Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16708–16723.
DOI: 10.3934/math.2023854
Received: 15 November 2022
Revised: 07 April 2023
Accepted: 20 April 2023
Published: 12 May 2023

Research article

Graphs with mixed metric dimension three and related algorithms

Dalal Awadh Alrowaili[1], **Uzma Ahmad**[2], **Saira Hameeed**[2], **Muhammad Javaid**[3,*]

[1] Mathematics Department, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia

[2] Department of Mathematics, University of the Punjab, Lahore, Pakistan

[3] Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan

* **Correspondence:** Email: muhammad.javaid@umt.edu.pk.

Abstract: Let $G = (V, E)$ be a simple connected graph. A vertex $x \in V(G)$ resolves the elements $u, v \in E(G) \cup V(G)$ if $d_G(x, u) \neq d_G(x, v)$. A subset $S \subseteq V(G)$ is a mixed metric resolving set for G if every two elements of G are resolved by some vertex of S. A set of smallest cardinality of mixed metric generator for G is called the mixed metric dimension. In this paper trees and unicyclic graphs having mixed dimension three are classified. The main aim is to investigate the structure of a simple connected graph having mixed dimension three with respect to the order of graph, maximum degree of basis elements and distance partite sets of basis elements. In particular to find necessary and sufficient conditions for a graph to have mixed metric dimension 3. Moreover three separate algorithms are developed for trees, unicyclic graphs and in general for simple connected graph $J_n \ncong P_n$ with $n \geq 3$ to determine "whether these graphs have mixed dimension three or not?". If these graphs have mixed dimension three, then these algorithms provide a mixed basis of an input graph.

Keywords: metric dimension; fractional metric dimension; modified prism networks
Mathematics Subject Classification: 05C12, 05C75

1. Introduction

One of the important features of graph metric generator is that its different version can be introduced according to the required scenario or application. Up till now, a lot of research work has been carried out on metric generators and its various versions starting from Slater [14], Haray [7] and then contributed by a number of authors [2–12]. The notion of graph metric generator was primarily studied due to its basic property of identification of intruder in the network as all the nodes in the network can be uniquely localized by a certain set of vertices called metric generator. However, in the situation, when an intruder can approach the system not only through nodes but also by manipulating the connections between the nodes (i.e., edges), then a basic metric generator may not be able to locate the intruder. This leads to the motivation of constructing a metric generator having capabilities of distinguishing both vertices and edges so that this type of situation can be handled. Kelenc et al. [9] proposed a metric generator variant referred as mixed metric generator which can identified both vertices and edges of graph simultaneously. They analyzed different properties of mixed metric generator. In particular, they characterized the graphs of order n having mixed dimension 2 and n. They proved that graph has mixed dimension two if and only if it is a path graph and has mixed dimension n if and only if it is a complete graph. They also determined mixed dimension of some well- known families of graph like path graph, cycle graph, cartesian product with path graphs, etc. The mixed metric dimension of petersen graph was determined by Raza and Ji [13]. The mixed metric dimension for unicyclic graphs was investigated in [15]. In [1], the necessary and sufficient conditions for graphs of order at least 3 having mixed fault-tolerant generators are established. Moreover, a mixed fault-tolerant metric generator is determined for graphs having shortest cycle length at least 4. Danas et al. [6] presented three general lower bounds for mixed metric dimension of graphs. They also compare the new bounds with already existing bounds in literature.

As the minimum mixed dimension for a simple connected graph $J \ncong P_n$ with $n \geq 3$ is three and one of such graph is cycle graph [9], so it is natural to seek all graphs having mixed dimension three. The focus of this paper is to characterize such graphs and to develop an algorithm to determine "whether a simple connected graph J with $n \geq 3$ vertices such that $J \ncong P_n$ has mixed dimension three or not?" We also classify all unicyclic graphs with mixed dimension three. To characterize the graphs having mixed dimension three and to develop algorithm for this, we use the idea of neihbourhood of vertices and vertex distance partitions which was used in [16] for characterization of graphs having metric dimension two. To recall, for a graph Q, the distance between two vertices is the length of the shortest path between them, whereas, the distance between a vertex x and an edge $e = yz$ is given as $d(x, e) = \min\{d(x, y), d(x, z)\}$. The subset $M = \{m_1, m_2, \cdots, m_k\} \subseteq V(Q)$ is referred as mixed metric generator or mixed generator, if distance vectors of any two members $x, y \in V(Q) \cup E(Q)$ are distinct, i.e., $r(x|M) \neq r(y|M)$, where distance vector $r(x|M) = (d(x, m_1), d(x, m_2, \cdots, d(x, m_k))$. The smallest mixed generator is called mixed basis and the number of elements in mixed basis is called mixed dimension. It is represented as $\dim_m(Q)$. For $x \in V(Q)$, the collection $\{X_0, X_1, \cdot, X_k\}$ is referred as distance partition of $V(Q)$ relative to the vertex x if $X_0 = \{x\}$and $X_j = \{y \in V(Q) : d(x, y) = j$, for $1 \leq j \leq k$, where k is the eccentricity of x in Q. The sets $X_0, X_1, \cdot, X_k$ are referred as distance partite sets. The open neighbourhood $N(v)$ of $v \in V(G)$ is defined as $N(v) = \{x \in V(G) : x \sim v\}$. The eccentricity $e(v)$ of any vertex $v \in V(G)$ is the maximum distance of v in $V(G)$, i.e., $e(v) = \max_{x \in V(G)} d(x, v)$. the vertex of degree one is referred as pendant vertex and vertex adjacent to pendant vertex is called support vertex. A unicyclic graph is a simple graph with exactly one cycle.

2. Mixed basis for trees and unicyclic graphs

Definition 2.1. Consider a graph G of order $r \geq 1$ with $V(G) = \{v_1, v_2, \cdots, v_r\}$ and r paths P_{s_i} of length s_i with $s_i \geq 1$. Then the graph obtained from G by identifying a pendant vertex of a path P_{s_i} with $v_i \in V(G)$ such that $i = 1, \ldots, r$. Then this new graph is denoted by $\Gamma_G(P_{s_1}, P_{s_2}, \cdots, P_{s_r})$.

Example 2.1. *Consider a wheel graph of order 8, where* $V(W_8) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.

Then G is the new graph obtained by identifying the pendent vertices of paths of lengths P_2, P_2, P_3, P_3 with vertices v_2, v_3, v_5, v_7 and path of length one with all remaining vertices $\Gamma_{W_8}(P_1, P_2, P_2, P_1, P_3, P_1, P_3, P_1)$ as shown in Figure 1.

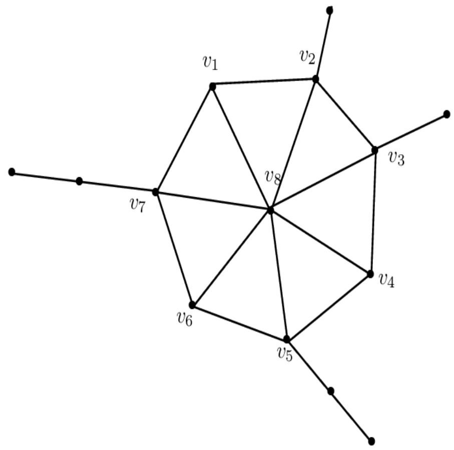

Figure 1. Γ_{W_8} obtained by W_8.

