1}, \frac{3}{2} - \frac{5}{11}\right\} +$

$\frac{1}{6} \times 2 > 0$ by R10, R11, R13 and R15. Otherwise, $D(v) \leq \Delta + 5$, then v is light and the vertices adjacent to v are all heavy. Hence, the 2-neighbor of v is adjacent to a Δ-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. This together with R11, R12 and R15 yields that $\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 2 - \left(\frac{5}{\Delta-1} - \frac{1}{4}\right) \times 2 + \frac{1}{6} \times 2 > 0$.

Case 4. $n_2(v) = 3$. If $D(v) \geq \Delta + 6$, then v is heavy and $d(v_4) = \Delta$. If the 2-vertices adjacent to v are all light and v is incident with at most a 5-face, then $\omega'(v) \geq 1 + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \frac{11}{12} \times 3 + \frac{1}{6} \times 3 \geq 0$ by R10, R11 and R15. If the 2-vertices adjacent to v are all light and v is incident with two 5-faces, then $\omega'(v) \geq 1 + \frac{3}{2} - \frac{5}{\Delta} - \frac{11}{12} \times 3 + \frac{1}{6} \times 2 \geq -\frac{1}{6}$ by R10, R11 and R15. If the 2-vertices adjacent to v are all heavy, then $\omega'(v) \geq 1 + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 + \frac{1}{6} \times 2 \geq \frac{5}{6}$ by R10, R11 and R15. If the 2-vertices adjacent to v are not all light, let v_1 be a light 2-vertex, then we have that $\omega'(v) \geq 1 + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \left(\frac{1}{3} + \frac{5}{\Delta}\right) - \frac{11}{12} \times 2 + \frac{1}{6} \times 2 > 0$ by R10, R11 and R15. Otherwise, $D(v) \leq \Delta + 5$, then v is light and the vertices adjacent to v are all heavy. Hence, the 2-neighbor of v is adjacent to a Δ-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. This together with R11, R12 and R15 yields that $\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 - \left(\frac{5}{\Delta-1} - \frac{1}{4}\right) + \frac{1}{6} \times 2 \geq -\frac{49}{114}$.

Case 5. $n_2(v) = 4$. Clearly, $D(v) \leq \Delta + 5$, which implies that the 2-neighbor of v is adjacent to a Δ-vertex. By R11 and R15, $\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 4 + \frac{1}{6} \times 2 \geq -1$.

(4) $k = 5$, $\omega(v) = \frac{5}{2}$.

Case 1. $n_2(v) \leq 2$. It follows from R11, R13 and R15 that $\omega'(v) \geq \frac{5}{2} - \frac{11}{12} \times 2 - \frac{1}{3} \times 3 + \frac{1}{6} \times 3 > 0$.

Case 2. $n_2(v) = 3$. If $D(v) \geq \Delta + 7$, then v is heavy and $d(v_4) + d(v_5) \geq \Delta + 1$. Therefore, v_5 is a 11^+-vertex. This means that $\omega'(v) \geq \frac{5}{2} - \frac{11}{12} \times 3 + \min\left\{-\frac{1}{3} + \left(\frac{3}{2} - \frac{5}{\Delta-2}\right), \frac{3}{2} - \frac{5}{11}\right\} + \frac{1}{6} \times 3 > 0$ by R10, R11, R13 and R15. Otherwise, $D(v) \leq \Delta + 6$, then v is light and the vertices adjacent to v are all heavy. So the 2-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 2)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. By R11, R12 and R15, $\omega'(v) \geq \frac{5}{2} - \left(\frac{1}{3} + \frac{5}{\Delta-1}\right) \times 3 - \left(\frac{5}{\Delta-2} - \frac{1}{4}\right) \times 2 + \frac{1}{6} \times 3 > 0$.

