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Research article

Stability and bifurcation of a delayed prey-predator eco-epidemiological model with the impact of media

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Abstract: In this paper, a delayed prey-predator eco-epidemiological model with the nonlinear media is considered. First, the positivity and boundedness of solutions are given. Then, the basic reproductive number is showed, and the local stability of the trivial equilibrium and the disease-free equilibrium are discussed. Next, by taking the infection delay as a parameter, the conditions of the stability switches are given due to stability switching criteria, which concludes that the delay can generate instability and oscillation of the population through Hopf bifurcation. Further, by using normal form theory and center manifold theory, some explicit expressions determining direction of Hopf bifurcation and stability of periodic solutions are obtained. What's more, the correctness of the theoretical analysis is verified by numerical simulation, and the biological explanations are also given. Last, the main conclusions are included in the end.

Keywords: eco-epidemiology; media; delay; stability switches; Hopf bifurcation
Mathematics Subject Classification: 37G15, 91D99, 92D25, 92D30

1. Introduction

One of the main fields of mathematics in biology is to explore the interaction among species. The model can analyze population dynamics and predict the tendency. Considering the impact of diseases or the interaction between populations based on epidemiology, the model combining infectious diseases and population is called eco-epidemiological model. In 1989, Harder and Freedman [1] first simulated the spread of disease between species. They assumed that parasites spread to habitats through the infected prey, and susceptible predator were infected by eating the infected prey. Chattopadhyay and Bairagi [2] established a prey-predator model in which diseases are transmitted in prey populations, and first proposed "ecological epidemics". Since then, many scholars have investigated the predator-prey model with disease transmission in prey population in reference [2–9]. Of course, some researchers have focused on the predator-prey model with disease in predator population in reference [10–17]. For example, Li et al. [11] showed a spatial eco-epidemiological system with disease spread within the predator population in open advective environments, and obtained some dynamics of such model, such as, the impacts of advection rate on the net reproductive rate, sufficient conditions for the prevalence of disease and the local stability of disease-free attractor. Meng et al. [14] considered a food chain model with two infected predators, and discussed the stability of equilibrium and back bifurcation.

The prevention and control of infectious diseases is one of the top concerns of the health sector. The study found that there are lots of factors affecting the spread of the disease, including the interaction between susceptible and infected people (effective contact rate), media reports, vaccination, migration, etc. With the progress of economic globalization and technology, the dissemination of information is more convenient and faster, and the role of information in controlling the spread of infectious diseases is becoming more and more important. Therefore, many experts has paid attention to the infectious disease models. During the outbreak of infectious diseases, people received relevant information about the disease from media reports, and then took related measures to avoid infection, such as wearing masks, washing hands frequently, maintaining social distance and vaccinating [18–21]. Hence, media reports can lead to change the personal behavior, and further affect the spread of infectious diseases. For example, in the early days of the outbreak of the coronavirus disease 2019 (COVID-19) in December 2019, the media reported less on the epidemic and did not attract people's attention. Later, a large number of reports on the spread of the disease, which make people understand the main transmission routes of the COVID-19, and enhance the people's sense of self-protection and reduce the chance of infection [22]. It can be seen that media reports play an important role in controlling the spread of the epidemic.

The media coverage is not only the most important factor affecting the spread of infectious diseases, but also is a very important issue. Due to a large number of infection cases, media reports may cause social panic, but they certainly reduce the probability of contact and transmission among susceptible populations, which is positive to control the spread of the disease [23]. Thus, mathematical modeling plays an important role to understand the potential impact of media coverage on the spread of infectious diseases. After the outbreak of Severe Acute Respiratory Syndrome (SARS) in 2002, the impact of media reports on the spread of infectious diseases was introduced. Many scholars studied the psychological impact of the SARS outbreak on the public and the impact of the media on the control and prevention of SARS [24–30]. In recent years, there are many studies on the media effect about the transmission of infectious diseases. For example, Liu et al. [26] described the impact of media reports on communication dynamics by introducing a media function $e^{-a_1E-a_2I-a_3H}$ into the transmission coefficient, here E, I, H represent the number of reported exposed, infected and hospital individuals, respectively, $a_i(i = 1, 2, 3)$ is a constant. Cai and Li [31] considered a new incidence function that reflects the impact of media coverage on disease spread and control, and built a compartment model with an incidence rate. Li and Cui [32] constructed an SIS epidemiological model with the incidence rate, here $\beta_2\frac{I}{m+I}$ reflects the reduction in contact rate caused by media reports. Xiao et al. [33] used a piecewise smooth function $\beta(t) = \beta_0 e^{-\varepsilon M(t)}$ to describe the media effect, here $M(t) = max\{0, p_1I(t) + q_1I_q(t) + p_2\frac{dI}{dt} + q_2\frac{dI_q}{dt}\}$, $I(t)$ and $I_q(t)$, represent the infected and the isolated infected, respectively, p_1, p_2, q_1 and q_2 are non-negative parameters.

Next, a common assumption in the mediation functions that the influence of the media on the transmission is instantaneous, that is, the number of the infected people at a certain moment affects the transmission coefficient at that moment, resulting in a decrease in the incidence rate. However, there is a time delay in the impact of media coverage on transmission dynamics, which describes the timing of an individual's response to a reported infection. Song and Xiao [34] considered the time delay of media report in the infectious disease model and studied the global Hopf bifurcation. Mathur et al. [35] considered the impact of media coverage on the population balance and the size of the infected population. The media induced response function is established by combining the media influence and the predator. The model is as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = rx(t)\Big(1 - \dfrac{x(t)}{K}\Big) - \beta e^{-mz(t)}x(t)y(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = k\beta e^{-mz(t)}x(t)y(t) - k\beta e^{-d_1\tau}e^{-mz(t-\tau)}x(t-\tau)y(t-\tau) - d_1y(t), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = k\beta e^{-d_1\tau}e^{-mz(t-\tau)}x(t-\tau)y(t-\tau) - d_2z(t), \end{cases}$$

where $x(t), y(t)$ and $z(t)$ are the densities of the prey, susceptible predator and infected predator, respectively. The term $\beta e^{-mz(t)}x(t)$ is the media induced response function, here β is the predation rate and the constant m is the media awareness corresponding to the predator population. When media awareness increases, the predation rate decreases exponentially. Lots of factors such as media coverage, lifestyle and population density may directly or indirectly affect the incidence. Therefore, the non-linear incidence $\beta_1 - \beta_2\frac{I}{m+I}$ reflects the characteristics of media reports, where β_1 is the normal contact rate and β_2 is the minimum effective exposure rate reduced through media reports, and m is the half-saturation constant. Thus, $\beta_2\frac{I}{m+I}$ reflects the reduced contact rate through media reports. Since media reports cannot completely block the spread of disease, we suppose that $\beta_1 > \beta_2$. The infected people will reduce their contact with others in a certain area. The more infected people are reported, the less chance of the susceptible people contacting with the infected are.

On the basis of literature [35], we will introduce a nonlinear media influence term and study the impact of media on the spread of disease. The transition from the susceptible to the infected is not instantaneous, which takes some time to become the infected one. Thus, the lag stage in the model divides the population into two stages: early stage and late stage. It is assumed that early individuals can be transferred to the later. When the susceptible people eat the adulterated food, they are recorded as the early stage. Because the susceptible population turn into the infection needs a constant period, and then transfers to the later stage.

The remainder of the paper is organized as follows. In Section 2, we introduce an eco-epidemic model with media influence and present the positivity and boundedness of solutions. In Section 3, we give the basic reproduction number and discuss the asymptotical stability of the non-trivial equilibriums. We also study the existence of the endemic equilibriums and the stability of the positive equilibrium by using the stability switching criteria given by Kuang and Beretta [36]. In Section 4, we obtain formulations to determine the direction of Hopf bifurcation and the stability of periodic solutions. In Section 5, some numerical simulations are carried out. In the last section, we give some discussions.

2. Mathematical model

2.1. Model description

In this part, we will give a prey-predator model including adulterated food interacts with a human population and nonlinear media effects, where the adulterated food is the prey population and the human is the predator. In addition, predators are divided as susceptible predators and infected predators. It is assumed that humans can only be infected by eating the adulterated food, but are not infectious. Let $X(t)$, $S(t)$ and $I(t)$ be the density of prey population, susceptible population and infected population at time t, respectively.

(A1) The prey population grows in a logistic form with an intrinsic growth rate r and the environmental carrying capacity K. The growth of susceptible predators is completely dependent on the prey. Thus, the growth of the susceptible population is proportional to predation with the proportionality constant k denoting the conversion rate and the natural mortality rate d_1.

(A2) The susceptible predation is affected by the information transmitted by the media. We use the nonlinear influence term $\beta_1 - \beta_2 \frac{I(t)}{m+I(t)}$, here β_1 is the general predation rate, β_2 is the predation rate affected by the media, and m is the half-saturation constant reflecting the effect of media reports on disease transmission. The term $\beta_2 \frac{I(t)}{m+I(t)}$ indicates the impact of media reports on communication. When the awareness of media reports increases, the predation rate decreases.

(A3) The term $(\beta_1 - \beta_2 \frac{I(t)}{m+I(t)})X(t)S(t)$ represents predation, and then $k(\beta_1 - \beta_2 \frac{I(t)}{m+I(t)})X(t)S(t)$ represents the growth of susceptible predators in the form of energy coverion. Moreover, since the predator is not instantaneous from the predation stage to the infection stage, it becomes infected after the delay τ. Thus, the term $ke^{-d_1\tau} f(X(t-\tau), S(t-\tau), I(t-\tau))$ represents the number of infected persons joining the susceptible predator at time $t-\tau$. The term $e^{-d_1\tau}$ denotes the maximum survival probability, here $0 < e^{-d_1\tau} < 1$.

Based on the above assumptions, the flow chart of the model is as follows (see Figure 1):

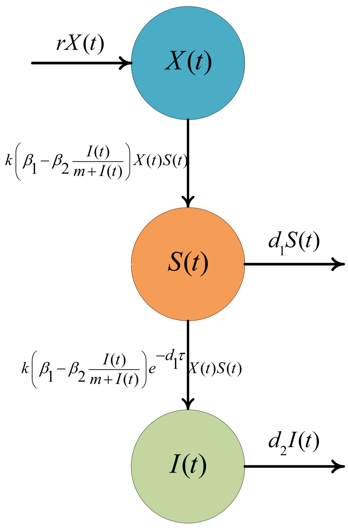


Figure 1. Flow chart of the epidemic model with the impact of media.