Example 2.2. *The unicycle graph $U_n = \Gamma_{C_8}(P_3, P_1, P_2, P_1, P_1, P_3, P_1, P_1)$ is obtained by C_8 and identifying the pendant vertices of paths P_3, P_2 and P_3 with vertices u_1, u_3, and u_6, respectively. This is shown in Figure 2.*

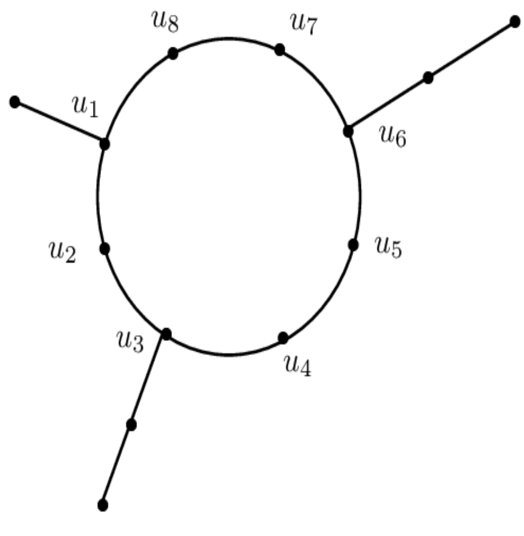

Figure 2. The unicycle graph $U_n = \Gamma_{C_8}(P_3, P_1, P_2, P_1, P_1, P_3, P_1, P_1)$.

Remark. If $P_m : v_1 - v_2 - \cdots - v_m$ is a path with end vertices v_1 and v_m. Then the graph $\Gamma_{p_m}(P_1, \cdots, P_1, P_r)$ is a path on $m + r - 1$ vertices for $r \geq 1$, whereas for $r \geq 2$ the graph $\Gamma_{p_m}(P_1, P_r, P_1 \cdots, P_1)$ is a tree which is not a path. The tree $\Gamma_{p_m}(P_1, P_2, P_1 \cdots, P_1)$ is shown in the Figure 3.

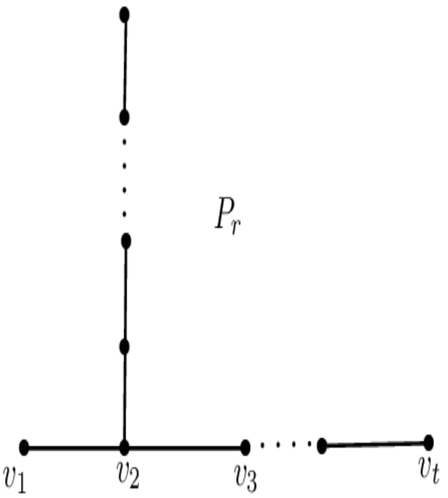

Figure 3. The tree $\Gamma_{p_m}(P_1, P_r, P_1 \cdots, P_1)$.

Remark. It can be easily seen that $\Gamma_G(P_{s_1}, P_{s_2}, \cdots, P_{s_r}) = G \cup_{i=1}^{r} P_{s_i}$ such that $G \cap P_{s_i} = v_i$ for all $i = 1, \cdots, r$ and $P_{s_i} \cap P_{s_j} = \phi$ for all $i \neq j$. Moreover $\Gamma_G(P_{s_1}, P_{s_2}, \cdots, P_{s_r})$ is subgraph of rooted product of $G \circ_a P_t$, where $t = \max_{i=1}^{r} s_i$ and a is a pendant vertex of P_t.

Theorem 2.1. *[9] If a graph G contains pendant vertices, then any mixed metric generator of G must contain all pendant vertices of G.*

Theorem 2.2. *Let $T_n \ncong P_n$ be a tree with $n \geq 3$ vertices. Then $dim_m(T_n) = 3$ if and only if $T_n \cong \Gamma_{P_t}(P_1, \cdots, P_1, P_r, P_1, \cdots, P_1)$, where P_r is attached to a vertex which is not pendant vertex such that $r \geq 2$ and $n = t + r - 1$ and $t \geq 3$.*

Proof. Suppose $dim_m(T_n) = 3$. Since $T_n \ncong P_n$, T_n contains at least three pendant vertices. As from Theorem 2.1, any mixed metric contains all the pendant vertices, T_n has mixed dimension three only if it contains exactly three pendant vertices, say v_1, v_2 and v_3. If v_1, v_2 and v_3 are attached to exactly one support vertex, say s, then we claim that $T_n \cong K_{1,3}$ for otherwise there is a vertex s_1 different from v_1, v_2 and v_3 but adjacent to s. If s_1 is a pendant vertex, then we have four pendant vertex, a contradiction. Thus there must be a vertex adjacent to s_1, say s_2. Now $s_2 \neq v_1, v_2, v_3$ or s for otherwise we have a cycle which contradict the fact that T_n is a tree. Continuing in this way, we have distinct vertices $s_1, s_2, \cdots$. But as we have finite number of vertices, this process must end and there ia an index k such that either $s_k = s_j$ for some $1 \leq j < k$ or $s_k = v_1, v_2, v_3$, or s. In either case, we have a cycle, a contradiction. Hence if there is exactly one support vertex, then $T_n \cong K_{1,3}$. Now we suppose that s and s' are two support vertices attached to vertices v_1 and v_2, respectively. Let $d(s, s') = h$, then as v_1 and v_2 are pendant vertices, so $d(v_1, v_2) = h + 2$. Let $t = h + 2$. Then we have a path P_t of length t from v_1 to v_2. Since T_n is connected and v_3 does not lie on the path P_t so the vertex v_3 is linked to every vertex of path P_t through some path. Now let v_i be the vertex on P_t which is nearer to v_3. Clearly $v_i \neq v_1, v_2$. Let $d(v_i, v_3) = r$ i.e., we have a path P_r whose end vertex v_i coincides with $i - th$ vertex of path P_t. Then the subgraph induced by $V(P_t)$ and $V(P_{r_i})$ is isomorphic to $\Gamma_{P_t}(P_1, \cdots, P_{r_i}, \cdots P_1)$. Clearly v_i is not a pendant vertex of P_t. Now we claim that all the vertices of T_n either lie on path P_t or on P_r. For if $(V(P_t) \cup V(P_{r_i})) \cap V(T_n) \neq \phi$, then there exists a vertex w_1 adjacent to some vertex of

$V(P_t) \cup V(P_{r_i})$. Without loss of generality, assume that w_1 is adjacent to some vertex of P_t. If degree of w_1 is one, than T_n has four pendant vertices, a contradiction. Therefore $deg(w_1) \geq 2$. Since T_n does not contain any cycle, w_1 is not adjacent to any vertex of $V(P_t)$ and $V(P_{r_i})$. Thus we have a vertex $w_2 \in T_n \setminus (V(P_t) \cup V(P_r))$. By continuing this process, we get vertices $w_1, w_2, \cdots \in T_n \setminus (V(P_t) \cup V(P_r))$. Since T_n is finite, there exists some index k such that $w_j = w_k$ for $1 \leq j < k$. But then T_n contains a cycle, a contradiction. Hence $G \cong \Gamma_{P_t}(P_1, \cdots, P_1, P_r, P_1, \cdots, P_1)$ such that P_r is not attached to pendant vertex of P_t and $n = t + r - 1$. □

Corollary 1. *A tree T_n has mixed dimension 3 if and only if degree sequence of T_n is* $(1, 1, 1\, \underbrace{2, \cdots, 2}_{(n-4) times}, 3)$.

