Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 17027–17037.
DOI:10.3934/math.2023869
Received: 08 March 2023
Revised: 27 April 2023
Accepted: 09 May 2023
Published: 16 May 2023

Research article

Hermite–Hadamard type inequalities for harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions

Chun-Ying He[1], **Aying Wan**[1,*] **and Bai-Ni Guo**[2,3,*]

[1] School of Mathematics and Statistics, Hulunbuir University, Hulunbuir 021008, Inner Mongolia, China
[2] School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
[3] Independent researcher, Dallas, TX 75252-8024, USA

Dedicated to retired Professor Bo-Yan Xi at Inner Mongolia University for Nationalities in China

* **Correspondence:** Email: wanaying1@qq.com, bai.ni.guo@gmail.com.

Abstract: In this paper, the authors define the notion of harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions, establish a new integral identity, present some new Hermite–Hadamard type integral inequalities for harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions, and derive some known results.

Keywords: Hermite–Hadamard type integral inequality; harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex function; integral identity
Mathematics Subject Classification: 26A51, 26D15, 26D20, 26E60, 41A55

1. A brief review

The following definitions are well known in the literature.

Definition 1.1. *A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([9]). *For $b > 0$ and some fixed number $m \in (0, 1]$, let $f : [0, b] \to \mathbb{R}_0 = [0, \infty)$. If*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

Definition 1.3 ([7]). *For $b > 0$ and some fixed tuple $(\alpha, m) \in (0,1]^2$, let $f : [0,b] \to \mathbb{R}_0$. If*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha) f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0,b]$.

Definition 1.4 ([2,6]). *Let $s \in (0,1]$ be a real number. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be s-convex (in the second sense) if*

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.5 ([16]). *For some number $s \in [-1,1]$, a function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be extended s-convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

is valid for all $x, y \in I$ and $\lambda \in (0,1)$.

Definition 1.6 ([17]). *For some numbers $m, \alpha \in (0,1]$, a function $f : (0,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_0$ is said to be harmonically (α, m)-convex if*

$$f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \le t^\alpha f(x) + m(1-t^\alpha) f(y)$$

is valid for all $x, y \in I$ and $t \in (0,1)$.

Definition 1.7 ([3,4]). *A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ with $a < b$ and $c < d$ is said to be convex on the coordinates on Δ if the partial mappings*

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f_y(u, y)$$

and

$$f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f_x(x, v)$$

are both convex for $x \in (a,b)$ and $y \in (c,d)$.

A formal definition for coordinated convex functions may be stated as follows.

Definition 1.8 ([3, 4]). *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the coordinates on $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ if the inequality*

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x,y) + t(1-\lambda) f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda) f(z,w)$$

holds for all $t, \lambda \in [0,1], (x,y), (z,w) \in \Delta$.

Definition 1.9 ([12]). *For $(s_1, m_1), (s_2, m_2) \in [-1,1] \times (0,1]$, a function $f : [0,b] \times [0,d] \to \mathbb{R}$ is said to be extended $((s_1, m_1)$-$(s_2, m_2))$-convex on the coordinates on $[0,b] \times [0,d]$ if the inequality*

$$\begin{aligned} f(tx + m_1(1-t)z, \lambda y + m_2(1-\lambda)w) \le{}& t^{s_1}\lambda^{s_2} f(x,y) + m_2 t^{s_1}(1-\lambda)^{s_2} f(x,w) \\ &+ m_1(1-t)^{s_1}\lambda^{s_2} f(z,y) + m_1 m_2 (1-t)^{s_1}(1-\lambda)^{s_2} f(z,w) \end{aligned}$$

holds for all $t, \lambda \in (0,1)$ and $(x,y), (z,w) \in [0,b] \times [0,d]$.

Definition 1.10 ([5]). *For $f : (0, b] \subseteq \mathbb{R}_+ \to \mathbb{R}$, $m \in (0, 1]$, and $s \in [-1, 1]$, the function f is said to be harmonically extended (s, m)-convex on $(0, b]$ if*

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \leq t^s f(x) + m(1-t)^s f(y)$$

holds for all $x, y \in (0, b]$ and $t \in (0, 1)$.

In a previous paper [3], Dragomir established the following theorem.