Case 3. $n_2(v) = 4$. If $D(v) \geq \Delta + 7$, then v is heavy and $d(v_5) \geq \Delta - 1$. By R10, R11 and R15, $\omega'(v) \geq \frac{5}{2} - \frac{11}{12} \times 4 + \frac{3}{2} - \frac{5}{\Delta-1} + \frac{1}{6} \times 3 > 0$. Otherwise, $D(v) \leq \Delta + 6$, then v is light and the vertices adjacent to v are all heavy. Hence, the 2-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 2)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. By R11, R12 and R15, $\omega'(v) \geq \frac{5}{2} - \left(\frac{1}{3} + \frac{5}{\Delta-1}\right) \times 4 - \left(\frac{5}{\Delta-2} - \frac{1}{4}\right) + \frac{1}{6} \times 3 > 0$.

Case 4. $n_2(v) = 5$. Clearly, $D(v) \leq \Delta + 6$, which means that the 2-neighbor of v is adjacent to a $(\Delta - 1)^+$-vertex. So $\omega'(v) \geq \frac{5}{2} - \left(\frac{1}{3} + \frac{5}{\Delta-1}\right) \times 5 + \frac{1}{6} \times 3 > 0$ by R11 and R15.

(5) $k = 6$, $\omega(v) = 4$.

Case 1. $n_2(v) \leq 4$. It is clear that $\omega'(v) \geq 4 - \frac{11}{12} \times 4 - \frac{1}{3} \times 2 + \frac{1}{6} \times 3 > 0$ by R11, R13 and R15.

Case 2. $n_2(v) = 5$. If $D(v) \geq \Delta + 8$, then v is heavy and $d(v_6) \geq \Delta - 2$. This together with R10, R11 and R15 implies that $\omega'(v) \geq 4 - \frac{11}{12} \times 5 + \frac{3}{2} - \frac{5}{\Delta-2} + \frac{1}{6} \times 3 > 0$. Otherwise, $D(v) \leq \Delta + 7$, then v is light and the vertices adjacent to v are all heavy. Therefore, the 2-neighbor of v is adjacent to a $(\Delta - 2)^+$-vertex and the 3(1)-neighbor of v is adjacent to a $(\Delta - 3)^+$-vertex. If the 3(1)-neighbor of v is adjacent to a 21^+-vertex, then v sends 0 to this 3(1)-vertex. By R11, R12 and R15, $\omega'(v) \geq 4 - \left(\frac{1}{3} + \frac{5}{\Delta-2}\right) \times 5 - \left(\frac{5}{\Delta-3} - \frac{1}{4}\right) + \frac{1}{6} \times 3 > 0$.

Case 3. $n_2(v) = 6$. It is easy to see that $D(v) \leq \Delta + 7$, which implies that the 2-neighbor of v is adjacent to a $(\Delta - 2)^+$-vertex. By R11 and R15, $\omega'(v) \geq 4 - \left(\frac{1}{3} + \frac{5}{\Delta-2}\right) \times 6 + \frac{1}{6} \times 3 > 0$.

(6) $k = 7$, $\omega(v) = \frac{11}{2}$.

Case 1. $n_2(v) \leq 6$. By R11, R13 and R15, $\omega'(v) \geq \frac{11}{2} - \frac{11}{12} \times 6 - \frac{1}{3} + \frac{1}{6} \times 4 > 0$.

Case 2. $n_2(v) = 7$. It is clear that $D(v) \leq \Delta + 7$, which means that the 2-neighbor of v is adjacent to a $(\Delta - 3)^+$-vertex. By R11 and R15, $\omega'(v) \geq \frac{11}{2} - \left(\frac{1}{3} + \frac{5}{\Delta-3}\right) \times 7 + \frac{1}{6} \times 4 > 0$.

(7) $k = 8$, $\omega(v) = 7$. Observe that $\omega'(v) \geq 7 - \frac{11}{12} \times 8 + \frac{1}{6} \times 4 > 0$ by R11, R13 and R15.

(8) $k = 9$, $\omega(v) = \frac{17}{2}$. By R11, R13 and R15, $\omega'(v) \geq \frac{17}{2} - \frac{11}{12} \times 9 + \frac{1}{6} \times 5 > 0$.