Corollary 2. *Given the degree sequence of tree T_n, it is decidable in time $O(1)$ whether T_n has mixed metric dimension three.*

Proof. From Corollary 1, it can be seen that if we have degree sequence of a tree, we only need to check the first four elements and last two elements of degree sequence. Since a tree has always at least two pendant vertices, first two entries will always contain one's for otherwise it is not a degree sequence of a tree. Thus to determine whether a tree has mixed dimension three, we only have to check the third, fourth and last two elements of degree sequence. This completes the proof. □

Lemma 2.3. *Let U be a unicyclic graph with $dim_m(U) = 3$. Then any cycle vertex has degree at most 3 and any non-cycle vertex has degree at most 2.*

Proof. Consider a unicyclic graph U, with a unique cycle of length n. Suppose that there exists a cycle vertex c_1 such that $deg(c_1) > 3$. Now label the cycle vertices as $\{c_1, c_2, \cdots, c_n\}$. Let v_1 and v_2 be two non-cycle vertices attached to cycle vertex c_1. Then there exists two distinct pendant vertices l_1 and l_2 such that $c_1 - v_1 - \cdots - l_1$ and $c_1 - v_2 - \cdots - l_2$ are the shortest paths having lengths r and s, respectively. Then l_1, l_2 must belong to M. Since $dim_m(U) = 3$, there exists $l_1, l_2 \neq m \in M$ such that $M = \{l_1, l_2, m\}$ is a mixed metric basis. First assume that m is any vertex of the maximal subtree containing c_1. Then consider $x = c_1$ and $y = c_1c_2$, where $c_2 \in N(c_1)$ be a cycle vertex. Since the vertex c_1 is more closer to l_1, l_2 and m than c_2, the coordinate vectors of x and y with reference to M are $r(x|M) = (r, s, t) = r(y|M)$, a contradiction. Hence we may assume that m does not belong to the maximal subtree containing c_1.
Now suppose n is odd, then $c = c_{\frac{n-1}{2}}$ and $c' = c_{\frac{n+1}{2}}$ (antipodal vertices) are equidistant from c_1. Without loss of generality, assume $c = c_{\frac{n-1}{2}}$ is more closer to m than c'. Then for a pair of elements $x = c$ and $y = c'c$, $d(m, c) = d(m, cc')$. Since c, c' are antipodal vertices, $d(cc', c_1) = d(c, c_1) = d(c', c_1)$. This further implies that $d(l_1, x) = d(l_1, c_1) + d(c_1, c) = d(l_1, c_1) + d(c_1, cc') = d(l_1, y)$. Similarly $d(l_2, x) = d(l_2, c_1) + d(c_1, c) = d(l_2, c_1) + d(c_1, cc') = d(l_2, y)$, i.e., $r(c|M) = r(y|M)$, a contradiction. Now if n is even then the vertices $c_{\frac{n}{2}}$ and $c_{\frac{n}{2}+2}$ are equidistant from c_1. Clearly $e_1 = c_{\frac{n}{2}}c_{\frac{n}{2}+1}, e_2 = c_{\frac{n}{2}+1}c_{\frac{n}{2}+2} \in E(U)$ and $d(e_1, l_i) = d(e_2, l_i) = d(l_i, c_{\frac{n}{2}}) = d(l_i, c_{\frac{n}{2}+2})$ for $i = 1, 2$. Now if $d(m, c_{\frac{n}{2}+1}) < d(m, c_{\frac{n}{2}}), d(c_{\frac{n}{2}+2}, m)$, then $d(e_1, m) = d(m, e_2)$ which implies that $r(e_1|M) = r(e_2|M)$, a contradiction. Thus without loss of generality, assume that m is more closer to $c_{\frac{n}{2}}$ than $c_{\frac{n}{2}+1}$ and $c_{\frac{n}{2}+2}$. But then $r(e_1|M) = r(c_{\frac{n}{2}}|M)$.
All cases lead to a contradiction. Thus degree of cycle vertex is at most 3.

Now let y be any non-cycle vertex with degree greater than 2. Then there exist at least two distinct pendant vertices l_1, l_2 such that there is exactly one path from y to l_i for $i = 1, 2$. By Theorem,

$l_1, l_2 \in M$. Since $dim_m(U) = 3$, $M = \{l_1, l_2, m\}$ is a mixed metric basis. If c_1 is a first cycle vertex closer to y. Then using the same arguments as above, it can be shown that M is not mixed resolving set. □

Theorem 2.4. *Let U_n be a unicycle graph. Then $dim_m(U_n) = 3$ if and only if either $U_n \cong C_n$ or $U_n \cong \Gamma_{C_m}(P_r, \underbrace{P_1, \cdots, P_1}_{m-1\ times})$, where $m + r - 1 = n$ (i.e., tadpole graph) or $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{m-2-i\ times})$, where $i \geq 0$ and $m + r_1 + r_2 - 2 = n$ or $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{j\ times}, P_{r_3}, \underbrace{P_1, \cdots P_1}_{m-i-j-3\ times})$ such that $m + r_1 + r_2 + r_3 - 3 = n$ and $i + j + 3 \geq s$, where $s = m/2 + 1$ if m is even and $s = (m + 1)/2$ otherwise.*

Proof. Suppose $dim_m(U_n) = 3$. As from Theorem 2.1, any mixed metric contains all the pendant vertices, U_n has metric dimension three only if it contains at most three pendant vertices. Now by Lemma 2.3, degree of each cycle vertex is at most 3. If all the cycle vertices have degree two than $U_n \cong C_n$. Thus we may assume that there exists at least one cycle vertex of degree three. We claim that there are at most three cycle vertices whose degree is 3, for if there are more than three vertices, then there must exists at least four pendant vertices, a contradiction. Hence, there are at most three cycle vertices whose degree is 3.

Let x be the cycle vertex of $deg(x) \geq 3$. Suppose T_x be the subtree attached at x containing unique cycle vertex x. We claim that T_x is a path, for if T_x is not a path then there must exists at least one vertex $w \in T_x$ such that $deg(w) \geq 3$. If $w = x$ then $deg(w)$ in U_n is at least 5, which contradicts Lemma 2.3. If w is a non- cycle vertex with $deg(w) \geq 3$, then it again contradicts Lemma2.3. This further shows that U_n is a graph in which a path is attached to all cycle vertices having degree at least 3. If there is only one cycle vertex with degree three, then it is easy to see that U_n is a tadpole and $U_n \cong \Gamma_{C_m}(P_r, \underbrace{P_1, \cdots, P_1}_{m-1\ times})$. If U_n contains two cycle vertex of degree three, then clearly, $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{m-2-i\ times})$ such that $m + r + t - 2 = n$. Now suppose U_n contains exactly three cycle vertex of degree three. Then $M = \{u, v, w\}$ consisting of pendant vertices and $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{j\ times}, P_{r_3}, \underbrace{P_1, \cdots P_1}_{m-i-j-3\ times})$, where $m + r_1 + r_2 + r_3 - 3 = n$. Let the vertices of C_m are ordered as $v = v_1, \cdots v_m$ such that $u = v_i, w = v_j$. Suppose m is even but both $i, j \leq m/2$. Now consider the edge $e = v_{m/2}v_{m/2+1}$ and the vertex $v_{m/2}$. As $d(v_1, v_{m/2}) = m/2$ and $d(v_1, v_{m/2+1}) = m/2 + 1$, so $d(v_1, e) = d(v_1, v_{m/2})$. Also as $i, j \leq m/2$, so $v_{m/2}$ is closer to $v_i = u$ and $v_j = w$ as compared to $v_{m/2+1}$. Hence $r(v_{m/2}|M) = r(e|M)$, a contradiction. Now suppose m is odd but both $i, j \leq (n + 1)/2$. Using same arguments, it can be shown that $r(v_{(m+1)/2}|M) = r(v_{(m+1)/2}V_{(m+3)/2}|M)$. It completes the proof □