Theorem 1.1 ([3, Theorem 1]). *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the coordinates on Δ. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\,x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\,y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y) \,\mathrm{d}\,y \,\mathrm{d}\,x \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x, c) + f(x, d)] \,\mathrm{d}\,x + \frac{1}{d-c}\int_c^d [f(a, y) + f(b, y)] \,\mathrm{d}\,y\right] \\ &\leq \frac{1}{4}[f(a, c) + f(b, c) + f(a, d) + f(b, d)]. \end{aligned}$$

There are many other new conclusions in the literature [1, 8, 10, 11, 13–15, 18].

The main purpose of this paper is to introduce the notion of harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions and to establish some new Hermite–Hadamard type integral inequalities for this class of convex functions.

2. A definition and a lemma

We now introduce the concept of harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2) coordinated convex functions.

Definition 2.1. *For $(s_1, m_1), (s_2, m_2) \in [-1, 1]\times(0, 1]$, a function $f : \Delta = (0, b]\times(0, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ is said to be harmonic-arithmetic extended (s_1, m_1)-(s_2, m_2)-convex on the coordinates on Δ if the inequality*

$$\begin{aligned} &f\left(\left(\frac{t}{x} + \frac{m_1(1-t)}{z}\right)^{-1}, \left(\frac{\lambda}{y} + \frac{m_2(1-\lambda)}{w}\right)^{-1}\right) \\ &\quad \leq t^{s_1}\lambda^{s_2} f(x, y) + m_2 t^{s_1}(1-\lambda)^{s_2} f(x, w) + m_1(1-t)^{s_1}\lambda^{s_2} f(z, y) + m_1 m_2(1-t)^{s_1}(1-\lambda)^{s_2} f(z, w) \end{aligned}$$

holds for all $t, \lambda \in (0, 1)$ and $(x, y), (z, w) \in \Delta$.

Example 2.1. *Let $f(x, y) = \frac{1}{(xy)^r}$ for $x, y \in \mathbb{R}_+$ and $r \geq 1$. For all tuples $(s_1, m_1), (s_2, m_2) \in [-1, 1]\times(0, 1]$ and $(x, y), (z, w) \in \mathbb{R}^2_+$, we have*

$$\begin{aligned} f\left(\left(\frac{t}{x} + \frac{m_1(1-t)}{z}\right)^{-1}, \left(\frac{\lambda}{y} + \frac{m_2(1-\lambda)}{w}\right)^{-1}\right) &\leq \frac{tz^r + (1-t)(m_1 x)^r}{(xz)^r}\frac{\lambda w^r + (1-\lambda)(m_2 y)^r}{(yw)^r} \\ &\leq \left(\frac{t^{s_1}}{x^r} + \frac{m_1(1-t)^{s_1}}{z^r}\right)\left(\frac{\lambda^{s_2}}{y^r} + \frac{m_2(1-\lambda)^{s_2}}{w^r}\right) \end{aligned}$$

$$= t^{s_1}\lambda^{s_2} f(x,y) + m_2 t^{s_1}(1-\lambda)^{s_2} f(x,w) + m_1(1-t)^{s_1}\lambda^{s_1} f(z,y) + m_1 m_2 (1-t)^{s_1}(1-\lambda)^{s_2} f(z,w).$$

Therefore, the function $f(x,y) = \frac{1}{(xy)^r}$ is harmonic-arithmetic extended (s_1,m_1)-(s_2,m_2) coordinated convex on $\mathbb{R}^2_+$.

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f : \Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a four-time partial differentiable function on $\Delta = [a,b] \times [c,d]$ with $a < b$ and $c < d$. If $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in L_1(\Delta)$, then*

$$\begin{aligned} H(f) := &\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} - \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,\mathrm{d}\,x \\ &- \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,\mathrm{d}\,y + \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\ = &\frac{(b-a)^2(d-c)^2}{4(abdc)^2}\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4} \\ &\times \frac{\partial^4}{\partial t^2 \partial \lambda^2} f\bigg(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}, \Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\bigg)\mathrm{d}\,t\,\mathrm{d}\,\lambda. \end{aligned}$$

Proof. Let $x = (\frac{t}{a} + \frac{1-t}{b})^{-1}$ and $y = (\frac{\lambda}{c} + \frac{1-\lambda}{d})^{-1}$ for $t, \lambda \in [0,1]$. Integrating by parts, we have

$$\begin{aligned} H(f) = &\frac{(b-a)(d-c)}{4abdc}\int_c^d\int_a^b (xy)^2\Big[\frac{a(b-x)}{x(b-a)} - \frac{a^2(b-x)^2}{x^2(b-a)^2}\Big]\Big[\frac{c(d-y)}{y(d-c)} - \frac{c^2(d-y)^2}{y^2(d-c)^2}\Big]\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y \\ = &\frac{(b-a)^2(d-c)^2}{4(abdc)^2}\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4} \\ &\times \frac{\partial^4}{\partial t^2 \partial \lambda^2} f\bigg(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1}, \Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\bigg)\mathrm{d}\,t\,\mathrm{d}\,\lambda. \end{aligned}$$

