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Research article

Complete convergence of moving average processes produced by negatively dependent random variables under sub-linear expectations

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Abstract: Suppose that $\{a_i, -\infty < i < \infty\}$ is an absolutely summable set of real numbers, $\{Y_i, -\infty < i < \infty\}$ is a subset of identically distributed, negatively dependent random variables under sub-linear expectations. Here, we get complete convergence and Marcinkiewicz-Zygmund strong law of large numbers for the partial sums of moving average processes $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ produced by $\{Y_i, -\infty < i < \infty\}$ of identically distributed, negatively dependent random variables under sub-linear expectations, complementing the relevant results in probability space.

Keywords: complete moment convergence; complete convergence; negatively dependent random variables; sub-linear expectations
Mathematics Subject Classification: 60F15, 60F05

1. Introduction

Peng [1, 2] introduced basic concepts of the sub-linear expectations space to describe the uncertainty in probability. Stimulated by the works of Peng [1, 2], many scholars tried to discover the results under sub-linear expectations space, similar to those in classic probability space. Zhang [3, 4] got exponential inequalities and Rosenthal's inequality under sub-linear expectations. Xu et al. [5], Xu and Kong [6] investigated complete convergence and complete moment convergence of weighted sums of negatively dependent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the readers could refer to Zhang [7], Xu and Zhang [8, 9], Wu and Jiang[10], Zhang and Lin [11], Zhong and Wu [12], Gao and Xu [13], Kuczmaszewska [14], Xu and Cheng [15, 16], Zhang [17], Chen [18], Zhang [19], Chen and Wu [20], Xu et al. [5], Xu and Kong [6], and references therein.

In classic probability space, Chen et al. [21] obtained limiting behavior of moving average processes under φ-mixing assumption. For references on complete moment convergence and complete convergence in probability space, the reader could refer to Hsu and Robbins [22], Chow [23], Hosseini and Nezakati [24], Meng et al. [25] and refercences therein. Inspired by the works of Chen et al. [21], Xu et al. [5], Xu and Kong [6], we try to discuss complete convergence for the partial sums of moving average processes generated by negatively dependent random variables under sub-linear expectations, and the relevant Marcinkiewicz-Zygmund strong law of large number, which complements the corresponding results in Chen et al. [21]. We also establish Conjecture 3.1 given by Xu and Kong [6] in some sense.

We organize the remainders of this article as follows. We give relevant basic notions, concepts and properties, and cite relevant lemmas under sub-linear expectations in Section 2. In Section 3, we present our main results, Theorems 3.1–3.4, the proofs of which are postponed in Section 4.

2. Preliminary

Hereafter, we use notions similar to that in the works by Peng [2], Zhang [4]. Assume that $(\Omega, \mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a set of all random variables on $(\Omega, \mathcal{F})$ fulfilling $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for $X_1, \cdots, X_n \in \mathcal{H}$, and each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is the set of φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for $C > 0$, $m \in \mathbb{N}$ relying on φ.

Definition 2.1. *A sub-linear expectation* $\mathbb{E}$ *on* $\mathcal{H}$ *is a functional* $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ *fulfilling the following: for every* $X, Y \in \mathcal{H}$,

(a) $X \geq Y$ *implies* $\mathbb{E}[X] \geq \mathbb{E}[Y]$*;*

(b) $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$*;*

(c) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$*;*

(d) $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ *whenever* $\mathbb{E}[X] + \mathbb{E}[Y]$ *is not of the form* $\infty - \infty$ *or* $-\infty + \infty$.

$V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.
Furthermore, if V is continuous, then V obey

(c) $A_n \uparrow A$ yields $V(A_n) \uparrow V(A)$.

(d) $A_n \downarrow A$ yields $V(A_n) \downarrow V(A)$.

V is said to be sub-additive when $V(A + B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

Under $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (cf. Zhang [3]). $\mathbb{V}$ is a sub-additive capacity. Write

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

As in 4.3 of Zhang [3], throughout this paper, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf \{\mathbb{E}[Y] : X \leq Y, Y \in \mathcal{H}\}.$$

Then $\mathbb{E}^*$ is a sublinear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$.

Suppose $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is named to be negatively dependent to $\mathbf{X}$, if for ψ_1 on $C_{l,Lip}(\mathbb{R}^m)$, ψ_2 on $C_{l,Lip}(\mathbb{R}^n)$, $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \leq \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \geq 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \geq 0$, $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [3], Definition 1.5 of Zhang [4]). $\{X_n\}_{n=-\infty}^{\infty}$ is said to be negatively dependent, if X_{n+l} is negatively dependent to $(X_l, X_{l+1}, \cdots, X_{l+n-1})$ for each $n \geq 1$, $-\infty < l < \infty$. The existence of negatively dependent random variables $\{X_n\}_{n=-\infty}^{\infty}$ under sub-linear expectations could be guaranteed by Example 1.6 of Zhang [4] and Kolmogorov's existence theorem in classic probabililty space.

Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors under $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are said to be identically distributed if for every $\psi \in C_{l,Lip}(\mathbb{R}^n)$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)].$$

$\{X_n\}_{n=1}^{\infty}$ is called to be identically distributed if for every $i \geq 1$, X_i and X_1 are identically distributed.

Throughout this paper, we suppose that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$ could be implied by $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Therefore $\mathbb{E}^*$ is also countably sub-additive. Write $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Let C denote a positive constant which may change from line to line. $I(A)$ or I_A is the indicator function of A. The symbol $a_x \approx b_x$ means that there exists two positive constants C_1, C_2 fulfilling $C_1|b_x| \leq |a_x| \leq C_2|b_x|$, x^+ stands for $\max\{x, 0\}$, for $x \in \mathbb{R}$.

As in Zhang [4], if $X_1, X_2, \ldots, X_n$ are negatively dependent random variables and $f_1, f_2, \ldots, f_n$ are all non increasing (or non decreasing) functions, then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are negatively dependent random variables.

We cite the following under sub-linear expectations.

Lemma 2.1. *(Cf. Lemma 4.5 (iii) of Zhang [3]) If $\mathbb{E}$ is countably sub-additive under $(\Omega, \mathcal{H}, \mathbb{E})$, then for $X \in \mathcal{H}$,*

$$\mathbb{E}|X| \leq C_{\mathbb{V}}(|X|).$$

Lemma 2.2. *(Cf. Theorem 2.1 of Zhang [4] and its proof there) Assume that $p > 1$ and $\{Y_n; n \geq 1\}$ is a sequence of negatively dependent random varables with $\mathbb{E}[Y_k] \leq 0$, $k \geq 0$, under $(\Omega, \mathcal{H}, \mathbb{E})$. Then for every $n \geq 1$, there exists a positive constant $C = C(p)$ relying on p such that for $p \geq 2$,*

$$\mathbb{E}\left[\left|\max_{1\leq i\leq n} \sum_{j=i}^{n} Y_j\right|^p\right] \leq C\left\{\sum_{i=1}^{n} \mathbb{E}|Y_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}Y_i^2\right)^{p/2}\right\}.$$

$$\mathbb{E}\left[\left(\left(\sum_{j=1}^{n} Y_j\right)^+\right)^p\right] \leq C\left\{\sum_{i=1}^{n} \mathbb{E}|Y_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}Y_i^2\right)^{p/2}\right\}. \tag{2.1}$$

By (2.1) of Lemma 2.2 and similar proof of Lemma 2.4 of Xu et al. [5], we could get the following.

Lemma 2.3. *Assume that $p > 1$ and $\{Y_n; n \geq 1\}$ is a sequence of negatively dependent random varables with $\mathbb{E}[Y_k] \leq 0$, $k \geq 0$, under $(\Omega, \mathcal{H}, \mathbb{E})$. Then for every $n \geq 1$, there exists a positive constant $C = C(p)$ relying on p such that for $p \geq 2$,*

$$\mathbb{E}\left[\max_{1\leq i\leq n}\left(\left(\sum_{j=1}^{i} Y_j\right)^+\right)^p\right] \leq C(\log n)^p \left\{\sum_{i=1}^{n} \mathbb{E}\,|Y_i|^p + \left(\sum_{i=1}^{n} \mathbb{E}Y_i^2\right)^{p/2}\right\}.$$

Lemma 2.4. *(Cf. Lemma 2.2 and its proof of Zhong and Wu [12])If $X \in \mathcal{H}$, $\alpha > 0, \beta > 0, \gamma > 0, \eta > 0$, $C_{\mathbb{V}}\left(|X|^\alpha h(|X|^\beta)(\log(1+|X|))^\eta\right) < \infty$, $h(\cdot)$ is a slowly varying function, then there exist two positive constants C_1, C_2 relying on $\alpha, \beta, \gamma, \eta$ such that*

$$\begin{aligned} C_1 C_{\mathbb{V}}\left(|X|^\alpha h(|X|^\beta)(\log(1+|X|))^\eta\right) &\leq \int_0^\infty \mathbb{V}\{|X| > \gamma y\}\, y^{\alpha-1} h(y^\beta) \mathrm{d}y \\ &\leq C_2 C_{\mathbb{V}}\left(|X|^\alpha h(|X|^\beta)(\log(1+|X|))^\eta\right) < \infty. \end{aligned}$$