3. Structure of graphs having mixed dimension three

Lemma 3.1. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G. Then $|N(v_i) \cap N(v_j)| \leq 3$ for all distinct pair of vertices $v_i, v_j \in M$. Moreover any two distinct vertices in $N(v_i) \cap N(v_j)$ for $i \neq j$ are not adjacent.*

Proof. Consider distinct pair of vertices $v_i, v_j \in M$. Suppose $d(v_i, v_k) = r$ for $v_k \neq v_i, v_j$. Now for any vertex $w \in N(v_i) \cap N(v_j)$, $d(w, v_i) = 1 = d(w, v_j)$ so we have

$$d(w, v_k) \leq 1 + r. \tag{3.1}$$

Now clearly, $d(w, v_k) \geq r - 1$ for otherwise we have a path $v_i - w - \cdots - v_k$ of length less than r between v_i and v_k, a contradiction. Hence

$$r - 1 \leq d(w, v_k) \leq r + 1. \tag{3.2}$$

This further implies that only possibilities are $d(w, v_k) = r - 1, r, r + 1$. Now if there are more than three vertices in $N(v_i) \cap N(v_j)$, then as $d(w, v_i) = 1 = d(w, v_j)$, the coordinate vectors of at least two vertices in $N(v_i) \cap N(v_j)$ with respect to M are same. This leads to a contradiction.
Now for $i \neq j$, let $x, y \in N(v_i) \cap N(v_j)$ be two distinct vertices such that $x \sim y$. As $d(x, v_i) = d(y, v_i) = 1$ and $d(x, v_j) = d(y, v_j) = 1$. Since M is mixed basis, we have $d(x, v_k) \neq d(y, v_k)$. Suppose $t = d(x, v_k) < d(y, v_k)$. Then $r(x|M) = (1, 1, t)$ and as x is more closer to v_k than y so $d(xy, v_k) = d(x, v_k) = t$. But then $d(xy|M) = (1, 1, t) = d(x|M)$, i.e., x and the edge xy are not resolved by M, a contradiction. □

Lemma 3.2. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$, where $e(v_i)$ is the eccentricity of v_i. Then*
$|V_{1i} \cap V_{2j} \cap V_{3k}| \leq 1$ for all $1 \leq i \leq e(v_1)$, $1 \leq j \leq e(v_2)$ and $1 \leq k \leq e(v_3)$

Proof. Suppose on contrary there exists some integers $r \leq e(v_1)$, $s \leq e(v_2)$ and $t \leq e(v_3)$ such that $|V_{1r} \cap V_{1s} \cap V_{1t}| > 1$. Then there exists $a, b \in V(G)$ such that $a, b \in V_{1r} \cap V_{2s} \cap V_{3t}$. Then clearly

$$d(a, v_1) = r = d(b, v_1),$$
$$d(a, v_2) = s = d(b, v_1),$$
$$d(a, v_3) = t = d(b, v_3),$$

i.e., $r(a|M) = r(b|M)$. This leads to a contradiction. □

Lemma 3.3. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{X_{i0}, X_{i1}, \cdots, X_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. If for some $i, j \in \{1, 2, 3\}$, $0 \leq r \leq e(v_i)$ and $0 \leq s \leq e(v_j)$, the vertices $a, b \in V_{ir} \cap V_{js}$, then a and b are not adjacent.*

Proof. Suppose the vertices a and b are adjacent, i.e., $e = ab \in E(G)$ and $d(a, v_k) = t$ for $v_k \neq v_i, v_j$. Then using the same arguments as used in the proof of Lemma 3.1, it can be seen that $d(b, v_k) = t - 1, t, t + 1$. Now $d(b, v_k) \neq t$ for otherwise $a, b \in V_{ir} \cap V_{js} \cap V_{kt}$ which contradicts Lemma 3.2. If $d(b, v_k) = t - 1$, then b is more closer to v_k than a. In this case, $d(b, v_k) = d(e, v_k)$. Now as $a, b \in V_{ir} \cap V_{js}$, therefore $d(b, v_i) = r = d(e, v_i)$ and $d(b, v_j) = s = d(e, v_j)$, i.e., the vertex b and the edge $e = ab$ are not resolved by M. Similarly if $d(b, v_k) = t + 1$, then it can be shown that the vertex a and the $e = ab$ cannot be distinguished by M. This leads to a contradiction. □

Lemma 3.4. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. If for some $0 \leq r \leq e(v_i)$, $0 \leq s \leq e(v_j)$ and $0 \leq t \leq e(v_3)$, the vertex $a \in V_{1r} \cap V_{2s} \cap V_{3t}$ and $b \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$, then a and b are not adjacent.*

Proof. As $a \in V_{1r} \cap V_{2s} \cap V_{3t}$ and $b \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$, so $d(a, v_i) \leq d(b, v_i)$ for $i = 1, 2, 3$. Thus if a is adjacent to b, i.e., $e = ab \in E(G)$, then $d(e, v_i) = d(a, v_i)$ for all i. This further implies that $r(a|M) = r(e = ab|M)$ which contradicts the fact that M is mixed basis. □

Lemma 3.5. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. If for $a \in V_{1r} \cap V_{2s} \cap V_{3t}$ ($0 \leq r \leq e(v_i)$, $0 \leq s \leq e(v_j)$ and $0 \leq t \leq e(v_3)$), there exist $b, c \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$ such that at least one of b and c must belong to v_{1r}, v_{2s} and v_{3t} then b and c are not adjacent.*

Proof. If a is equal to one of b and c, say $a = c$ then from Lemma 3.4, $a = c$ is not adjacent to b. Thus we may assume that $a \neq b, c$. Suppose on contrary $b \sim c$, i.e., $e = bc \in E(G)$. Now $d(a, v_1) = r$, $d(a, v_2) = s$ and $d(a, v_3) = t$. Since at least one of b and c (end points of edge e) must belong to v_{1r}, v_{2s} and v_{3t}, $d(e, v_1) = r$, $d(e, v_2) = s$ and $d(e, v_3) = t$. This implies that $r(a|M) = (r, s, t) = r(e = ab|M)$, a contradiction to the assumption that M is a mixed basis of G. □

