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Research article

Solution of integral equations for multivalued maps in fuzzy b-metric spaces using Geraghty type contractions

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Abstract: In this article, the notion of Hausdorff fuzzy b-metric space is studied. Some fixed point results for multivalued mappings using Geraghty type contractions in G-complete fuzzy b-metric spaces are established. To strengthen the results, an illustrative example is furnished. A fuzzy integral inclusion is constructed as an application of fixed point result which shows the validity of the proved results. The presented outcomes are the generalization of the existing results in literature.

Keywords: Geraghty type contractions, Hausdorff fuzzy b-metric space, fuzzy b-metric space, fuzzy metric space
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1. Introduction and preliminaries

Fixed point theory plays a fundamental role in mathematics and applied science, such as optimization, mathematical models and economic theories. Also, this theory has been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches of mathematics, see [1, 2]. A prominent result in fixed point theory is the Banach contraction principle [3]. Since the appearance of this principle, there has been a lot of activity in this area. Bakhtin [4] in 1989 introduced the notion of a b-metric space (Bms). Shoaib et. al [5] proved certain fixed point results in rectangular metric spaces. Multivalued mappings in various types of metric spaces have been extensively studied by many researchers to establish fixed point results and their applications, see for instance [6–12].

In 1965, Zadeh [13] introduced the concept of a fuzzy set theory to deal with the unclear or inexplicit situations in daily life. Using this theory, Kramosil and Michálek [14] defined the concept of a fuzzy metric space (Fms). Grabiec [15] gave contractive mappings on a Fms and extended fixed point theorems of Banach and Edelstein in such a space. Successively, George and Veeramani [16] slightly modified the notion of a Fms introduced by Kramosil and Michálek [14] and then obtained a Hausdorff topology and a first countable topology on it. Many fixed point results have been established in a Fms. For instance, see [17–25] and the references therein. Recently, some coupled fuzzy fixed-point results on closed ball are established in fuzzy metric spaces [26]. The notion of generalized fuzzy metric spaces is studied in [27].

The notion of a fuzzy b-metric space (Fbms) was defined in [28]. The notion of a Hausdorff Fms is introduced in [29]. Fixed point theory for multivalued mapping in fuzzy metric spaces has been extended in many directions. For a multivalued mapping (Mvp) in a complete Fms, some fixed point results are establish in [30]. Some fixed point results for a Mvp in a Hausdorff fuzzy b-metric space (Hfbms) are proved in [31]. In this article, we prove some fixed point results for a Mvp using Geraghty type contractions in a Hfbms. Results in [31, 32] and [30] turn out to be special cases of our results.

Throughout the article, $\mho$ refers to a non-empty set, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ corresponds to the collection of real numbers, $CB(\mho)$ and $\hat{C}_0(\mho)$ represent the collection of closed and bounded subsets and compact subsets of $\mho$, respectively.

Let us have a look at some core concepts that will be helpful for the proof of our main results.

Definition 1.1. [33] For a real number $b \geq 1$, the triplet $(\mho, \Theta_{fb}, *)$ is called a Fbms on $\mho$ if for all $\psi_1, \psi_2, \psi_3 \in \mho$ and $\gamma > 0$, the following axioms hold, where $*$ is a continuous t-norm and Θ_{fb} is a fuzzy set on $\mho \times \mho \times (0, \infty)$:

$[Fb1 :]$ $\Theta_{fb}(\psi_1, \psi_2, \gamma) > 0$;

$[Fb2 :]$ $\Theta_{fb}(\psi_1, \psi_2, \gamma) = 1$ if and only if $\psi_1 = \psi_2$;

$[Fb3 :]$ $\Theta_{fb}(\psi_1, \psi_2, \gamma) = \Theta_{fb}(\psi_2, \psi_1, \gamma)$;

$[Fb4 :]$ $\Theta_{fb}(\psi_1, \psi_3, b(\gamma + \beta)) \geq \Theta_{fb}(\psi_1, \psi_2, \gamma) * \Theta_{fb}(\psi_2, \psi_3, \beta)\ \forall\ \gamma, \beta \geq 0$;

$[Fb5 :]$ $\Theta_{fb}(\psi_1, \psi_2, .) \colon (0, \infty) \to [0, 1]$ is left continuous.

The notion of a Fms in the sense of George and Veeramani [16] can be obtained by taking $b = 1$ in the above definition.

Example 1.1. For a Bms $(\mho, \Theta_b, \wedge)$, define a mapping $\Theta_{fb} \colon \mho \times \mho \times (0, \infty) \to [0, 1]$ by

$$\Theta_{fb}(\psi_1, \psi_2, \gamma) = \frac{\gamma}{\gamma + d_b(\psi_1, \psi_1)}.$$

Then $(\mho, \Theta_{fb}, \wedge)$ is a Fbms.

Following Grabiec [15], the notions of G-Cauchyness and completeness are defined as follows:

Definition 1.2. [15]

(i) If for a sequence $\{\psi_n\}$ in a Fbms $(\mho, \Theta_{fb}, *)$, there is $\psi \in \mho$ such that

$$\lim_{n\to\infty} \Theta_{fb}(\psi_n, \psi, \gamma) = 1, \ \forall\ \gamma > 0,$$

then $\{\psi_n\}$ is said to be convergent.

(ii) If for a sequence $\{\psi_n\}$ in a Fbms $(\mho, \Theta_{fb}, *)$, $\lim_{n\to\infty} \Theta_{fb}(\psi_n, \psi_{n+q}, \gamma) = 1$ then $\{\psi_n\}$ is a G-Cauchy sequence for all $\gamma > 0$ and positive integer q.

(iii) A Fbms is G-complete if every G-Cauchy sequence is convergent.

Definition 1.3. [30] Let B be any nonempty subset of a Fms $(\mho, \Theta_{fb}, *)$ and $\gamma > 0$, then we define $F_{\Theta_{fb}}(\varrho_1, B, \gamma)$, the fuzzy distance between an element $\varrho_1 \in \mho$ and the subset B, as follows:

$$F_{\Theta_{fb}}(\varrho_1, B, \gamma) = \sup\{\Theta_f(\varrho_1, \varrho_2, \gamma) \colon \varrho_2 \in B\}.$$

Note that $F_{\Theta_{fb}}(\varrho_1, B, \gamma) = F_{\Theta_{fb}}(B, \varrho_1, \alpha)$.