Lemma 2.1 is proved. □

3. Some integral inequalities of Hermite–Hadamard type

In this section, we establish Hermite–Hadamard type integral inequalities for harmonic-arithmetic extended (s_1,m_1)-(s_2,m_2) coordinated convex functions.

Theorem 3.1. *Let $f : \Delta = (0,b] \times (0,d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a four-time partial differentiable function on Δ such that $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in L_1(\Delta)$. If $\big|\frac{\partial^4 f}{\partial x^2 \partial y^2}\big|^q$ is a harmonic-arithmetic extended (s_1,m_1)-(s_2,m_2) convex function on the coordinates on Δ for $q \ge 1$ and for some fixed tuples $(s_1,m_1),(s_2,m_2) \in [-1,1]\times(0,1]$ and $(a,c),(b,d) \in \Delta$ with $a < b$ and $c < d$, then*

$$\begin{aligned} |H(f)| \le &\frac{(b-a)^2(d-c)^2}{144[(s_1+2)(s_1+3)(s_2+2)(s_2+3)]^{1/q}}\Big(\frac{6ac}{bd}\Big)^{2/q}\bigg[\bigg\{S(2,2)\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big|^q + m_2 S(2,s_2+2) \\ &\times \Big|\frac{\partial^4 f(a,m_2 d)}{\partial x^2\partial y^2}\Big|^q + m_1 S(s_1+2,2)\Big|\frac{\partial^4 f(m_1 b,c)}{\partial x^2\partial y^2}\Big|^q + m_1 m_2 S(s_1+2,s_2+2)\Big|\frac{\partial^4 f(m_1 b,m_2 d)}{\partial x^2\partial y^2}\Big|^q\bigg\}^{1/q}, \end{aligned} \tag{3.1}$$

where

$$S(u,v) = {}_2F_1\Big(4, u, s_1+4, \frac{b-a}{b}\Big){}_2F_1\Big(4, v, s_2+4, \frac{d-c}{d}\Big)$$

and ${}_2F_1(c,d;e;z)$ is the Gauss hypergeometric function which has the integral representation

$${}_2F_1(c,d,e;z) = \frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,\mathrm{d}\,t$$

for $e > d > 0$, $|z| < 1$, $c \in \mathbb{R}$, and $-1 \le u, v \le 1$ with

$$\Gamma(w) = \int_0^\infty t^{w-1}\,\mathrm{e}^{-t}\,\mathrm{d}\,t, \quad \Re(w) > 0.$$

Proof. By Lemma 2.1, by Hölder's integral inequality, and by the coordinated harmonic-arithmetic extended $((s_1,m_1)$-$(s_2,m_2))$-convexity of $\left|\frac{\partial^4 f}{\partial x^2\partial y^2}\right|^q$, we have

$$\begin{aligned}
|H(f)| \le{}& \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\\
&\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\left|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\bigg(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\bigg)\right|\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
\le{}& \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\bigg[\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\\
&\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg]^{1-1/q}\bigg[\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\\
&\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\left|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\bigg(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\bigg)\right|^q\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg]^{1/q}\\
\le{}& \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\bigg[\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\\
&\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg]^{1-1/q}\bigg\{\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\\
&\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\bigg[t^{s_1}\lambda^{s_2}\left|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\right|^q + m_2 t^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^4 f(a,m_2d)}{\partial x^2\partial y^2}\right|^q\\
&+ m_1(1-t)^{s_1}\lambda^{s_2}\left|\frac{\partial^4 f(m_1b,c)}{\partial x^2\partial y^2}\right|^q + m_1m_2(1-t)^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^4 f(m_1b,m_2d)}{\partial x^2\partial y^2}\right|^q\bigg]\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg\}^{1/q}.
\end{aligned}$$