Thus, the corresponding model is given as

$$\begin{cases} \dfrac{dX}{dt} =rX(t)\Big(1-\dfrac{X(t)}{K}\Big)-\Big(\beta_1-\beta_2\dfrac{I(t)}{m+I(t)}\Big)X(t)S(t), \\ \dfrac{dS}{dt} =k\Big(\beta_1-\beta_2\dfrac{I(t)}{m+I(t)}\Big)X(t)S(t)-d_1S(t) \\ \qquad -ke^{-d_1\tau}\Big(\beta_1-\beta_2\dfrac{I(t-\tau)}{m+I(t-\tau)}\Big)X(t-\tau)S(t-\tau), \\ \dfrac{dI}{dt} =ke^{-d_1\tau}\Big(\beta_1-\beta_2\dfrac{I(t-\tau)}{m+I(t-\tau)}\Big)X(t-\tau)S(t-\tau)-d_2I(t). \end{cases} \tag{2.1}$$

All the parameters are positive and their descriptions are given in Table 1. Suppose that $C = C([-\tau, 0], \mathbb{R}^3)$ is the Banach space of continuous functions $\phi = (\phi_1, \phi_2, \phi_3) : [-\tau, 0] \to \mathbb{R}^3$ with norm $\|\phi\| = sup_{-\tau\le\theta\le0}\{|\ \phi_1(\theta)\ |,|\ \phi_2(\theta)\ |,|\ \phi_3(\theta)\ |\}$. The initial conditions of the model (2.1) are given as

$$X(\theta) = \phi_1(\theta) \ge 0, \quad S(\theta) = \phi_2(\theta) \ge 0, \quad I(\theta) = \phi_3(\theta) \ge 0, \quad \theta \in [-\tau, 0], \tag{2.2}$$

where the initial function ϕ belongs to the Banach space C. By the fundamental theory of functions differential equations [37], there is a unique solution $(X(t), S(t), I(t))$ of model (2.1) with initial conditions (2.2).

Table 1. The parameters of model (2.1).

Parameters	Descriptions	Unit
r	The intrinsic growth rate of prey	time^{-1}
K	The carrying capacity of prey	ind
β_1	The general predation efficiency of the predator	ind^{-1} time^{-1}
β_2	The maximum effective predation efficiency after media influence	ind^{-1} time^{-1}
m	The half-saturation constant reflecting the impact of media coverage on the spread of the disease	−
k	The conversion rate	−
d_1	The death rate of susceptible predator	time^{-1}
d_2	The death rate of infected	time^{-1}
τ	Time period	time

2.2. *Preliminaries*

In order to show that model (2.1) is epidemiologically meaningful, we will present some preliminaries, including positive invariance and boundedness of solutions. We have the following results.

2.2.1. Positivity of solutions

Theorem 2.1. *The solutions of model (2.1) with the initial condition (2.2) are positive.*

Proof. First, we give some symbols as follows:

$$Y = col(X, S, I) \in \mathbb{R}^3_+, (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C_+ = ([-\tau, 0], \mathbb{R}^3_+), \phi_1(0), \phi_2(0), \phi_3(0) > 0,$$

$$F(Y)=\begin{pmatrix} F_1(Y)\\ F_2(Y)\\ F_3(Y)\end{pmatrix}=\begin{pmatrix} rX(1-\frac{X}{K})-(\beta_1-\beta_2\frac{I}{m+I})XS\\ k(\beta_1-\beta_2\frac{I}{m+I})XS-k(\beta_1-\beta_2\frac{I}{m+I})e^{-d_1\tau}XS-d_1S\\ k(\beta_1-\beta_2\frac{I}{m+I})e^{-d_1\tau}XS-d_2I\end{pmatrix}.$$

Then, model (2.1) can be written in the following form

$$\dot{Y}=F(Y) \tag{2.3}$$

with $Y(\theta)=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))\in C_+$ and $\phi_i(0)>0, i=1,2,3$. It is easy to check that whenever choosing $y(\theta)\in\mathbb{R}_+$ in model (2.3) such that $X=0, S=0, I=0$, then

$$F_i(Y)\mid_{y_i(t)=0,Y\in\mathbb{R}^3_+}\geq 0,$$

where $y_1(t)=X(t), y_2(t)=S(t), y_3(t)=I(t)$. Due to lemma in [38], any solution of (2.3) with $Y(\theta)\in C_+$, defined $Y(t)=Y(t,Y(\theta))$ such that $Y(t)\in\mathbb{R}^3_+$ for all $t\geq 0$ is positive.

2.2.2. Boundedness of solutions

Theorem 2.2. *All positive solutions of model (2.1) with initial condition (2.2) are ultimately bounded.*

Proof. Let $(X(t),S(t),I(t))$ be any solution of model (2.1) with initial condition (2.2). We suppose that

$$V(t)=kX(t)+S(t)+I(t).$$

Calculating the derivative of V along solutions of (2.1), we obtain

$$\begin{aligned}\dot{V}(t) &= rkX(t)\Big(1-\frac{X(t)}{K}\Big)+d_1S(t)+d_2I(t)\\ &\leq (r+1)kX(t)-\frac{rkX^2(t)}{K}-hV(t)\\ &\leq M_0-hV(t),\end{aligned}$$

where $M_0=\frac{kK(r+1)^2}{4r}, h=min\{1,d_1,d_2\}$. Then there exists an $M>0$, depending only on the parameters of model (2.1), such that $V(t)<M$ for all t large enough. Then $X(t),S(t),I(t)$ have an ultimately bound. This proof is completed.

3. Analysis of the model

3.1. The basic reproductive number

Supposing that the right-hand sides of model (2.1) are zero, we have that

$$\begin{cases} rX(1-\frac{X}{K})-(\beta_1-\beta_2\frac{I}{m+I})XS=0,\\ k(\beta_1-\beta_2\frac{I}{m+I})XS-k(\beta_1-\beta_2\frac{I}{m+I})e^{-d_1\tau}XS-d_1S=0,\\ k(\beta_1-\beta_2\frac{I}{m+I})e^{-d_1\tau}XS-d_2I=0.\end{cases} \tag{3.1}$$

It is straightforward to see that model (2.1) has a trivial equilibrium $E_0=(0,0,0)$ and a disease-free equilibrium $E_1=(K,0,0)$.

Next, we use the next generation matrix method [39] to obtain the basic reproduction number R_0 of model (2.1). Here, we have the new infection matrix $\mathcal{F}(x)$ and the transition matrix $\mathcal{V}(x)$. Let $x = (I,\ S)^T$, then model (2.1) can be rewritten as:

$$\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F}(x) = \begin{pmatrix} k(\beta_1 - \beta_2 \frac{I}{m+I})e^{-d_1\tau}XS \\ 0 \end{pmatrix},$$

$$\mathcal{V}(x) = \begin{pmatrix} d_2 I \\ d_1 S - k(\beta_1 - \beta_2 \frac{I}{m+I})XS + k(\beta_1 - \beta_2 \frac{I}{m+I})e^{-d_1\tau}XS \end{pmatrix}.$$

The Jacobian matrices $\mathcal{F}(x)$ and $\mathcal{V}(x)$ at the disease-free equilibrium E_1 are, respectively,

$$D\mathcal{F}(E_1) = \begin{pmatrix} 0 & kK\beta_1 e^{-d_1\tau} \\ 0 & 0 \end{pmatrix}, \qquad D\mathcal{V}(E_1) = \begin{pmatrix} d_2 & 0 \\ 0 & d_1 - kK\beta_1 + kK\beta_1 e^{-d_1\tau} \end{pmatrix},$$

$$D\mathcal{V}(E_1)^{-1} = \begin{pmatrix} 0 & \frac{kK\beta_1 e^{-d_1\tau}}{d_1 - kK\beta_1(1-e^{-d_1\tau})} \\ 0 & 0 \end{pmatrix}.$$

Therefore, the basic reproductive number R_0 is

$$R_0 = \rho(D\mathcal{V}(E_1)^{-1}) = \frac{kK\beta_1 e^{-d_1\tau}}{d_1 - kK\beta_1(1 - e^{-d_1\tau})}. \tag{3.2}$$

3.2. Stability of the non-trivial equilibria

Let $\widetilde{E}(\widetilde{X}, \widetilde{S}, \widetilde{I})$ be any arbitrary equilibrium. The Jacobian matrix evaluated at $\widetilde{E}$ leads us to the following characteristic equation:

$$det(\lambda I - M_1 - M_2 e^{-\lambda\tau}) = 0, \tag{3.3}$$

where I is an identity matrix and

$$M_1 = \begin{pmatrix} g_1 - g_2\widetilde{S} & -g_2\widetilde{X} & g_3\widetilde{X}\widetilde{S} \\ kg_2\widetilde{S} & kg_2\widetilde{X} - d_1 & -kg_3\widetilde{X}\widetilde{S} \\ 0 & 0 & -d_2 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 \\ -kg_2 e^{-d_1\tau}\widetilde{S} & -kg_2 e^{-d_1\tau}\widetilde{X} & kg_3 e^{-d_1\tau}\widetilde{X}\widetilde{S} \\ kg_2 e^{-d_1\tau}\widetilde{S} & kg_2 e^{-d_1\tau}\widetilde{X} & -kg_3 e^{-d_1\tau}\widetilde{X}\widetilde{S} \end{pmatrix},$$

here $g_1 = r(1 - \frac{2\widetilde{X}}{K})$, $g_2 = \beta_1 - \frac{\beta_2\widetilde{I}}{m+\widetilde{I}}$, $g_3 = \frac{\beta_2 m}{(m+\widetilde{I})^2}$.

For the trivial equilibrium $E_0 = (0, 0, 0)$, Eq (3.3) reduces to

$$(\lambda - r)(\lambda + d_1)(\lambda + d_2) = 0. \tag{3.4}$$

The characteristic roots of Eq (3.4) are $r, -d_1$ and $-d_2$. Since the characteristic root r is positive, E_0 is unstable.

Theorem 3.1. *The disease-free equilibrium E_1 of model (2.1) is locally asymptotically stable when $R_0 < 1$, but is unstable when $R_0 > 1$.*

Proof. The characteristic equation of model (2.1) at the disease-free equilibrium E_1 is

$$(\lambda + r)(\lambda + d_2)\left[\lambda - kK\beta_1(1 - e^{-d_1\tau}e^{-\lambda\tau}) + d_1\right] = 0. \tag{3.5}$$

Thus, the two eigenvalues of Eq (3.5) are that $\lambda_1 = -r$, $\lambda_2 = -d_2$ and the other are determined by

$$\lambda - kK\beta_1(1 - e^{-d_1\tau}e^{-\lambda\tau}) + d_1 = 0. \tag{3.6}$$

If $R_0 < 1$. one can easily check that the graph $f_1(\lambda) = \lambda$ and $f_2(\lambda) = kK\beta_1(1 - e^{-d_1\tau}e^{-\lambda\tau}) - d_1$, must intersect at a negative value of λ, and hence the disease-free equilibrium E_1 is locally asymptotically stable provided that $R_0 < 1$, i.e., for all $\tau \in (0, \widehat{\tau})$.

Therefore, when $R_0 < 1$, the disease-free equilibrium E_1 is locally asymptotically stable, but the diseased-free equilibrium E_1 is unstable when $R_0 > 1$.

3.3. Existence of the endemic equilibrium

Theorem 3.2. *If model (2.1) satisfies the following condition:*

$$K(\beta_1 - \beta_2)(1 - e^{-d_1\tau}) > d_1,$$

then model (2.1) has the endemic equilibrium.