Proof. Here we give a detailed proof. By Lemma 2.1 of Zhong and Wu [12], $h(x) = c(x)\exp\left\{\int_0^x \frac{f(u)}{u}\mathrm{d}u\right\}$, where $\lim_{x\to\infty} c(x) = c > 0$, $c(x) \geq 0$, $\lim_{x\to\infty} f(x) = 0$. Set $Z(x) = |x|^\alpha h(|x|^\beta)(\log(1+|x|))^\eta$ and write the inverse function of $Z(x)$ to be $Z^{-1}(x)$. We get

$$\begin{aligned} &\int_0^\infty \mathbb{V}\{|X| > \gamma y\}\, y^{\alpha-1} h(y^\beta)(\log(1+y))^\eta \mathrm{d}y \\ &\approx \int_0^\infty \mathbb{V}\{|X| > \gamma y\}(1/\alpha)\left(\alpha\gamma^\alpha y^{\alpha-1} h((\gamma y)^\beta) + \beta\gamma^\alpha y^{\alpha-1} h((\gamma y)^\beta) f((\gamma y)^\beta)\right)(\log(1+\gamma y))^\eta \mathrm{d}y \\ &\approx \int_0^\infty \mathbb{V}\left(|X| > Z^{-1}(x) := \gamma y\right)\mathrm{d}x \\ &= \int_0^\infty \mathbb{V}\left(|X^\alpha| h(|X|^\beta)(\log(1+|X|))^\eta > x\right)\mathrm{d}x = C_{\mathbb{V}}\left(|X^\alpha| h(|X|^\beta)(\log(1+|X|))^\eta\right) < \infty. \end{aligned}$$

□

3. Main results

Our main results are below.

Theorem 3.1. *Assume that h is a slowly varying function, $1 \leq p < 2$, and $r > 1$. Suppose $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process produced by a sequence of negatively dependent random varables $\{Y_i, -\infty < i < \infty\}$ with $\sum_{i=-\infty}^{\infty} a_i < \infty$, $\{a_i, -\infty < i < \infty\}$ is a subset of numbers being all non-negative, and for fixed $-\infty < i < \infty$, Y_i is identically distributed as Y under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Suppose that for some $q > \max\{2, rp\}$, $C_{\mathbb{V}}\left(|Y|^{rp} h(|Y|^p)(\log(1+|Y|))^q\right) < \infty$. Then for all $\varepsilon > 0$,*

(i) $\sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right) \geq \varepsilon n^{1/p}\right\} < \infty$,
$\sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(-X_i - \mathbb{E}(-X_i))\right) \geq \varepsilon n^{1/p}\right\} < \infty$,
and
(ii) $\sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{\sup_{k\geq n}\left(\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right)/k^{1/p} \geq \varepsilon\right\} < \infty$,
$\sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{\sup_{k\geq n}\left(\sum_{i=1}^{k}(-X_i - \mathbb{E}(-X_i))\right)/k^{1/p} \geq \varepsilon\right\} < \infty$.

Moreover, if $\mathbb{E}(X_i) = -\mathbb{E}(-X_i)$*, then for all* $\varepsilon > 0$,
(iii) $\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right| \ge \varepsilon n^{1/p}\right\} < \infty$,
$\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\sup_{k\ge n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right|/k^{1/p} \ge \varepsilon\right\} < \infty$

Remark 3.1. *Letting* $a_0 = 1$*,* $a_i = 0$ *for* $i \neq 0$*, and* $h(x) = 1$ *in Theorem 3.1, and by the similar proof of Corollary 3.1 of Xu and Kong [6], we deduce that Conjecture 3.1 of Xu and Kong [6] holds in some sense. Adapting the proof of Theorem 3.1, we see that* (iii) *of Theorem 3.1 still holds when the condition that for some* $q > \max\{2, rp\}$*,* $C_{\mathbb{V}}\left(|Y|^{rp}h(|Y|^p)(\log(1+|Y|))^q\right) < \infty$ *is reduced to that* $C_{\mathbb{V}}\left(|Y|^{rp}h(|Y|^p)\right) < \infty$*, and the other conditions remained unchanged. The above discussion also could applies to that in Theorems 3.2, 3.3, 3.4.*

By Theorem 2.1 (b) of Zhang [4] and its proof there, similar proof of Theorem 3.1, we could get the following.

Theorem 3.2. *Suppose that in Theorem 3.1, with the condition that* Y_m *is negatively dependent to* $(Y_{m+1}, \ldots, Y_{m+l})$ *for each* $-\infty < m < \infty$ *and* $l \ge 1$ *in place of the assumption that* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of negatively dependent random varables, the other conditions remained unchanged. Suppose that* $\mathbb{E}(Y) = 0$ *and for some* $q > \max\{2, rp\}$*,* $C_{\mathbb{V}}\left(|Y|^{rp}h(|Y|^p)(\log(1+|Y|))^q\right) < \infty$*. Then all conclusions in Theorem 3.1 also hold.*

We study the occation $r = 1$ in the following.