Direct computation gives

$$\begin{aligned}
\int_0^1 t(1-t)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\mathrm{d}\,t &= \frac{(ab)^2}{6},\\
\int_0^1 t^{s+1}(1-t)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\mathrm{d}\,t &= \frac{a^4}{(s+2)(s+3)}{}_2F_1\Big(4,2,s+4,\frac{b-a}{b}\Big),\\
\int_0^1 t(1-t)^{s+1}\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4}\mathrm{d}\,t &= \frac{a^4}{(s+2)(s+3)}{}_2F_1\Big(4,s+2,s+4,\frac{b-a}{b}\Big).
\end{aligned}$$

Combining the last three equalities with the above inequality leads to the inequality (3.1). Theorem 3.1 is proved. □

Remark 3.1. *Under the conditions of Theorem 3.1, if we put $q = 1$ in Theorem 3.1, then*

$$|H(f)| \leq \frac{[ac(b-a)(d-c)]^2}{4(bd)^2(s_1+2)(s_1+3)(s_2+2)(s_2+3)}\Big[S(2,2)\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big| + m_2 S(2,s_2+2)\Big|\frac{\partial^4 f(a,m_2 d)}{\partial x^2\partial y^2}\Big| + m_1 S(s_1+2,2)\Big|\frac{\partial^4 f(m_1 b,c)}{\partial x^2\partial y^2}\Big| + m_1 m_2 S(s_1+2,s_2+2)\Big|\frac{\partial^4 f(m_1 b,m_2 d)}{\partial x^2\partial y^2}\Big|\Big].$$

In particular, when $s_1 = s_2 = s$ and $m_1 = m_2 = m$, we have

$$|H(f)| \leq \frac{[ac(b-a)(d-c)]^2}{4(bd)^2(s+2)^2(s+3)^2}\Big[S(2,2)\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big| + mS(2,s+2)\Big|\frac{\partial^4 f(a,md)}{\partial x^2\partial y^2}\Big| + mS(s+2,2)\Big|\frac{\partial^4 f(mb,c)}{\partial x^2\partial y^2}\Big| + m^2 S(s+2,s+2)\Big|\frac{\partial^4 f(mb,md)}{\partial x^2\partial y^2}\Big|\Big].$$

Theorem 3.2. *For some fixed tuples $(s_1,m_1),(s_2,m_2) \in [-1,1]\times(0,1]$, let $f : \Delta = (0,b]\times(0,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ be a four-time partial differentiable function on Δ such that $\frac{\partial^4 f}{\partial x^2\partial y^2} \in L_1(\Delta)$. If $\big|\frac{\partial^4 f}{\partial x^2\partial y^2}\big|^q$ is a harmonic-arithmetic extended (s_1,m_1)-(s_2,m_2) convex function on the coordinates on Δ for $q > 1$ and $(a,c),(b,d) \in \Delta$ with $a < b$ and $c < d$, then*

$$|H(f)| \leq \frac{1}{4}\Big[\frac{(b-a)(d-c)}{abdc}\Big]^{2/q}\frac{[T(a,b)T(c,d)]^{1-1/q}}{[(s_1+2)(s_1+3)(s_2+2)(s_2+3)]^{1/q}}\Big[\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big|^q + m_2\Big|\frac{\partial^4 f(a,m_2 d)}{\partial x^2\partial y^2}\Big|^q + m_1\Big|\frac{\partial^4 f(m_1 b,c)}{\partial x^2\partial y^2}\Big|^q + m_1 m_2\Big|\frac{\partial^4 f(m_1 b,m_2 d)}{\partial x^2\partial y^2}\Big|^q\Big]^{1/q}, \tag{3.2}$$

where

$$T(a,b) = \frac{(q-1)^2\{[(q+3)b-(3q+1)a]b^{2(q+1)/(q-1)} + [(3q+1)b-(q+3)a]a^{2(q+1)/(q-1)}\}}{2(q+1)(q+3)(3q+1)(b-a)}.$$

Proof. Using Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}|H(f)| &\leq \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4} \\ &\quad\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4}\Big|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\Big)\Big|\,\mathrm{d}t\,\mathrm{d}\lambda \\ &\leq \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\Big[\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4q/(q-1)} \\ &\quad\times\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4q/(q-1)}\mathrm{d}t\,\mathrm{d}\lambda\Big]^{1-1/q}\Big[\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda) \\ &\quad\times\Big|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\Big)\Big|^q\,\mathrm{d}t\,\mathrm{d}\lambda\Big]^{1/q}\end{aligned}$$

and, by the coordinated harmonic-arithmetic extended $((s_1,m_1)$-$(s_2,m_2))$-convexity of $\big|\frac{\partial^4 f}{\partial x^2\partial y^2}\big|^q$,