(9) $k \geq 10$, $\omega(v) = \frac{3}{2}k - 5$. By R10, $\omega'(v) \geq \frac{3}{2}k - 5 - \left(\frac{3}{2} - \frac{5}{k}\right) \times k = 0$.

After the first step, $\omega'(x) \geq 0$ for each $x \in V \cup F$ except some 3-vertices and 4-vertices. For $f \in F(G)$, if $d(f) \geq 5$, then $\omega'(f) \geq \min\left\{0, d(f) - 5 - \frac{d(f)-5}{d(f)} \cdot d(f)\right\} = 0$ by R15. For convenience, a vertex v is called *bad* if $\omega'(v) < 0$. If $uv \in E(G)$ and u,v are all 10^+-vertices, then uv is called special edge.

There are four bad vertices:

I-vertex: 3-vertex v is adjacent to three 3(1)-vertices and incident with a 5-face, two 6-faces or a 5-face, a 6-face and a 7-face, $\omega'(v) \geq -\frac{1}{6}$.

II-vertex: 4-vertex v is adjacent to three light 2-vertices, a Δ-vertex and incident with two 5-faces, $\omega'(v) \geq -\frac{1}{6}$.

III-vertex: light 4-vertex v is adjacent to three 2-vertices, $\omega'(v) \geq -\frac{49}{114}$.

IV-vertex: 4-vertex v is adjacent to four 2-vertices, $\omega'(v) \geq -1$.

Obviously, bad vertex is not adjacent to bad vertex.

The second step

R20. Every 10^+-vertex v sends $\frac{1}{2}$ to incident face f through its special edge.

R21. Every 9^--vertex sends its remained positive weight averagely to each incident face.

R22. Every 5^+-face sends its remained positive weight averagely to each incident bad vertex.

(1) v is I-vertex, see Figure 1a. Clearly, v is a light 3-vertex. By Lemma 2.1, $D(v_i) \geq \Delta + 5$, which means that $d\left(v_i'\right) = \Delta$ for $1 \leq i \leq 3$.

Let $f_1 = v_1 v v_2 x y$ be a 5-face with $d(x), d(y) \in \{2, \Delta\}$, f_2 and f_3 be 6^+-faces. By Lemma 2.2, at least one of x and y is not 2-vertex. Note that $\{d(x), d(y)\} \neq \{2, \Delta\}$ by Lemma 2.5. Hence, $d(x) = d(y) = \Delta$, which means that $v_1'v_2'$ is a special edge. By R20, f_1 receives $\frac{1}{2} \times 2$ from v_1', v_2'. Obviously, f_1 is only incident with a bad vertex v, which together with R22 shows that $\omega^*(v) \geq -\frac{1}{6} + 1 > 0$.

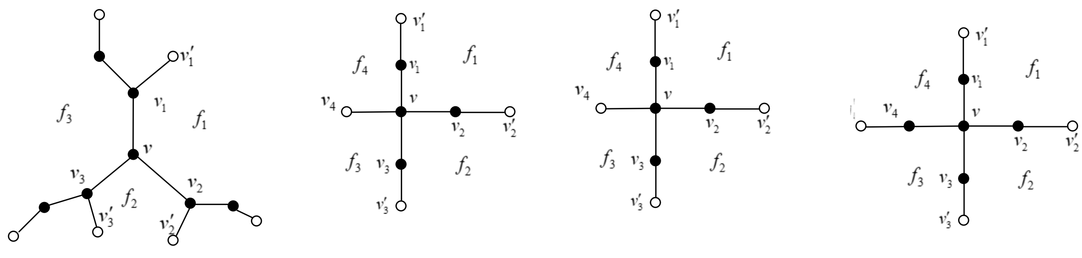

Figure 1. The configurations of bad vertices. (The degrees of black nodes are the actual degrees in the figure)

(2) v is II-vertex, see Figure 1b. By Lemma 2.1, $D(v'_i) \geq \Delta + 2 + d(v'_i)$. Let f_1 and f_3 be 5-faces.