Lemma 3.6. *Let M be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. If for $a, c \in V_i$, there exists $b, d \in (V_{ir} \cup V_{i(r+1)}) \cap (V_{js} \cup V_{j(s+1)}) \cap (V_{kt} \cup V_{k(t+1)})$ ($i \neq j \neq k$) such that $a, d \in V_{js}$ and $b, c \in V_{kt}$, where $0 \leq s \leq e(v_j)$, $0 \leq t \leq e(v_k)$ then either $a \nsim b$ or $c \nsim d$.*

Proof. Suppose on contrary, for $a, c \in V_i$, there exists $b, d \in (V_{ir} \cup V_{i(r+1)}) \cap (V_{js} \cup V_{j(s+1)}) \cap (V_{kt} \cup V_{k(t+1)})$ ($i \neq j \neq k$) such that $a, d \in V_{js}$, $b, c \in V_{kt}$ but $a \sim b$ and $c \sim d$, i.e., $e_1 = ab, e_2 = cd \in E(G)$. Since $a, c \in V_i$, a and c are more closer to v_i, than b and d, respectively. This shows that $d(e_1, v_i) = d(a, v_i) = r$ and $d(e_2, v_i) = d(c, v_i) = r$. Now as $a, d \in V_{js}$, so a and d are more closer to v_j, than b and c, respectively. This implies that $d(e_1, v_j) = d(a, v_j) = s$ and $d(e_2, v_j) = d(d, v_j) = s$. Similarly we can see that $d(e_1, v_k) = d(b, v_k) = t$ and $d(e_2, v_k) = d(c, v_j) = t$. Thus if $M = \{v_i, v_2, v_3$ is ordered mixed basis of G, then $r(e_1|M) = (r, s, t) = r(e_2|M)$, a contradiction. □

Lemma 3.7. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. Then induced subgraph of any distance partition V_{ij} of any vertex $v_i \in M$ is triangle free.*

Lemma 3.8. *Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G and $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ be the distance partition of $V(G)$ with respect to $v_i \in M$. Then the maximum degree of any vertex in induced subgraph of any distance partition V_{ij} is at most 2.*

Lemma 3.9. *Let M be a mixed metric basis of a graph G with $|M| = 3$. Then the maximum degree of any vertex of M is at most 3.*

Proof. Let $v_i \in M$ be any vertex and let $M = \{v_i, v_j, v_k\}$ be ordered basis of G. Suppose $d(v_i, v_j) = r$ and $d(v_i, v_k) = t$. Suppose $a \in N(v_i)$. Then clearly $d(a, v_j) = r, r-1$ or $r+1$ and $d(a, v_k) = t, t-1$ or $t+1$. We claim that if $d(a, v_j) = r$, then $d(a, v_k)$ must be $t-1$ for otherwise $r(v_i|M) = (0, r, t) = r(av_i|M)$, i.e., the vertex v_i and edge av_i are not resolved by M. Similarly if $d(a, v_j) = r+1$, then $d(a, v_k) = t-1$. Also we claim that the distinct vertices $a, c \in N(v_i)$ with representation $r(a|M) = (1, r, t-1)$ and $r(c|M) = (1, r+1, t-1)$ do not occur simultaneously for otherwise $a, c \in V_{k(t-1)} \cap V_{i1}$, $b = v_i = d \in V_{kt} \cap V_{i0}$ and $a, d \in V_{jr}$. By Lemma 3.3, either a is not adjacent to $b = v_i$ or c is not adjacent to $d = v_i$, a contradiction as both $a, c \in N(v_i)$. Now we claim that there are at most two vertices in $N(v_i)$

with distance $r-1$ from v_j, for if there are at least three distinct vertices $a, b, c \in N(v_j)$, then their distance representation must be $(1, r-1, t-1)$, $(1, r-1, t)$ and $(1, r-1, t+1)$, respectively. But then $b, c \in V_{j(r-1)} \cap V_{i1}$, $e = v_i = f \in V_{jr} \cap V_{i0}$ and $b, e, f \in V_{kt}$. By Lemma 3.3, either b is not adjacent to $e = v_i$ or c is not adjacent to $f = v_i$, a contradiction as both $b, c \in N(v_i)$. Thus $N(v_i)$ can contain at most three vertices; maximum 2 with distance $r-1$ and one with distance r or $r+1$ (but not both). Thus maximum number of vertices in $N(v_i)$ is 3. □

Theorem 3.10. *Let G be a connected graph which is not a tree having mixed metric dimension* 3 *and diameter D. Then G contains at most* $(D^3 + 12D^2)/2$ *vertices.*

Proof. For each vertex except basis vertex, the associated coordinate vector with positive coordinates (α, β, γ), where $1 \le \alpha, \beta, \gamma \le D$ can be chosen in one of D^3 ways. The coordinate vector corresponding to basis elements are $(0, \beta_1, \gamma_1)$, $(\alpha_2, 0, \gamma_2)$ and $(\alpha, \beta, 0)$ for some $1 \le \alpha, \beta, \gamma \le D$. On the other hand, the coordinate vectors are also assigned to edges and these must be distinct from the coordinate vectors of vertices. All the edges except those which are incident with basis elements have positive integers in each coordinate of coordinate vectors. Therefore, these again must be chosen from possible D^3 coordinate vectors. From Lemma 3.9, the maximum number of edges incident to three basis vertices are 9. Hence maximum number of coordinate vectors which can be assigned to edges and vertices must not exceed $D^3 + 3D^2 + 9D^2$. Then this further implies that

$$|V(G)| + |E(G)| \le D^3 + 12D^2 \tag{3.3}$$

As G is connected but not a tree, so it must contain at least $|V(G)|$ edges. Thus (3.3) becomes

$$\begin{aligned} |V(G)| &\le D^3 + 12D^2 - |E(G)| \\ &\le D^3 + 12D^2 - |V(G)| \end{aligned}$$

Hence

$$|V(G)| \le (D^3 + 12D^2)/2$$

□

Theorem 3.11. *Let G be a graph with* $n \ge 3$ *vertices such that G is not a tree and* $\{V_{i0}, V_{i1}, \cdots, V_{ie(v_i)}\}$ *be the distance partition of* $V(G)$ *with respect to* $v_i \in V(G)$*, where* $e(v_i)$ *is the eccentricity of* v_i*. Then mixed metric dimension of G is three if and only if there exists three vertices* v_1, v_2 *and* v_3 *which satisfy the following conditions:*

1. $|V_{1i} \cap V_{2j} \cap V_{3k}| \le 1$ *for all* $1 \le i \le e(v_1)$, $1 \le j \le e(v_2)$ *and* $1 \le k \le e(v_3)$.
2. *If for* $a \in V_{1r} \cap V_{2s} \cap V_{3t}$ $(0 \le r \le e(v_i)$, $0 \le s \le e(v_j)$ *and* $0 \le t \le e(v_3))$, *there exists* $b, c \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$ *such that at least one of b and c must belong to* v_{1r}, v_{2s} *and* v_{3t} *then b and c are not adjacent.*
3. *If for* $a, c \in V_i$, *there exists* $b, d \in (V_{ir} \cup V_{i(r+1)}) \cap (V_{js} \cup V_{j(s+1)}) \cap (V_{kt} \cup V_{k(t+1)})$ $(i \ne j \ne k)$ *such that* $a, d \in V_{js}$ *and* $b, c \in V_{kt}$, *where* $0 \le s \le e(v_j)$, $0 \le t \le e(v_k)$ *then either* $a \nsim b$ *or* $c \nsim d$.