Lemma 1.1. *[31] Consider a Fbms $(\mho, \Theta_{fb}, *)$ and let $CB(\mho)$ be the collection of closed bounded subsets of $\mho$. If $A \in CB(\mho)$ then $\psi \in A$ if and only if $F_{\Theta_{fb}}(A, \psi, \gamma) = 1\ \forall\ \gamma > 0$.*

Definition 1.4. [31] Let $(\mho, \Theta_{fb}, *)$ be a Fbms. Define $\mathcal{H}_{F_{\Theta_{fb}}}$ on $\hat{C}_0(\mho) \times \hat{C}_0(\mho) \times (0, \infty)$ by

$$\mathcal{H}_{F_{\Theta_{fb}}}(A, B, \gamma) = \min\{ \inf_{\psi\in A} F_{\Theta_{fb}}(\psi, B, \gamma), \inf_{\varrho\in B} F_{\Theta_{fb}}(A, \varrho, \gamma)\},$$

for all $A, B \in \hat{C}_0(\mho)$ and $\gamma > 0$.

For Geraghty type contractions, follow [33] to define a class $\mathbb{F}_{\Theta b}$ of all functions $\beta : [0, \infty) \to [0, \frac{1}{b})$ for $b \geq 1$, as

$$\mathbb{F}_{\Theta b} = \left\{\beta : [0, \infty) \to [0, \tfrac{1}{b}) \mid \lim_{n\to\infty} \beta(\gamma_n) = \tfrac{1}{b} \Rightarrow \lim_{n\to\infty} \gamma_n = 0\right\}. \tag{1.1}$$

Lemma 1.2. *[31] Let $(\mho, \Theta_{fb}, *)$ be a G-complete Fbms. If $\psi, \varrho \in \mho$ and for a function $\beta \in \mathbb{F}_{\Theta_{fb}}$*

$$\Theta_{fb}(\psi, \varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) \geq \Theta_{fb}\,(\psi, \varrho, \gamma)\,,$$

then $\psi = \varrho$.

Lemma 1.3. *[31] Let $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ be a Hfbms where $(\Theta_{fb}, *)$ is a Fbm on $\mho$. If for all $A, B \in \hat{C}_0(\mho)$, for each $\psi \in A$ and for $\gamma > 0$ there exists $\varrho_\psi \in B$, satisfying $F_{\Theta_{fb}}(\psi, B, \gamma) = \Theta_{fb}(\psi, \varrho_\psi, \gamma)$, then*

$$\mathcal{H}_{F_{\Theta_{fb}}}(A, B, \gamma) \leq \Theta_{fb}(\psi, \varrho_\psi, \gamma).$$

2. Main results

In this section, we develop some fixed point results by using the idea of a Hfbms. Furthermore, an example is also presented for a deeper understanding of our results.

Recall that, given a multivalued mapping $\Xi : \mho \to \hat{C}_0(\mho)$, a point ψ is said to be a fixed point of Ξ if $\psi \in \Xi\psi$.

Theorem 2.1. *Let* $(\mho, \Theta_{fb}, *)$ *be a G-complete Fbms and* $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ *be a Hfbms. Let* $\Xi\colon \mho \to \hat{C}_0(\mho)$ *be a Mvp satisfying*

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) \geq \Theta_{fb}(\psi, \varrho, \gamma), \tag{2.1}$$

for all $\psi, \varrho \in \mho$*, where* $\beta \in \mathbb{F}_{\Theta_{fb}}$ *as defined in* (1.1)*. Then* Ξ *has a fixed point.*

Proof. Choose $\{\psi_n\}$ for $\psi_0 \in \mho$ as follows: Let $\psi_1 \in \mho$ such that $\psi_1 \in \Xi\psi_0$ by the application of Lemma 1.3, we can choose $\psi_2 \in \Xi\psi_1$ such that for all $\gamma > 0$,

$$\Theta_{fb}(\psi_1, \psi_2, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0, \Xi\psi_1, \gamma).$$

By induction, we have $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma) \ \forall \ r \ \in \mathbb{N}.$$

By the application of (2.1) and Lemma 1.3, we have

$$\begin{aligned}
\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) &\geq \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma) \geq \Theta_{fb}\left(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\right) \\
&\geq \mathcal{H}_{F_{\Theta_{fb}}}\left(\Xi\psi_{r-2}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\right) \\
&\geq \Theta_{fb}\left(\psi_{r-2}, \psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))\beta(\Theta_{fb}(\psi_{r-2}, \psi_{r-1}, \gamma))}\right) \\
&\vdots \\
&\geq \mathcal{H}_{F_{\Theta_{fb}}}\left(\Xi\psi_0, \Xi\psi_1, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))\beta(\Theta_{fb}(\psi_{r-2}, \psi_{r-1}, \gamma))\ldots\beta(\Theta_{fb}(\psi_1, \psi_2, \gamma))}\right) \\
&\geq \Theta_{fb}\left(\psi_0, \psi_1, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))\beta(\Theta_{fb}(\psi_{r-2}, \psi_{r-1}, \gamma))\ldots\beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right).
\end{aligned} \tag{2.2}$$

For any $q \in \mathbb{N}$, writing $q(\frac{\gamma}{q}) = \frac{\gamma}{q} + \frac{\gamma}{q} + \ldots + \frac{\gamma}{q}$ and using $[Fb4]$ repeatedly,

$$\begin{aligned}
&\Theta_{fb}(\psi_r, \psi_{r+q}, \gamma) \\
&\geq \Theta_{fb}\left(\psi_r, \psi_{r+1}, \frac{\gamma}{qb}\right) * \Theta_{fb}\left(\psi_{r+1}, \psi_{r+2}, \frac{\gamma}{qb^2}\right) * \Theta_{fb}\left(\psi_{r+2}, \psi_{r+3}, \frac{\gamma}{qb^3}\right) * \ldots * \Theta_{fb}\left(\psi_{r+q-1}, \psi_{r+q}, \frac{\gamma}{qb^q}\right).
\end{aligned}$$