Case 1. If there is a bad vertex in $\left\{v'_1, v'_2, v'_3\right\}$, then it can not be I, III, IV-vertex. If v'_1 is a bad vertex, then v'_2 is a Δ-vertex by Lemma 2.2, which means that v_2 is a heavy 2-vertex, a contradiction. Hence, v'_1 can not be a bad vertex. Similarly, v'_2 can not be a bad vertex. Next, suppose that v'_3 is a bad vertex, see Figure 2a. If u is a 2-vertex, then u is a heavy 2-vertex. This is contradict with II-vertex v'_3 only adjacent to light 2-vertex. If u is a Δ-vertex, then uv_4 is a special edge. By R20, f_3 gets 1 from uv_4. Obviously, f_3 is only incident with two bad vertices v and v'_3, which implies that $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{2} > 0$ by R22.

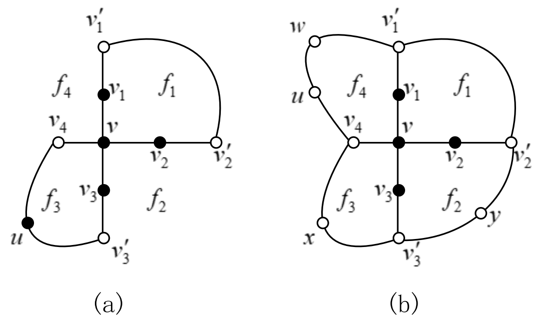

Figure 2. II-vertex.

Case 2. If v'_1, v'_2, v'_3 are not bad vertices, then f_4 is incident with at most $\left\lfloor \frac{d(f_4)-3}{2} \right\rfloor + 1$ bad vertices.

Case 2.1. Suppose that f_4 is a 7^+-face. Since $d(v_4) = \Delta$, we can get $\omega'(f_4) \geq \frac{d(f_4)-5}{d(f_4)}$. By the first step and R22, $\omega^*(v) \geq 1 + \frac{3}{2} - \frac{5}{\Delta} - \frac{11}{12} \times 3 + \frac{d(f_4)-5}{d(f_4)} + \frac{1}{6} + \frac{d(f_4)-5}{d(f_4)} \times \frac{1}{\left\lfloor \frac{d(f_4)-3}{2} \right\rfloor + 1} > 0$.

Case 2.2. Suppose that f_2 is a 7^+-face. Since $d(v_4) = \Delta$, we can get $\omega'(f_4) \geq \frac{d(f_4)-5}{d(f_4)}$. By the first step and R22, $\omega^*(v) \geq 1 + \frac{3}{2} - \frac{5}{\Delta} - \frac{11}{12} \times 3 + \frac{d(f_4)-5}{d(f_4)} + \frac{2}{7} + \frac{d(f_4)-5}{d(f_4)} \times \frac{1}{\left\lfloor \frac{d(f_4)-3}{2} \right\rfloor + 1} > 0$.

Case 2.3. Suppose that f_2 and f_4 are all 6-faces, see Figure 2b. It is easy to see that f_2 and f_3 are only incident with a bad vertex v, f_4 is incident with at most two bad vertices.

If v'_1 is a 10^+-vertex, then $\omega'(f_4) \geq \frac{1}{6} \times 2$ by $d(v_4) = \Delta$. By R22, $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} > 0$. If v'_2 is a 10^+-vertex, then $\omega'(f_2) \geq \frac{1}{6} \times 2$. By $d(v_4) = \Delta$, $\omega'(f_4) \geq \frac{1}{6}$. This means that $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + \frac{1}{6} > 0$ by R22. If v'_3 is a 10^+-vertex, then $\omega'(f_2) \geq \frac{1}{6}$. By $d(v_4) = \Delta$, $\omega'(f_4) \geq \frac{1}{6}$, which implies that $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + \frac{1}{6} > 0$.