Proof. We assume that $E^* = (X^*, S^*, I^*)$ is a solution of Eq (3.1), that is,

$$\begin{cases} rX^*(1 - \frac{X^*}{K}) - (\beta_1 - \beta_2\frac{I^*}{m+I^*})X^*S^* = 0, \\ k(\beta_1 - \beta_2\frac{I^*}{m+I^*})X^*S^* - k(\beta_1 - \beta_2\frac{I^*}{m+I^*})e^{-d_1\tau}X^*S^* - d_1S^* = 0, \\ k(\beta_1 - \beta_2\frac{I^*}{m+I^*})e^{-d_1\tau}X^*S^* - d_2I^* = 0. \end{cases} \tag{3.7}$$

From the second and third equation of (3.7), we have that

$$S^* = \frac{d_2}{d_1}(e^{-d_1\tau} - 1)I^*, \quad I^* = \frac{k\beta_1 m(1 - e^{-d_1\tau})X^* - d_1 m}{d_1 - k(\beta_1 - \beta_2)(1 - e^{-d_1\tau})X^*}.$$

Substituting S^* and I^* into the first equation of (3.7), we have

$$X^3 + AX^2 + BX + C = 0, \tag{3.8}$$

where $A = K - \frac{d_1}{(\beta_1-\beta_2)(1-e^{-d_1\tau})}$, $B = \frac{d_1K}{k(\beta_1-\beta_2)(1-e^{-d_1\tau})} - \frac{K\beta_1 md_2}{rke^{-d_1\tau}(\beta_1-\beta_2)}$, $C = \frac{d_1d_2mK}{rk^2e^{-d_1\tau}(\beta_1-\beta_2)(1-e^{-d_1\tau})}$. According to the theory of the distribution of cubic equations roots [40], we have the following results.

Let $D = (\frac{q}{2})^2 + (\frac{p}{3})^3$, where $q = \frac{2}{27}A^3 - \frac{1}{3}AB + C, p = B - \frac{1}{3}A^2$. Thus, there are three cases for the existence of roots of (3.8).

(a) Equation (3.8) has a pair of complex roots and a real root when $D > 0$. The positivity condition of the real root is given by

$$X = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}} - \frac{A}{3}.$$

(b) When $D = 0$, Eq (3.8) has all real roots with two being equal. Moreover, if $A > 0$, then Eq (3.8) has only one positive real root $X_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{A}{3}$; if $A < 0$, then Eq (3.8) has only one positive real root $X_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{A}{3}$ when $\sqrt[3]{-\frac{q}{2}} > \frac{A}{3}$, and there exist two positive roots when $\frac{A}{6} < \sqrt[3]{-\frac{q}{2}} < \frac{A}{3}$, defined as

$$X_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{A}{3}, \quad X_2 = -\sqrt[3]{-\frac{q}{2}} - \frac{A}{3}.$$

(c) If $D < 0$, then there are three distinct real roots given as

$$\begin{aligned} X_1 &= 2\sqrt{-\frac{|p|}{2}}cos(\frac{\varphi}{3}) - \frac{A}{3}, \\ X_2 &= 2\sqrt{-\frac{|p|}{2}}cos(\frac{\varphi + 2\phi}{3}) - \frac{A}{3}, \\ X_3 &= 2\sqrt{-\frac{|p|}{2}}\cos(\frac{\varphi + 4\phi}{3}) - \frac{A}{3}, \end{aligned}$$

where $\varphi = \arccos(\frac{-q}{2\sqrt{(\frac{|p|}{3})^3}})$, has to be calculated in radians. Furthermore, if $A > 0$, then there exists only one positive root.

Furthermore, the endemic equilibrium is $E^*(X^*, S^*, I^*)$, where

$$X^* = 2\sqrt[3]{-\frac{q}{2}} - \frac{A}{3},\ S^* = \frac{d_1}{d_2}e^{-d_1\tau}(1 - e^{-d_1\tau})I^*,\ I^* = \frac{k\beta_1 m(1 - e^{-d_1\tau})X^* - d_1 m}{d_1 - k(\beta_1 - \beta_2)(1 - e^{-d_1\tau})X^*}.$$

Remark 1. Theorem 3.2 indicates that a unique endemic equilibrium exists, whenever the delay parameter cross a threshold $\tau > \widehat{\tau}$, here

$$\widehat{\tau} = \frac{1}{d_1}\log\Big(\frac{K(\beta_1 - \beta_2)}{K(\beta_1 - \beta_2) - d_1}\Big).$$

3.4. Stability switches at the endemic equilibrium

In the following, we will use the stability switching criteria given by Kuang and Beretta [36], which provides practical guidelines for combining graphical information with analytical work to study effectively the local stability of those models involving delay-dependent parameters. Specifically, we will show that the stability of a given steady state is determined only graphically by a number of functions including delays which can be explicitly represented.

First, we give the stability switching criterion for the third-order characteristic equation. The characteristic equation of the model (2.1) can be rewritten as:

$$\lambda^3 + A_1(\tau)\lambda^2 + A_2(\tau)\lambda + A_3(\tau) + \big(B_1(\tau)\lambda^2 + B_2(\tau)\lambda + B_3(\tau)\big)e^{-\lambda\tau} = 0, \tag{3.9}$$

where

$$\begin{aligned}
A_1(\tau) &= -g_1 - kg_2X^* + g_2S^* + d_1 + d_2,\\
A_2(\tau) &= kg_1g_2X^* - (g_1 - g_2S^*)(d_1 + d_2) - kd_2g_2X^* + kg_2(g_2 - 1)X^*S^* - d_1d_2,\\
A_3(\tau) &= kd_2g_1g_2X^* - g_1d_1d_2 + d_1d_2g_2S^* - k^2g_3{g_2}^2X^{*2}S^{*2},\\
B_1(\tau) &= kg_2e^{-d_1\tau}X^* + kg_3e^{-d_1\tau}X^*S^*,\\
B_2(\tau) &= kd_2g_2e^{-d_1\tau}X^* + kd_1g_3e^{-d_1\tau}X^*S^* - kg_1g_2e^{-d_1\tau}X^* - kg_1g_2g_3e^{-d_1\tau}X^*S^*,\\
B_3(\tau) &= k^2g_3{g_2}^2e^{-d_1\tau}X^{*2}S^{*2} - kd_1g_1g_3e^{-d_1\tau}X^*S^* - kd_2g_1g_2e^{-d_1\tau}X^*.
\end{aligned}$$

In order to use the criterion due to Kuang and Beretta [36], we need to verify the following conditions for all $\tau \in (\widehat{\tau}, +\infty)$.

(1) $P(0, \tau) + Q(0, \tau) \neq 0$;

(2) $P(i\omega, \tau) + Q(i\omega, \tau) \neq 0$;

(3) $\limsup\{|\frac{P(\lambda,\tau)}{Q(\lambda,\tau)}| : |\lambda| \to \infty, Re\lambda \geqslant 0\} < 1$;

(4) $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ has a finite number of zeros;

(5) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

For the endemic equilibrium E^*, Eq (3.9) reduces to the following equation as

$$H(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{3.10}$$

where

$$P(\lambda, \tau) = \lambda^3 + A_1(\tau)\lambda^2 + A_2(\tau)\lambda + A_3(\tau), \tag{3.11}$$

$$Q(\lambda, \tau) = B_1(\tau)\lambda^2 + B_2(\tau)\lambda + B_3(\tau). \tag{3.12}$$

In the following, we investigate the existence of purely imaginary roots $\lambda = i\omega(\omega > 0)$ of Eq (3.10) taking the form of a third-degree exponential polynomial in λ, where all the coefficients of $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ depend on τ.

Let $\tau \in (\widehat{\tau}, +\infty)$, it is easy to see that

$$P(0, \tau) + Q(0, \tau) = A_3(\tau) + B_3(\tau) \neq 0.$$

This implies that condition (1) is satisfied. And condition (2) is obviously true because

$$P(i\omega, \tau) + Q(i\omega, \tau) = [A_3(\tau) + B_3(\tau) - A_1(\tau) - B_1(\tau)\omega^2] + i\omega[A_2(\tau) + B_2(\tau) - \omega^2] \neq 0.$$

From (3.11) and (3.12), we know that

$$\lim_{|\lambda|\to+\infty} \left|\frac{Q(\lambda, \tau)}{P(\lambda, \tau)}\right| = 0.$$

Thus, condition (3) is true.

Let $F(\omega, \tau)$ be defined as in condition (4). From

$$\begin{aligned}
|P(i\omega, \tau)|^2 &= [-\omega^3 + A_2(\tau)\omega]^2 + [-A_2(\tau)\omega^2 + A_3(\tau)]^2\\
&= \omega^6 + [-2A_2(\tau) + A_1^2(\tau)]\omega^4 + [A_2^2(\tau) - 2A_1(\tau)A_3(\tau)]\omega^2 + A_3^2(\tau),
\end{aligned}$$

and

$$|Q(i\omega,\tau)|^2 = B_1^2(\tau)\omega^4 - B_2^2(\tau)\omega^2 - 2B_1(\tau)B_3(\tau)\omega^2 + B_3^2(\tau),$$

we have that

$$F(\omega,\tau) = \omega^6 + a_1(\tau)\omega^4 + a_2(\tau)\omega^2 + a_3(\tau),$$

where

$$\begin{aligned} a_1(\tau) &= A_1^2(\tau) - 2A_2(\tau) - B_1^2(\tau), \quad a_3(\tau) = A_3^2(\tau) - B_3^2(\tau), \\ a_2(\tau) &= A_2^2(\tau) + 2B_1(\tau)B_3(\tau) - 2A_1(\tau)A_3(\tau) - B_2^2(\tau). \end{aligned}$$

It is obvious that condition (4) is satisfied. By implicit function theorem, condition (5) is also satisfied.

Now let $\lambda = i\omega(\omega > 0)$ be a root of Eq (3.10). Substituting it into Eq (3.10) and separating the real and imaginary parts yields

$$-A_1(\tau)\omega^2 + A_3(\tau) = -[-B_1(\tau)\omega^2 + B_3(\tau)]\cos(\omega\tau) - B_2(\tau)\omega\sin(\omega\tau), \tag{3.13}$$

$$-\omega^3 + A_2(\tau)\omega = -B_2(\tau)\omega\cos(\omega\tau) + [-B_1(\tau)\omega^2 + B_3(\tau)]\sin(\omega\tau). \tag{3.14}$$

One can check that, if (ω,τ) is a solution of the Eqs (3.13) and (3.14), then so is $(-\omega,\tau)$. Hence, if $i\omega$ is a purely imaginary characteristic root of (3.10), then its conjugate has the same property. Consequently, we only look in the following for purely imaginary roots of (3.10) with positive imaginary part.

Equations (3.13) and (3.14) yield

$$\cos(\omega\tau) = \frac{(B_2 - A_1B_1)\omega^4 + (A_1B_3 + A_3B_1 - A_2B_2)\omega^2 - A_3B_3}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2}, \tag{3.15}$$

$$\sin(\omega\tau) = \frac{B_1\omega^5 + (A_1B_2 - A_2B_1 - B_3)\omega^3 + (A_2B_3 - A_3B_2)\omega}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2}, \tag{3.16}$$

where we deliberately omit the dependence of the A_i and B_i on τ.