Using (2.2) and $[Fb5]$, we get

$$
\begin{aligned}
&\Theta_{fb}(\psi_r, \psi_{r+q}, \gamma) \\
&\geq \Theta_{fb}\left(\psi_0, \psi_1, \frac{\gamma}{qb\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))\beta(\Theta_{fb}(\psi_{r-2}, \psi_{r-1}, \gamma)) \ldots \beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right) \\
&* \Theta_{fb}\left(\psi_0, \psi_1, \frac{\gamma}{qb^2\beta(\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma))\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma)) \ldots \beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right) \\
&* \Theta_{fb}\left(\psi_0, \psi_1, \frac{\gamma}{qb^3\beta(\Theta_{fb}(\psi_{r+1}, \psi_{r+2}, \gamma))\beta(\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma)) \ldots \beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right) * \ldots * \\
&\Theta_{fb}\left(\psi_0, \psi_1, \frac{\gamma}{qb^q\beta(\Theta_{fb}(\psi_{r+q-2}, \psi_{r+q-1}, \gamma))\beta(\Theta_{fb}(\psi_{r+q-3}, \psi_{r+q-2}, \gamma)) \ldots \beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\Theta_{fb}(\psi_r, \psi_{r+q}, \gamma) \\
&\geq \Theta_{fb}\left(\psi_0, \psi_1, \frac{b^{r-1}\gamma}{q}\right) * \Theta_{fb}\left(\psi_0, \psi_1, \frac{b^{r-1}\gamma}{q}\right) * \Theta_{fb}\left(\psi_0, \psi_1, \frac{b^{r-1}\gamma}{q}\right) * \ldots * \Theta_{fb}\left(\psi_0, \psi_1, \frac{b^{r-1}\gamma}{q}\right).
\end{aligned}
$$

Taking limit as $r \to \infty$, we get

$$
\lim_{n\to\infty} \Theta_{fb}(\psi_r, \psi_{r+q}, \gamma) = 1 * 1 * \ldots * 1 = 1.
$$

Hence, $\{\psi_r\}$ is G-Cauchy sequence. By the G-completeness of $\mho$, there exists $\phi \in \mho$ such that

$$
\begin{aligned}
\Theta_{fb}(\phi, \Xi\phi, \gamma) &\geq \Theta_{fb}\left(\phi, \psi_{r+1}, \frac{\gamma}{2b}\right) * \Theta_{fb}\left(\psi_{r+1}, \Xi\phi, \frac{\gamma}{2b}\right) \\
&\geq \Theta_{fb}\left(\phi, \psi_{r+1}, \frac{\gamma}{2b}\right) * \mathcal{H}_{F_{\Theta_{fb}}}\left(\Xi\psi_r, \Xi\phi, \frac{\gamma}{2b}\right) \\
&\geq \Theta_{fb}\left(\phi, \psi_{r+1}, \frac{\gamma}{2b}\right) * \Theta_{fb}\left(\psi_r, \phi, \frac{\gamma}{2b\beta(\Theta_{fb}(\psi_r, \phi, \gamma))}\right) \\
&\longrightarrow 1 \quad \text{as} \quad r \to \infty.
\end{aligned}
$$

By Lemma 1.1, it follows that $\phi \in \Xi\phi$. That is, ϕ is a fixed point for Ξ. □

Remark 2.1.

(1) If we take $\beta(\Theta_{fb}(\psi, \varrho, \gamma)) = k$ with $bk < 1$, we get Theorem 3.1 of [31].
(2) By setting $\hat{C}_0(\mho) = \mho$ the mapping $\Xi\colon \mho \to \hat{C}_0(\mho)$ becomes a single valued and we get Theorem 3.1 of [32]. Notice that when Ξ is a singlevalued map, $\Xi\psi$ becomes a singleton set and the fact that $\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \gamma) = \Theta_{fb}(\Xi\psi, \Xi\varrho, \gamma)$ indicates that the fixed point will be unique as proved in [32].
(3) Set $b = 1$ and $\hat{C}_0(\mho) = \mho$ and let $k \in (0, 1)$ be such that $\beta(\Theta_{fb}(\psi, \varrho, \gamma)) = k$ then we get the main result of [15].

The next example illustrates Theorem 2.1.

Example 2.1. Let $\mho = [0, 1]$ and define a G-complete Fbms by

$$\Theta_{fb}(\psi, \varrho, \gamma) = \frac{\gamma}{\gamma + (\psi - \varrho)^2},$$

with $b \geq 1$. For $\beta \in \mathbb{F}_b$, define a mapping $\Xi\colon \mho \to \hat{C}_0(\mho)$ by

$$\Xi\psi = \begin{cases} \{0\} & \text{if } \psi = 0, \\ \{0, \frac{\sqrt{\beta(\Theta_{fb}(\psi,\varrho,\gamma))}\psi}{2}\} & \text{otherwise.} \end{cases}$$

For $\psi = \varrho$,

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) = 1 = \Theta_{fb}(\psi, \varrho, \gamma).$$

If $\psi \neq \varrho$, then following cases arise.

For $\psi = 0$ and $\varrho \in (0, 1]$, we have

$$\begin{aligned}
&\mathcal{H}_{F_{\Theta_{fb}}}(\Xi 0, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) \\
&= \min\{ \inf_{a\in\Xi 0} F_{\Theta_{fb}}(a, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma), \inf_{b\in\Xi\varrho} F_{\Theta_{fb}}(\Xi 0, b, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma))\} \\
&= \min\left\{ \inf_{a\in\Xi 0} F_{\Theta_{fb}}\left(a, \left\{0, \frac{\sqrt{\beta(\Theta_{fb}(\psi, \varrho, \gamma))}\varrho}{2}\right\}, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma\right), \inf_{b\in\Xi\varrho} F_{\Theta_{fb}}\left(\{0\}, b, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma\right) \right\} \\
&= \min\left\{ \inf\left\{ F_{\Theta_{fb}}\left(0, \{0, \frac{\sqrt{\beta(\Theta_{fb}(\psi, \varrho, \gamma))}\varrho}{2}\}, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma\right)\right\}, \right. \\
&\qquad \left. \inf\left\{ F_{\Theta_{fb}}\left(\{0\}, 0, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma\right), F_{\Theta_{fb}}\left(\{0\}, \frac{\sqrt{\beta(\Theta_{fb}(\psi, \varrho, \gamma))}\varrho}{2}, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma\right)\right\}\right\} \\
&= \min\left\{ \inf\left\{ \sup\left\{ F_{\Theta_{fb}}(0, 0, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma), F_{\Theta_{fb}}(0, \frac{\sqrt{\beta(\Theta_{fb}(\psi, \varrho, \gamma))}\varrho}{2}, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma)\right\}\right\}, \right. \\
&\qquad \left. \inf\left\{ F_{\Theta_{fb}}(0, 0, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma), F_{\Theta_{fb}}(0, \frac{\sqrt{\beta(\Theta_{fb}(\psi, \varrho, \gamma))}y}{2}, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma)\right\}\right\} \\
&= \min\left\{ \inf\left\{ \sup\left\{ 1, \frac{\gamma}{\gamma + \frac{\varrho^2}{4}} \right\}\right\}, \inf\left\{ 1, \frac{\gamma}{\gamma + \frac{\varrho^2}{4}} \right\}\right\} \\
&= \min\left\{ \inf\{1\}, \frac{\gamma}{\gamma + \frac{\varrho^2}{4}} \right\} = \min\left\{ 1, \frac{\gamma}{\gamma + \frac{\varrho^2}{4}} \right\} = \frac{\gamma}{\gamma + \frac{\varrho^2}{4}}.
\end{aligned}$$