Theorem 3.3. *Assume that* h *is a slowly varying function and* $1 \le p < 2$*. Suppose that* $\{a_i, -\infty < i < \infty\}$ *is a subset of numbers being all non-negative,* $\sum_{i=-\infty}^{\infty} a_i^{\theta} < \infty$*, where* $\theta \in (0,1)$ *if* $p = 1$ *and* $\theta = 1$ *if* $1 < p < 2$*. Assume that* $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \ge 1\}$ *is a moving average process produced by a sequence of negatively dependent random varables* $\{Y_i, -\infty < i < \infty\}$*, and for fixed* $-\infty < i < \infty$*,* Y_i *is identically distributed as* Y *under* $(\Omega, \mathcal{H}, \mathbb{E})$*. Suppose that for some* $q > \max\{2, rp\}$*,* $C_{\mathbb{V}}\left(|Y|^p h(|Y|^p)(\log(1+|Y|)^q)\right) < \infty$*. Then for all* $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty}\frac{h(n)}{n}\mathbb{V}\left\{\max_{1\le k\le n}\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i)) \ge \varepsilon n^{1/p}\right\} < \infty,$$

$$\sum_{n=1}^{\infty}\frac{h(n)}{n}\mathbb{V}\left\{\max_{1\le k\le n}\sum_{i=1}^{k}(-X_i - \mathbb{E}(-X_i)) \ge \varepsilon n^{1/p}\right\} < \infty.$$

In particular, if $\mathbb{E}Y = -\mathbb{E}(-Y)$*,* $C_{\mathbb{V}}\left(|Y|^p\right) < \infty$ *and* $\mathbb{V}$ *is continuous, then* $S_n/n^{1/p} \to \mathbb{E}(Y)$ *a.s.* $\mathbb{V}$*, i.e.,*

$$\mathbb{V}\left\{\Omega \setminus \{\lim_{n\to\infty} S_n/n^{1/p} = \mathbb{E}(Y)\}\right\} = 0,$$

which is called the Marcinkiewicz-Zygmund type strong law of large numbers under sub-linear expectations,

By Theorem 2.1 (b) of Zhang [4] and its proof there, similar proof of Theorem 3.3, we could get the following.

Theorem 3.4. *Suppose that in Theorem 3.1, with the condition that* Y_m *is negatively dependent to* $(Y_{m+1}, \ldots, Y_{m+l})$ *for each* $-\infty < m < \infty$ *and* $l \ge 1$ *in place of the assumption that* $\{Y_i, -\infty < i < \infty\}$

is a sequence of negatively dependent random varables, the other conditions remained unchanged. Suppose that for some $q > \max\{2, rp\}$,

$$C_{\mathbb{V}}\left(|Y|^{rp}h(|Y|^p)(\log(1+|Y|)^q)\right) < \infty.$$

Then all conclusions in Theorem 3.3 also hold.

Remark 3.2. *Theorems 3.3, 3.4 complement Theorem 1 for identically distributed, independent random variables under sub-linear expectations in Zhang and Lin [11].*

4. Proofs of main results

We obtain helpful lemmas firstly.

Lemma 4.1. *Suppose* $r > 1$, *and* $1 \le p < 2$. *Then for all* $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\sup_{k\ge n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)/k^{1/p}\ge\varepsilon\right\}\\
&\quad\le \sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\le k\le n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\ge\left(\varepsilon/2^{2/p}\right)n^{1/p}\right\}.
\end{aligned}$$

Proof. We get

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\sup_{k\ge n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)/k^{1/p}\ge\varepsilon\right\}\\
&=\sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} n^{r-2}h(n)\mathbb{V}\left\{\sup_{k\ge n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)/k^{1/p}\ge\varepsilon\right\}\\
&\le C\sum_{m=1}^{\infty}\mathbb{V}\left\{\sup_{k\ge 2^{m-1}}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)/k^{1/p}\ge\varepsilon\right\}\sum_{n=2^{m-1}}^{2^m-1}2^{m(r-2)}h(2^m)\\
&\le C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)\mathbb{V}\left\{\sup_{k\ge 2^{m-1}}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)/k^{1/p}\ge\varepsilon\right\}\\
&=C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)\mathbb{V}\left\{\sup_{l\ge m}\max_{2^{l-1}\le k<2^l}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\ge\varepsilon 2^{(l-1)/p}\right\}\\
&\le C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)\sum_{l=m}^{\infty}\mathbb{V}\left\{\max_{1\le k<2^l}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\ge\varepsilon 2^{(l-1)/p}\right\}\\
&=C\sum_{l=1}^{\infty}\mathbb{V}\left\{\max_{1\le k<2^l}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\ge\varepsilon 2^{(l-1)/p}\right\}\sum_{m=1}^{l}2^{m(r-1)}h(2^m)\\
&\le C\sum_{l=1}^{\infty}2^{l(r-1)}h(2^l)\mathbb{V}\left\{\max_{1\le k<2^l}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\ge\varepsilon 2^{(l-1)/p}\right\}
\end{aligned}$$

$$\leq C\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\geq (\varepsilon/2^{2/p})n^{1/p}\right\}$$
$$\leq C\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(X_i-\mathbb{E}(X_i))\right)\geq (\varepsilon/2^{2/p})n^{1/p}\right\}.$$