$$\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\Big|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\Big)\Big|^q\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$\leq \int_0^1 \int_0^1 t\lambda(1-t)(1-\lambda)\left[t^{s_1}\lambda^{s_2}\left|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\right|^q + m_2 t^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^4 f(a,m_2d)}{\partial x^2\partial y^2}\right|^q \right.$$
$$\left. + m_1(1-t)^{s_1}\lambda^{s_2}\left|\frac{\partial^4 f(m_1b,c)}{\partial x^2\partial y^2}\right|^q + m_1m_2(1-t)^{s_1}(1-\lambda)^{s_2}\left|\frac{\partial^4 f(m_1b,m_2d)}{\partial x^2\partial y^2}\right|^q\right] \mathrm{d}\,t\,\mathrm{d}\,\lambda.$$

Furthermore, a straightforward computation gives

$$\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4q/(q-1)} \mathrm{d}\,t = \frac{(ab)^2}{(b-a)^2}T(a,b)$$

and, for $s \geq -1$,

$$\int_0^1 t^{s+1}(1-t)\,\mathrm{d}\,t = \frac{1}{(s+2)(s+3)}.$$

Combining the last two equalities and the above inequalities gives the desired result (3.2). The proof of Theorem 3.2 is complete. □

Remark 3.2. *Under the conditions of Theorem 3.2, if $s_1 = s_2 = s$ and $m_1 = m_2 = m$, we have*

$$|H(f)| \leq \left[\frac{(b-a)(d-c)}{2^q abdc}\right]^{2/q} \frac{[T(a,b)T(c,d)]^{1-1/q}}{[(s+2)(s+3)]^{2/q}} \left\{\left|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\right|^q \right.$$
$$\left. + m\left|\frac{\partial^4 f(a,md)}{\partial x^2\partial y^2}\right|^q + m\left|\frac{\partial^4 f(mb,c)}{\partial x^2\partial y^2}\right|^q + m^2\left|\frac{\partial^4 f(mb,md)}{\partial x^2\partial y^2}\right|^q\right\}^{1/q}.$$

Theorem 3.3. *Under the conditions of Theorem 3.2, we have*

$$|H(f)| \leq \frac{1}{4}\left[\frac{(b-a)(d-c)}{abdc}\right]^{1/q+1} \left[\frac{(q-1)^2}{(3q+1)^2}\left(b^{(3q+1)/(q-1)} - a^{(3q+1)/(q-1)}\right)\left(d^{(3q+1)/(q-1)} - c^{(3q+1)/(q-1)}\right)\right]^{1-1/q}$$
$$\times\left[R(s_1,0,s_2,0)\left|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\right|^q + m_2R(s_1,0,0,s_2)\left|\frac{\partial^4 f(a,m_2d)}{\partial x^2\partial y^2}\right|^q\right.$$
$$\left. + m_1R(0,s_1,s_2,0)\left|\frac{\partial^4 f(m_1b,c)}{\partial x^2\partial y^2}\right|^q + m_1m_2R(0,s_1,0,s_2)\left|\frac{\partial^4 f(m_1b,m_2d)}{\partial x^2\partial y^2}\right|^q\right]^{1/q},$$

where

$$R(u,v,e,\ell) = B(u+q+1, v+q+1)B(e+q+1, \ell+q+1)$$

for $u, v, e, \ell \geq 0$ and $B(x,y)$ is the beta function, which can be defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t, \quad x,y > 0.$$

Proof. Using Lemma 2.1 and Hölder's integral inequality, we have

$$|H(f)| \leq \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\int_0^1\int_0^1 t\lambda(1-t)(1-\lambda)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4}$$
$$\times\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-4}\left|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1},\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-1}\right)\right|\mathrm{d}\,t\,\mathrm{d}\,\lambda$$

$$\leq \frac{(b-a)^2(d-c)^2}{4(abdc)^2}\Big[\int_0^1\int_0^1\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4q/(q-1)}\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-4q/(q-1)}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big]^{1-1/q}$$
$$\times\Big[\int_0^1\int_0^1[t\lambda(1-t)(1-\lambda)]^q\Big|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\Big)\Big|^q\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\Big]^{1/q},$$

where

$$\int_0^1\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-4q/(q-1)}\,\mathrm{d}\,t=\frac{(q-1)ab}{(3q+1)(b-a)}(b^{(3q+1)/(q-1)}-a^{(3q+1)/(q-1)}).$$