Equations (3.15) and (3.16) can be written

$$\cos(\omega\tau) = Im\left(\frac{P(i\omega,\tau)}{Q(i\omega,\tau)}\right), \quad \text{and} \quad \sin(\omega\tau) = -Re\left(\frac{P(i\omega,\tau)}{Q(i\omega,\tau)}\right). \tag{3.17}$$

A necessary condition for this model to have solutions is that the sum of the squares of the right hand side terms equals one. Adding the squares of both sides of (3.13) and (3.14), a necessary condition for the existence of solution is that

$$[-A_1(\tau)\omega^2 + A_3(\tau)]^2 + [-\omega^3 + A_2(\tau)\omega]^2 = [-B_1(\tau)\omega^2 + B_3(\tau)]^2 + [B_2(\tau)\omega]^2.$$

That is,

$$F(\omega,\tau) = 0. \tag{3.18}$$

Let $z = \omega^2$. The polynomial function F can be written as

$$h(z,\tau) = z^3 + a_1(\tau)z^2 + a_2(\tau)z + a_3(\tau). \tag{3.19}$$

Thus, Eq (3.18) is defined by $h(z,\tau) = 0$. We set $\Delta(\tau) = a_1^2(\tau) - 3a_2(\tau)$. When $\Delta(\tau) > 0$, we have that

$$z_0 = \frac{-a_1(\tau) \pm \sqrt{\Delta(\tau)}}{3}. \tag{3.20}$$

In the following, we will use the following lemma [41].

Lemma 3.1. *Let $\tau \in (\widehat{\tau}, +\infty)$ and $z_0(\tau)$ be defined by (3.20). Then Eq (3.19) has positive roots if and only if*

$$a_3 < 0 \quad or \quad a_3 \geq 0, \quad \Delta(\tau) \geq 0, \quad z_0(\tau) > 0 \quad and \quad h(z_0(\tau), \tau) < 0. \tag{3.21}$$

We set $I = (\widehat{\tau}, +\infty)$ as an interval in which (3.9) is satisfied. For $\tau \in I$, there exists at least $\omega = \omega(\tau) > 0$ such that $F(\omega, \tau) = 0$.

Then, $\theta(\tau) \in [0, 2\pi]$ can be defined for $\tau \in I$ by

$$\cos(\theta(\tau)) = \frac{(B_2 - A_1B_1)\omega^4 + (A_1B_3 + A_3B_1 - A_2B_2)\omega^2 - A_3B_3}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2},$$
$$\sin(\theta(\tau)) = \frac{B_1\omega^5 + (A_1B_2 - A_2B_1 - B_3)\omega^3 + (A_2B_3 - A_3B_2)\omega}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2}.$$

Since $F(\omega, \tau) = 0$ for $\tau \in I$, it follows that θ is well and uniquely defined for all $\tau \in I$. Using (3.13) and (3.19), $i\omega^*$ with $\omega^* = \omega(\tau^*)$ is a purely imaginary characteristic root of Eq (3.10) if and only if τ^* is a zero point of the function $S_n(\tau)$ defined by

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau \in I, \quad with \quad n \in N. \tag{3.22}$$

The following theorem is true due to Kuang and Beretta [36].

Theorem 3.3. *Assume that $\omega(\tau)$ is a positive root of (3.18) defined for $\tau^* \in I$, $I \subseteq R_{+0}$, and at some $\tau^* \in I$, $S_n(\tau^*) = 0$ for some $n \in N_0$. Then, a pair of simple conjugate purely imaginary roots $\lambda = \pm i\omega$ at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where*

$$\delta(\tau^*) = sign\{F'_\omega(\omega(\tau^*), \tau^*)\} sign\Big\{\frac{dS_n(\tau)}{d\tau}\Big|_{\tau=\tau^*}\Big\}. \tag{3.23}$$

Since $\frac{\partial F}{\partial \omega}(\omega, \tau) = 2\omega \frac{\partial h}{\partial \omega}(\omega^2, \tau)$, the condition (3.23) is equivalent to

$$\sin\Big\{\frac{dRe(\lambda)}{d\tau}\Big|_{\lambda=i\omega(\tau^*)}\Big\} = sign\{F'_\omega(\omega(\tau^*), \tau^*)\} sign\Big\{\frac{dS_n(\tau)}{d\tau}\Big|_{\tau=\tau^*}\Big\}.$$

We can easily observe that $S_n(0) < 0$. Moreover, $S_n(\tau) > S_{n+1}(\tau)$ with $n \in N$ for all $\tau \in I$. Therefore, if S_0 has no root in I, then the functions S_n have no root in I. If the function $S_n(\tau)$ has positive zeros, denoted τ_n^i for some $\tau \in I$, $n \in N$, then without loss of generality, we may assume that

$$\frac{dS_n(\tau_n^i)}{d\tau} > 0.$$

Applying the similar method as in [36], it is obtained that the stability switches occur at the zeros of $S_0(\tau)$, denoted by τ_0^i. We can conclude the existence of a Hopf bifurcation which is stated in the next theorem.

Theorem 3.4. *Assume that the endemic equilibrium E^* exists.*

(i) If function $S_0(\tau)$ has no positive zero point in I, then the endemic equilibrium E^* is locally asymptotically stable for all $\tau \geq \widehat{\tau}$.

(ii) If function $S_0(\tau)$ has at least one positive zero point in I, then there exists $\tau_1^*, \tau_1^* \in I$ such that the endemic equilibrium E^* is locally asymptotically stable for $\tau \in (\widehat{\tau}, \tau_1^*)$ and $\tau \in (\tau_2^*, +\infty)$, but unstable for $\tau \in (\tau_1^*, \tau_2^*)$. That is, a Hopf bifurcation occurs when $\tau = \tau_1^*$ (or $\tau = \tau_2^*$) if and only if

$$\frac{\partial h}{\partial \omega}(\omega^2(\tau_1^*), \tau_1^*) > 0 \ \ (\ or\ \frac{\partial h}{\partial \omega}(\omega^2(\tau_2^*), \tau_2^*) < 0).$$

4. The direction of Hopf bifurcation and stability of periodic solutions

In the previous section, we obtained the conditions that the periodic solution bifurcates from the endemic equilibrium E^* at the critical value τ_1^*. In this section, we follow the idea of Hassard et al. [42] derive an normal formula determining the properties of Hopf bifurcation at the critical values of τ_1^*. Thus, the direction, stability, and period of the periodic solution bifurcated from the endemic equilibrium E^* are determined. We always assume that model (2.1) has purely imaginary roots $\pm i\omega^*$ at $\tau = \tau_1^*$ corresponding to the characteristic equation of the endemic equilibrium E^*.

Let $x_1 = X - X^*$, $x_2 = S - S^*$, $x_3 = I - I^*$, $\bar{x}_i = x_i(\tau t)$, $\tau = \tau_1^* + \mu$. Dropping the bars for simplification of notations, model (2.1) is transformed into a functional differential equation in $C = C([-1, 0], \mathbb{R}^3)$ as

$$x(t) = L_\mu(x_t) + f(\mu, x_t), \tag{4.1}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$, and $L_\mu : C \to R$, $f : R \times C \to \mathbb{R}$ are given by

$$\begin{aligned}
L_\mu(\phi) &= (\tau_1^* + \mu)G_1\phi(0) + (\tau_1^* + \mu)G_2\phi(-1),\\
G_1 &= \begin{pmatrix} g_1 - g_2S^* & -g_2X^* & g_3X^*S^* \\ kg_2S^* & kg_2X^* - d_1 & -kg_3X^*S^* \\ 0 & 0 & -d_2 \end{pmatrix},\\
G_2 &= \begin{pmatrix} 0 & 0 & 0 \\ -kg_2e^{-d_1\tau_1^*}S^* & -kg_2e^{-d_1\tau_1^*}X^* & kg_3e^{-d_1\tau_1^*}X^*S^* \\ kg_2e^{-d_1\tau_1^*}S^* & kg_2e^{-d_1\tau_1^*}X^* & -kg_3e^{-d_1\tau_1^*}X^*S^* \end{pmatrix},
\end{aligned}$$

and

$$f(\mu, x_t) = (\tau_1^* + \mu)\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} + H.O.T, \tag{4.2}$$

where

$$\begin{aligned}
\Delta_1 =& -\frac{r}{K}\phi_1^2(0) + g_3S^*\phi_1(0)\phi_2(0) + g_3X^*\phi_2(0)\phi_3(0) - g_2\phi_1(0)\phi_2(0),\\
\Delta_2 =& kg_2\phi_1(0)\phi_2(0) - kg_3S^*\phi_1(0)\phi_2(0) - kg_3X^*\phi_2(0)\phi_3(0) + kg_2e^{-d_1\tau_1^*}[g_3S^*\phi_1(-1)\phi_3(-1)\\
&+ g_3x^*\phi_2(-1)\phi_3(-1) - \phi_1(-1)\phi_2(-1)],\\
\Delta_3 =& -kg_2e^{-d_1\tau_1^*}[g_3S^*\phi_1(-1)\phi_3(-1) + g_3X^*\phi_2(-1)\phi_3(-1) - \phi_1(-1)\phi_2(-1)].
\end{aligned}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta), \quad for \quad \phi \in C.$$

In fact, we choose

$$\begin{aligned}\eta(\theta,\mu) =&(\tau^*+\mu)\begin{pmatrix} g_1-g_2S^* & -g_2X^* & g_3X^*S^* \\ kg_2S^* & kg_2X^*-d_1 & -kg_3X^*S^* \\ 0 & 0 & -d_2 \end{pmatrix}\delta(\theta)\\ &-(\tau^*+\mu)\begin{pmatrix} 0 & 0 & 0 \\ -g_2e^{-d_1\tau_1^*}S^* & -kg_2e^{-d_1\tau_1^*}X^* & kg_3e^{-d_1\tau_1^*}X^*S^* \\ kg_2e^{-d_1\tau_1^*}S^* & kg_2e^{-d_1\tau_1^*}X^* & -kg_3e^{-d_1\tau_1^*}X^*S^* \end{pmatrix}\delta(\theta+1)\end{aligned}$$

where δ is the Dirac delta function. For $\phi \in C^1([-1,0],\mathbb{R}^3)$, we define the operator $A(\mu)$ as

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0. \end{cases} \tag{4.3}$$

Further, we can define the operator $R(\mu)$ as

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{4.4}$$

Then, we can rewrite model (4.1) as the following operator equation

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{4.5}$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1,0]$.

For $\psi \in C^1([-1,0],(\mathbb{R}^3)^*)$, the adjoint operator $A_0{}^*$ of A_0 can be define as

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1,0], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0, \end{cases} \tag{4.6}$$

where η^T is the transpose of η, and the domains of A_0 and $A_0{}^*$ are $C^1([-1,0],C^2)$ and $C^1([0,1],C^2)$, respectively.