Proof. Let $M = \{v_1, v_2, v_3\}$ be a mixed metric basis of a graph G. suppose on contrary condition (1) is not satisfied. Then there exists some integers $r \leq e(v_1)$, $s \leq e(v_2)$ and $t \leq e(v_3)$ such that $|V_{1r} \cap V_{1s} \cap V_{1t}| > 1$. Then there exists $a, b \in V(G)$ such that $a, b \in V_{1r} \cap V_{2s} \cap V_{3t}$. Then clearly

$$d(a, v_1) = r = d(b, v_1),$$
$$d(a, v_2) = s = d(b, v_1),$$
$$d(a, v_3) = t = d(b, v_3),$$

i.e., $r(a|M) = r(b|M)$. This leads to a contradiction. Hence condition (1) holds. Also conditions (2) and 3 are satisfied from Lemma 3.5 and Lemma 3.6.

Conversely suppose that there exists three vertices v_1, v_2 and v_3 that satisfy all the conditions. We will show that the set $M = \{v_1, v_2, v_3\}$ is mixed metric generator for G. Suppose on contrary there exists distinct elements $x, y \in V(G) \cup E(G)$ such that x and y are not resolved by any member of M. Then

$$d(x, v_1) = d(y, v_1) = r, d(x, v_2) = d(y, v_2) = s, d(x, v_3) = d(y, v_3) = t. \tag{3.4}$$

If $x, y \in V(G)$, then $x, y \in V_{1r} \cap V_{2s} \cap V_{3t}$, i.e., $|V_{1r} \cap V_{2s} \cap V_{3t}| \geq 2$, which contradicts condition (1). Hence for at least one $v_i \in M$, $d(x, v_i) \neq d(y, v_i)$. Now suppose $x \in V(G)$ and $y = y_1y_2 \in E(G)$. Then from (3.4), we have $x \in V_{1r} \cap V_{2s} \cap V_{3t}$ and $y_1, y_2 \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$ such that at least one of y_1 and y_2 must belong to V_{1r}, V_{2s} and V_{3t}. But then from condition (2), $y_1 \nsim y_2$, which contradicts the assumption that $y = y_1y_2$ is an edge. Finally, assume that $x = x_1x_2$ and $y = y_1y_2$ are two distinct edges. There arise two cases:
Case 1: The edges x and y are not adjacent, i.e., $x_1 \neq x_2 \neq y_1 \neq y_2$. Then from (3.4),

$$x_1, x_2, y_1, y_2 \in (V_{1r} \cup V_{1(r+1)} \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)}). \tag{3.5}$$

Consider the edge $x = x_1x_2$, by pigeon hole principle, one of x_1 and x_2 is more closer to two of v_1, v_2 and v_3. Without lose of generality, assume that x_1 is more closer to v_1 and v_2 than x_2, i.e., $d(x, v_1) = d(x_1, v_1) = r$ and $d(x, v_2) = d(x_1, v_2) = s$. This implies that

$$x_1 \in V_{1r} \cap V_{2s}. \tag{3.6}$$

Now we claim that $x_1 \notin V_{3t}$ for otherwise $x_1 \in V_{1r} \cap V_{2s} \cap V_{3t}$ and $x_2 \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$. Then by condition (2), for $a = x_1 = b$ and $c = x_2$, $x_1 = b \nsim c = x_2$, which contradicts the assumption that $x = x_1x_2$ is an edge. Thus

$$x_2 \in V_{3t}, x_1 \in V_{3(t+1)}. \tag{3.7}$$

Now consider the edge $y = y_1y_2$ and again by similar arguments, one of the end points of edge y, say, y_1 is more closer to two of members of M than y_2. There arise the following sub-cases:
Case 1a:
If y_1 is more closer to v_1 and v_2 as compared to y_2, then $y_1, x_1 \in V_{1r} \cap V_{2s}$. Again by using condition (2), it can be shown that $y_2 \in V_{3t}$ and $y_1 \in V_{3(t+1)}$. But these along with Eq (3.7) imply that $x_1, y_1 \in V_{3(t+1)}$ so that $x_1, y_1 \in V_{1r} \cap V_{2s} \cap V_{3(t+1)}$. This contradicts condition (1).

Case1b:
Now suppose $y_1 \in V_{3t}$ and y_1 belong to exactly one of V_{1r} and V_{2r}. Without loss of generality assume that $y_1 \in V_{1r}$. Then

$$y_1 \in V_{1r} \cap V_{3t}. \tag{3.8}$$

But by using condition (2), we have

$$y_2 \in V_{2s}, y_1 \in V_{2(s+1)}. \tag{3.9}$$

Now from Eqs (3.5), (3.6), (3.7), (3.8) and (3.9), we have for $x_1, y_1 \in V_{1r}$, there exists $x_2, y_2 \in (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$ such that $x_1, y_2 \in V_{2s}$ and $x_2, y_1 \in V_{3t}$. But then from condition (3), either $x_1 \nsim x_2$ or $y_1 \nsim y_2$, a contradiction.
Case 2:
Suppose the edges $x = x_1x_2$ and y_1y_2 are adjacent with their common vertex $x_1 = y_1$. Then from (3.4),

$$x_1, x_2, y_2 \in (V_{1r} \cup V_{1(r+1)} \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)}). \tag{3.10}$$

We claim that the common vertex x_1 is nearer to at least two of the vertices of M as compared to other vertices x_2 and y_2 for otherwise from (3.10), either $x_1 \in V_{1(r+1)} \cap V_{2(s+1)} \cap V_{3(t+1)}$ or x_1 belongs to exactly one of V_{1r}, V_{2s} and V_{3t}, say V_{1r}. If $x_1 \in V_{1(r+1)} \cap V_{2(s+1)} \cap V_{3(t+1)}$, then from 3.4, $d(x_2, v_1) = d(y_2, v_1) = r$, $d(x_2, v_2) = d(y_2, v_2) = s$ and $d(x_2, v_3) = d(y_2, v_3) = t$. This further implies that $x_2, y_2 \in V_{1r} \cap V_{2s} \cap V_{3t}$, a contradiction. Also if $x_1 \in V_{1r}$ but $x_1 \notin V_{2s}, V_{3t}$, then as from (3.4), x and y are at same distance from v_2 and v_3, and from (3.10), the only possibility is

$$x_2, y_2 \in V_{2s} \cap V_{3t}. \tag{3.11}$$

Now we claim that none of x_2 and y_2, belongs to V_{1r} for if one of x_2 and y_2, say x_2 belong to V_{1r}, then from (3.11), $x_2 \in V_{1r} \cap V_{2s} \cap V_{3t}$ and $x_1, x_2 \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3(t+1)} \cup V_{V_{3t}}$. Using condition (2), $x_1 \nsim x_2$, a contradiction as $x = x_1x_2$ is an edge. Hence none of x_2 and y_2, belongs to V_{1r}. Then from (3.10), $x_2, y_2 \in V_{1(r+1)}$. But then $x_2, y_2 \in V_{1(r+1)} \cap V_{2s)} \cap V_{3t}$ which contradicts condition (1). Hence x_1 is closer to two of the vertices of M, say v_1 and v_2 as compared to vertices x_2 and y_2. Now using condition (2) and (3.10), we can write