By the coordinated harmonic-arithmetic extended $((s_1,m_1)$-$(s_2,m_2))$-convexity of $\left|\frac{\partial^4 f}{\partial x^2\partial y^2}\right|^q$, we obtain

$$\int_0^1\int_0^1[t\lambda(1-t)(1-\lambda)]^q\Big|\frac{\partial^4}{\partial t^2\partial\lambda^2}f\Big(\Big(\frac{t}{a}+\frac{1-t}{b}\Big)^{-1},\Big(\frac{\lambda}{c}+\frac{1-\lambda}{d}\Big)^{-1}\Big)\Big|^q\,\mathrm{d}\,t\,\mathrm{d}\,\lambda$$
$$\leq\int_0^1\int_0^1[t\lambda(1-t)(1-\lambda)]^q\Big[t^{s_1}\lambda^{s_2}\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big|^q+m_2t^{s_1}(1-\lambda)^{s_2}\Big|\frac{\partial^4 f(a,m_2d)}{\partial x^2\partial y^2}\Big|^q$$
$$+m_1(1-t)^{s_1}\lambda^{s_2}\Big|\frac{\partial^4 f(m_1b,c)}{\partial x^2\partial y^2}\Big|^q+m_1m_2(1-t)^{s_1}(1-\lambda)^{s_2}\Big|\frac{\partial^4 f(m_1b,m_2d)}{\partial x^2\partial y^2}\Big|^q\Big]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda$$
$$=R(s_1,0,s_2,0)\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big|^q+m_2R(s_1,0,0,s_2)\Big|\frac{\partial^4 f(a,m_2d)}{\partial x^2\partial y^2}\Big|^q$$
$$+m_1R(0,s_1,s_2,0)\Big|\frac{\partial^4 f(m_1b,c)}{\partial x^2\partial y^2}\Big|^q+m_1m_2R(0,s_1,0,s_2)\Big|\frac{\partial^4 f(m_1b,m_2d)}{\partial x^2\partial y^2}\Big|^q.$$

Combining the above equality and the above two inequalities, we complete the proof of Theorem 3.3. □

Remark 3.3. *Under the conditions of Theorem 3.3, if $s_1=s_2=s$ and $m_1=m_2=m$, then*

$$|H(f)|\leq\frac{1}{4}\Big[\frac{(b-a)(d-c)}{abdc}\Big]^{1/q+1}\Big[\frac{(q-1)^2}{(3q+1)^2}(b^{(3q+1)/(q-1)}-a^{(3q+1)/(q-1)})(d^{(3q+1)/(q-1)}-c^{(3q+1)/(q-1)})\Big]^{1-1/q}$$
$$\times\Big[R(s,0,s,0)\Big|\frac{\partial^4 f(a,c)}{\partial x^2\partial y^2}\Big|^q+mR(s,0,0,s)\Big|\frac{\partial^4 f(a,md)}{\partial x^2\partial y^2}\Big|^q$$
$$+mR(0,s,s,0)\Big|\frac{\partial^4}{\partial x^2\partial y^2}f(mb,c)\Big|^q+m^2R(0,s,0,s)\Big|\frac{\partial^4 f(mb,md)}{\partial x^2\partial y^2}\Big|^q\Big]^{1/q}.$$

Theorem 3.4. *Let $f:\Delta=(0,b]\times(0,d]\subseteq\mathbb{R}^2_+\to\mathbb{R}$ and $f\in L_1(\Delta)$. Denote $H(a,b)=\frac{2ab}{a+b}$. If f is a harmonic-arithmetic extended (s_1,m_1)-(s_2,m_2)-convex function on the coordinates on Δ for some fixed tuples $(s_1,m_1),(s_2,m_2)\in(-1,1]\times(0,1]$ and $(a,c),(b,d)\in\Delta$ with $a<b$ and $c<d$, then*

$$f(H(a,b),H(c,d))$$
$$\leq\frac{1}{2^{s_1+s_2}}\frac{abcd}{(b-a)(d-c)}\int_c^d\int_a^b\frac{f(x,y)+m_1f(m_1x,y)+m_2f(x,m_2y)+m_1m_2f(m_1x,m_2y)}{x^2y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y$$

and

$$\int_c^d\int_a^b\frac{f(x,y)}{x^2y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y\leq\frac{(b-a)(d-c)}{abcd}\frac{f(a,c)+m_2f(a,m_2d)+m_1f(m_1b,c)+m_1m_2f(m_1b,m_2d)}{(s_1+1)(s_2+1)}.$$