□

Lemma 4.2. *Assume that Y is a random variable fulfilling $C_{\mathbb{V}}(|Y|^{rp}h(|Y|^p)) < \infty$, for some $r \geq 1$, $p \geq 1$. Write $Y' = -n^{-1/p}I\{Y < -n^{-1/p}\} + YI\{|Y| \leq n^{1/p}\} + n^{1/p}I\{Y > n^{1/p}\}$. Suppose $q > rp$. Then*

$$\sum_{n=1}^{\infty} n^{r-1-q/p}h(n)(\log n)^q\mathbb{E}|Y'|^q \leq CC_{\mathbb{V}}(|Y|^{rp}h(|Y|^p)(\log(1+|Y|))^q).$$

Proof. Since $r - q/p < 0$, from Lemma 2.1 and Lemma 2.4, follows that

$$\sum_{n=1}^{\infty} n^{r-1-q/p}h(n)(\log n)^q\mathbb{E}|Y'|^q \leq \sum_{n=1}^{\infty} n^{r-1-q/p}h(n)(\log n)^q C_{\mathbb{V}}\{|Y'|^q\}$$
$$\leq \sum_{n=1}^{\infty} n^{r-1-q/p}h(n)(\log n)^q\int_0^{n^{1/p}}\mathbb{V}\{|Y'|^q > x^q\}qx^{q-1}\mathrm{d}x$$
$$\leq C\int_1^{\infty} y^{r-1-q/p}h(y)(\log y)^q\left[\int_0^1+\int_1^{y^{1/p}}\right]\mathbb{V}\{|Y'|^q > x^q\}x^{q-1}\mathrm{d}x\mathrm{d}y$$
$$\leq C\int_0^1\mathbb{V}\{|Y|^q > x\}\mathrm{d}x\int_1^{\infty} y^{r-1-q/p}h(y)(\log y)^q\mathrm{d}y$$
$$+ C\int_1^{\infty}\mathbb{V}\{|Y| > x\}x^{q-1}\int_{x^p}^{\infty} y^{r-1-q/p}h(y)(\log y)^q\mathrm{d}y\mathrm{d}x$$
$$\leq C + C\int_1^{\infty}\mathbb{V}\{|Y| > x\}h(x^p)x^{rp-1}(\log x)^q\mathrm{d}x$$
$$\leq CC_{\mathbb{V}}(|Y|^{rp}h(|Y|^p)(\log(1+|Y|))^q) < \infty.$$

□

In the rest of this paper, let $\frac{1}{2} < \mu < 1$, $g(y) \in C_{l,Lip}(\mathbb{R})$ fulfilling $0 \leq g(y) \leq 1$ for all y and $g(y) = 1$ if $|y| \leq \mu$, $g(y) = 0$, if $|y| > 1$. We assume $g(y)$ to be a decreasing function for $y \geq 0$. The next lemma gives a useful fact in the proofs of Theorems 3.1 and 3.3.

Lemma 4.3. *Assume that h is a slowly varying function and $p \geq 1$. Assume that $\{X_n, n \geq 1\}$ is a moving average process produced by a sequence of negatively dependent random varables $\{Y_i, -\infty < i < \infty\}$, $\{a_i, -\infty < i < \infty\}$ is a subset of numbers being all non-negative, and for fixed $-\infty < i < \infty$, Y_i is identically distributed as Y with $\mathbb{E}(Y) = 0$, $C_{\mathbb{V}}(|Y|^p) < \infty$ under $(\Omega, \mathcal{H}, \mathbb{E})$. For all $\varepsilon > 0$, write*

$$I := \sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\leq k\leq n}\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} Y''_j \geq \varepsilon n^{1/p}/2\right\},$$

and

$$II := \sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{ \sum_{i=-\infty}^{\infty} a_i \max_{1\le k\le n} \left(\sum_{j=i+1}^{i+k} (Y_j' - \mathbb{E}[Y_j']) \right)^+ \ge \varepsilon n^{1/p}/4 \right\},$$

where

$$Y_j' = -n^{1/p} I\{Y_j < -n^{-1/p}\} + |Y_j| I\{|Y_j| \le n^{1/p}\} + n^{1/p} I\{Y_j > n^{1/p}\},$$

$$Y_j'' = Y_j - Y_j' = (Y_j + n^{1/p}) I\{Y_j < -n^{1/p}\} + (Y_j - n^{1/p}) I\{Y_j > n^{1/p}\}.$$