It follows that

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi 0, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) > \Theta_{fb}(0, \varrho, \gamma) = \frac{\gamma}{\gamma + \varrho^2}.$$

For ψ and $\varrho \in (0, 1]$, after simplification we have

$$\mathcal{H}_{F_{\Theta_{fb}}}(S(\psi), \Xi\varrho, \beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) = \min\left\{\sup\left\{\frac{\gamma}{\gamma+\frac{\psi^2}{4}}, \frac{\gamma}{\gamma+\frac{(\psi-\varrho)^2}{4}}\right\}, \sup\left\{\frac{\gamma}{\gamma+\frac{\varrho^2}{4}}, \frac{\gamma}{\gamma+\frac{(\psi-\varrho)^2}{4}}\right\}\right\}$$
$$\geq \frac{\gamma}{\gamma+\frac{(\psi-\varrho)^2}{4}} > \frac{\gamma}{\gamma+(\psi-\varrho)^2} = \Theta_{fb}(\psi,\varrho,\gamma).$$

Thus, for all cases, we have

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) \geq \Theta_{fb}(\psi,\varrho,\gamma).$$

Since all conditions of Theorem 2.1 are satisfied and 0 is a fixed point of Ξ.

Theorem 2.2. *Let $(\mho, \Theta_{fb}, *)$ be a G-complete Fbms with $b \geqslant 1$ and $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ be a Hfbms. Let $\Xi\colon \mho \to \hat{C}_0(\mho)$ be a Mvp satisfying*

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) \geq \min\left\{\frac{F_{\Theta_{fb}}(\varrho, \Xi\varrho, \gamma)\left[1+F_{\Theta_{fb}}(\psi,\Xi\psi,\gamma)\right]}{1+\Theta_{fb}(\psi,\varrho,\gamma)}, \Theta_{fb}(\psi,\varrho,\gamma)\right\}, \tag{2.3}$$

for all $\psi, \varrho \in \mho$, where $\beta \in \mathbb{F}_{\Theta_{fb}}$ as given in (1.1)*. Then Ξ has a fixed point.*

Proof. Choose $\{\psi_n\}$ for $\psi_0 \in \mho$ as follows: Let $\psi_1 \in \mho$ such that $\psi_1 \in \Xi\psi_0$. By the application of Lemma 1.3 we can choose $\psi_2 \in \Xi\psi_1$ such that

$$\Theta_{fb}(\psi_1,\psi_2,\gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0, \Xi\psi_1, \gamma), \quad \forall\, \gamma > 0.$$

By induction, we have $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma), \quad \forall\, r \in \mathbb{N}.$$

By the application of (2.3) and Lemma 1.3 we have

$$\begin{aligned}
&\Theta_{fb}(\psi_r,\psi_{r+1},\gamma) \geq \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma)\\
&\geq \min\Bigg\{\frac{F_{\Theta_{fb}}\left(\psi_r, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\left[1+F_{\Theta_{fb}}\left(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\right]}{1+\Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)},\\
&\qquad \Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\Bigg\}\\
&\geq \min\Bigg\{\frac{\Theta_{fb}\left(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\left[1+\Theta_{fb}\left(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\right]}{1+\Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)},\\
&\qquad \Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\Bigg\},
\end{aligned}$$

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma) \geq \min\left\{\Theta_{fb}\left(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right),\Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\right\}. \tag{2.4}$$

If

$$\min\left\{\Theta_{fb}\left(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right),\Theta_{fb}\left(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right)\right\}$$
$$= \Theta_{fb}\left(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\right),$$

then (2.4) implies

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma) \geq \Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big).$$

The result is obvious by Lemma 1.2.

If

$$\min\left\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\right\}$$
$$= \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),$$

then from (2.4) we have

$$\begin{aligned}
\Theta_{fb}(\psi_r,\psi_{r+1},\gamma) &\geq \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\\
&\geq \Theta_{fb}\Big(\psi_{r-2},\psi_{r-1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))\beta(\Theta_{fb}(\psi_{r-2},\psi_{r-1},\gamma))}\Big)\\
&\vdots\\
&\geqslant \Theta_{fb}\Big(\psi_0,\psi_1,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))\beta(\Theta_{fb}(\psi_{r-2},\psi_{r-1},\gamma))\ldots\beta(\Theta_{fb}(\psi_0,\psi_1,\gamma)}\Big).
\end{aligned}$$

The rest of the proof can be done by proceeding same as in Theorem 2.1. □

Remark 2.2.

(1) If we take $\beta(\Theta_{fb}(\psi,\varrho,\gamma)) = k$ with $bk < 1$, we get Theorem 3.2 of [31].
(2) By taking $b = 1$ and for some $0 < k < 1$ setting $\beta(\Theta_{fb}(\psi,\varrho,\gamma)) = k$ in Theorem 2.2, we get the main result of [30].

Theorem 2.3. *Let $(\mho,\Theta_{fb},*)$ be a G-complete Fbms with $b \geqslant 1$ and $(\hat{C}_0(\mho),\mathcal{H}_{F_{\Theta_{fb}}},*)$ be a Hfbms. Let $\Xi\colon \mho \to \hat{C}_0(\mho)$ be a Mvp satisfying*

$$\begin{aligned}
&\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi,\Xi\varrho,\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma)\\
&\geq \min\left\{\frac{F_{\Theta_{fb}}(\varrho,\Xi\varrho,\gamma)\left[1+F_{\Theta_{fb}}(\psi,\Xi\psi,\gamma)+F_{\Theta_{fb}}(\varrho,\Xi\psi,\gamma)\right]}{2+\Theta_{fb}(\psi,\varrho,\gamma)},\Theta_{fb}(\psi,\varrho,\gamma)\right\}
\end{aligned} \tag{2.5}$$

for all $\psi,\varrho \in \mho$, where $\beta \in \mathbb{F}_{\Theta_{fb}}$, the class of functions defined in (1.1)*. Then Ξ has a fixed point.*

Proof. Choose $\{\psi_n\}$ for $\psi_0 \in \mho$ as follows: Let $\psi_1 \in \mho$ such that $\psi_1 \in \Xi\psi_0$. by the application of Lemma 1.3 we can choose $\psi_2 \in \Xi\psi_1$ such that

$$\Theta_{fb}(\psi_1, \psi_2, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0, \Xi\psi_1, \gamma), \quad \forall\, \gamma > 0.$$