Proof. Since

$$H(a,b)=\frac{2}{\frac{t}{a}+\frac{1-t}{b}+\frac{1-t}{a}+\frac{t}{b}}$$

for $t\in[0,1]$, by the coordinated harmonic-arithmetic extended $((s_1,m_1)$-$(s_2,m_2))$-convexity of f, we have

$$\begin{aligned} f(H(a,b),H(c,d)) \le \frac{1}{2^{s_1+s_2}}\int_0^1\int_0^1\Bigg[& f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1},\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-1}\right)\\ &+m_1 f\left(m_1\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1},\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-1}\right)\\ &+m_2 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1},m_2\left(\frac{1-\lambda}{c}+\frac{\lambda}{d}\right)^{-1}\right)\\ &+m_1m_2 f\left(m_1\left(\frac{1-t}{a}+\frac{t}{b}\right)^{-1},m_2\left(\frac{1-\lambda}{c}+\frac{\lambda}{d}\right)^{-1}\right)\Bigg]\mathrm{d}\,t\,\mathrm{d}\,\lambda. \end{aligned}$$

Setting $x=(\frac{t}{a}+\frac{1-t}{b})^{-1}$ and $y=(\frac{\lambda}{c}+\frac{1-\lambda}{d})^{-1}$ for $t,\lambda\in(0,1)$, we have

$$\int_0^1\int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1},\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-1}\right)\mathrm{d}\,t\,\mathrm{d}\,\lambda=\frac{abcd}{(b-a)(d-c)}\int_c^d\int_a^b\frac{f(x,y)}{x^2y^2}\mathrm{d}\,x\mathrm{d}\,y.$$

Combining the above equality and inequality yields the first inequality in Theorem 3.4.

On the other hand, we have

$$\begin{aligned} \frac{abcd}{(b-a)(d-c)}\int_c^d\int_a^b\frac{f(x,y)}{x^2y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y &=\int_0^1\int_0^1 f\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1},\left(\frac{\lambda}{c}+\frac{1-\lambda}{d}\right)^{-1}\right)\mathrm{d}\,t\,\mathrm{d}\,\lambda\\ \le\int_0^1\int_0^1\big[&t^{s_1}\lambda^{s_2}f(a,c)+m_2t^{s_1}(1-\lambda)^{s_2}f(a,m_2d)+m_1(1-t)^{s_1}\lambda^{s_2}f(m_1b,c)\\ &+m_1m_2(1-t^{s_1})(1-\lambda)^{s_2}f(m_1b,m_2d)\big]\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\ =\frac{f(a,c)+m_2f(a,m_2d)+m_1f(m_1b,c)+m_1m_2f(m_1b,m_2d)}{(s_1+1)(s_2+1)}&. \end{aligned}$$

The second inequality in Theorem 3.4 is proved. □

Corollary 3.1. *Under the conditions of Theorem 3.4, if* $m_1=m_2=1$*, then*

$$\begin{aligned} f(H(a,b),H(c,d)) &\le \frac{1}{2^{s_1+s_2-2}}\frac{abcd}{(b-a)(d-c)}\int_c^d\int_a^b\frac{f(x,y)}{x^2y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{2^{s_1+s_2-2}(s_1+1)(s_2+1)}. \end{aligned}$$

Next, we give an application.

Theorem 3.5. *Let* $b>a>0,\ d>c>0,$ *and* $r\ge1$*. Then*

$$\left[\frac{(a+b)(c+d)}{4}\right]^r\le\frac{\left(b^{r+1}-a^{r+1}\right)\left(d^{r+1}-c^{r+1}\right)}{(r+1)^2(b-a)(d-c)}\le\frac{(a^r+b^r)(c^r+d^r)}{4}.$$

In particular, when $r = 2$, we have

$$\frac{(a+b)^2(c+d)^2}{16} \le \frac{\left(a^2+ab+b^2\right)\left(c^2+cd+d^2\right)}{9} \le \frac{\left(a^2+b^2\right)\left(c^2+d^2\right)}{4}.$$

Proof. Using Example 2.1 and Corollary 3.1, we obtain

$$\left[\frac{(a+b)(c+d)}{4abcd}\right]^r \le \frac{abcd\left(b^{-r-1}-a^{-r-1}\right)\left(d^{-r-1}-c^{-r-1}\right)}{(r+1)^2(b-a)(d-c)} \le \frac{(ac)^{-r}+(ad)^{-r}+(bc)^{-r}+(bd)^{-r}}{4}.$$

After simplification, Theorem 3.5 is proved. □

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant No. 12061033) and by the Science Research Fund of Hulunbuir University (Grant No. 2022ZKYB06), China.