Suppose $I < \infty$ and $II < \infty$. Then

$$\sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{ \max_{1\le k\le n} S_k \ge \varepsilon n^{1/p} \right\} \le I + II < \infty.$$

Proof. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j.$$

By $\sum_{i=-\infty}^{\infty} a_i < \infty$, $\mathbb{E}(Y_j) = 0$, and $|\mathbb{E}(X) - \mathbb{E}(Y)| \le \mathbb{E}|X - Y|$, Lemma 2.1, we get

$$\begin{aligned}
& n^{-1/p} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \left|\mathbb{E} Y_j'\right| = n^{-1/p} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \left| \mathbb{E}\left[Y_j'\right] - \mathbb{E}[Y_j] \right| \\
& \le n^{-1/p} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E}|Y_j - Y_j'| \le C n^{-1/p} \mathbb{E}|Y_1''| = C n^{-1/p} \mathbb{E}|Y''| \le C n^{-1/p} \mathbb{E}(n^{-1/p})^{p-1} |Y''|^p \\
& \le C n^{1-1/p} \mathbb{E}|Y|^p \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \le C C_{\mathbb{V}} \left\{ |Y|^p \left(1 - g\left(\frac{|Y|}{n^{1/p}}\right)\right) \right\} \\
& \le C C_{\mathbb{V}} \left\{ |Y|^p I\{|Y| \ge \mu n^{1/p}\} \right\} \to 0,\ n \to 0,
\end{aligned}$$

where Y'' and Y' is defined as Y_1'' and Y_1' only with Y in place of Y_1 throughout this paper. Therefore for n sufficiently large, we obtain

$$n^{-1/p} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \left| \mathbb{E} Y_j' \right| < \varepsilon/4.$$

Then

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{ \max_{1\le k\le n} S_k \ge \varepsilon n^{1/p} \right\} \\
& \quad \le C \sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V}\left\{ \max_{1\le k\le n} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j'' \ge \varepsilon n^{1/p}/2 \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\le k\le n}\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}(Y_j'-\mathbb{E}[Y_j'])\ge \varepsilon n^{1/p}/4\right\}\\
&\le C\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\max_{1\le k\le n}\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_j''\ge \varepsilon n^{1/p}/2\right\}\\
&\quad+\sum_{n=1}^{\infty} n^{r-2}h(n)\mathbb{V}\left\{\sum_{i=-\infty}^{\infty}a_i\max_{1\le k\le n}\left(\sum_{j=i+1}^{i+k}(Y_j'-\mathbb{E}[Y_j'])\right)^{+}\ge \varepsilon n^{1/p}/4\right\}\\
&=: I+II.
\end{aligned}
$$

□

Proof of Theorem 3.1. By Lemma 4.1, it is sufficient to establish that (i) holds. Without loss of restrictions, we assume that $\mathbb{E}(Y) = 0$. By Lemma 4.3, we just need to deduce that $I < \infty$ and $II < \infty$.

For I, combining Markov inequality under sub-linear expectations, Lemma 2.1, and Lemma 2.4 results in

$$
\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-1/p}\mathbb{E}^*\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_j''\right|\\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\mathbb{E}^*|Y_1''| = C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\mathbb{E}|Y_1''| = C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\mathbb{E}|Y''|\\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)C_{\mathbb{V}}\{|Y''|\}\\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\int_0^{\infty}\mathbb{V}\{|Y''|>x\}\,\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\left[\mathbb{V}\left\{|Y|>n^{1/p}\right\}n^{1/p}+\int_{n^{1/p}}^{\infty}\mathbb{V}\{|Y|>x\}\,\mathrm{d}x\right]\\
&\le C\int_1^{\infty}x^{r-1}h(x)\mathbb{V}\left\{|Y|>x^{1/p}\right\}\mathrm{d}x + C\int_1^{\infty}y^{r-1-1/p}h(y)\int_{y^{1/p}}^{\infty}\mathbb{V}\{|Y|>x\}\,\mathrm{d}x\mathrm{d}y\\
&\le C\int_1^{\infty}\mathbb{V}\{|Y|^{pr}h(|Y|^p)>x^r h(x)\}\,\mathrm{d}(x^r h(x)) + C\int_1^{\infty}\mathbb{V}\{|Y|>x\}\,\mathrm{d}x\int_1^{x^p}y^{r-1-1/p}h(y)\mathrm{d}y\\
&\le CC_{\mathbb{V}}\{|Y|^{pr}h(|Y|^p)\} + C\int_1^{\infty}\mathbb{V}\{|Y|>x\}\,x^{rp-1}h(x^p)\mathrm{d}x\\
&\le CC_{\mathbb{V}}\{|Y|^{pr}h(|Y|^p)\} < \infty.
\end{aligned}
$$