For $\phi \in C^1([-1,0],C^2)$ and $\psi \in C^1([0,1],C^2)$, a bilinear from is given by

$$\langle\psi,\phi\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{4.7}$$

where $\eta(\theta) = \eta(\theta,0)$. Then A_0 and $A_0{}^*$ are a pair of adjoint operators. By the results obtained in the previous section, we know that $\pm i\omega^*$ are the eigenvalues of A_0. Thus they are also eigenvalues of $A_0{}^*$. What's more, we suppose that $q(\theta)$ is the eigenvector of A_0 corresponding to $i\omega^*$, then

$$A_0q(\theta) = i\omega^*q(\theta). \tag{4.8}$$

By Eq (4.3), Eq (4.8) can be rewritten as

$$\begin{cases} \frac{dq(\theta)}{d\theta} = i\omega^*q(\theta), & \theta \in [-1,0), \\ L_0q(0) = i\omega^*q(0), & \theta = 0. \end{cases} \tag{4.9}$$

From Eq (4.9) we can get

$$q(\theta) = Ve^{i\omega^*\theta}, \quad \theta \in [-1, 0], \tag{4.10}$$

where $V = (1, v_1, v_2)^T \in C^3$ is a constant vector. By virtue of Eq (4.9), we obtain

$$v_1 = \frac{-d_2 - kg_3 e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*} X^* S^* - kg_2 e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*} S^*}{kg_2 e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*} X^*}, \quad v_2 = \frac{ke^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*}(g_1 - i\omega^*)}{d_2 - i\omega^*}.$$

On the other hand, $-i\omega^*$ is the eigenvalue of A_0^*. Thus we have $A_0^* q^*(\xi) = -i\omega^* q^*(\xi)$. For the non-zero vector $q^*(\xi)$, $\xi \in [0, \tau_1^*]$, we have that

$$G_1^T V + G_2^T e^{i\omega^*\tau_1^*} V^* + i\omega^* I V^* = 0.$$

Let $q^*(\xi) = DV^* e^{i\omega^*\xi}$, here $\xi \in [0, \tau_1^*]$, and $V^* = (1, v_1^*, v_2^*)^T$ be a constant vector. Similarly, we get that

$$v_1^* = \frac{(g_1 - i\omega^*)X^*}{(d_1 - i\omega^*)S^*}, \quad v_2^* = \frac{g_2 X^* S^* - v_1^*(kg_2 X^* - d_1 - kg_2 e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*} + i\omega^*)S^*}{kg_2 e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*} X^* S^*}.$$

In order to assure $\langle q^*(\xi), q(\theta)\rangle = 1$, we need to determine the value of D. From Eq (4.7), we have

$$\begin{aligned}
\langle q^*(\xi), q(\theta)\rangle =& \bar{q}^{*T}(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi - \theta)d\eta(\theta)q(\xi)d\xi \\
=& \bar{D}[\bar{V}^{*T} V - \int_{-1}^{0} \bar{V}^{*T} \theta e^{i\omega^*\tau_1^*\theta} d\eta(\theta) V e^{i\omega^*\xi} d\xi] \\
=& \bar{D}\{1 + v_1\bar{v}_1^* + v_2\bar{v}_2^* + \tau_1^* e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*}[(\bar{v}_1^* S^* - \bar{v}_2^* S^* + v_1\bar{v}_1^* X^*)kg_2 + kg_3(v_2\bar{v}_1^* - v_2\bar{v}_2^*)]\}.
\end{aligned}$$

Therefore, we can choose $\bar{D} = [1 + v_1\bar{v}_1^* + v_2\bar{v}_2^* + \tau_1^* e^{-d_1\tau_1^*} e^{-i\omega^*\tau_1^*}\zeta]^{-1}$, here $\zeta = (\bar{v}_1^* S^* - \bar{v}_2^* S^* + v_1\bar{v}_1^* X^*)kg_2 + (v_2\bar{v}_1^* - v_2\bar{v}_2^*)kg_3$.

Next, we use the same notations as the previous part to study the stability of bifurcating periodic solution. We will calculate the coordinates to express the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Eq (4.5) when $\mu = 0$. Define

$$z_t = \langle q^*, x_t\rangle, \quad W(t, \theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{4.11}$$

On the center manifold C_0, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{4.12}$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}z^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{4.13}$$

z and $\bar{z}$ are the local coordinates for the center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. Thus we only consider the real solution.

For the solution $x_t \in C_0$ of Eq (4.5), since $\mu = 0$, we obtain

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}(t)\rangle = \langle A_0^* q^*, x_t\rangle + \bar{q}^*(0) f_0(0, x_t)\\ &= i\omega^*\tau_1^* z + \bar{q}^* f(0, W(z,\bar{z},0) + 2Re\{zq(\theta)\}) = i\omega^*\tau_1^* z + \bar{q}^*(0) f_0(z,\bar{z}).\end{aligned}$$

That is

$$\dot{z}(t) = i\omega^*\tau_1^* z(t) + g(z,\bar{z}), \tag{4.14}$$

where

$$g(z,\bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + g_{30}(\theta)\frac{z^3}{6} + \cdots. \tag{4.15}$$

It follows from (4.11) that

$$\begin{aligned}x_t(\theta) &= W(t,\theta) + 2Re\{z(t)q(t)\}\\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + (1, v_1, v_2)^T e^{i\omega^*\tau_1^*\theta} z + (1, \bar{v}_1, \bar{v}_2)^T e^{-i\omega^*\tau_1^*\theta}\bar{z} + \cdots.\end{aligned} \tag{4.16}$$

It follows (4.2) that

$$\begin{aligned}g(z,\bar{z}) &= \bar{q}^*(0) f_0(z,\bar{z}) = \bar{q}^*(0) f(0, x_t) = \bar{D}\bar{V}^*\,(\Delta_1 \ \ \Delta_2 \ \ \Delta_3)^T\\
&= -\tau_1^*\bar{D}\frac{r}{K}[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]^2\\
&+ \bar{D}g_3 S^*[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]\\
&\times [v_1 z + \bar{v}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}]\\
&+ \bar{D}g_3 X^*[v_1 z + \bar{v}_1\bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]\\
&\times [v_2 z + \bar{v}_2\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}]\\
&- \bar{D}g_2[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]\\
&\times [v_1 z + \bar{v}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}]\\
&+ \bar{D}kg_2\bar{v}_1^*[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]\\
&\times [v_1 z + \bar{v}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}]\\
&- \bar{D}kg_3\bar{v}_1^* S^*[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}]\\
&\times [v_1 z + \bar{v}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}]\\
&- \bar{D}kg_3\bar{v}_1^* X^*[v_1 z + \bar{v}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}]\end{aligned}$$

$$
\begin{aligned}
&\times [v_2 z + \bar{v_2}\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}]\\
&+ k\bar{D}g_2\bar{v_1^*}e^{-d_1\tau_1^*}\Big\{g_3S^*[e^{-i\omega^*\tau_1^*}z + e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [e^{-i\omega^*\tau_1^*}v_2 z + e^{-i\omega^*\tau_1^*}\bar{v_2}\bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2}]\\
&+ g_3X^*[v_1e^{-i\omega^*\tau_1^*}z + \bar{v_1}e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [e^{-i\omega^*\tau_1^*}v_2 z + e^{-i\omega^*\tau_1^*}\bar{v_2}\bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2}]\\
&- [e^{-i\omega^*\tau_1^*}z + e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [v_1e^{-i\omega^*\tau_1^*}z + \bar{v_1}e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}]\Big\}\\
&- k\bar{D}g_2\bar{v_2^*}e^{-d_1\tau_1^*}\Big\{g_3S^*[e^{-i\omega^*\tau_1^*}z + e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [e^{-i\omega^*\tau_1^*}v_2 z + e^{-i\omega^*\tau_1^*}\bar{v_2}\bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2}]\\
&+ g_3X^*[v_1e^{-i\omega^*\tau_1^*}z + \bar{v_1}e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [e^{-i\omega^*\tau_1^*}v_2 z + e^{-i\omega^*\tau_1^*}\bar{v_2}\bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2}]\\
&- [e^{-i\omega^*\tau_1^*}z + e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}]\\
&\times [v_1e^{-i\omega^*\tau_1^*}z + \bar{v_1}e^{-i\omega^*\tau_1^*}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2}]\Big\}.
\end{aligned}
$$

Comparing the coefficients with Eq (4.15), we have

$$
\begin{aligned}
g_{20} =&2\tau_1^*\bar{D}\Big\{-\frac{r}{K} + g_3S^*v_1 + g_3X^*v_1v_2 - g_2v_1 + kv_1^2\bar{v^*}(g_2 - g_3S^*) - kg_3X^*\bar{v_1^*}v_1v_2\\
&+ kg_2e^{-d_1\tau_1^*}\bar{v^*}[g_3e^{-i\omega^*\tau_1^*}v_2(S^* - v_1) - v_1e^{-i\omega^*\tau_1^*}]\\
&- kg_2\bar{v_2^*}e^{-d_1\tau_1^*}[g_3v_2e^{-i\omega^*\tau_1^*}(S^* + v_1X^*) - v_1e^{-i\omega^*\tau_1^*}]\Big\},\\
g_{11} =&\tau_1^*\bar{D}\Big\{-\frac{2r}{K} + 2(g_3S^* - g_2 + kg_2\bar{v_1^*} - kg_3\bar{v_1^*}S^* - kg_2e^{-d_1\tau_1^*}(\bar{v_1^*} - \bar{v_2^*}))Re\{v_1\}\\
&+ g_3X^*(1 - k\bar{v_1^*})(\bar{v_1}v_2 + v_1\bar{v_2}) + 2kg_2g_3S^*Re\{v_2\}e^{-d_1\tau_1^*}(\bar{v_1^*} - \bar{v_2^*})\\
&+ kg_2g_3e^{-d_1\tau_1^*}X^*(\bar{v_1^*} - \bar{v_2^*})(\bar{v_1}v_2 + v_1\bar{v_2})\Big\},\\
g_{02} =&2\tau_1^*\bar{D}\Big\{-\frac{r}{K} + g_3S^*\bar{v_1} + g_3X^*\bar{v_1}\bar{v_2} - g_2\bar{v_1} + k\bar{v_1}\bar{v^*}\bar{v_1}(g_2 - g_3S^*) - kg_3X^*\bar{v_1^*}\bar{v_1}\bar{v_2}\\
&+ kg_2e^{-d_1\tau_1^*}\bar{v_2^*}[g_3e^{-i\omega^*\tau_1^*}\bar{v_2}(S^* - \bar{v_1}) - \bar{v_1}e^{-i\omega^*\tau_1^*}]\\
&- kg_2\bar{v_2^*}e^{-d_1\tau_1^*}[g_3\bar{v_2}e^{-i\omega^*\tau_1^*}(S^* + \bar{v_1}X^*) - \bar{v_1}e^{-i\omega^*\tau_1^*}]\Big\},\\
g_{21} =&2\tau_1^*\bar{D}\Big\{-\frac{r}{K}[2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)] + (g_3S^* - g_2 + kg_2\bar{v_1^*} - kg_3\bar{v_1^*}S^*)[w_{11}^{(2)}(0)
\end{aligned}
$$

$$
\begin{aligned}
&+ v_1 W_{11}^{(1)}(0) + \frac{1}{2} W_{20}^{(2)}(0) + \frac{\bar{v_1}}{2} W_{20}^{(1)}(0)] + (g_3 X^* - k g_3 \bar{v_1^*} X^*)\Big[v_2 W_{11}^{(3)}(0) \\
&+ \frac{\bar{v_1}}{2} W_{20}^{(3)}(0) + v_1 W_{11}^{(3)}(0) + \frac{\bar{v_2}}{2} W_{20}^{(3)}(0)\Big] + k g_2 g_3 S^* e^{-d_1 \tau_1^*}(\bar{v_1^*} - \bar{v_2^*})\Big[e^{-\omega^* \tau_1^*} W_{11}^{(2)}(-1) \\
&+ \frac{1}{2} e^{\omega^* \tau_1^*} W_{20}^{(2)}(-1) + \frac{\bar{v_2}}{2} e^{\omega^* \tau_1^*} W_{20}^{(1)}(-1) + v_2 e^{-\omega^* \tau_1^*} W_{11}^{(1)}(-1)] \\
&+ k g_2 g_3 X^* e^{-d_1 \tau_1^*}(\bar{v_1^*} - \bar{v_2^*})[v_1 e^{-\omega^* \tau_1^*} W_{11}^{(3)}(-1) + \bar{v_1} e^{\omega^* \tau_1^*} W_{20}^{(3)}(-1) + \bar{v_2} e^{\omega^* \tau_1^*} W_{20}^{(1)}(-1)]\Big\}.
\end{aligned}
$$