By induction, we have $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma), \quad \forall\, r \in \mathbb{N}.$$

By the application of (2.5) and Lemma 1.3, we have

$$\begin{aligned}
&\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geq \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma)\\
&\geq \min\left\{\frac{F_{\Theta_{fb}}(\psi_r, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[1 + F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}) + F_{\Theta_{fb}}(\psi_r, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{2 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},\right.\\
&\qquad\left.\Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\right\}\\
&\geq \min\left\{\frac{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[1 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}) + \Theta_{fb}(\psi_r, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{2 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},\right.\\
&\qquad\left.\Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\\
&\geq \min\left\{\frac{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[1 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}) + 1\right]}{2 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})}, \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\\
&\geq \min\left\{\frac{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[2 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{2 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})}, \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\\
&\geq \min\left\{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}), \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}.
\end{aligned} \tag{2.6}$$

If

$$\begin{aligned}
&\min\left\{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}), \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\\
&= \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big),
\end{aligned}$$

then (2.6) implies

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geq \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big),$$

and the proof follows by Lemma 1.2.

If

$$\begin{aligned}
&\min\left\{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}), \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\\
&= \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).
\end{aligned}$$

Then from (2.6) we have

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geq \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})$$
$$\geqslant \ldots \geqslant \Theta_{fb}\left(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))\beta(\Theta_{fb}(\psi_{r-2}, \psi_{r-1}, \gamma)) \ldots \beta(\Theta_{fb}(\psi_0, \psi_1, \gamma))}\right).$$

The rest of the proof is same as in Theorem 2.1. □

Remark 2.3. Theorem 3.3 of [31] becomes a special csae of the above theorem by setting $\beta(\Theta_{fb}(\psi, \varrho, \gamma)) = k$ where k is chosen such that $bk < 1$.

Theorem 2.4. *Let $(\mho, \Theta_{fb}, *)$ be a G-complete Fbms with $b \geqslant 1$ and $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ be a Hfbms. Let $\Xi\colon \mho \to \hat{C}_0(\mho)$ be a multivalued mapping satisfying*

$$\begin{aligned}&\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma)\\ &\geq \min\left\{\frac{F_{\Theta_{fb}}(\psi, \Xi\psi, \gamma)\left[1 + F_{\Theta_{fb}}(\varrho, \Xi\varrho, \gamma)\right]}{1 + F_{\Theta_{fb}}(\Xi\psi, \Xi\varrho, \gamma)}, \frac{F_{\Theta_{fb}}(\psi, \Xi\varrho, \gamma)\left[1 + F_{\Theta_{fb}}(\psi, \Xi\psi, \gamma)\right]}{1 + \Theta_{fb}(\psi, \varrho, \gamma)},\right.\\ &\left.\frac{F_{\Theta_{fb}}(\psi, \Xi\psi, \gamma)\left[2 + F_{\Theta_{fb}}(\psi, \Xi\varrho, \gamma)\right]}{1 + \Theta_{fb}(\psi, \Xi\varrho, \gamma) + F_{\Theta_{fb}}(\varrho, \Xi\psi, \gamma)}, \Theta_{fb}(\psi, \varrho, \gamma)\right\},\end{aligned} \tag{2.7}$$

for all $\psi, \varrho \in \mho$, where $\beta \in \mathbb{F}_{fb}$, the class of functions defined in (1.1). *Then Ξ has a fixed point.*

Proof. In the same way as Theorem 2.1, we have

$$\Theta_{fb}(\psi_1, \psi_2, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0, \Xi\psi_1, \gamma), \quad \forall\, \gamma > 0.$$

By induction, we obtain $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma), \quad \forall\, n\ \in \mathbb{N}.$$

Now, by (2.7) together with Lemma 1.3, we have

$$\begin{aligned}\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) &\geq \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma)\\ &\geq \min\left\{\frac{F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[1 + F_{\Theta_{fb}}(\psi_r, S_{\psi_r}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{1 + F_{\Theta_{fb}}(\Xi\psi_{r-1}, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},\right.\\ &\frac{F_{\Theta_{fb}}(\psi_r, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[1 + F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{1 + \Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},\\ &\frac{F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\left[2 + F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right]}{1 + F_{\Theta_{fb}}(\psi_{r-1}, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}) + F_{\Theta_{fb}}(\psi_r, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},\\ &\left.\Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\right\}\end{aligned}$$

$$\geq \min\Bigg\{\frac{\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big[1+\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big]}{1+\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},$$
$$\frac{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big[1+\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big]}{1+\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},$$
$$\frac{\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big[2+\Theta_{fb}(\psi_{r-1},\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big]}{1+\Theta_{fb}(\psi_{r-1},\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})+\Theta_{fb}(\psi_r,\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})},$$
$$\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Bigg\},$$

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq \min\left\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right\}. \quad (2.8)$$

If

$$\min\left\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right\}$$
$$=\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),$$

then (2.8) implies

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big).$$

Then the proof follows by Lemma 1.2.

If

$$\min\left\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\right\}$$
$$=\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),$$

then from (2.6) we have

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)$$
$$\vdots$$
$$\geqslant\Theta_{fb}\Big(\psi_0,\psi_1,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))\beta(\Theta_{fb}(\psi_{r-2},\psi_{r-1},\gamma))\ldots\beta(\Theta_{fb}(\psi_0,\psi_1,\gamma))}\Big).$$

The rest of the proof is similar as in Theorem 2.1. □

Remark 2.4. Again by taking $\beta(\Theta_{fb}(\psi,\varrho,\gamma))=k$ with $kb<1$, we get Theorem 3.4 of [31].