For II, by Markov inequality under sub-linear expectations, Hölder inequality, Lemma 2.3, we have for all $q > 2$,

$$II \le C \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-q/p} \mathbb{E}^* \left| \sum_{i=-\infty}^{\infty} a_i \max_{1\le k\le n} \left(\sum_{j=i+1}^{i+k} (Y'_j - \mathbb{E}[Y'_j]) \right)^+ \right|^q$$

$$\le C \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-q/p} \mathbb{E}^* \left[\sum_{i=-\infty}^{\infty} a_i^{1-1/q} \left(a_i^{1/q} \left| \max_{1\le k\le n} \left(\sum_{j=i+1}^{i+k} (Y'_j - \mathbb{E}[Y'_j]) \right)^+ \right| \right) \right]^q$$

$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) \left(\sum_{i=-\infty}^{\infty} a_i \right)^{q-1} \sum_{i=-\infty}^{\infty} a_i \mathbb{E}^* \left| \max_{1\le k\le n} \left(\sum_{j=i+1}^{i+k} (Y'_j - \mathbb{E}[Y'_j]) \right)^+ \right|^q$$

$$= C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) \left(\sum_{i=-\infty}^{\infty} a_i \right)^{q-1} \sum_{i=-\infty}^{\infty} a_i \mathbb{E} \max_{1\le k\le n} \left(\left(\sum_{j=i+1}^{i+k} (Y'_j - \mathbb{E}[Y'_j]) \right)^+ \right)^q$$

$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) (\log n)^q \left(n \mathbb{E}|Y'_1|^2 \right)^{q/2} + C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) (\log n)^q \mathbb{E}|Y'_1|^q$$

$$=: II_1 + II_2.$$

To get $II_1 < \infty$, we study two cases. If $rp < 2$, take $q > 2$, observe that in this case $r-2+q/2-rq/2 < -1$. By Lemma 2.1, we obtain

$$II_1 = C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) (\log n)^q n^{q/2} \left(\mathbb{E}|Y'_1|^2 \right)^{q/2}$$

$$= C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) (\log n)^q n^{q/2} \left(\mathbb{E}|Y'|^2 \right)^{q/2}$$

$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) (\log n)^q \left(\mathbb{E}|Y'|^{rp} |Y'|^{2-rp} \right)^{q/2}$$

$$\le C \sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) (\log n)^q \left(C_{\mathbb{V}} \left(|Y|^{rp} \right) \right)^{q/2} n^{\frac{2-rp}{p} \frac{q}{2}}$$

$$\le C \sum_{n=1}^{\infty} n^{r-2+q/2-rq/2} h(n) (\log n)^q < \infty.$$

If $rp \ge 2$, take $q > pr$. Note in this case $\mathbb{E}|Y|^2 < C_{\mathbb{V}}(|Y|^2) < \infty$. We get

$$II_1 = C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) (\log n)^q \left(\mathbb{E}|Y'_1|^2 \right)^{q/2} = C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) (\log n)^q \left(\mathbb{E}|Y'|^2 \right)^{q/2}$$

$$\le C \sum_{n=1}^{\infty} n^{r-1-q/p} (\log n)^q h(n) < \infty.$$

By Lemma 4.2, we conclude that $II_2 < \infty$. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.3. Without loss of restrictions, we assume that $\mathbb{E}(Y) = 0$. By Lemma 4.3, we just need to establish that $I < \infty$ and $II < \infty$ with $r = 1$. For I, by Markov inequality under sub-linear expectations, C_r inequality, Lemma 2.1, and Lemma 2.4 (observe that $\theta < 1$), we get