According to Eqs (4.5) and (4.11), we can have

$$
\dot{W} = \dot{x_t} - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} A_0 W - 2Re[\bar{q^*}(0) f_0 q(\theta)], & \theta \in [-1, 0), \\ A_0 W - 2Re[\bar{q^*}(0) f_0 q(0)] + f_0, & \theta = 0. \end{cases} \tag{4.17}
$$

That is

$$
\dot{W} = A_0 W + H(z, \bar{z}, \theta), \tag{4.18}
$$

where

$$
H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{30}\frac{z^3}{6} + \cdots. \tag{4.19}
$$

Substituting Eqs (4.3) and (4.13) into Eq (4.18) and comparing the coefficients, we obtain

$$
(A_0 - 2i\omega^* I) W_{20}(\theta) = -H_{20}(\theta), A_0 W_{11} = -H_{11}(\theta), (A_0 + 2i\omega^* I) W_{02}(\theta) = -H_{02}(\theta). \tag{4.20}
$$

Due to $W_{20}(\theta)$ and $W_{11}(\theta)$ appearing in g_{21}, we need to compute them. From Eqs (4.17) and (4.18), we have that for $\theta \in [-1, 0)$,

$$
H(z, \bar{z}, \theta) = -2Re[\bar{q^*} f_0(z, \bar{z}) q(\theta)] = -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \bar{q}(\theta). \tag{4.21}
$$

Comparing the coefficients with Eq (4.20) gives that

$$
H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \tag{4.22}
$$

$$
H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta). \tag{4.23}
$$

From Eqs (4.6) and (4.20), we get

$$
\dot{W}_{20}(\theta) = 2i\omega^* W_{20} + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta), \quad \dot{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta). \tag{4.24}
$$

Noticing that $q(\theta) = V^T e^{i\omega^* \tau_1^* \theta}$, we have

$$
W_{20}(\theta) = \frac{i g_{20}}{\omega^* \tau_1^*} q(\theta) e^{i\omega^* \theta} + \frac{i \bar{g_{02}} \bar{q}(0)}{3\omega^* \tau_1^*} e^{-i\omega^* \theta} + U_1 e^{2i\omega^* \theta}, \tag{4.25}
$$

where $U_1 = (U_1^{(1)}, U_2^{(1)}, U_3^{(1)}) \in \mathbb{R}^3$ is a constant vector.

Similarly, from Eqs (4.20) and (4.23), we obtain

$$
W_{11}(\theta) = \frac{i g_{11}}{\omega^* \tau_1^*} q(\theta) e^{i\omega^* \theta} + \frac{i \bar{g_{11}} \bar{q}(0)}{\omega^* \tau_1^*} e^{-i\omega^* \theta} + U_2 e^{2i\omega^* \theta}, \tag{4.26}
$$

where $U_2 = (U_1^{(2)}, U_2^{(2)}, U_3^{(2)}) \in \mathbb{R}^3$ is a constant vector.

In what follows, we will seek U_1 and U_2. From Eqs (4.3) and (4.20), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20} = 2i\omega^*\tau_1^* W_{20}(0) - H_{20}(0), \tag{4.27}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11} = -H_{11}(0), \tag{4.28}$$

where $\eta(\theta) = \eta(0, \theta)$.

By Eqs (4.20) and (4.23), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_1^*(H_1, H_2, H_3)^T, \tag{4.29}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}\bar{q}(0) + 2\tau_1^*(-P_1, -P_2, -P_3)^T,$$

where

$$\begin{aligned}
H_1 &= -\frac{r}{K} + g_3v_1(v_2X^* + S^*) - g_2v_1,\\
H_2 &= kv_1(g_2 - g_3S^*) - kg_3v_1v_2X^* + kg_2e^{-d_1\tau_1^*}[g_3v_2e^{-i\omega^*\tau_1^*}(S^* - v_1) - v_1e^{-i\omega^*\tau_1^*}],\\
H_3 &= kg_2e^{-d_1\tau_1^*}[g_3v_2e^{-i\omega^*\tau_1^*}(S^* + v_1X^*) - v_1e^{-i\omega^*\tau_1^*}],\\
P_1 &= \frac{r}{K} + 2Re\{v_1\}[g_3S^* - g_2 + g_3X^*(\bar{v_1}v_2 + v_1\bar{v_2})],\\
P_2 &= 2Re\{v_1\}[kg_2(1 - e^{-d_1\tau_1^*}) - kg_3S^*] - (kg_2g_3e^{-d_1\tau_1^*}X^* - kg_3X^*)(\bar{v_1}v_2 + v_1\bar{v_2}),\\
P_3 &= 2kg_2Re\{v_1\}e^{-d_1\tau_1^*} - 2kg_2g_3Re\{v_2\}e^{-d_1\tau_1^*} - kg_2g_3e^{-d_1\tau_1^*}X^*(\bar{v_1}v_2 + v_1\bar{v_2}).
\end{aligned}$$

Substituting Eqs (4.25) and (4.29) into Eq (4.27), we obtain

$$\left(2i\omega^*\tau_1^*I - \int_{-1}^{0} e^{2i\omega^*\tau_1^*\theta}d\eta(\theta)\right)U_1 = 2\tau_1^*(H_1, H_2, H_3)^T,$$

which leads to

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix} U_1 = 2\tau_1^*(H_1, H_2, H_3)^T$$

where

$$\begin{aligned}
&\sigma_{11} = 2i\omega^* - g_1 + g_2S^*,\ \ \sigma_{12} = g_2X^*,\ \ \sigma_{13} = g_3X^*S^*,\ \ \sigma_{21} = -kg_2S^* + kg_2e^{-d_1\tau_1^*}e^{2i\omega^*\tau_1^*},\\
&\sigma_{22} = 2i\omega^* - kg_2X^* + d_1 + kg_2e^{-d_1\tau_1^*}e^{2i\omega^*\tau_1^*}X^*,\ \ \sigma_{23} = kg_3X^*S^* - kg_3e^{-d_1\tau_1^*}e^{2i\omega^*\tau_1^*}X^*S^*,\\
&\sigma_{31} = -kg_2e^{-d_1\tau_1^*}e^{2i\omega^*\tau_1^*}S^*,\ \ \sigma_{32} = -kg_2e^{-d_1\tau^*}e^{2i\omega^*\tau_1^*}X^*,\ \ \sigma_{33} = 2i\omega^*\tau_1^* + kg_3e^{-d_1\tau_1^*}e^{2i\omega^*\tau_1^*}X^*S^* + d_2.
\end{aligned}$$

It follows that

$$U_1^{(1)} = \frac{\Delta_{11}}{\sigma},\quad U_1^{(2)} = \frac{\Delta_{12}}{\sigma},\quad U_1^{(3)} = \frac{\Delta_{13}}{\sigma},$$

where

$$\Delta_{11} = 2\begin{pmatrix} H_1 & \sigma_{12} & \sigma_{13} \\ H_2 & \sigma_{22} & \sigma_{23} \\ H_3 & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad \Delta_{12} = 2\begin{pmatrix} \sigma_{11} & H_1 & \sigma_{13} \\ \sigma_{21} & H_2 & \sigma_{23} \\ \sigma_{31} & H_3 & \sigma_{33} \end{pmatrix}, \quad \Delta_{13} = 2\begin{pmatrix} \sigma_{11} & \sigma_{12} & H_1 \\ \sigma_{21} & \sigma_{22} & H_2 \\ \sigma_{31} & \sigma_{32} & H_3 \end{pmatrix}.$$

Similarly, it follows that

$$U_2^{(1)} = \frac{\Delta_{21}}{\delta}, \quad U_2^{(2)} = \frac{\Delta_{22}}{\delta}, \quad U_2^{(3)} = \frac{\Delta_{23}}{\delta},$$

where

$$\Delta_{21} = \begin{pmatrix} -P_1 & \delta_{12} & \delta_{13} \\ -P_2 & \delta_{22} & \delta_{23} \\ -P_3 & \delta_{32} & \delta_{33} \end{pmatrix}, \quad \Delta_{22} = \begin{pmatrix} \delta_{11} & -P_1 & \delta_{13} \\ \delta_{21} & -P_2 & \delta_{23} \\ \delta_{31} & -P_3 & \delta_{33} \end{pmatrix}, \quad \Delta_{23} = \begin{pmatrix} \delta_{11} & \delta_{12} & -P_1 \\ \delta_{21} & \delta_{22} & -P_2 \\ \delta_{31} & \delta_{32} & -P_3 \end{pmatrix},$$

where

$$\begin{aligned} &\delta_{11} = g_1 - g_2 S^*, \ \ \delta_{12} = -g_2 X^*, \ \ \delta_{13} = -g_3 X^* S^*, \ \ \delta_{21} = k g_2 S^* - k g_2 e^{-d_1 \tau_1^*} S^*, \\ &\delta_{22} = k g_2 X^* - d_1 - k g_2 e^{-d_1 \tau_1^*} X^*, \ \ \delta_{23} = -k g_3 X^* S^* + k g_3 e^{-d_1 \tau_1^*} X^* S^*, \\ &\delta_{31} = k g_2 e^{-d_1 \tau_1^*} S^*, \ \ \delta_{32} = k g_2 e^{-d_1 \tau_1^*} X^*, \ \ \delta_{33} = -k g_3 e^{-d_1 \tau_1^*} X^* S^* - d_2. \end{aligned}$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from Eqs (4.25) and (4.26). Furthermore, g_{21} can be expressed by the parameters and delay. Thus, we can compute the followings values:

$$\begin{aligned} c_1(0) &= \frac{1}{2i\omega^* \tau_1^*}\Big(g_{20} g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau_1^*)\}}, \\ \beta_2 &= 2Re(c_1(0)), \\ T_2 &= -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_1^*)\}}{\omega^* \tau_1^*}, \end{aligned} \tag{4.30}$$

which determine the properties of bifurcating periodic solution in the center manifold at the critical value τ_1^*. From the conclusion of Hassard et al. [42], we give the following main findings.