If v'_1, v'_2, v'_3 are all 9^--vertices, then we discuss the classification of $d(x)$. We can obtain that $\omega'(f_4) \geq \frac{1}{6}$ since $d(v_4) = \Delta$.

If $d(x) \geq 10$, then xv_4 is a special edge. By R20, each of x and v_4 sends $\frac{1}{2}$ to f_3. Hence, $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + 1 > 0$ by R22. If $5 \leq d(x) \leq 9$, then $\omega^*(x) \geq \frac{3}{2}d(x) - 5 + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \frac{11}{12}(d(x) - 1) + \frac{1}{6} \times \left\lceil \frac{d(x)}{2} \right\rceil \geq \frac{2}{3}d(x) - \frac{17}{6}$ by R10, R11 and R15. This means that f_3 gets $\left(\frac{2}{3}d(x) - \frac{17}{6}\right)\frac{1}{d(x)} \geq \frac{1}{10}$ by R21. Hence, $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + \frac{1}{10} > 0$ by R22.

If $d(x) = 4$, then $\omega'(x) \geq 1 + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \frac{11}{12} \times 2 - \left(\frac{5}{\Delta - 1} - \frac{1}{4}\right) + \frac{1}{6} \times 2 = \frac{14}{19}$ by R10, R11, R12 and R15. This implies that f_3 gets $\frac{7}{38}$ by R21. Hence, $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + \frac{7}{38} > 0$ by R22. If $d(x) = 3$, then

$\omega'(x) \geq -\frac{1}{2} + \left(\frac{3}{2} - \frac{5}{\Delta}\right) - \frac{11}{12} + \frac{1}{6} \times 2 = \frac{1}{6}$ by R10, R11, R12 and R15. This means that f_3 gets $\frac{1}{18}$ by R21. Similarly, we consider the degree of u. If $d(u) \geq 3$, then f_4 gets at least $\min\left\{1, \frac{1}{10}, \frac{1}{18}\right\} = \frac{1}{18}$. It follows from R22 that $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} \times \frac{1}{2} + \frac{1}{18} + \frac{1}{18} \times \frac{1}{2} = 0$. If $d(u) = 2$, then f_4 is only incident with a bad vertex v. By R22, $\omega^*(v) \geq -\frac{1}{6} + \frac{1}{6} = 0$.

(3) v is III-vertex, see Figure 1c. Observe that $d\left(v_i'\right) = \Delta$ (i=1,2,3) by Lemma 2.1.

Suppose that f_1 or f_2 is a 5-face. Without loss of generality, let f_1 be a 5-face, then $v_1'v_2'$ is a special edge. By R20, f_1 gets at least 1. Clearly, f_1 is only incident with a bad vertex v. This together with R22 implies that $\omega^*(v) \geq -\frac{49}{114} + 1 > 0$. Next, we consider the case when f_1 and f_2 are 6^+-faces.

Suppose that f_3 or f_4 is a 5-face. Without loss of generality, let f_4 be a 5-face, then f_3 is a 6^+-face. After the first step, $\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 - \left(\frac{5}{\Delta-1} - \frac{1}{4}\right) + \frac{d(f_1)-5}{d(f_1)} + \frac{d(f_2)-5}{d(f_2)} + \frac{d(f_3)-5}{d(f_3)}$ (observe that v_4 is adjacent to a $(\Delta-1)^+$-vertex if v_4 is a 3(1)-vertex). Since $d\left(v_i'\right) = \Delta$ for $1 \leq i \leq 3$, $\omega'\left(f_j\right) \geq \frac{2\left(d\left(f_j\right)-5\right)}{d\left(f_j\right)}$ for $1 \leq j \leq 2$, $\omega'(f_3) \geq \frac{d(f_3)-5}{d(f_3)}$. It is easy to see that f_j is incident with at most $\left\lfloor \frac{d\left(f_j\right)-4}{2} \right\rfloor + 1$ bad vertices for $1 \leq j \leq 2$, f_3 is incident with at most $\left\lfloor \frac{d(f_3)-3}{2} \right\rfloor + 1$ bad vertices. Claim that $d(f_i) \geq 6$, $\frac{d(f_i)-5}{d(f_i)} + \frac{2(d(f_i)-5)}{d(f_i)} \times \frac{1}{\left\lfloor \frac{d(f_i)-4}{2} \right\rfloor + 1} \geq \frac{1}{3}$ for $i = 1, 2$; $\frac{d(f_3)-5}{d(f_3)} + \frac{d(f_3)-5}{d(f_3)} \times \frac{1}{\left\lfloor \frac{d(f_3)-3}{2} \right\rfloor + 1} \geq \frac{1}{4}$. By R22,