Theorem 2.5. *Let* $(\mho,\Theta_{fb},*)$ *be a G-complete Fbms with* $b\geqslant 1$ *and* $(\hat{C}_0(\mho),\mathcal{H}_{F_{\Theta_{fb}}},*)$ *be a Hfbms. Let* $\Xi\colon \mho\to\hat{C}_0(\mho)$ *be a Mvp satisfying*

$$\begin{aligned}&\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi,\Xi\varrho,\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma)\\&\geq \frac{\min\{\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi,\Xi\varrho,\gamma).\Theta_{fb}(\psi,\varrho,\gamma),\mathcal{H}_{F_{\Theta_{fb}}}(\psi,\Xi\psi,\gamma).\mathcal{H}_{F_{\Theta_{fb}}}(\varrho,\Xi\varrho,\gamma)\})}{\max\{\mathcal{H}_{F_{\Theta_{fb}}}(\psi,\Xi\psi,\gamma),\mathcal{H}_{F_{\Theta_{fb}}}(\varrho,\Xi\varrho,\gamma)\}},\end{aligned}\tag{2.9}$$

for all $\psi,\varrho\in\mho$*, where* $\beta\in\mathbb{F}_{fb}$*. Then* Ξ *has a fixed point.*

Proof. In the same way as Theorem 2.1, we have

$$\Theta_{fb}(\psi_1,\psi_2,\gamma)\geqslant\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0,\Xi\psi_1,\gamma),\quad\forall\,\gamma>0.$$

By induction we have $\psi_{r+1}\in\Xi\psi_r$ satisfying

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geqslant\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1},\Xi\psi_r,\gamma),\quad\forall\,n\ \in\mathbb{N}.$$

Now by (2.7) together with Lemma 1.3 and some obvious simplification step, we have

$$\begin{aligned}&\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1},\Xi\psi_r,\gamma)\\&\geq\frac{\min\Big\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}).\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}).\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\Big\}}{\max\{\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\}}\\&\geq\frac{\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}).\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})}{\max\{\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\}}\end{aligned}\tag{2.10}$$

If

$$\begin{aligned}&\max\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\}\\&=\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\end{aligned}$$

then (2.10) implies

$$\Theta_{fb}(\psi_r,\psi_{r+1},t)\geq\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)$$

Then the proof follows by Lemma 1.2.

If

$$\begin{aligned}&\max\{\Theta_{fb}(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}),\Theta_{fb}(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})\}\\&=\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\end{aligned}$$

then from (2.10) we have

$$\begin{aligned}\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)&\geq\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\\&\ \vdots\\&\geqslant\Theta_{fb}\Big(\psi_0,\psi_1,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))\beta(\Theta_{fb}(\psi_{r-2},\psi_{r-1},\gamma))\ldots\beta(\Theta_{fb}(\psi_0,\psi_1,\gamma))}\Big).\end{aligned}$$

The remaining proof follows in the same way as in Theorem 2.1. □

Theorem 2.6. *Let* $(\mho, \Theta_{fb}, *)$ *be a G-complete Fbms with* $b \geqslant 1$ *and* $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ *be a Hfbms. Let* $\Xi\colon \mho \to \hat{C}_0(\mho)$ *be a Mvp satisfying*

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) \geq \Gamma_1(\psi,\varrho,\gamma) * \Gamma_2(\psi,\varrho,\gamma), \tag{2.11}$$

where,

$$\left.\begin{cases}\Gamma_1(\psi,\varrho,\gamma) & = \min\{\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi,\Xi\varrho,\gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\psi,\Xi\psi,\gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\varrho,\Xi\varrho,\gamma), \Theta_{fb}(\psi,\varrho,\gamma)\} \\ \Gamma_2(\psi,\varrho,\gamma) & = \max\{\mathcal{H}_{F_{\Theta_{fb}}}(\psi,\Xi\varrho,\gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi,\varrho,\gamma)\}\end{cases}\right\}, \tag{2.12}$$

for all $\psi, \varrho \in \mho$, *and* $\beta \in \mathbb{F}_{fb}$. *Then* Ξ *has a fixed point.*

Proof. In the same way as Theorem 2.1, we have

$$\Theta_{fb}(\psi_1,\psi_2,\gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0,\Xi\psi_1,\gamma), \quad \forall\, \gamma > 0.$$

By induction we have $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\begin{aligned}\mathcal{H}_{F_{\Theta_{fb}}}(\psi_r,\psi_{r+1},\gamma) &= F_\theta(\Xi\psi_{r-1},\Xi\psi_r,\gamma) \\ &\geq \Gamma_1\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big) * \Gamma_2\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\end{aligned} \tag{2.13}$$

Now,

$$\begin{aligned}&\Gamma_1\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big) \\ &= \min\Big\{\mathcal{H}_{F_{\Theta_{fb}}}\Big(\Xi\psi_{r-1},\Xi\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_{r-1},\Xi\psi_{r-1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \\ &\qquad \mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_r,\Xi\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\} \\ &= \min\Big\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \\ &\qquad \Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}.\end{aligned}$$

$$\begin{aligned}&\Gamma_1(\psi_{r-1},\psi_{r-1},\frac{\gamma}{\beta(\Theta_{fb}\big(\psi_{r-1},\psi_r,\gamma))}) \\ &= \min\Big\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}.\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\Gamma_2\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big) \\ &= \max\Big\{\mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_{r-1},\Xi\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big), \mathcal{H}_{F_{\Theta_{fb}}}\Big(\Xi\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}\end{aligned}$$

$$= \max\Big\{\Theta_{fb}\Big(\psi_{r-1},\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_r,\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}$$
$$= \max\Big\{\Theta_{fb}\Big(\psi_{r-1},\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),1\Big\}.$$

$$\Gamma_2\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)=1. \tag{2.15}$$

Using (2.14) and (2.15) in (2.13) we have

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq \min\Big\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}*1,$$
$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq \min\Big\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}. \tag{2.16}$$

If

$$\min\left\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\right\}$$
$$=\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),$$

then (2.16) implies

$$\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)\geq\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)$$

Then the proof follows by Lemma 1.2

If

$$\min\Big\{\Theta_{fb}\Big(\psi_r,\psi_{r+1},\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\Big\}$$
$$=\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big),$$

then from (2.16), we have

$$\begin{aligned}\Theta_{fb}(\psi_r,\psi_{r+1},\gamma)&\geq\Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}\Big)\\ &\vdots\\ &\geqslant\Theta_{fb}\Big(\psi_0,\psi_1,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))\beta(\Theta_{fb}(\psi_{r-2},\psi_{r-1},\gamma))\ldots\beta(\Theta_{fb}(\psi_0,\psi_1,\gamma))}\Big).\end{aligned}$$

The remaining proof is similar as in Theorem 2.1. □

Remark 2.5. If we set $\hat{C}_0(\mho)=\mho$ the map Ξ becomes a singlevalued and we get Theorem 3.11 of [32]. Again as stated in Remark 2.1, the corresponding fixed point will be unique.