Theorem 4.1. *The values of the parameters μ_2,β_2 and T_2 of (4.30) will determine the properties of Hopf bifurcation of system.*

(i) The sign of μ_2 determines the direction of Hopf bifurcation: The Hopf bifurcation is supercritical if $\mu_2 > 0$ and the Hopf bifurcation is subcritical if $\mu_2 < 0$.

(ii) The sign of β_2 determines the stability of bifurcation periodic solution: The periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$.

(iii) The sign of T_2 determines the period of bifurcation periodic solution: The period increases if $T_2 > 0$ and the period decreases if $T_2 < 0$.

5. Numerical simulation

In this section, with the help of Matlab software, we will show some numerical simulations that support and illustrate our theoretical results. By satisfying the biological significance of the model and the condition that the model has a unique endemic equilibrium, we choose a set of the following parameters: $r = 0.1,\ K = 6,\ \beta_1 = 0.24,\ \beta_2 = 0.15,\ m = 2,\ k = 0.54,\ d_1 = 0.2,\ d_3 = 0.3$. Thus, model (2.1) is

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = 0.1X(t)\Big(1 - \dfrac{X(t)}{6}\Big) - \Big(0.24 - \dfrac{0.15I(t)}{2 + I(t)}\Big)X(t)S(t), \\ \dfrac{\mathrm{d}S}{\mathrm{d}t} = 0.54\Big(0.24 - \dfrac{0.15I(t)}{2 + I(t)}\Big)X(t)S(t) - 0.2S(t) \\ \qquad - 0.54e^{-0.2\tau}\Big(0.24 - \dfrac{0.15I(t-\tau)}{2 + I(t-\tau)}\Big)X(t-\tau)S(t-\tau), \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = 0.54e^{-0.2\tau}\Big(0.24 - \dfrac{0.15I(t-\tau)}{2 + I(t-\tau)}\Big)X(t-\tau)S(t-\tau) - 0.3I(t). \end{cases}$$

We can see that all key parameters including predation rate β, media influence coefficient m, carrying capacity K, growth rate r, death rates d_1, d_2, and conversion rate k are involved in the above threshold condition. This allows us to address the control parameter τ as well as the impact of adulterated food on human health. It is easy to verify that $R_0 = 0.917 < 1$, when $\tau = 1.4$. Further, we have the unique disease-free equilibrium $E_1 = (6, 0, 0)$ of model (2.1). Then, from Theorem 3.1, the disease-free equilibrium $E_1 = (6, 0, 0)$ is locally asymptotically stable when $R_0 = 0.917 < 1$ (see Figure 2).

The disease-free equilibrium $E_1(6, 0, 0)$ is not feasible because the human may never be extinct in reality. Therefore, our main purpose is to discuss the human living conditions under the condition that adulterated food has minimal impact on human health. Furthermore, the endemic equilibrium exists when $\tau > \widehat{\tau} = 1.63$. From Theorem 3.3, the characteristic equation (3.10) has at least one pair of purely imaginary roots. According to Eq (3.22), the function image of $S_n(\tau)$ is drawn by Matlab software. Therefore, when $n = 0$, the function $S_0(\tau)$ has two zeros points at $\tau_1^* = 1.85$ and $\tau_2^* = 5.36$ (see Figure 3); When $n = 1$, the function $S_1(\tau)$ has no zero point (see Figure 4). Therefore, Theorem 3.4 ensures that the endemic equilibrium E^* is unstable for all $\tau \in (\tau_1^*, \tau_2^*)$, but the endemic equilibrium E^* is locally asymptotically stable when $\tau \in (\widehat{\tau}, \tau_1^*)$ and $\tau \in (\tau_2^*, +\infty)$.

In addition, the periodic fluctuations of the disease occur in a different range of the delay parameter τ, which ensures that the endemic equilibrium changes from stable to unstable to stable (see Figure 3). It can be seen that there are two critical values at which stability switching occurs. According to Theorem 3.4, if τ is approximately equal to τ_1^*, the Hopf bifurcation will occur and periodic solution will appear. In Figure 4, it can be checked that S_1 has no positive root in I. Therefore, there are only two delay thresholds at which stability switching occurs at τ_1^* and τ_2^*.

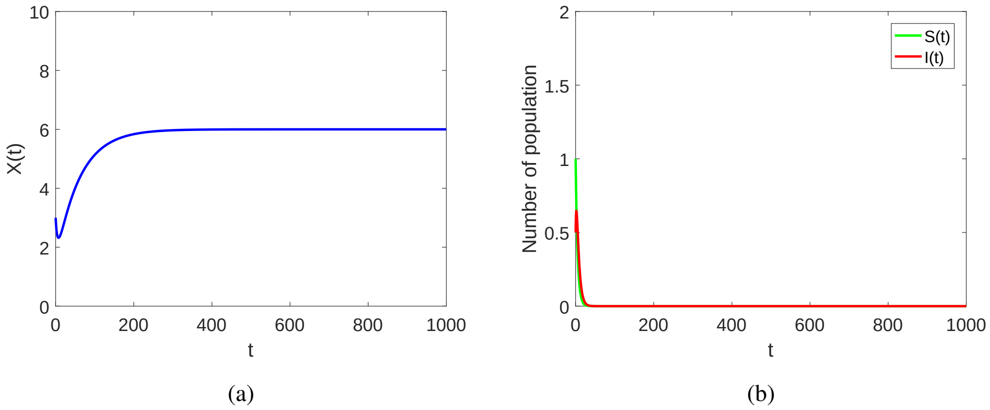


Figure 2. The disease-free equilibrium $E_1 = (6, 0, 0)$ of model (2.1) is locally asymptotically stable when $R_0 = 0.917 < 1$. (a) prey, (b) predator.

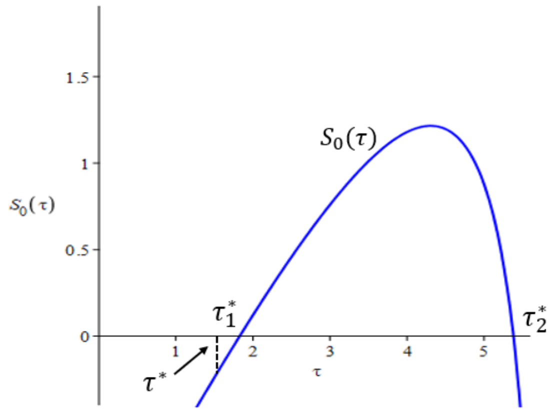


Figure 3. Graph of function $S_0(\tau)$.

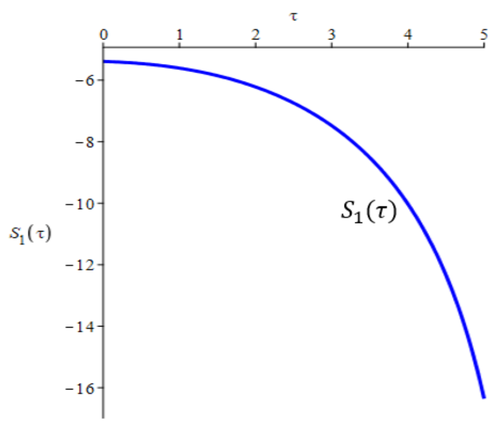


Figure 4. Graph of function $S_1(\tau)$ with the same values as in Figure 3.

Next, when $\tau = 1.7 > \widehat{\tau}$, there is a unique endemic equilibrium $E^*(4.3396, 0.1201, 0.1361)$, which is locally asymptotically stable for $R_0 = 1.837 > 1$ (see Figure 5).

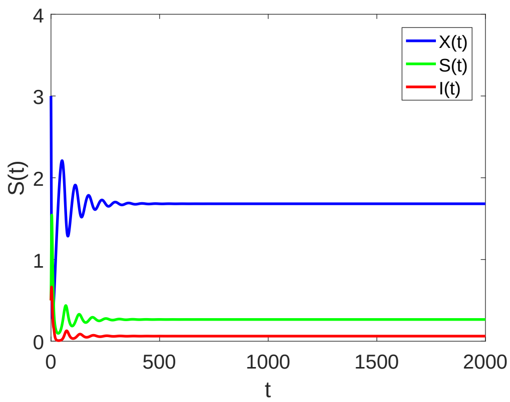
As τ increases, the oscillation period increases. Thus, the steady-state becomes unstable (see Figure 6). By using Theorem 4.1, when $\tau = \tau_1^*$, Hopf bifurcation occurs. Furthermore, we can obtain

$$c_1(0) = -0.3402 - 0.5561i,\ \mu_2 = 2.1733 > 0,\ \beta_2 = -0.68040 < 0,\ T_2 = -0.6495 < 0$$

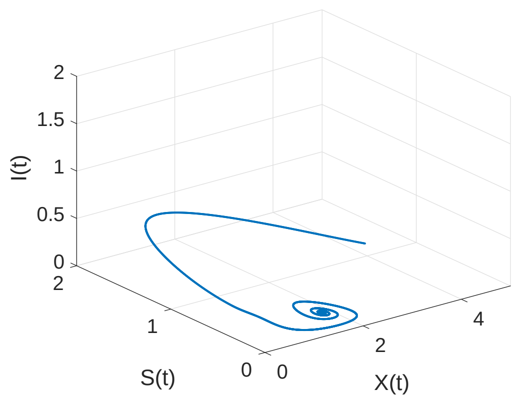
by using Eq (4.30). Therefore, the Hopf bifurcation of the model (2.1) at the endemic equilibrium E^* is supercritical and the bifurcating periodic solutions are orbitally asymptotically stable (see Figure 6). When $\tau = 5.4 > \tau_2^* = 5.36$, the Hopf bifurcation occurring, the solution is stable and converges to the endemic equilibrium (see Figure 7). As shown in Figure 8, the chaotic behavior takes place when $\tau = 4.2$. The change of the media influence coefficient m reflects the dynamics of the model.

With the increase of m, the susceptible population increases and the infected population decreases. According to Figure 9(a), when m increases, the number of the susceptible population increases and the number of people who eat adulterated food decreases. Figure 9(b) shows that the number of the infected population gradually decreases with the increase of m, which means that the number of the susceptible population decreases and the intake of adulterated food decreases. Therefore, increasing the media influence coefficient reduces the intake of adulterated food.

Further, when $m = 1.2$, $\tau = 1.74$, the endemic equilibrium E^* loses stability (see Figure 10). Thus, the low media awareness damages the stability of the model. As the improve of disease depends on many factors, we spread disease news through the media, which has a positive impact on controlling the spread of disease.



(a)



(b)

Figure 5. The endemic equilibrium $E^* = (4.3396, 0.1201, 0.1361)$ of model (2.1) is locally asymptotically stable when $R_0 = 1.837 > 1$: (a) prey and predator, and (b) phase plot.