$$\begin{aligned}
I &\le \sum_{n=1}^{\infty} n^{-1}h(n)n^{-\theta/p}\mathbb{E}^* \max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j''\right|^{\theta} \\
&\le C\sum_{n=1}^{\infty} h(n)n^{-\theta/p}\mathbb{E}^*|Y_1''|^{\theta} = C\sum_{n=1}^{\infty} h(n)n^{-\theta/p}\mathbb{E}|Y_1''|^{\theta} = C\sum_{n=1}^{\infty} h(n)n^{-\theta/p}\mathbb{E}|Y''|^{\theta} \\
&\le C\sum_{n=1}^{\infty} h(n)n^{-\theta/p}C_{\mathbb{V}}\left(|Y''|^{\theta}\right) \\
&\le C\sum_{n=1}^{\infty} h(n)n^{-\theta/p}C_{\mathbb{V}}\left\{|Y|^{\theta}I\{|Y| > n^{1/p}\}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{-\theta/p}h(n)\int_0^{\infty}\mathbb{V}\left\{|Y|^{\theta}I\{|Y| > n^{1/p}\} > x\right\}\mathrm{d}x \\
&\le C\int_1^{\infty} y^{-\theta/p}h(y)\int_0^{\infty}\mathbb{V}\left\{|Y|^{\theta}I\{|Y| > y^{1/p}\} > x\right\}\mathrm{d}x\mathrm{d}y \\
&\le C\int_1^{\infty} y^{-\theta/p}h(y)\left[\int_0^{y^{\theta/p}} + \int_{y^{\theta/p}}^{\infty}\right]\mathbb{V}\left\{|Y|^{\theta}I\{|Y| > y^{1/p}\} > x\right\}\mathrm{d}x\mathrm{d}y \\
&\le C\int_1^{\infty}\mathbb{V}\left\{|Y| > y^{1/p}\right\}h(y)\mathrm{d}y \\
&\quad + C\int_1^{\infty}\mathbb{V}\left\{|Y|^{\theta} > x\right\}\int_1^{x^{p/\theta}} y^{-\theta/p}h(y)\mathrm{d}y\mathrm{d}x \\
&\le CC_{\mathbb{V}}\left(|Y|^p h(|Y|^p)\right) + C\int_1^{\infty}\mathbb{V}\left\{|Y|^{\theta} > x\right\}x^{p/\theta-1}h(x^{p/\theta})\mathrm{d}x \\
&\le CC_{\mathbb{V}}\left(|Y|^p h(|Y|^p)\right) < \infty.
\end{aligned}$$

For II, from Markov inequality under sub-linear expectations, Hölder inequality, and Lemmas 2.1, 2.3, follows

$$\begin{aligned}
II &\le C\sum_{n=1}^{\infty} n^{-1}h(n)n^{-2/p}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i \max_{1\le k\le n}\left(\sum_{j=i+1}^{i+k}(Y_j' - \mathbb{E}[Y_j'])\right)^{+}\right|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-1}h(n)n^{-2/p}\mathbb{E}^*\left(\sum_{i=-\infty}^{\infty} a_i^{1/2}\left(a_i^{1/2}\max_{1\le k\le n}\left(\sum_{j=i+1}^{i+k}(Y_j' - \mathbb{E}[Y_j'])\right)^{+}\right)\right)^2 \\
&\le C\sum_{n=1}^{\infty} n^{-1-2/p}h(n)\sum_{i=-\infty}^{\infty} a_i\sum_{i=-\infty}^{\infty} a_i\mathbb{E}^*\left(\max_{1\le k\le n}\left(\sum_{j=i+1}^{i+k}(Y_j' - \mathbb{E}[Y_j'])\right)^{+}\right)^2 \\
&= C\sum_{n=1}^{\infty} n^{-1-2/p}h(n)\sum_{i=-\infty}^{\infty} a_i\sum_{i=-\infty}^{\infty} a_i\mathbb{E}\max_{1\le k\le n}\left(\left(\sum_{j=i+1}^{i+k}(Y_j' - \mathbb{E}[Y_j'])\right)^{+}\right)^2
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{-1-2/p}h(n)(\log n)^2\left[n\mathbb{E}[|Y_1'|^2]\right]$$
$$= C\sum_{n=1}^{\infty} n^{-2/p}h(n)(\log n)^2\mathbb{E}[|Y_1'|^2] =: II_1.$$

By Lemma 4.2, we get $II_1 < \infty$. Now we will get almost sure convergence under $\mathbb{V}$. Without loss of restrictions, we assume $\mathbb{E}(Y_1) = \mathbb{E}(-Y_1) = 0$. By $C_{\mathbb{V}}(|Y|^p) < \infty$, we have

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{V}\left\{\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right\} < \infty, \text{ for all } \varepsilon > 0.$$

Therefore,

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{-1}\mathbb{V}\left\{\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right\}\\ &= \sum_{k=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1} n^{-1}\mathbb{V}\left\{\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right\}\\ &\geq \frac{1}{2}\mathbb{V}\left\{\max_{1\leq m\leq 2^{k-1}}|S_m| > \varepsilon 2^{k/p}\right\}.\end{aligned}$$

By Borel-Cantelli lemma under sub-linear expectations (cf. Lemma 1 of Zhang and Lin [11]), we get

$$2^{-k/p}\max_{1\leq m\leq 2^k}|S_m| \to 0, \text{ a. s. } \mathbb{V},$$

which yields $S_n/n^{1/p} \to 0$, a. s. $\mathbb{V}$. □

5. Conclusions

We have obtained new results about complete convergence for moving average processes produced by negatively dependent random variables under sub-linear expectations. Results obtained in our article extend those for negatively dependent random variables under classical probability space, and Theorems 3.1–3.4 complement the results of Xu et al. [5], Xu and Kong [6], and in Remark 3.1 we establish Conjecture 3.1 of Xu and Kong in some sense.

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Conflict of interest

All authors state no conflict of interest in this article.

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