The authors appreciate anonymous referees for their valuable comments, careful corrections, and helpful suggestions to the original version of this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. O. Y. Mohammed Alabdali, A. Guessab, On the approximation of strongly convex functions by an upper or lower operator, *Appl. Math. Comput.*, **247** (2014), 1129–1138. http://dx.doi.org/10.1016/j.amc.2014.09.007
2. W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen (German), *Publications de l'Institut Mathématique*, **23** (1978), 13–20.
3. S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese J. Math.*, **5** (2001), 775–788. http://dx.doi.org/10.11650/twjm/1500574995
4. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite–Hadamard type inequalities and applications*, RGMIA Monographs, Victoria University, 2000. Available from: `http://rgmia.org/monographs/hermite_hadamard.html`
5. C.-Y. He, B.-Y. Xi, B.-N. Guo, Inequalities of Hermite–Hadamard type for extended harmonically (s,m)-convex functions, *Miskolc Math. Notes*, **22** (2020), 245–258. http://dx.doi.org/10.18514/MMN.2021.3080
6. H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aeq. Math.*, **48** (1994), 100–111. http://dx.doi.org/10.1007/BF01837981

7. V. G. Miheşan, A generalization of the convexity (Romania), *Proceedings of Seminar on Functional Equations*, 1993.

8. F. Qi, Z.-L. Wei, Q. Yang, Generalizations and refinements of Hermite–Hadamard's inequality, *Rocky Mountain J. Math.*, **35** (2005), 235–251. http://dx.doi.org/10.1216/rmjm/1181069779

9. G. Toader, Some generalizations of the convexity, *Proceedings of the Colloquium on Approximation and Optimization (Cluj-Napoca)*, 1985, 329–338.

10. B.-Y. Xi, R.-F. Bai, F. Qi, Hermite–Hadamard type inequalities for the m- and (α, m)-geometrically convex functions, *Aequat. Math.*, **84** (2012), 261–269. http://dx.doi.org/10.1007/s00010-011-0114-x

11. B.-Y. Xi, D.-D. Gao, F. Qi, Integral inequalities of Hermite–Hadamard type for (α, s)-convex and (α, s, m)-convex functions, *Ital. J. Pure Appl. Math.*, **44** (2020), 499–510.

12. B.-Y. Xi, C.-Y. He, F. Qi, Some new inequalities of the Hermite–Hadamard type for extended $((s_1, m_1)$-$(s_2, m_2))$-convex functions on co-ordinates, *Cogent Math.*, **3** (2016), 1267300. http://dx.doi.org/10.1080/23311835.2016.1267300

13. B.-Y. Xi, F. Qi, Hermite–Hadamard type inequalities for geometrically r-convex functions, *Stud. Sci. Math. Hung.*, **51** (2014), 530–546. http://dx.doi.org/10.1556/sscmath.51.2014.4.1294

14. B.-Y. Xi, F. Qi, Some Hermite–Hadamard type inequalities for differentiable convex functions and applications, *Hacet. J. Math. Stat.*, **42** (2013), 243–257.

15. B.-Y. Xi, F. Qi, Some integral inequalities of Hermite–Hadamard type for s-logarithmically convex functions(Chinese), *Acta Math. Sci. Ser. A Chin. Ed*, **35** (2015), 515–524.

16. B.-Y. Xi, F. Qi, Inequalities of Hermite–Hadamard type for extended s-convex functions and applications to means, *J. Nonlinear Convex Anal.*, **16** (2015), 873–890.

17. B.-Y. Xi, F. Qi, T.-Y. Zhang, Some inequalities of Hermite–Hadamard type for m-harmonic-arithmetically convex functions, *ScienceAsia*, **41** (2015), 357–361. http://dx.doi.org/10.2306/scienceasia1513-1874.2015.41.357

18. B.-Y. Xi, Y. Wang, F. Qi, Some integral inequalities of Hermite–Hadamard type for extended (s, m)-convex functions, *Transylv. J. Math. Mech.*, **5** (2013), 69–84.

AIMS Press