$$\omega^*(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 - \left(\frac{5}{\Delta - 1} - \frac{1}{4}\right) + \frac{d(f_1) - 5}{d(f_1)} + \frac{d(f_2) - 5}{d(f_2)} + \frac{d(f_3) - 5}{d(f_3)}$$
$$+ \frac{2(d(f_1) - 5)}{d(f_1)} \times \frac{1}{\left\lfloor \frac{d(f_1)-4}{2} \right\rfloor + 1} + \frac{2(d(f_2) - 5)}{d(f_2)} \times \frac{1}{\left\lfloor \frac{d(f_2)-4}{2} \right\rfloor + 1} + \frac{d(f_3) - 5}{d(f_3)} \times \frac{1}{\left\lfloor \frac{d(f_3)-3}{2} \right\rfloor + 1}$$
$$> 0.$$

Suppose $d(f_i) \geq 6$ for $1 \leq i \leq 4$. After the first step,

$$\omega'(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 - \left(\frac{5}{\Delta - 1} - \frac{1}{4}\right) + \frac{d(f_1) - 5}{d(f_1)} + \frac{d(f_2) - 5}{d(f_2)} + \frac{d(f_3) - 5}{d(f_3)} + \frac{d(f_4) - 5}{d(f_4)}.$$

Since $d\left(v_i'\right) = \Delta$ for $1 \leq i \leq 3$, we can get $\omega'(f_i) \geq \frac{2(d(f_i)-5)}{d(f_i)}$ for $1 \leq i \leq 2$, $\omega'(f_k) \geq \frac{d(f_k)-5}{d(f_k)}$ for $3 \leq k \leq 4$. It is clear that f_i is incident with at most $\left\lfloor \frac{d(f_i)-4}{2} \right\rfloor + 1$ bad vertices (i=1,2), f_k is incident with at most $\left\lfloor \frac{d(f_k)-3}{2} \right\rfloor + 1$ bad vertices (k=3,4). By R22,

$$\omega^*(v) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 3 - \left(\frac{5}{\Delta - 1} - \frac{1}{4}\right) + \frac{d(f_1) - 5}{d(f_1)} + \frac{d(f_2) - 5}{d(f_2)} + \frac{d(f_3) - 5}{d(f_3)} + \frac{d(f_4) - 5}{d(f_4)} + \frac{2(d(f_1) - 5)}{d(f_1)}$$
$$\times \frac{1}{\left\lfloor \frac{d(f_1)-4}{2} \right\rfloor + 1} + \frac{2(d(f_2) - 5)}{d(f_2)} \times \frac{1}{\left\lfloor \frac{d(f_2)-4}{2} \right\rfloor + 1} + \frac{d(f_3) - 5}{d(f_3)} \times \frac{1}{\left\lfloor \frac{d(f_3)-3}{2} \right\rfloor + 1} + \frac{d(f_4) - 5}{d(f_4)} \times \frac{1}{\left\lfloor \frac{d(f_4)-3}{2} \right\rfloor + 1}$$
$$> 0.$$

(4) v is IV-vertex, see Figure 1d. By Lemma 2.1, $d\left(v_i'\right) = \Delta$ for $1 \leq i \leq 4$.