$$x_1 \in V_{3(t+1)}, x_2, y_2 \in V_{3t} \tag{3.12}$$

Also from (3.10), $x_2, y_2 \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)})$. If $x_2, y_2 \in V_{1(r+1) \cap V_{2(s+1)}}$, then this along with (3.12) imply that $x_2, y_2 \in V_{1(r+1) \cap V_{2(s+1)}} \cap V_{3t}$. This contradicts condition (1). Therefore one of x_2 and y_2 must belong to one of V_{1r} and V_{2s}. Suppose without loss of generality that $x_2 \in V_{1r}$. Consider $y_1 = x_1, x_2 \in V_{1r}$, then there exists $x_1, y_2 \in (V_{1r} \cup V_{1(r+1)}) \cap (V_{2s} \cup V_{2(s+1)}) \cap (V_{3t} \cup V_{3(t+1)})$ such that $y_1, x_1 \in V_{2s}$ and $x_2, y_2 \in V_{3t}$. By condition (3), either $y_1 = x_1 \nsim toy_2$ or $x_2 \nsim x_1$. This leads to a contradiction as $x = x_1x_2$ and $y_1y_2 = x_1y_2$ are edges.

All the possible cases lead to a contradiction. Hence M is a mixed metric generator. Therefore $dim_m(G) \leq 3$. As $G \ncong P_n$, $dim_m \geq 3$. Hence $dim_m(G) = 3$. □

4. Development of algorithms

In this section, three algorithms are developed for determine whether or not “a tree, unicyclic graph and in general a simple connected graphs has mixed dimension three?”. The algorithm for tree to have mixed metric dimension three using Theorem 2.2 and Corollary 1 is given in Algorithm 1.

Theorem 4.1. *The complexity of the algorithm to determine a mixed basis of dimension three for a graph J of order n and diameter D is* $D^3(\frac{n(n-1)^4(n-2)}{6})$.

Proof. It is noted that the number of sets in any distance partition of $V(J)$ is maximum δ. In the proposed algorithm, three distance partitions of $V(J)$ with respect to three distinct vertices are considered at each step. Each set in one partition is compared with two distinct sets from two other partitions(one from each partition). Thus total number of set comparisons is $D^3(\frac{n(n-1)(n-2)}{6})$. Moreover, in each comparison of three sets, the element wise comparison is at most $(n-1)^3$. Thus the complexity of the algorithm is $D^3(\frac{n(n-1)^4(n-2)}{6})$. □

Lemma 2.3 and Theorem 2.4 are used to develop an algorithm to find unicyclic graphs having mixed dimension 3. It is given in Algorithm 2.

The Lemma 3.9, Lemma3.1, Theorem 3.10, and Theorem 3.11 provide the criteria to determine whether a simple connected graph has mixed metric dimension 3 or not. If a graph has mixed metric dimension three, then Theorem 3.11 also provide the mixed metric basis for the graph. With the help of these results, an algorithm is developed which determines whether a graph has mixed metric dimension three or not and if it exists then it also finds its mixed basis. The algorithm requires the distance matrix of a simple connected graph which is not a path graph. It results in a mixed basis of dimension three of graph, if exists. Otherwise indicates that graph has mixed metric greater than three.

Algorithm 1 Determination of Tree having mixed dimension 3.

Require: Degree sequence S of T_n
Ensure:“Mixed Basis with three elements” or “the graph has mixed dimension 2 or greater than 3”
2. **if**$(S(3) > 1) \vee (n = 2)$ **then**
3. Tree is path graph having mixed dimension 2
4. **else if**$(S(3) = 1) \wedge (S(4)) > 1 \wedge (S(n) = 3) \wedge (S(n-1) < 3)$ **then**
5. Mixed metric basis consists of three pendant vertices 6. **else**
7. Tree has mixed dimension greater than 3
8. **end if**

Algorithm 2 Determination of Unicycle graph having mixed dimension 3.

Require: cycle length r, Adjacency matrix of U_n such that first r rows are cycle vertices
Ensure:"Mixed Basis with 3 elements" or " $dim_m(U_n) \geq 3$ "
2. $n \leftarrow$ number of rows of adjacency matrix
3. **for** $i = 1 : n$ **do** 4. $s \leftarrow$ number of one's in $i - th$ row
5. **if** $s = 3$ **then** 6. $S \leftarrow$ the set of cycle vertices i with degree 3
7. **else if** $s = 1$ **then**
8. $P \leftarrow$ the set of pendant vertices i
9. **else if** $s > 3$ **then**
10. $L \leftarrow$ the set of vertices i with degree greater than 3
11. **end if**
12. **end for**
13. **if** $|P| = 0 = |S| = |L|$ **then**
14. print "$M = \{1, \lceil r/2 \rceil, \lfloor (r+4)/2 \rfloor\}$ is mixed metric basis"
15. **else if** $(1 \leq |P| \leq 3) \wedge (1 \leq |S| \leq 3) \wedge (|L| = 0)$ **then**
16. $C(1) \leftarrow S(1)$ *Relabeling of cycle starting from $S(1)$ (first cycle vertex of degree three) in array C
17. **for** $k = 1 : r$ **do**
18. **if** $C(k) \leq r - 1$ **then**
19. $C(k+1) \leftarrow C(k) + 1$
20. **else**
21. $C(k+1) \leftarrow C(k) + 1 - r$
22. **end if**
23. **end for**
24. **if** $(|P| = 1 = |S|) \wedge (|L| = 0)$ **then**
25. Print "$M \leftarrow \{C(1), C(\lceil r/2 \rceil), C(\lfloor (r+4)/2 \rfloor)\}$ is mixed metric basis"
26. **else if** $(|P| = 2) \wedge (|S \cap C| = 2) \wedge (|L| = 0)$ **then**
27. **for** $j = 1 : r$ **do**
28. **if** $C(j) = S(2) \wedge (j \leq \lceil r/2 \rceil)$ **then**
29. Print "$M = P(1), P(2), C(\lfloor (r+4)/2 \rfloor)$ is mixed basis"
30. **else**
31. Print "$M = P(1), P(2), C(\lceil r/2 \rceil)$ is mixed basis"
32. **end if**
33. **end for**
34. **else if** $(|P| = 3) \wedge (|S \cap C| = 3) \wedge (|L| = 0)$ **then**
35. **for** $j = 1 : r$ **do**
36. **if** $(S(2), S(3) \in C$ for $j \geq \lceil (r+4)/2 \rceil)) \vee (S(2), S(3) \in C$ for $j \leq \lceil r/2 \rceil)$ **then**
37. Print "Graph has mixed dimension greater than three"
38. **else**
39. Print "$M = \{P(1), P(2), P(3)\}$ is mixed basis"
40. **end if**
41. **end for**
42. **end if**
43. **else**
44. Print "Graph has mixed dimension greater than 3" 45. **end if**

Algorithm 3 Graph with mixed dimension 3.