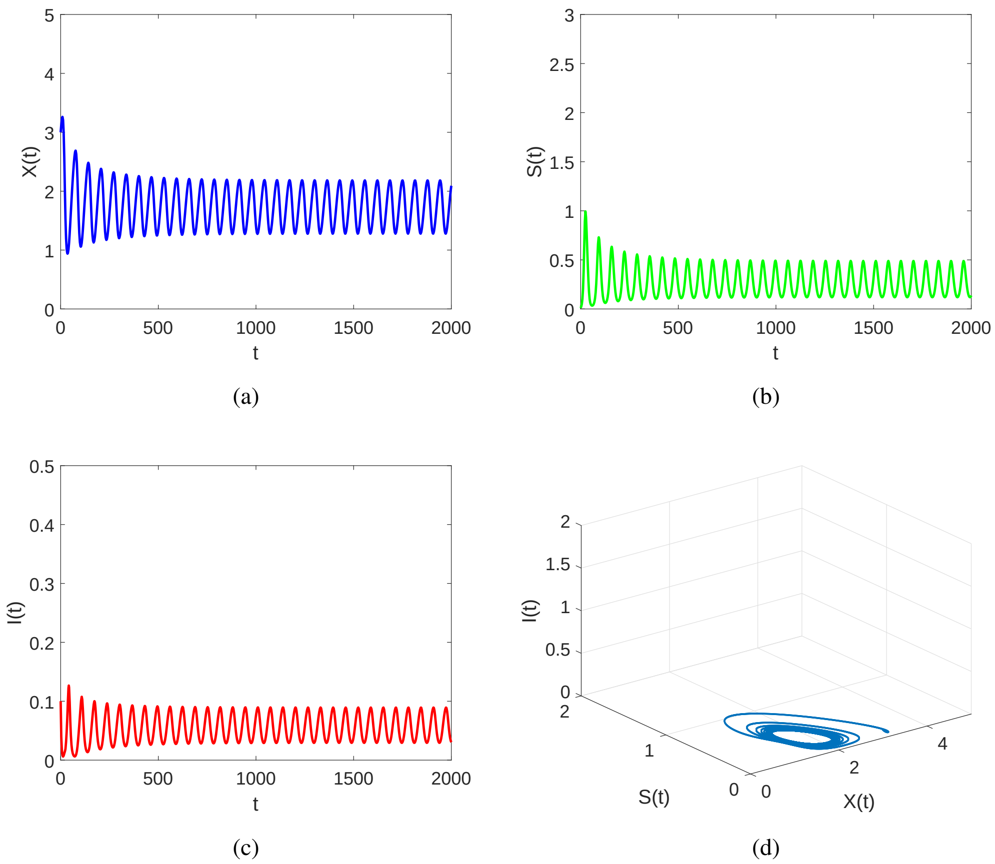


Figure 6. When $\tau = 2.1 \in (\tau_1^*, \tau_2^*)$, model (2.1) is unstable and Hopf bifurcation takes place: (a) prey, (b) susceptible predator, (c) infected predator, and (d) phase plot.

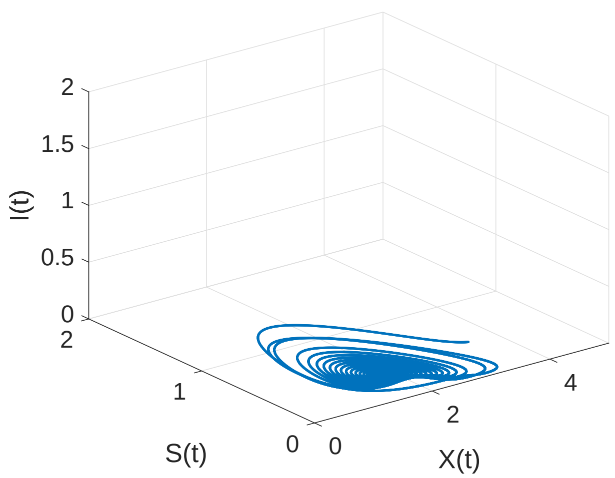


Figure 7. When $\tau = 5.4 > \tau_2^* = 5.36$, the positive equilibrium E^* of model (2.1) is stable again.

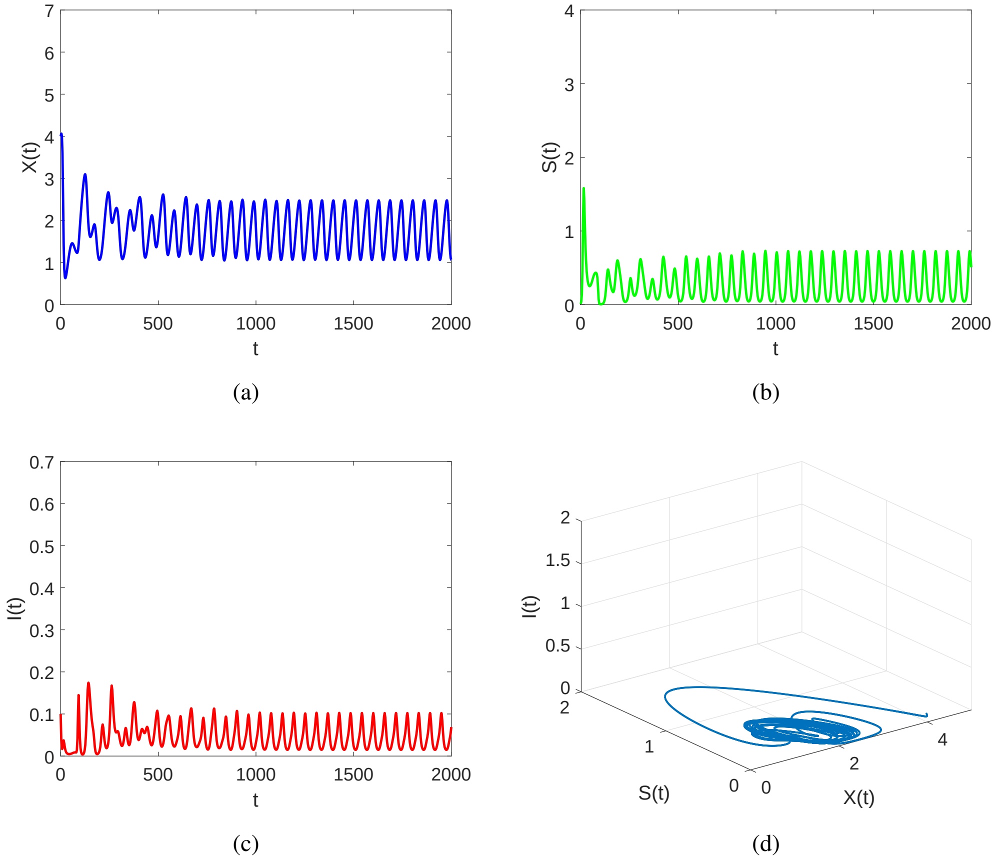


Figure 8. When $\tau = 4.2 \in (\tau_1^*, \tau_2^*)$, chaotic behavior takes place in model (2.1): (a) prey, (b) susceptible predator, (c) infected predator, and (d) phase plot.

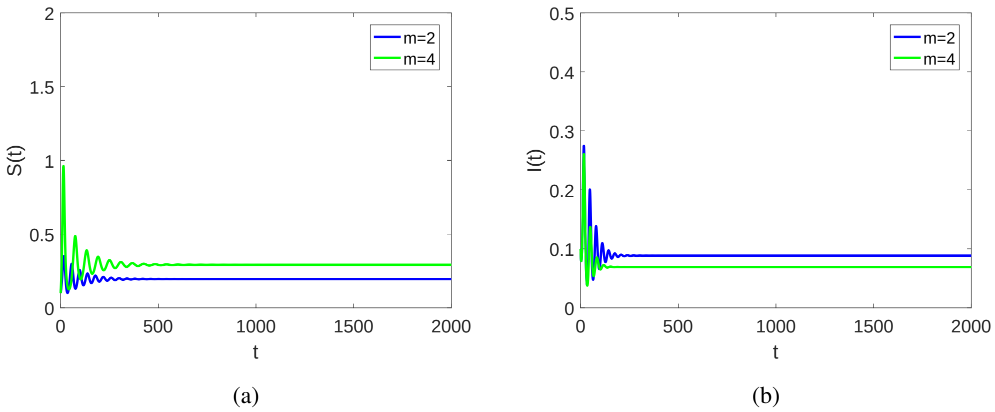
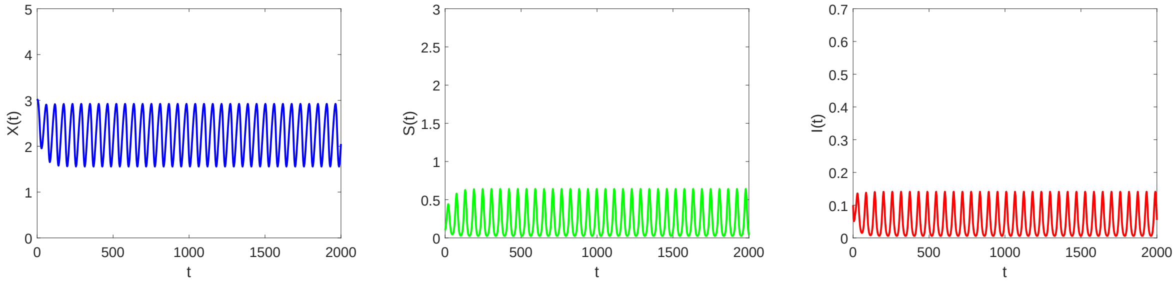


Figure 9. Declination of infected population with increasing media effect for $\tau = 1.73$: (a) susceptible predator, and (b) infected predator.



(a) (b) (c)

Figure 10. Approaching towards the periodic fluctuations in disease occurrence with media awareness coefficient $m = 1.2$: (a) prey, (b) susceptible predator, and (c) infected predator.

6. Discussion and conclusions

In this paper, we give an eco-epidemic model with adulterated food and media effects, where human growth is entirely dependent on adulterated food. By using adulterated food, humans are divided into susceptible and infected populations. The model (2.1) has three equilibria, namely the trivial equilibrium, the disease-free equilibrium and the endemic equilibrium. The trivial equilibrium is an unstable saddle point. When $R_0 < 1$, the disease-free equilibrium is locally asymptotically stable. This suggests that due to adulteration, the susceptible population becomes infected within a short period of time and both populations become extinct. The time delay τ is taken as the bifurcation parameter to obtain the local stable switching conditions of the endemic equilibrium. By determining the stability switching conditions, the time delay can cause a small oscillation of the population density, that is, Hopf bifurcation takes place. It means that time delay plays an important role in the spread of disease, which cases the endemic equilibrium take the form of periodic fluctuations. In addition, by using the center manifold theory, we obtain the direction of Hopf bifurcation and stability of periodic solutions.

This media influence helps people to realize the various hazards of adulterated food. When the media increases, the susceptible population increases and the infected population decreases, which indicates that the media influence makes people reduce their intake of adulterated food and reduce infection. The threshold values of media influence coefficient m indicate that higher intake of adulterated food can lead to instability of the model.

In the future, we will not only take media reports as an independent variable, that is, the term $M(t)$ represents the average number of news related to the epidemic, but also introduce the time delay caused by the response time of individuals to current media reports.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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