There is exactly one 5-face in f_1, f_2, f_3 and f_4. Without loss of generality, let f_1 be a 5-face, then $v_1'v_2'$ is a special edge. By R20, f_1 gets at least 1. It is easy to see that f_1 is only incident with a bad vertex v. By R22, $\omega^*(v) \geq -1 + 1 = 0$. Next, we consider f_i $(1 \leq i \leq 4)$ is 6^+-face.

Suppose v is incident with at least a 6-face. Without loss of generality, let f_1 be a 6-face. It is clear that f_1 is only incident with a bad vertex v. Since $d\left(v_1'\right) = d\left(v_2'\right) = d\left(v_3'\right) = \Delta$, we can get $\omega'\left(f_1\right) \geq \frac{1}{6}$. We use m_6 to denote the number of 6-faces which is incident with v. After the first step and the second step, $\omega^*\left(v\right) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 4 + \frac{1}{6} \times m_6 + \frac{2}{7} \times \left(4 - m_6\right) + \frac{2}{6} \times m_6 = -\frac{4}{21} + \frac{3}{14} m_6 \geq \frac{5}{21} > 0$.

Suppose v is incident with four 7^+-faces. Note that f_i is incident with at most $\left\lfloor \frac{d(f_i)-4}{2} \right\rfloor + 1$ $(1 \leq i \leq 4)$ bad vertices. Since $d\left(v_i'\right) = \Delta$, we can get $\omega'\left(f_i\right) \geq \frac{2(d(f_i)-5)}{d(f_i)}$ $(1 \leq i \leq 4)$. After the first step,

$$\omega'\left(v\right) \geq 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 4 + \frac{d\left(f_1\right) - 5}{d\left(f_1\right)} + \frac{d\left(f_2\right) - 5}{d\left(f_2\right)} + \frac{d\left(f_3\right) - 5}{d\left(f_3\right)} + \frac{d\left(f_4\right) - 5}{d\left(f_4\right)}.$$

By R22,

$$\begin{aligned}\omega^*\left(v\right) \geq{} & 1 - \left(\frac{1}{3} + \frac{5}{\Delta}\right) \times 4 + \frac{d\left(f_1\right) - 5}{d\left(f_1\right)} + \frac{d\left(f_2\right) - 5}{d\left(f_2\right)} + \frac{d\left(f_3\right) - 5}{d\left(f_3\right)} + \frac{d\left(f_4\right) - 5}{d\left(f_4\right)} + \frac{2\left(d\left(f_1\right) - 5\right)}{d\left(f_1\right)} \times \frac{1}{\left\lfloor \frac{d(f_1)-4}{2} \right\rfloor + 1} \\ & + \frac{2\left(d\left(f_2\right) - 5\right)}{d\left(f_2\right)} \times \frac{1}{\left\lfloor \frac{d(f_2)-4}{2} \right\rfloor + 1} + \frac{2\left(d\left(f_3\right) - 5\right)}{d\left(f_3\right)} \times \frac{1}{\left\lfloor \frac{d(f_3)-4}{2} \right\rfloor + 1} + \frac{2\left(d\left(f_4\right) - 5\right)}{d\left(f_4\right)} \times \frac{1}{\left\lfloor \frac{d(f_4)-4}{2} \right\rfloor + 1} \\ & > 0.\end{aligned}$$

Now we have checked that the final weight $w^*(x) \geq 0$ for each $x \in V \cup F$. Then,

$$0 \leq \sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) = -10,$$

which is a contradiction.

3. Conclusions

In this paper, we consider the injective chromatic index of planar graphs without adjacent 5-cycles and proved that such graphs have $\chi_i(G) \leq \Delta(G) + 2$ if $g(G) \geq 5$ and $\Delta\left(G\right) \geq 20$. A natural problem in context of our main result is the following: What is the optimal constant c such that $\chi_i(G) \leq \Delta(G) + 2$ for every planar graph G with $g(G) \geq 5$ and $\Delta\left(G\right) \geq c$.

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Conflict of interest

The authors declare no conflicts of interest.

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