Require: $(n \times n)$ distance matrix D of graph $G \ncong P_n$

Ensure:"Mixed Basis with three elements" or "the graph has mixed dimension greater than 3"

$n \leftarrow$ number of rows of distance matrix

$d \leftarrow$maximum entry in distance matrix

$r \leftarrow$ No. of rows with at most three $1's$. (To determine elements with degrees at most 3)

if$(n \leq (d^3 + 12d)/2) \wedge (r \geq 3)$**then**

Construct the set $W \leftarrow \{j : Ones(D(j, :)) \leq 3\}$

for$j, k \in W$ **do**

$N_j \leftarrow \{i : D(j, i) = 1\}$,

$N_k \leftarrow \{i : D(k, i) = 1\}$

if $|N_j \cap N_k| > 3$ or for $x \in N(j), y \in N(k,)\ D(x, y) = 1$ **then**

$W \leftarrow W \setminus \{i, j\}$

end if

end for

for $i, j, k \in W, a = 1 : \max(D(i, :)), b = 1 : \max(D(j, :)), c = 1 : \max(D(k, :))$ **do**

$V_{ia} = \{s : D(s, j) = a\}, V_{jb} = \{s : D(s, j) = b\}, V_{kc} = \{s : D(s, j) = c\}$

$V_{i(a+1)} = \{s : D(s, j) = a + 1\}, V_{j(b+1)} = \{s : D(s, j) = b + 1\}, V_{k(c+1)} = \{s : D(s, j) = c + 1\}$

$U \leftarrow (V_{ia} \cup V_{i(a+1)}) \cap (V_{jb} \cup V_{j(b+1)}) \cap (V_{kc} \cup V_{k(c+1)})$

if $V_{ia} \cap V_{jb} \cap V_{kc} \geq 2$ **then**

$W \leftarrow W \setminus \{i, j, k\}$

else

for$(s \in V_{ia} \cap V_{jb} \cap V_{kc}) \wedge (t, u \in U)$ **do**

$d = D(u, t)$

if $((u \vee t) \in (V_{ia} \wedge V_{jb} \wedge V_{kc})) \wedge (d = 1)$ **then**

$W \leftarrow W \setminus \{i, j, k\}$

end if

end for

for $(u, v \in V_{ia}) \wedge (w, x \in U)$ **do**

$d1 = D(u, w), d2 = D(v, x)$

if $(u, x \in V_{jb}) \wedge (v, w \in V_{kc}) \wedge (d1 = 1) \wedge (d2 = 1)$ **then**

$W \leftarrow W \setminus \{i, j, k\}$

end if

end for

end if

end if

if$|W| \geq 3$ **then**

List all 3-subsets of W. Each 3-subset is possible mixed basis for G

else

Print "Graph has mixed metric basis greater than 3"

end if

else

Print "Graph has mixed metric dimension greater than 3"

end if

5. Conclusions

In this paper, it is shown that the mixed metric dimension of tree $T_n \ncong P_n$ ($n \geq 3$) is 3 if and only if $T_n \cong \Gamma_{P_t}(P_1, \cdots, P_1, P_r, P_1, \cdots, P_1)$, where P_r is attached to a vertex which is not pendant vertex such that $r \geq 2$ and $n = t + r - 1$ and $t \geq 3$. For unicycle graph U_n, it is proved that $dim_m(U_n) = 3$ if and only if either $U_n \cong C_n$ or $U_n \cong \Gamma_{C_m}(P_r, \underbrace{P_1, \cdots, P_1}_{m-1\ times})$, where $m + r - 1 = n$ (i.e., tadpole graph) or $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{m-2-i\ times})$, where $i \geq 0$ and $m + r_1 + r_2 - 2 = n$ or $U_n \cong \Gamma_{C_m}(P_{r_1}, \underbrace{P_1 \cdots, P_1}_{i\ times}, P_{r_2}, \underbrace{P_1, \cdots P_1}_{j\ times}, P_{r_3}, \underbrace{P_1, \cdots P_1}_{m-i-j-3\ times})$ such that $m + r_1 + r_2 + r_3 - 3 = n$ and $i + j + 3 \geq s$, where $s = m/2 + 1$ if m is even and $s = (m + 1)/2$ otherwise. Moreover, several lemmas related to order of graph, maximum degree of basis elements and distance partite sets of basis elements are presented. These lemmas are then used to find the necessary and sufficient conditions for a graph to have mixed metric dimension 3. Finally, three separate algorithms are developed for tree, unicyclic graphs and in general for simple connected graph $J_n \ncong P_n$ with $n \geq 3$ to determine "whether these graphs have mixed dimension three or not?". If these graphs have mixed dimension three, then these algorithms provide the mixed basis of input graphs.

Acknowledgments

The authors appreciate the valuable comments and remarks of anonymous referees which helped to greatly improve the quality of the paper. The corresponding author(Muhammad Javaid) is supported by the Higher Education Commission of Pakistan through the National Research Program for Universities (NRPU) Grant NO. $20 - 16188/NRPU/R\&D/HEC/20212021$.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. U. Ahmad, S. Ahmed, Muhammad Javaid, M. N. Alam, Computing Fault-tolerant Metric Dimension of Connected Graphs, *J. Math.*, **2022** (2022), Article ID 9773089. https://doi.org/10.1155/2022/97730892022
2. R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, *Math. Bohem.*, **128** (2003), 325–364. https://doi.org/10.21136/MB.2003.134179
3. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.,* **105** (2000), 99–113.
4. G. Chartrand, V. Saenpholphat, P. Zhang, The independent resolving number of a graph, *Math. Bohem.*, **128** (2003), 379–393. https://doi.org/10.21136/MB.2003.134003

5. A. Estrada-Moreno, J. A. Rodriguez-Velazquez, I. G. Yero, *The k-metric dimension of a graph*, *Appl. Math. Inf. Sci.*, **9** (2015), 2829–2840.
6. M. M. Danasa, J. Kraticab, A. Savic, Z. Maksimovic, Some new general lower bounds for mixed metric dimension of graphs, *Filomat*, **35** (2021). https://doi.org/10.2298/FIL2113275M
7. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, **2** (1976), 191–195.
8. A. Kelenc, N. Tratnik, I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, *Discrete Appl. Math.*, **251** (2018), 204–220. https://doi.org/10.1016/j.dam.2018.05.052
9. A. Kelenc, D. Kuzia, A. Taranenko, I. G. Yero, Mixed metric dimension of graphs, *Appl. Math. Comput.,* **314** (2017), 429–438. https://doi.org/10.1016/j.amc.2017.07.027
10. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.,* **70** (1996), 217–229.
11. O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, *Discret Appl. Math.*, **155** (2007), 356–364. https://doi.org/10.1016/j.dam.2006.06.009
12. F. Okamoto, B. Phinezyn, P. Zhang, The local metric dimension of a graph, *Math. Bohem.*, **135** (2010), 239–255. https://doi.org/10.21136/MB.2010.140702
13. H. Razza, Y. Ji, Computing the Mixed Metric Dimension of a Generalized Petersen Graph $P(n, 2)$, *Front. Phys.*, 28 July 2020. https://doi.org/10.3389/fphy.2020.00211
14. P. J. Slater, Leaves of trees, Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congressus Numerantium*, **14** (1975), 549–559.
15. J. Sedlar, R. Škrekovsk, Mixed metric dimension of graphs with edge disjoint cycles, *Discrete Appl. Math.*, **300** (2021), 1–8. https://doi.org/10.1016/j.dam.2021.05.004
16. G. Sudhakara, A. R. Hemanth Kumar, Graphs with metric dimension two a characterization, *Adv. Appl. Discrete Math.*, **4** (2009), 169–186.