Theorem 2.7. *Let $(\mho, \Theta_{fb}, *)$ be a G-complete Fbms with $b \geqslant 1$ and $(\hat{C}_0(\mho), \mathcal{H}_{F_{\Theta_{fb}}}, *)$ be a Hfbms. Let $\Xi\colon \mho \to \hat{C}_0(\mho)$ be a Mvp satisfying*

$$\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \beta(\Theta_{fb}(\psi, \varrho, \gamma))\gamma) \geq \frac{\Gamma_1(\psi, \varrho, \gamma) * \Gamma_2(\psi, \varrho, \gamma)}{\Gamma_3(\psi, \varrho, \gamma)}, \tag{2.17}$$

where

$$\begin{cases} \Gamma_1(\psi, \varrho, \gamma) & = \min\{\mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi, \Xi\varrho, \gamma).\Theta_{fb}(\psi, \varrho, \gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\psi, \Xi\psi, \gamma).\mathcal{H}_{F_{\Theta_{fb}}}(\varrho, \Xi\varrho, \gamma)\} \\ \Gamma_2(\psi, \varrho, \gamma) & = \max\{\mathcal{H}_{F_{\Theta_{fb}}}(\psi, \Xi\psi, \gamma).\mathcal{H}_{F_{\Theta_{fb}}}(\psi, \Xi\varrho, \gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\varrho, \Xi\psi, \gamma))^2\} \\ \Gamma_3(\psi, \varrho, \gamma) & = \max\{\mathcal{H}_{F_{\Theta_{fb}}}(\psi, \Xi\psi, \gamma), \mathcal{H}_{F_{\Theta_{fb}}}(\varrho, \Xi\varrho, \gamma)\} \end{cases}, \tag{2.18}$$

for all $\psi, \varrho \in \mho$, and $\beta \in \mathbb{F}_{fb}$. Then Ξ has a fixed point.

Proof. In the same way as Theorem 2.1, we have

$$\Theta_{fb}(\psi_1, \psi_2, \gamma) \geqslant \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_0, \Xi\psi_1, \gamma), \quad \forall\, \gamma > 0.$$

By induction we have $\psi_{r+1} \in \Xi\psi_r$ satisfying

$$\begin{aligned} \Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) &= \mathcal{H}_{F_{\Theta_{fb}}}(\Xi\psi_{r-1}, \Xi\psi_r, \gamma) \\ &\geq \frac{\Gamma_1(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))}) * \Gamma_2(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})}{\Gamma_3(\psi, \varrho, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma))})}. \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\Gamma_1\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big) \\ &= \min\Big\{\mathcal{H}_{F_{\Theta_{fb}}}\Big(\Xi\psi_{r-1}, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).F_\theta\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \\ &\qquad \mathcal{H}_{F_{\Theta_{fb}}}(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}).\mathcal{H}_{F_{\Theta_{fb}}}(\psi_r, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\Big\} \\ &= \min\Big\{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}).\Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}), \\ &\qquad \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).\Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big\} \\ &= \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).\Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big). \end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned} &\Gamma_2\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big) \\ &= \max\Big\{\mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_{r-1}, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \end{aligned}$$

$$\Big(\mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_r, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big)^2\Big\}$$

$$= \max\Big\{\Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1}, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big),$$

$$\Big(\Theta_{fb}\Big(\psi_r, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big)^2\Big\}$$

$$= \max\Big\{\Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1}, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), 1\Big\}.$$

It follows that

$$\Gamma_2\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big) = 1. \tag{2.21}$$

$$\Gamma_3\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)$$

$$= \max\Big\{\mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_{r-1}, \Xi\psi_{r-1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \mathcal{H}_{F_{\Theta_{fb}}}\Big(\psi_r, \Xi\psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big\}$$

$$= \max\Big\{\Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big\}. \tag{2.22}$$

Using (2.20), (2.21) and (2.22) in (2.19), we have

$$\Theta_{fb}(\psi_r, \psi_{r+1}, t) \geq \frac{\Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}).\Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})}{\max\{\Theta_{fb}(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}), \Theta_{fb}(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))})\}}. \tag{2.23}$$

If

$$\max\Big\{\Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big\}$$

$$= \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big),$$

then (2.23) implies

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geq \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).$$

It is obvious by Lemma 1.2.

If

$$\max\Big\{\Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big), \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big)\Big\}$$

$$= \Theta_{fb}\Big(\psi_r, \psi_{r+1}, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big),$$

then from (2.23), we have

$$\Theta_{fb}(\psi_r, \psi_{r+1}, \gamma) \geq \Theta_{fb}\Big(\psi_{r-1}, \psi_r, \frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1}, \psi_r, \gamma))}\Big).$$

Continuing in this way, we will get

$$\begin{aligned}\Theta_{fb}(\psi_r,\psi_{r+1},t) &\geq \Theta_{fb}\Big(\psi_{r-1},\psi_r,\frac{\gamma}{\beta(F_\theta(\psi_{r-1},\psi_r,\gamma))}\Big)\\ &\vdots\\ &\geq \Theta_{fb}\Big(\psi_0,\psi_1,\frac{\gamma}{\beta(\Theta_{fb}(\psi_{r-1},\psi_r,\gamma)).\beta(\Theta_{fb}(\psi_{n-2},\psi_{r-1},\gamma))\dots\beta(\Theta_{fb}(\psi_0,\psi_1,\gamma))}\Big).\end{aligned}$$

The rest of the proof follows in the same way as in Theorem 2.1. □

Remark 2.6. By setting $\hat{C}_0(\mho) = \mho$, the mapping $\Xi\colon \mho \to \hat{C}_0(\mho)$ becomes a self (singlevalued) mapping and we get Theorem 3.13 of [32].

3. Application

An application of Theorem 2.1 is presented here. Recall that the space of all continuous realvalued functions on $[0,1]$ is denoted by $C([0,1],\mathbb{R})$. Now set $\mho = C([0,1],\mathbb{R})$ and define the G-complete Fbm on $\mho$ by

$$\Theta_{fb}(\psi,\varrho,\gamma) = e^{-\frac{\sup_{u\in[0,1]}|\psi(u)-\varrho(u)|^2}{\gamma}}, \quad \forall\ \gamma > 0 \text{ and } \psi,\varrho \in \mho.$$

Consider

$$\psi(u) \in \int_0^u G(u,v,\psi(v))dv + h(u) \quad \text{for all } u,v \in [0,1], \quad \text{whereas} \quad h,\psi \in C([0,1],\mathbb{R}). \tag{3.1}$$

Here $G\colon [0,1]\times[0,1]\times\mathbb{R} \to P_{cv}(\mathbb{R})$ is multivalued function and $P_{cv}(\mathbb{R})$ represents the collections of convex and compact subsets of $\mathbb{R}$. Moreover, for each ψ in $C([0,1],\mathbb{R})$ the operator $G(\cdot,\cdot,\psi)$ is lower semi-continuous.

For the integral inclusion given in (3.1), define a multivalued operator $S : \mho \to \hat{C}_0(\mho)$ by

$$S\psi(u) = \Big\{w \in \mho : w \in \int_0^u G(u,v,\psi(v))dv + h(u), \quad u \in [0,1]\Big\}.$$

Now for arbitrary $\psi \in (C([0,1],\mathbb{R})$, denote $G_\psi(u,v) = G(u,v,\psi(v))$ where $u,v \in [0,1]$. For the multivalued map $G_\psi : [0,1]\times[0,1] \to P_{cv}(\mathbb{R})$, by Michael selection theorem [34], there exists a continuous selection $g_\psi : [0,1]\times[0,1] \to \mathbb{R}$ such that $g_\psi(u,v) \in G_\psi(u,v)$ for each $u,v \in [0,1]$. It follows that

$$\int_0^u g_\psi(u,v)dv + h(u) \in S\psi(u).$$

Since g_ψ is continuous on $[0,1]\times[0,1]$ and h is continuous on $[0,1]$, therefore both g_ψ and h are bounded realvalued functions. It follows that, the operator $S\psi$ is nonempty and $S\psi \in \hat{C}_0(\mho)$.

With the above setting, the upcoming outcome shows the existence of a solution of the integral inclusion (3.1).

Theorem 3.1. *Let $\mho = C([0,1],\mathbb{R})$ and define the multivalued operator $S : \mho \to \hat{C}_0(\mho)$ by*

$$S\psi(u) = \left\{ w \in \mho : w \in \int_0^u G(u,v,\psi(v))dv + h(u), \quad u \in [0,1] \right\},$$

where $h : [0,1] \to \mathbb{R}$ is continuous and the map $G : [0,1] \times [0,1] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ is defined in such a way that for every $\psi \in C([0,1],\mathbb{R})$, the operator $G(\cdot,\cdot,\psi)$ is lower semi-continuous. Assume further that the given terms are satisfied:

(i) There exists a continuous mapping $f\colon [0,1] \times [0,1] \to [0,\infty)$ such that

$$\mathcal{H}_{F_{\Theta_{fb}}}(G(u,v,\psi(v)) - G(u,v,\varrho(v)) \leq f^2(u,v)|\psi(v) - \varrho(v)|^2,$$

for each $\psi, \varrho \in \mho$ and $u, v \in [0,1]$.

(ii) There exists $\beta \in \mathbb{F}_{\Theta 2}$, such that

$$\sup_{u \in [0,1]} \int_0^u f^2(u,v)dv \leq \beta(\Theta_{fb}(\psi,\varrho,\gamma)).$$

Then (3.1) *has a solution in* $\mho$.

Proof. We will show that the operator S satisfies the conditions of Theorem 2.1. In particular we prove (2.1) as follows:

Let $\psi, \varrho \in \mho$ be such that $q \in S\psi$. As stated earlier, by selection theorem there is $g_\psi(u,v) \in G_\psi(u,v) = G(u,v,\psi(v))$ for $u, v \in [0,1]$ such that

$$q(u) = \int_0^u g_\psi(u,v)dv + h(u), \quad u \in [0,1].$$

Further, the condition (i) ensures that there is some $g(u,v) \in G_\varrho(u,v)$ such that

$$|g_\psi(u,v) - g(u,v) \leq f^2(u,v)|\psi(v) - \varrho(v)|^2, \quad \forall\, u, v \in [0,1].$$

Now consider the multivalued operator T defined as follows:

$$T(u,v) = G_\varrho(u,v) \cap \left\{ w \in \mathbb{R} : \left| g_\psi(u,v) - w \right| \leq f^2(u,v)|\psi(v) - \varrho(v)|^2 \right\}.$$

Since, by construction, T is lower semi-continuous, it follows again by the selection theorem that there is continuous function $g_\varrho(u,v) : [0,1] \times [0,1] \to \mathbb{R}$ such that for each $u, v \in [0,1]$, $g_\varrho(u,v) \in T(u,v)$.

Thus, we have

$$r(u) = \int_0^u g_\varrho(u,v)dv + h(u) \in \int_0^u G(u,v,\varrho(v))dv + h(u), \quad u \in [0,1].$$

Therefore, for each $u \in [0,1]$ we get

$$
\begin{aligned}
e^{-\frac{\sup\limits_{t\in[0,1]}|q(u)-r(u))|^2}{\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma}} &\geq e^{-\frac{\sup\limits_{u\in[0,1]}\int_0^u |g_\psi(u,v)-g_\varrho(u,v)|^2dv}{\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma}} \\
&\geq e^{-\frac{\sup\limits_{u\in[0,1]}\int_0^u f^2(u,v)|\psi(v)-\varrho(v)|^2dv}{\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma}} \\
&\geq e^{-\frac{|\psi(v)-\varrho(v)|^2\sup\limits_{u\in[0,1]}\int_0^u f^2(u,v)dv}{\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma}} \\
&\geq e^{-\frac{\beta(\Theta_{fb}(\psi,\varrho,\gamma))|\psi(v)-\varrho(v)|^2}{\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma}} \\
&= e^{-\frac{|\psi(v)-\varrho(v)|^2}{\gamma}} \\
&\geq e^{-\frac{\sup\limits_{v\in[0,1]}|\psi(v)-\varrho(v)|^2}{\gamma}} \\
&= \Theta_{fb}(\psi,\varrho,\gamma).
\end{aligned}
$$

This implies that,

$$\Theta_{fb}(q,r,\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) \geq \Theta_{fb}(\psi,\varrho,\gamma).$$

Interchanging the roles of ψ and ϱ, we get

$$\mathcal{H}_{F_{\Theta_{fb}}}(S\psi,S\varrho,\beta(\Theta_{fb}(\psi,\varrho,\gamma))\gamma) \geq \Theta_{fb}(\psi,\varrho,\gamma).$$

Hence, by Theorem 2.1, the operator S has a fixed point which in turn proves the existence of a solution of integral inclusion (3.1). □

4. Conclusions

In the present work, in the setting of a Hausdorff Fbms, some fixed fixed point results for multivalued mappings are established. The main result, that is, Theorem 2.1 shows that a multivalued mapping satisfying Geraghty type contractions on G-complete Hfbms has a fixed point. Example 2.1 illustrates the main result. Some other interesting fixed point theorems are also proved for the multivalued mappings satisfying certain contraction condition on G-complete Hfbms. The results proved in [30–32] turn out to be special cases of the results established in this work. For the significance of our results, an application is presented to prove the existence of solution of an integral inclusion.

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Conflict of interest

The authors declare that they have no conflict of interest.

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